

On weighted Blaschke–Santaló and strong Brascamp–Lieb inequalities

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Abstract

In this paper, we study new extensions of the functional Blaschke–Santaló inequalities, and explore applications of such new inequalities beyond the classical setting of the standard Gaussian measure. In particular, we study functionals of the type

$$\left(\int_{\mathbb{R}^n} e^{-\Phi} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} e^{-\frac{1}{p-1} \Phi^*(\nabla V)} dx \right)^{\frac{p-1}{p}}, \quad (1)$$

where $p > 1$ and V is a p -homogeneous convex even function. The function Φ is assumed to be convex and even.

In particular, we prove that maximizers of (1) are p -homogeneous and solve an equation of the Monge–Ampère type appearing in the L^q -Minkowski problem. This gives a novel mass-transport approach to Blaschke–Santaló inequality, which is of independent interest even in the classical setting. We find sufficient conditions for $\Phi = V$ to be the maximizer of (1). In particular, these conditions are satisfied if $V = |x|_q^p$, where $p \geq q \geq 2$ and $|\cdot|_q$ is the l^q -norm. We prove that in general V fails to be the maximizer.

In addition, we prove that any maximizer of (1) satisfies a strong version of the Brascamp–Lieb inequality on the set of even functions. In particular, if V is the maximizer of (1), then $\mu = \frac{e^{-V} dx}{\int e^{-V} dx}$ satisfies the following strengthening of the Brascamp–Lieb inequality on the set of even functions:

$$\mathrm{Var}_\mu f \leq \lambda \int_{\mathbb{R}^n} \langle (D^2 V)^{-1} \nabla f, \nabla f \rangle d\mu \quad (2)$$

with the sharp value $\lambda = 1 - \frac{1}{p}$. We also estimate the best constant $\lambda < 1$ in (2) for probability measures of the form $\mu = C e^{-c|x|_q^p} dx$ for various values of p and q .

1 Introduction

The Blaschke–Santaló inequality

$$|K| |K^\circ| \leq |B_2^n|^2 \quad (3)$$

discovered in the beginning of XXth century by W. Blaschke in dimensions $n = 3$ and proved later by L. Santaló [60] for $n > 3$, is one of the fundamental and most celebrated results in convex geometry. Here $K \subset \mathbb{R}^n$ is a symmetric convex body, $|\cdot|$ is the n -dimensional volume, and

$$K^\circ = \{y : \langle x, y \rangle \leq 1 \ \forall x \in K\}$$

is the corresponding polar body. The notation B_p^n stands for the unit l^p -ball in \mathbb{R}^n :

$$B_p^n = \left\{ x : |x|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \leq 1 \right\}.$$

The standard l_p -norm of vector x is denoted by $|x|_p$. In the case $p = 2$ we omit subscript 2 and write $|x|$.

The classical proof of (3) goes via the Steiner symmetrization and the Brunn–Minkowski inequality see e.g. Artstein–Avidan, Giannopolous, Milman [1], Campi, Gronchi [17], Meyer, Pajor [53]. An important

feature of the *volume product* functional $|K||K^o|$, which appears on the left hand side of this inequality, is its affine invariance: for any linear transformation T on \mathbb{R}^n , we have

$$|K||K^o| = |T(K)||T(K)^o|.$$

A remarkable functional form of the Blaschke–Santaló inequality was discovered by K. Ball in [4]. Let Φ be an arbitrary proper even function on \mathbb{R}^n with values in $(-\infty, +\infty]$ and

$$\Phi^*(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - \Phi(x))$$

be its Legendre transform. Then

$$\int e^{-\Phi(x)} dx \int e^{-\Phi^*(y)} dy \leq (2\pi)^n. \quad (4)$$

The equality holds if and only if $\Phi = a + \langle Ax, x \rangle$ for some symmetric non-degenerate matrix A . This result was later generalized by Artstein-Avidan, Klartag, Milman [2], Fradelizi, Meyer [29], [30], [31], Lehec [49], Lin, Leng [50], Kolesnikov, Werner [48], Gozlan, Fradelizi, Sadovskiy, Zugmeyer [28]. Stability of the inequality has been studied by Böröczky [11]. A Fourier analytic proof is presented by Bianchi, Kelly [5]. A novel analytic approach (semigroups and a monotonicity property) to (4) is presented in the recent work of Nakamura and Tsuji [56]. We remark that both (3) and (4) have a conjectured reverse form, referred to as Mahler conjecture, which shall not be discussed in detail in this work. However, an intense research activity has been carried out in relation to this conjecture; we refer the interested reader to Fradelizi, Meyer, Zvavitch [32] and the references therein.

Recall that the standard Gaussian measure γ on \mathbb{R}^n is the measure with the density $(2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}}$. The inequality (4) states that the functional

$$\int e^{-\Phi(x)} dx \int e^{-\Phi^*(y)} dy$$

is maximized when Φ is the potential of a Gaussian measure (in particular, the standard one). Hence the inequality (4) seems like a purely Gaussian phenomenon. However, Fradelizi and Meyer [29] extended the formulation of (4) in a way so that other rotation-invariant measures appear as maximizers of similar inequalities.

In this paper, we study a different possibility to extend the Blaschke–Santaló inequality, with the goal of potentially getting a richer class of measures as maximizers of this type of functionals. Namely, we pose the following questions.

Question 1.1. *What are the maximizers for the generalized weighted Blaschke–Santaló functional*

$$\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi) = \left(\int e^{-\alpha\Phi(x)} \rho_1(x) dx \right)^{\frac{1}{\alpha}} \left(\int e^{-\beta\Phi^*(y)} \rho_2(y) dy \right)^{\frac{1}{\beta}}, \quad \alpha, \beta \in (0, +\infty),$$

with symmetric weights ρ_1, ρ_2 on the set of even functions? Of interest are existence, uniqueness and characterizations of the maximizers of $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}$, and especially situations when $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}$ admits a closed-form solution.

The setting of Question 1.1 is very general even for the problem of existence of maximizers. In the largest part of this work we deal mainly with a particular case of the generalized Blaschke–Santaló functional. Let $p > 1$ and let V be an even strictly convex p -homogeneous C^2 function on \mathbb{R}^n . We consider the functional

$$\mathcal{BS}_{p, V}(\Phi) = \int e^{-\Phi(x)} dx \left(\int e^{-\frac{1}{p-1}\Phi^*(\nabla V(y))} dy \right)^{p-1}.$$

Note that $\mathcal{BS}_{p,V}$ is a particular case of $\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}$, because after the change of variables $z = \nabla V(y)$ it can be rewritten as

$$\mathcal{BS}_{p,V}(\Phi) = \int e^{-\Phi(x)} dx \left(\int e^{-\frac{1}{p-1}\Phi^*(z)} \det(D^2V^*(z)) dz \right)^{p-1}.$$

A remarkable property of $\mathcal{BS}_{p,V}$ is the following homogeneity invariance:

$$\mathcal{BS}_{p,V}(\Phi(tx)) = \mathcal{BS}_{p,V}(\Phi(x))$$

for all Φ and $t > 0$. Another important property of $\mathcal{BS}_{p,V}$ is that the function V (more generally $V(tx)$) is a natural candidate to be a maximizer. Indeed, this hint comes from the observation that V satisfies the corresponding Euler–Lagrange equation for the minimization problem defining $\mathcal{BS}_{p,V}$ (see Proposition 7.2). Thus the following question appears naturally.

Question 1.2. *Let $p > 1$ and let V be an even strictly convex p -homogeneous C^2 function on \mathbb{R}^n . Under which conditions do we have*

$$\int e^{-\Phi(x)} dx \left(\int e^{-\frac{1}{p-1}\Phi^*(\nabla V(y))} dy \right)^{p-1} \leq \left(\int e^{-V(x)} dx \right)^p, \quad (5)$$

for an arbitrary proper convex even function Φ on \mathbb{R}^n with values in $(-\infty, +\infty]$?

As we shall explain in Section 6 (following the ideas of Artstein-Avidan, Klartag, Milman [2]), inequality (5) of Question 1.2 is equivalent to certain Blaschke–Santaló type inequality for sets.

Proposition 1.3. *Let $p > 1$ and let V be an even strictly convex p -homogeneous C^2 function on \mathbb{R}^n . Inequality (5) holds for arbitrary convex proper function Φ if and only if inequality*

$$|K| \cdot |\nabla V^*(K^\circ)|^{p-1} \leq \left| \left\{ V \leq \frac{1}{p} \right\} \right|^p \quad (6)$$

holds for arbitrary compact convex body K .

If inequality (6) holds, then equality is attained when K is a level set of V : $K = \{V \leq \alpha\}$.

Unfortunately, one can not hope for the affirmative answer to Question 1.2 in too great a generality. To see this, in view of Proposition 1.3, consider $V(x) = |x|_p^p$ with $p \in [1, 2]$. In this case, (6) states that the maximizer of the functional

$$|K| \left(\int_{K^\circ} \prod_{i=1}^n |x_i|^{\frac{2-p}{p-1}} dx \right)^{p-1}$$

among all symmetric convex sets is $K = \frac{1}{p} B_p^n$. In the limiting case $p \rightarrow 1$, this would be equivalent to the inequality

$$|K| \sup_{x \in K^\circ} \prod_{i=1}^n |x_i| \leq |B_1^n| \sup_{x \in B_\infty^n} \prod_{i=1}^n |x_i| = \frac{2^n}{n!}.$$

However, this is false: if we let $v = e_1 + \dots + e_n$, and $K_R^\circ = [-Rv, Rv] + B_2^n$ (which, for large R , is a long needle-like body pointing in the direction of v), then

$$\lim_{R \rightarrow \infty} \left(|K_R| \sup_{x \in K_R^\circ} \prod_{i=1}^n |x_i| \right) = \infty.$$

Remark 1.4. *It seems to be of independent interest to find minimizers and maximizers of the functional*

$$\inf_{T \in GL_n} \left(|T(K)| \left(\int_{[T(K)]^\circ} \prod_{i=1}^n |x_i|^{\frac{2-p}{p-1}} dx \right)^{p-1} \right),$$

since this may provide an interesting novel extension of the Blaschke–Santaló inequality which has not been previously studied.

In a similar manner, one may see that the lack of affine invariance of the functional $\mathcal{BS}_{p,V}$ makes (5) hopeless in many situations. See Section 5.4 for more details and other counterexamples. Nevertheless, in the following Theorem we were able to obtain some sufficient conditions for (5).

Theorem A. *Let $p > 1$ and let V be an even strictly convex p -homogeneous C^2 function on \mathbb{R}^n . Assume that V is an unconditional function, and that the function*

$$x = (x_1, \dots, x_n) \mapsto V\left(x_1^{\frac{1}{p}}, \dots, x_n^{\frac{1}{p}}\right)$$

is concave in

$$\mathbb{R}_+^n = \{(x_1, \dots, x_n) : x_i \geq 0 \forall i = 1, \dots, n\}.$$

Then inequality (5) holds for every unconditional convex Φ .

Assume, in addition, that for every coordinate hyperplane H , with unit normal e , and for every $x' \in H$, the function $\varphi: [0, +\infty) \rightarrow \mathbb{R}$ defined by

$$\varphi(t) = \det D^2 V^*(x' + te)$$

is decreasing. Then inequality (5) holds for every even convex Φ .

Corollary 1.5. *Let $V = c|x|_q^p$, $c \geq 0$. Then inequality (5) holds in the following cases:*

1. For $p \geq q > 1$ and unconditional Φ
2. For $p \geq q \geq 2$ and even Φ .

The proof of the “unconditional” part of Theorem A is based on the application of the Prékopa–Leindler inequality in the unconditional case via a change of variables on \mathbb{R}_+^n (see e.g. Fradelizi, Meyer [29] for another application of this idea). The general case follows by a symmetrization argument: we show that Steiner symmetrization increases the value of the functional and reduce the problem to the unconditional case.

In Section 5 we prove that under various assumptions of additional symmetries (for V and for Φ) the left hand side of our inequality admits a non-degenerate maximizer. See, for instance, Theorem 5.4 for the existence in the case of rotation-invariant weights while Φ is even, and Theorem 5.2 for the existence when both the weights and Φ are assumed to be 1-symmetric. In particular, in Section 4 we use symmetrization techniques to show existence and various properties of such maximizers.

From our perspective, one of the most exciting aspects of this work is the novel mass transport approach to proving the classical Blaschke–Santaló inequality. The starting point for our mass transport approach is an observation, going back to [48], that any maximizer Φ for the Blaschke–Santaló functional is a solution for a nonlinear PDE of the Monge–Ampère type.

Theorem 1.6 ([48]). *Let $\alpha, \beta > 0$, and let ρ_1, ρ_2 be positive even functions. Assume that Φ is a maximum point of the functional $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}$. Then $\nabla \Phi$ is the optimal transportation pushing forward μ onto ν , where*

$$d\mu = \frac{e^{-\alpha\Phi} \rho_1 dx}{\int e^{-\alpha\Phi} \rho_1 dx}, \quad d\nu = \frac{e^{-\beta\Phi^*} \rho_2 dy}{\int e^{-\beta\Phi^*} \rho_2 dy}.$$

This result implies, in particular, that Φ solves the following nonlinear PDE of the Monge–Ampère type.

$$\frac{e^{-\alpha\Phi}}{\int e^{-\alpha\Phi} \rho_1 dx} \rho_1 = \frac{e^{-\beta\Phi^*(\nabla\Phi)}}{\int e^{-\beta\Phi^*} \rho_2 dy} \rho_2(\nabla\Phi) \det D^2\Phi. \quad (7)$$

Of course, there is no hope to find a closed-form solution for this equation for general ρ_1, ρ_2 . However, in some particular cases there exists a natural candidate, as is demonstrated by our results. For the case $\alpha = \beta = 1, \rho_1 = \rho_2 = 1$ this equation was already studied in [48]. It was proved there that under additional

regularity assumptions all solutions to equation (7) are positive quadratic forms: $\Phi(x) = \langle Ax, x \rangle$. This would not prove the classical Blaschke–Santaló inequality, unless we know *a priori* that any solution to

$$\frac{e^{-\alpha\Phi}}{\int e^{-\alpha\Phi} dx} = \frac{e^{-\beta\Phi^*(\nabla\Phi)}}{\int e^{-\beta\Phi^*} dy} \det D^2\Phi \quad (8)$$

is sufficiently regular; in this paper we prove that any solution to (8) (understood in the mass transportation sense) is indeed regular, which gives, in particular, a transportation proof of the functional Blaschke–Santaló inequality (4) (see Remark 7.13).

When trying to extend these arguments to other cases, we face the difficulty that Monge–Ampère equation (7) may have many solutions. Indeed, let us consider the functional $\mathcal{BS}_{p,V}$. We prove the following result.

Theorem B. *Let V be convex and p -homogeneous. Then any maximum point Φ of the functional $\mathcal{BS}_{p,V}$ satisfying $\Phi(0) = 0$ is p -homogeneous.*

Using homogeneity one can reduce equation for the maximizer of $\mathcal{BS}_{1,V}$ to a Monge–Ampère equation on the unit sphere. More precisely, we get that it is equivalent to the so-called L^q -Minkowski problem for some corresponding $q(p, n)$. The latter is the following non-linear elliptic problem on the sphere: given a measure $\mu = f dx$ on \mathbb{S}^{n-1} solve equation of the type

$$h^{1-q} \det(h_{ij} + h\delta_{ij}) = f. \quad (9)$$

Uniqueness of solution to (9) would provide a natural way of establishing affirmative answer to Question 1.2. Unfortunately, it is known that in general equation (9) has many (even infinitely many) solutions for those values of q which are of interest for us. For instance, for $p = 2$ we get $q = -n$ and this is the so-called centro-affine Minkowski problem. We refer to [18], [33], [36], [57] for examples of non-uniqueness. It is known that uniqueness in Minkowski problem is closely related to the conjectured L^p -Brunn–Minkowski inequality. See, in particular, seminal paper [13] about log-Brunn–Minkowski inequality. Some uniqueness results via L^p -Brunn–Minkowski inequality can be found in [47], [34], [35]. More information the reader can find in the recent book [12]. See Subsection 7.5 for details.

Let us now discuss some possible motivations for studying the aforementioned questions. The inequality (4) is related to many other important and interesting results, such as the Reverse Log-Sobolev inequality of Artstein-Avidan, Klartag, Schütt, Werner [3] (see also Caglar, Fradelizi, Gozlan, Lehec, Schütt, Werner [15]), and the generalized Talagrand’s Transport-Entropy inequality due to Fathi [24]. Therefore one would hope that our results pave the way to discovering this type of phenomena also beyond the Gaussian setting.

In order to explain a particularly important motivation, let us recall the connection of the Blaschke–Santaló inequality to the sharp symmetric Gaussian Poincaré inequality. Inequality (4) implies that for any $t > 0$ and any *even* $f \in C^2(\mathbb{R}^n)$, the function

$$F(t) = \int e^{-(\frac{x^2}{2} + tf(x))} dx \int e^{-(\frac{y^2}{2} + tf(y))^*(y)} dy$$

is maximized at $t = 0$.

One may thus check that $F'(0) = 0$ and deduce that $F''(0) \leq 0$. Computing $F''(0)$ we obtain the “symmetric Gaussian Poincaré inequality”: for any even locally-Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, one has

$$\int f^2 d\gamma - \left(\int f d\gamma \right)^2 \leq \frac{1}{2} \int |\nabla f|^2 d\gamma. \quad (10)$$

(see more details e.g. at [51]). This fact can be also proven using Hermite polynomial decomposition, and in many other ways – see e.g. Cordero-Erausquin, Fradelizi, Maurey [20].

Using an analogous variational argument, one can get

Proposition 1.7. *Let Φ be the maximum point of $\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}$. Then $\mu = \frac{e^{-\alpha\Phi}\rho_1 dx}{\int e^{-\alpha\Phi}\rho_1 dx}$ satisfies*

$$\mathrm{Var}_\mu f \leq \frac{1}{\alpha + \beta} \int \langle (D^2\Phi)^{-1} \nabla f, \nabla f \rangle d\mu.$$

The result of Proposition 1.7 (for $\rho_1 = 1$) is an improvement of the Brascamp–Lieb inequality for maximizers of the Blaschke–Santaló functional on the set of even functions. Recall that a log-concave probability measure

$$d\mu = \frac{e^{-V} dx}{\int e^{-V} dx}$$

satisfies the famous Brascamp–Lieb inequality (see [14])

$$\mathrm{Var}_\mu f \leq \int \langle (D^2V)^{-1} \nabla f, \nabla f \rangle d\mu$$

for any smooth (in general, not symmetric) f , under the assumption that $V \in C^2(\mathbb{R}^n)$ and $D^2V(x)$ is positive definite for every x . The last inequality can be viewed as the infinitesimal version of the Prékopa–Leindler inequality (see [6]). The Brascamp–Lieb inequality is sharp with equality case $f = V_{x_i}$.

Can be the Brascamp–Lieb inequality improved on the set of even functions in the spirit of inequality (10)? One of the motivation for this question comes from the attempts to solve the so-called B -conjecture (see [20]). In particular, it was shown in [23] that the following inequality is equivalent to the B -conjecture:

$$\frac{1}{\mu(K)} \int_K \langle \nabla V, x \rangle^2 d\mu - \left(\frac{1}{\mu(K)} \int_K \langle \nabla V, x \rangle d\mu \right)^2 \leq \frac{1}{\mu(K)} \int_K (\langle \nabla V, x \rangle + \langle \nabla^2 V x, x \rangle) d\mu,$$

where K is an arbitrary even symmetric convex set. The latter inequality can be viewed as a particular case of a strengthening of the Brascamp–Lieb inequality for measure $\frac{1}{\mu(K)} I_K \cdot \mu$ for specific function $f(x) = \langle \nabla V(x), x \rangle$.

A reasonable guess of what can be the strengthening of the Brascamp–Lieb inequality is the following: let V be a p -homogeneous even convex function and $\mu = \frac{e^{-V} dx}{\int e^{-V} dx}$. Is it true that for every even function f the following holds?

$$\mathrm{Var}_\mu f \leq \left(1 - \frac{1}{p}\right) \int \langle (D^2V)^{-1} \nabla f, \nabla f \rangle d\mu. \quad (11)$$

Note that the conjectured inequality turns to be equality for $f(x) = \langle \nabla V(x), x \rangle = pV(x)$.

As a consequence of Proposition 1.7 together with Theorem A, we derive the following strengthening of the Brascamp–Lieb inequality:

Corollary 1.8. *Let $p > 1$ and let V be an even strictly convex p -homogeneous C^2 function on \mathbb{R}^n . Assume that V is an unconditional function, and that the function*

$$x = (x_1, \dots, x_n) \mapsto V\left(x_1^{\frac{1}{p}}, \dots, x_n^{\frac{1}{p}}\right)$$

is concave in \mathbb{R}_+^n . Then inequality (11) holds for every unconditional f .

Assume, in addition, that for every coordinate hyperplane H , with unit normal e and for every $x' \in H$, the function $\varphi: [0, +\infty) \rightarrow \mathbb{R}$ defined by

$$\varphi(t) = \det D^2V^*(x' + te)$$

is decreasing. Then inequality (11) holds for every even f .

Similarly to generalized Blaschke–Santaló inequality, inequality (11) fails to hold for arbitrary convex even p -homogeneous V . In Section 8 we study strengthening of the Brascamp–Lieb inequality for a specific family of measures.

Theorem C. Let $p, q > 1$. Consider probability measure μ such that

$$d\mu = Ce^{-\frac{1}{p}|x|_q^p} dx = Ce^{-\frac{1}{p}(\sum_{i=1}^n |x_i|^q)^{\frac{p}{q}}} dx,$$

where $C \geq 0$. Assume that

$$\lambda \geq \max\left(1 - \frac{1}{p}, 1 - \frac{1}{q}, \frac{1}{2(1 + \frac{q-2}{n})}\right).$$

Then inequality

$$\text{Var}_\mu f \leq \lambda \int \langle (D^2 V)^{-1} \nabla f, \nabla f \rangle d\mu, \quad (12)$$

holds on the set of even functions. This inequality is sharp and holds with $\lambda = 1 - \frac{1}{p}$ in the following cases:

1.

$$q \geq 2, \quad p \geq q,$$

2.

$$q \leq 2, \quad p \geq \frac{2(n+q-2)}{n+2(q-2)} = 2 - \frac{2(q-2)}{n+2(q-2)}.$$

Note that statement (1) of Theorem C follows from Corollary 1.5 and Proposition 1.7. Moreover, we prove the following result (see Subsection 8.2).

Proposition 1.9. Let $1 < p < 2$ and $\mu = Ce^{-\frac{1}{p}|x|_p^p} dx$. Then the best value of λ in inequality (12) satisfies $\lambda > 1 - \frac{1}{p}$. In particular, inequality (5) fails to hold in this case.

Remark 1.10. Let us stress the presence of a big discrepancy between the unconditional and symmetric cases. Indeed, according to Theorem A inequality (5) holds for $\mu = Ce^{-\frac{1}{p}|x|_p^p} dx$ and **all** p on the set of unconditional functions.

To conclude the discussion on the strong Brascamp–Lieb inequality, let us describe the main steps in the proof of Theorem C. First we make the change of variables pushing forward measure μ into a measure of the form

$$\prod_{i=1}^n |y_i|^{\frac{2}{p}-1} \cdot m_0,$$

where m_0 is a rotationally invariant measure. Then using homogeneity we show that our problem is equivalent to the spectral gap problem for the following operator on \mathbb{S}^{n-1} :

$$Lf = \Delta_{\mathbb{S}^{n-1}} f + \left(\frac{2}{p} - 1\right) \langle \omega, \nabla_{\mathbb{S}^{n-1}} f \rangle,$$

where $\Delta_{\mathbb{S}^{n-1}}, \nabla_{\mathbb{S}^{n-1}}$ are the spherical Laplacian and the spherical gradient, respectively, and

$$\omega = \left(\frac{1}{y_1}, \dots, \frac{1}{y_n}\right).$$

Note that a complete orthogonal system of eigenfunctions for L contains non-elementary functions. This is true even for $n = 2$, in this case the eigenfunctions (after appropriate change of variables) belong to the family of Legendre functions, which are non-elementary in general. Fortunately, it turns out that the eigenfunctions we need for establishing the sharp spectral gap estimate on the set of even functions are elementary and the expressions for corresponding eigenvalues have simple algebraic form.

The paper is organised as follows. Section 2 contains preliminary material. In Section 3 we study finiteness and continuity of the Blaschke–Santaló functional. Section 4 is devoted to the symmetrization approach and symmetric properties of the maximizers. In Section 5 we discuss the existence of maximizers

for the Blaschke–Santaló functional, we present some counterexamples, and we prove Theorem A (Theorems 5.19 and 5.21). In Section 6 we present the reduction to the convex body case. In Section 7 we outline the mass transport approach to the Blaschke–Santaló inequality and prove Theorem B (Theorem 7.9). Finally, in Section 8 we deal with the strong Brascamp–Lieb type inequalities and we prove Theorem C (Theorem 8.8).

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2 Preliminaries

As a rule we will omit the domain of integration of an integral if this domain is entire \mathbb{R}^n :

$$\int f dx := \int_{\mathbb{R}^n} f dx.$$

2.1 Convex bodies

By a *convex body*, we mean a convex and compact subset of \mathbb{R}^n with non empty interior. Our main reference for properties of convex bodies is the monograph [61]. Given a convex body K we shall consider its support function $h_K : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$h_K(x) = \max_{y \in K} \langle x, y \rangle.$$

The support function of a convex body is 1-homogeneous and convex.

The radial function ρ_K of a convex body K containing the origin, is defined, for $x \in \mathbb{R}^n$, by

$$\rho_K(x) = \sup\{t > 0 : tx \in K\}.$$

If K is an origin symmetric convex body, then $\rho_K = h_K^{-1}$.

We also recall the Minkowski functional of an origin symmetric convex body K :

$$|x|_K = \rho_K^{-1}(x) \quad \forall x \in \mathbb{R}^n.$$

2.2 Convex functions

We will consider convex functions $\Phi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$. The space of these functions will be denoted by $\text{Conv}(\mathbb{R}^n)$. Our general references on convex functions are the monographs [58] and [59]. Given a convex function Φ we define its domain as

$$\text{dom}(\Phi) = \{x \in \mathbb{R}^n : \Phi(x) < +\infty\}.$$

A convex function Φ is said *proper* if its domain is not empty, and *coercive* if

$$\lim_{|x| \rightarrow \infty} \Phi(x) = +\infty.$$

On the set of convex functions we fix the topology induced by epi-convergence. A sequence Φ_k , $k \in \mathbb{N}$, epi-converges to Φ if:

(i)

$$\liminf_{k \rightarrow +\infty} \Phi_k(x) \geq \Phi(x),$$

for every $x \in \mathbb{R}^n$;

(ii) for every $x \in \mathbb{R}^n$ there exists a sequence $x_k, k \in \mathbb{N}$, converging to x and such that

$$\liminf_{k \rightarrow +\infty} \Phi_k(x_k) = \Phi(x).$$

We note that for sequences of finite convex functions, epi-convergence to a finite convex function is equivalent to uniform convergence on compact subsets of \mathbb{R}^n .

2.3 The Legendre transform

Given a function $\Phi: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, $\Phi \not\equiv +\infty$, we denote by Φ^* its conjugate, or Legendre transform, which is defined as follows:

$$\Phi^*(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - \Phi(x)), \quad \forall y \in \mathbb{R}^n.$$

We collect some properties of the Legendre transform that will be used in this paper.

Proposition 2.1. *The following properties hold.*

- Φ is a finite convex function defined in \mathbb{R}^n if and only if Φ^* is a super-coercive convex function, that is:

$$\lim_{|x| \rightarrow \infty} \frac{\Phi^*(x)}{|x|} = \infty.$$

- a sequence $\Phi_k, k \in \mathbb{N}$, of finite convex functions defined in \mathbb{R}^n epi-converges to a finite convex function Φ if and only if the sequence Φ_k^* epi-converges to Φ^* .

Remark 2.2. *Let $A: \mathbb{R} \rightarrow \mathbb{R}$ be an invertible linear map, and $b \in \mathbb{R}$. Given a finite convex function Φ , consider the function $\bar{\Phi}: \mathbb{R}^n \rightarrow \mathbb{R}$ defined by*

$$\bar{\Phi}(x) = \Phi(Ax) + b.$$

Clearly $\bar{\Phi}$ is a finite convex function as well. Its conjugate verifies the relation:

$$\bar{\Phi}^*(y) = \Phi^*(A^{-T}y) - b, \quad y \in \mathbb{R}^n,$$

where A^{-T} is the inverse of the transpose of A .

Further important properties of the Legendre transform are contained in the following statement.

Proposition 2.3 (Legendre transform of smooth functions). *Let $V \in C^2(\mathbb{R}^n)$ be such that $D^2V(x)$ is positive definite for every x . Then the following properties hold, for every x .*

1. $V(x) + V^*(\nabla V(x)) = \langle x, \nabla V(x) \rangle$,
2. $\nabla V(\nabla V^*(x)) = x$, in other words $\nabla V \circ \nabla V^* = Id$.
3. $\nabla^2 V^*(\nabla V(x)) = (\nabla^2 V)^{-1}(x)$.

2.4 Optimal transportation

Let us consider two probability measures μ and ν on \mathbb{R}^n with finite second moments. We assume that both measures are absolutely continuous with respect to the Lebesgue measure, and we denote their respective densities by ρ_μ and ρ_ν . According to the celebrated Brenier theorem (see [63], [9]) there exists a lower semi-continuous convex function U such that ∇U pushes forward μ onto ν :

$$\int f(\nabla U) d\mu = \int f d\nu$$

for any test function f .

The mapping $T : x \rightarrow \nabla U(x)$ is known as the optimal transportation mapping. Note that T is well defined almost everywhere with respect to the Lebesgue measure, since U is almost everywhere differentiable as a convex function.

The optimal transport mapping ∇U is μ -a.e. unique, meaning that if $T_1 = \nabla U_1$ and $T_2 = \nabla U_2$ are pushing forward μ onto ν and U_1, U_2 are convex, then

$$T_1 = T_2$$

μ -almost everywhere.

The regularity of U is in general a difficult issue (see [25]). However for many purposes it is sufficient to know only the validity of the following change of variables formula, which holds in the non-smooth setting (see [63] and [52] for explanations):

$$\rho_\mu(x) = \rho_\nu(\nabla U(x)) \det D_a^2 U(x). \quad (13)$$

where $D_a^2 U$ is the absolutely continuous part of the distributional Hessian $D^2 U$ – in particular, $D_a^2 U$ is a symmetric and positive semi-definite matrix. Equation (13) holds for μ -almost all x .

Another important instrument related to optimal transportation is the so called Hessian metric. Assume that ρ_μ, ρ_ν are smooth and positive and U is smooth and strictly convex. We consider the following Riemannian metric

$$g(x) = D^2 U(x),$$

and the corresponding Dirichlet form

$$\mathcal{E}(f) = \int \langle (D^2 U)^{-1} \nabla f, \nabla f \rangle d\mu.$$

The generator of \mathcal{E}

$$Lf = \text{Tr}[(D^2 U)^{-1} D^2 f] - \langle \nabla f, \nabla W(\nabla U) \rangle$$

is a second-order elliptic differential operator, naturally related to (μ, ν, T) . L is symmetric with respect to μ : if f, g are smooth and supported on compact sets lying inside of $\text{supp}(\mu)$, then

$$- \int Lfg d\mu = \int \langle (D^2 U)^{-1} \nabla f, \nabla g \rangle d\mu.$$

This metric and its applications to convex geometry has been studied in [44], [46], [39], [40], [41], [42], [43], [45], [16]. Its counterpart on the sphere together with related elliptic operator (Hilbert operator) is a natural instrument for studying Minkowski-type problems (see [47], [45], [54], [55], [34], [35]). Finally, we remark that g is a particular (degenerated) example of a complex Kähler metric.

3 Finiteness and continuity conditions for the Blaschke-Santaló functional

Let μ_1 and μ_2 be non-negative Borel measures on \mathbb{R}^n . We will assume that μ_1 and μ_2 are absolutely continuous with respect to the Lebesgue measure, and denote by ρ_1 and ρ_2 their respective densities. We study the functional

$$\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(\Phi) = \left(\int e^{-\alpha\Phi} \rho_1 dx \right)^{\frac{1}{\alpha}} \left(\int e^{-\beta\Phi^*} \rho_2 dy \right)^{\frac{1}{\beta}},$$

where α, β are positive numbers.

Definition 3.1. We define \mathcal{C} as the class of convex functions Φ verifying the following properties.

1. $\Phi: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$;

2. Φ is even and

$$\text{int}(\text{dom}(\Phi)) \neq \emptyset,$$

where “int” denotes the interior;

3.

$$\lim_{|x| \rightarrow \infty} \Phi(x) = +\infty.$$

Remark 3.2. It can be proved that $\Phi \in \mathcal{C}$ if and only if $\Phi^* \in \mathcal{C}$.

Definition 3.3. Let μ be a Borel measure on \mathbb{R}^n . We say that μ is admissible if

1. μ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^n , and its density ρ is positive on \mathbb{R}^n ;

2. there exist positive constants A, B, p such that

$$\rho(x) \leq A + B|x|^p$$

for every $x \in \mathbb{R}^n$.

Proposition 3.4. Let μ_1 and μ_2 be admissible measures, with density ρ_1 and ρ_2 respectively. Then for every $\Phi \in \mathcal{C}$,

$$0 < \int_{\mathbb{R}^n} e^{-\Phi} d\mu_1, \int_{\mathbb{R}^n} e^{-\Phi^*} d\mu_2 < +\infty.$$

In particular

$$0 < \mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2} < +\infty.$$

Proof. As $\Phi \in \mathcal{C}$, its domain contains a neighborhood of the origin; hence $e^{-\Phi}$ is continuous (by the continuity of Φ in the interior of its domain) and strictly positive in a neighborhood of 0. As ρ_1 is positive everywhere, we obtain

$$0 < \int_{\mathbb{R}^n} e^{-\Phi} d\mu_1.$$

The condition

$$0 < \int_{\mathbb{R}^n} e^{-\Phi^*} d\mu_2$$

is obtained via the same argument, as $\Phi^* \in \mathcal{C}$ and μ_2 is admissible.

As $\Phi \in \mathcal{C}$, $\lim_{|x| \rightarrow \infty} \Phi(x) = +\infty$. This implies that there exist $a \in \mathbb{R}$ and $b > 0$ such that $\Phi(x) \geq a + b|x|$ for every $x \in \mathbb{R}^n$ (see for instance [19, Lemma 8]). By the growth condition verified by ρ_1 , we get

$$\int_{\mathbb{R}^n} e^{-\Phi} d\mu_1 < +\infty.$$

In a similar way

$$\int_{\mathbb{R}^n} e^{-\Phi^*} d\mu_2 < +\infty$$

can be proved. □

We will now show a continuity property of our functional.

Proposition 3.5. *Let μ_1 and μ_2 be admissible measure with density ρ_1 and ρ_2 , respectively. Let Φ , Φ_k , $k \in \mathbb{N}$, belong to \mathcal{C} . Assume that Φ_k epi-converges to Φ . Then*

$$\lim_{k \rightarrow +\infty} \mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi_k) = \mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi).$$

Proof. As Φ_k , $k \in \mathbb{N}$, and Φ are coercive, and Φ_k epi-converges to Φ , there exist $a > 0$ and $b \in \mathbb{R}$ such that

$$\Phi_k(x) \geq a|x| + b \tag{14}$$

for every $x \in \mathbb{R}^n$ and for every $k \in \mathbb{N}$, and the same property is verified by Φ (see for instance [19, Lemma 8]). Moreover, as Φ_k epi-converges to Φ , $\Phi_k(x)$ converges to $\Phi(x)$ for every x in the interior of $\text{dom}(\Phi)$. On the other hand, if $x \notin \text{dom}(\Phi)$, then

$$\lim_{k \rightarrow +\infty} \Phi_k(x) = +\infty = \Phi(x).$$

As the boundary of $\text{dom}(\Phi^*)$ has zero Lebesgue measure, we conclude that

$$\lim_{k \rightarrow +\infty} e^{-\Phi_k(x)} = e^{-\Phi(x)}$$

for almost every $x \in \mathbb{R}^n$. By (14) and the dominated convergence theorem, we obtain

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} e^{-\Phi_k} d\mu_1 = \int_{\mathbb{R}^n} e^{-\Phi} d\mu_1.$$

Next, note that Φ_k^* epi-converges to Φ^* ; moreover Φ^* and Φ_k^* , $k \in \mathbb{N}$, belong to \mathcal{C} . Therefore, we may repeat the same considerations of the previous part of this proof, and deduce that

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} e^{-\Phi_k^*} d\mu_2 = \int_{\mathbb{R}^n} e^{-\Phi^*} d\mu_2.$$

Finally, as Φ and Φ^* belong to \mathcal{C} ,

$$0 < \int_{\mathbb{R}^n} e^{-\Phi} d\mu_1, \int_{\mathbb{R}^n} e^{-\Phi^*} d\mu_2 < \infty.$$

This concludes the proof. □

4 Symmetrization

The symmetrization technique is the main tool for proving inequality of the Blaschke–Santaló type, it is therefore not surprising that they can be used also in the functional version of this result.

To symmetrize a function Φ we apply Steiner symmetrization to its level sets. It is a standard observation that the symmetrized Φ_H (where H is the hyperplane with respect to which we symmetrize) has the same distribution as Φ , thus the value of the integral $\int e^{-\Phi} dx$ is preserved under symmetrization. Moreover, we will show that under some natural assumptions on density ρ_1 (ρ_1 must be in a sense decreasing) the value of

$$\int e^{-\Phi} \rho_1 dx$$

is increasing under symmetrization. In addition, the value of

$$\int e^{-\Phi^*} \rho_2 dx$$

is increasing under symmetrization as well, if ρ_2 is log-concave and admits appropriate symmetries.

Let H be a hyperplane of \mathbb{R}^n , passing through the origin. Given a convex set $C \subset \mathbb{R}^n$, we denote by C_H the Steiner symmetral of C , with respect to H

Let Φ be a convex and coercive function defined in \mathbb{R}^n . We denote by Φ_H the Steiner symmetral of Φ with respect to H . One way of defining Φ_H is through its level sets:

$$\{x: \Phi_H(x) \leq s\} = (\{x: \Phi(x) \leq s\})_H, \quad \forall s \in \mathbb{R}$$

(with the convention that the Steiner symmetral of the empty set is the empty set).

4.1 Monotonicity results for $\int_{\mathbb{R}^n} e^{-\Phi^*} d\mu$

The main result of this part is the following theorem.

Theorem 4.1. *Let H be an hyperplane passing through the origin in \mathbb{R}^n . Let $\mu = \rho dx$ be a log-concave measure in \mathbb{R}^n . Assume that*

$$\rho(te + y) = \rho(te - y) \tag{15}$$

for every $t \in \mathbb{R}$ and $y \in H$, where e is a normal unit vector to H . Then for every even, proper, coercive convex function Φ defined on \mathbb{R}^n , we have

$$\int_{\mathbb{R}^n} e^{-\Phi^*} d\mu \leq \int_{\mathbb{R}^n} e^{-(\Phi_H)^*} d\mu.$$

We proceed with some corollaries. The next statements follow from Theorem 4.1 applied to the Lebesgue measure, and more generally to radially symmetric log-concave measures, and to measures with unconditional density.

Corollary 4.2. *Let $\Phi: \mathbb{R}^n \rightarrow (-\infty, \infty]$ be convex, even and coercive. Then, for every hyperplane H ,*

$$\int_{\mathbb{R}^n} e^{-\Phi^*(x)} dx \leq \int_{\mathbb{R}^n} e^{-(\Phi_H)^*(x)} dx.$$

Corollary 4.3. *Let μ be a log-concave measure on \mathbb{R}^n , with a radially symmetric density with respect to the Lebesgue measure. Let $\Phi: \mathbb{R}^n \rightarrow (-\infty, \infty]$ be convex, even and coercive. Then, for every hyperplane H ,*

$$\int_{\mathbb{R}^n} e^{-\Phi^*} d\mu \leq \int_{\mathbb{R}^n} e^{-(\Phi_H)^*} d\mu.$$

For the next result, we say that a function $\rho: \mathbb{R}^n \rightarrow \mathbb{R}$ is unconditional, if

$$\rho(x_1, \dots, x_n) = \rho(\pm x_1, \dots, \pm x_n)$$

for every choice of the signs $+$ and $-$ on the right hand side. This is equivalent to say that the graph of ρ is symmetric with respect to each coordinate hyperplane. Note that, if ρ is unconditional and H is a coordinate hyperplane, then condition (15) is verified.

Corollary 4.4. *Let μ be a log-concave measure on \mathbb{R}^n , with unconditional density with respect to the Lebesgue measure. Let $\Phi: \mathbb{R}^n \rightarrow (-\infty, \infty]$ be convex, even and coercive. Then, for every coordinate hyperplane H ,*

$$\int_{\mathbb{R}^n} e^{-\Phi^*} d\mu \leq \int_{\mathbb{R}^n} e^{-(\Phi_H)^*} d\mu.$$

4.1.1 Proof of Theorem 4.1

The idea of the proof of Theorem 4.1 is inspired by the argument of Meyer and Pajor in [53].

Given a function $f: \mathbb{R}^n \rightarrow (-\infty, +\infty]$, we denote by $\text{epi}(f)$ its epigraph:

$$\text{epi}(f) = \{(x, z) \in \mathbb{R}^{n+1} : z \geq f(x)\}.$$

Lemma 4.5. *Let Φ be a proper convex function defined in \mathbb{R}^n . Then*

$$\text{epi}(\Phi^*) = \{(y, w) \in \mathbb{R}^n \times \mathbb{R} : w + z \geq \langle x, y \rangle, \forall (x, z) \in \text{epi}(\Phi)\}.$$

Proof. Let

$$X = \{(y, w) \in \mathbb{R}^n \times \mathbb{R} : w + z \geq \langle x, y \rangle, \forall (x, z) \in \text{epi}(\Phi)\}.$$

Let $(y, w) \in \text{epi}(\Phi^*)$; then

$$w \geq \Phi^*(y) = \sup_x \langle y, x \rangle - \Phi(x),$$

whence

$$w + \Phi(x) \geq \langle y, x \rangle \quad \forall x.$$

If $z \geq \Phi(x)$, then

$$w + z \geq \langle y, x \rangle.$$

This proves that $(y, w) \in X$. Assume now that $(y, z) \in X$. Then, for every $x \in \mathbb{R}^n$,

$$w + \Phi(x) \geq \langle y, x \rangle,$$

so that

$$w \geq \sup_x \langle y, x \rangle - \Phi(x) = \Phi^*(y).$$

This proves that $(y, z) \in \text{epi}(\Phi^*)$. □

In the sequel, we choose a coordinate system so that

$$H = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n = 0\}.$$

The points of \mathbb{R}^n will be written in the form

$$(X, x) \in H \times \mathbb{R} \quad \text{or} \quad (Y, y) \in H \times \mathbb{R}.$$

Similarly, the points of \mathbb{R}^{n+1} will be written as

$$(X, x, z) \in H \times \mathbb{R} \times \mathbb{R} \quad \text{or} \quad (Y, y, w) \in H \times \mathbb{R} \times \mathbb{R}.$$

We also set

$$H' = \{(X, 0, z) : X \in H, z \in \mathbb{R}\} \subset \mathbb{R}^{n+1}.$$

Given the function Φ as in the statement of Theorem 4.1, we denote by P_Φ the orthogonal projection of $\text{epi}(\Phi)$ onto H' . We also note that the epigraph of Φ_H is the Steiner symmetral of the epigraph of Φ with respect to H' :

$$\text{epi}(\Phi_H) = (\text{epi}(\Phi))_{H'}.$$

We have

$$\text{epi}(\Phi_H) = \left\{ (X, x, z) : (X, z) \in P_\Phi, x = \frac{x_2 - x_1}{2}, (X, x_1, z), (X, x_2, z) \in \text{epi}(\Phi) \right\}.$$

From Lemma 4.5 we know that

$$\text{epi}(\Phi^*) = \{(Y, y, w) : zw \geq \langle X, Y \rangle + zw, \forall (X, x, z) \in \text{epi}(\Phi)\}.$$

Clearly here $\langle X, Y \rangle$ denotes the scalar product in $H = \mathbb{R}^{n-1}$. Moreover

$$\text{epi}((\Phi_H)^*) = \left\{ (Y, y, w) : zw \geq \langle X, Y \rangle + y \left(\frac{x_2 - x_1}{2} \right), \forall (X, x_i, w) \in \text{epi}(\Phi), i = 1, 2 \right\}.$$

For a general set $A \subset \mathbb{R}^{n+1}$, and $y \in \mathbb{R}$, we set

$$A(y) = \{(Y, 0, w) \in H' : (Y, y, w) \in A\}.$$

Proposition 4.6. *In the previous notations, for every $y \in \mathbb{R}$:*

$$\frac{1}{2}\text{epi}(\Phi^*)(y) + \frac{1}{2}\text{epi}(\Phi^*)(-y) \subset \text{epi}((\Phi_H)^*)(y).$$

Proof. Let

$$(Y', w') \in \text{epi}(\Phi^*)(y), \quad (Y'', w'') \in \text{epi}(\Phi^*)(-y),$$

and let

$$(X, x_1, z), (X, x_2, z) \in \text{epi}(\Phi).$$

Then

$$w'z \geq \langle Y', X \rangle + yx_2, \quad w''z \geq \langle Y'', X \rangle - yx_1,$$

whence

$$\left(\frac{w' + w''}{2} \right) z \geq \left\langle \left(\frac{Y' + Y''}{2} \right), X \right\rangle + y \left(\frac{x_2 - x_1}{2} \right).$$

It follows that

$$\left(\frac{Y' + Y''}{2}, \frac{w' + w''}{2} \right) \in \text{epi}((\Phi_H)^*)(y).$$

□

Let g be a function defined in $\mathbb{R}^n = H \times \mathbb{R}$; for every $y \in \mathbb{R}$ we denote by g_y the function defined on H by

$$g_y(Y) = g(Y, y).$$

Clearly

$$\text{epi}(g_y) = (\text{epi}(g))(y).$$

Proof of Theorem 4.1. By Proposition 4.6, we have

$$\frac{1}{2}\text{epi}((\Phi^*)_y) + \frac{1}{2}\text{epi}((\Phi^*)_{-y}) \subset \text{epi}(((\Phi_H)^*)_y)$$

for every $y \in \mathbb{R}$. Now let $\bar{\Phi}^*$ be the function defined by:

$$\bar{\Phi}^*(Y, y) = \Phi^*(Y, -y).$$

Note that as Φ is even, Φ^* and $\bar{\Phi}^*$ are even as well.

Given $(Y, w) \in H'$, we have $(Y, w) \in \text{epi}((\Phi^*)_{-y})$ if and only if

$$w \geq \Phi^*(Y, -y) = \Phi^*(-Y, y) = \bar{\Phi}^*(Y, y),$$

that is, if and only if $(Y, w) \in \text{epi}((\bar{\Phi}^*)_y)$. Therefore

$$\frac{1}{2}\text{epi}((\Phi^*)_y) + \frac{1}{2}\text{epi}((\bar{\Phi}^*)_y) \subset \text{epi}(((\Phi_H)^*)_y) \quad \forall y \in \mathbb{R}.$$

The set on the left hand side of the previous relation is the graph of the function

$$\begin{aligned} H \ni Y &\mapsto \sup \left\{ \frac{1}{2} \Phi^*(Y_1, y) + \frac{1}{2} \bar{\Phi}^*(Y - 2, y) : \frac{Y_1 + Y_2}{2} = Y \right\} \\ &= \frac{1}{2} \cdot (\Phi^*)_y \square \frac{1}{2} \cdot (\bar{\Phi}^*)_y, \end{aligned}$$

where \square denotes the sup convolution operation, and \cdot the corresponding product by non-negative coefficients (see [58], Section 16)).

Hence

$$((\Phi_H)^*)_y \leq \frac{1}{2} \cdot (\Phi^*)_y \square \frac{1}{2} \cdot (\bar{\Phi}^*)_y,$$

so that

$$e^{-((\Phi_H)^*)_y} \geq e^{-(\frac{1}{2} \cdot (\Phi^*)_y \square \frac{1}{2} \cdot (\bar{\Phi}^*)_y)}. \quad (16)$$

We recall that μ has a density ρ with respect to the Lebesgue measure; let μ_y be the measure on H , with density ρ_y (with respect to the $(n-1)$ dimensional Lebesgue measure on H). Using the log-concavity of μ , and then of μ_y , the inequality (16) and the Prékopa-Leindler inequality, we get

$$\int_H e^{-((\Phi_H)^*)_y} d\mu_y \geq \left(\int_H e^{-(\Phi^*)_y} d\mu_y \right)^{1/2} \left(\int_H e^{-(\bar{\Phi}^*)_y} d\mu_y \right)^{1/2}.$$

On the other hand

$$\begin{aligned} \int_H e^{-(\bar{\Phi}^*)_y} d\rho_y &= \int_H e^{-\bar{\Phi}^*(-Y, y)} \rho(Y, y) dY \\ &= \int_H e^{-\Phi^*(-Y, y)} \rho(-Y, y) dY \\ &= \int_H e^{-\Phi^*(Y, y)} \rho(Y, y) dY. \end{aligned}$$

We conclude that

$$\int_H e^{-((\Phi_H)^*)_y} d\mu_y \geq \int_H e^{-(\Phi^*)_y} d\mu_y$$

for every y . That is

$$\int_{\mathbb{R}^{n-1}} e^{-(\Phi_H)^*(Y, y)} \rho(Y, y) \geq \int_{\mathbb{R}^{n-1}} e^{-\Phi^*(Y, y)} \rho(Y, y), \quad \forall y \in \mathbb{R}^n.$$

The claim of the theorem follows from Fubini's theorem. \square

4.2 Monotonicity results for $\int_{\mathbb{R}^n} e^{-\Phi} d\mu$

The following statements are probably well-known within the area of rearrangements of functions; we include their proofs for completeness.

Proposition 4.7. *Let μ be a measure on \mathbb{R}^n , which is absolutely continuous with respect to the Lebesgue measure, with density ρ . Let H be a hyperplane through the origin with unit normal vector e . Assume that for every $x' \in H$, the function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$\varphi(t) = \rho(x' + te)$$

is even (in \mathbb{R}), and decreasing in $[0, \infty)$. Then, for every $\Phi \in \mathcal{C}$,

$$\int_{\mathbb{R}^n} e^{-\Phi(x)} d\mu(x) \leq \int_{\mathbb{R}^n} e^{-\Phi_H(x)} d\mu(x).$$

The next two results are consequences of Proposition 4.7 (the first one can be obtained also as an application of the Hardy-Littlewood inequality for decreasing rearrangements).

Proposition 4.8. *Let μ be a measure on \mathbb{R}^n , which is absolutely continuous with respect to the Lebesgue measure, with density ρ of the form*

$$\rho = \rho(x) = \varphi(|x|)$$

where $\varphi: [0, +\infty) \rightarrow [0, +\infty)$ is decreasing. Let $\Phi \in \mathcal{C}$ and let H be an hyperplane in \mathbb{R}^n , passing through the origin. Then

$$\int_{\mathbb{R}^n} e^{-\Phi(x)} d\mu(x) \leq \int_{\mathbb{R}^n} e^{-\Phi_H(x)} d\mu(x).$$

Proposition 4.9. *Let μ be a measure on \mathbb{R}^n , which is absolutely continuous with respect to the Lebesgue measure, with unconditional density ρ . Assume that for every coordinate hyperplane H with unit normal vector e , and for every $x' \in H$, the function $\varphi: [0, +\infty) \rightarrow \mathbb{R}$ defined by*

$$\varphi(t) = \rho(x' + te)$$

is decreasing. Let $\Phi \in \mathcal{C}$ and let H be a coordinate hyperplane. Then

$$\int_{\mathbb{R}^n} e^{-\Phi(x)} d\mu(x) \leq \int_{\mathbb{R}^n} e^{-\Phi_H(x)} d\mu(x).$$

4.2.1 Proof of Proposition 4.7

We will need the following one dimensional result.

Lemma 4.10. *Let μ be a measure on \mathbb{R} , absolutely continuous with respect to the Lebesgue measure; assume that the density of μ is a function $\psi: \mathbb{R} \rightarrow [0, +\infty)$ which is even and decreasing in $[0, +\infty)$. Then, for every $a, b \in \mathbb{R}$ with $a \leq b$,*

$$\mu([a, b]) \leq \mu\left(\left[-\frac{b-a}{2}, \frac{b-a}{2}\right]\right).$$

Proof. Let, for $x \in \mathbb{R}$,

$$F(x) = \int_0^x \psi(t) dt,$$

with the convention

$$\int_0^x \psi(t) dt = -\int_0^{-x} \psi(x) dx \quad \text{if } x \leq 0.$$

Note that F is odd in \mathbb{R} , and it is concave in $[0, +\infty)$, as ψ is decreasing in $[0, +\infty)$. If

$$a \leq 0 \leq b$$

then

$$\mu([a, b]) = \int_a^b \psi(t) dt = F(b) - F(a) = F(b) + F(-a).$$

On the other hand,

$$\begin{aligned} \mu\left(\left[-\frac{b-a}{2}, \frac{b-a}{2}\right]\right) &= \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \psi(t) dt = 2 \int_0^{\frac{b-a}{2}} \psi(t) dt \\ &= 2F\left(\frac{b-a}{2}\right). \end{aligned}$$

The inequality then follows from the concavity of F . The case $0 \leq a \leq b$ can be reduced to the previous one, observing that

$$\mu([a, b]) \leq \mu\left(\left[0, \frac{b-a}{2}\right]\right)$$

as ψ is decreasing in $[0, +\infty)$. The case $a \leq b \leq 0$ is completely analogous. \square

Lemma 4.11. *Let μ be a measure on \mathbb{R}^n , which is absolutely continuous with respect to the Lebesgue measure, with density ρ . Let H be a hyperplane through the origin with unit normal vector e . Assume that for every $x' \in H$, the function $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$\varphi(t) = \rho(x' + te)$$

is even, and decreasing in $[0, +\infty)$. Then, for every convex body K in \mathbb{R}^n ,

$$\mu(K) \leq \mu(K_H).$$

Proof. We may assume that

$$H = \{x = (x_1, \dots, x_n) = (x', x_n): x_n = 0\}.$$

Let K' be the orthogonal projection of K onto H . Then, by Fubini's theorem

$$\mu(K) = \int_{K'} \int_{K_{x'}} \rho(x', t) dt dx',$$

where

$$K_{x'} = \{t \in \mathbb{R}: (x', t) \in K\}.$$

The assert follows from Lemma 4.10, and the fact that for every $x' \in K'$ the function $t \rightarrow \rho(x', t)$ is even and decreasing for $t \geq 0$. \square

Proof of Proposition 4.7. Use the Layer Cake Principle and Lemma 4.11. \square

4.3 Applications to the Blaschke-Santaló functional

Let us turn back to our functional $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}$. We will derive from the previous results some consequences on the behavior of $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}$ under the action of Steiner symmetrizations, in the radially symmetric, unconditional, and 1-symmetric cases.

4.3.1 The radially symmetric case

Let us assume that $\rho_1, \rho_2: \mathbb{R}^n \rightarrow \mathbb{R}$ satisfy the following assumptions:

(R1) ρ_1 is of the form

$$\rho = \rho(x) = \varphi(|x|)$$

where $\varphi: [0, +\infty) \rightarrow [0, +\infty)$ is decreasing;

(R2) ρ_2 is radially symmetric and log-concave.

The next results follows from Corollary 4.3 and Proposition 4.8.

Proposition 4.12. *Assume that ρ_1 and ρ_2 verify assumptions (R1) and (R2). Then, for every hyperplane H through the origin, and for every $\Phi \in \mathcal{C}$*

$$\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi) \leq \mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi_H).$$

4.3.2 The unconditional case

Let us assume that $\rho_1, \rho_2: \mathbb{R}^n \rightarrow \mathbb{R}$ satisfy the following assumptions:

(U1) ρ_1 is unconditional, and for every coordinate hyperplane H , with unit normal e , for every $x' \in H$, the function $\varphi: [0, +\infty) \rightarrow \mathbb{R}$ defined by

$$\varphi(t) = \rho_1(x' + te)$$

is decreasing;

(U2) ρ_2 is unconditional and log-concave.

From Corollary 4.4 and Proposition 4.9 we deduce the following statement.

Proposition 4.13. *Assume that ρ_1 and ρ_2 verify assumptions (U1) and (U2). Then, for every coordinate hyperplane H and for every $\Phi \in \mathcal{C}$*

$$\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi) \leq \mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi_H).$$

Example 4.14. *If ρ_1 and ρ_2 are densities of the form*

$$\rho(x) = \rho(x_1, \dots, x_n) = e^{-(|x_1|^p + \dots + |x_n|^p)}$$

with $p \geq 1$, then they verify the assumptions.

4.3.3 The 1-symmetric case.

We recall that set or a function are said to be *1-symmetric* if they possess all the symmetries of the cube. More precisely, we give the following definition.

Definition 4.15. *A convex body K is 1-symmetric if for every $x = (x_1, \dots, x_n) \in K$, we have $(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in K$ for any choice of signs $\epsilon_i \in \{-1, 1\}$, and also $(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \in K$ for any permutation σ .*

A function $\Phi \in \mathcal{C}$ is 1-symmetric if

$$\Phi(x_1, \dots, x_n) = \Phi(\epsilon_1 x_1, \dots, \epsilon_n x_n) = \Phi(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for every $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and every permutation σ .

Let us denote by \mathcal{C}_{1s} the class of all functions $\Phi \in \mathcal{C}$ that are 1-symmetric.

Let $\mathcal{H} = \{H_1, \dots, H_N\}$ be a set of hyperplanes through the origin, in \mathbb{R}^n , such that a set (respectively, a function) is 1-symmetric if and only if it is symmetric (respectively, even) with respect to every $H \in \mathcal{H}$. Let us assume that $\rho_1, \rho_2: \mathbb{R}^n \rightarrow \mathbb{R}$ satisfy the following assumptions:

(S1) ρ_1 is 1-symmetric, and for every hyperplane $H \in \mathcal{H}$, with unit normal e , for every $x' \in H$, the function $\varphi: [0, +\infty) \rightarrow \mathbb{R}$ defined by

$$\varphi(t) = \rho_1(x' + te)$$

is decreasing;

(S2) ρ_2 is 1-symmetric and log-concave.

The following proposition is a consequence of Theorem 4.1 and Proposition 4.7.

Proposition 4.16. *Assume that ρ_1 and ρ_2 verify assumptions (S1) and (S2). Then, for every $H \in \mathcal{H}$ and for every $\Phi \in \mathcal{C}$*

$$\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi) \leq \mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi_H).$$

5 Existence of maximizers

In this section we present some results which guarantee the existence of maximizers of the Blaschke-Santaló functional $\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}$, under specific conditions on $\rho_1, \rho_2, \alpha, \beta$. The picture is completed by examples, collected at the end of this section, showing that existence of maximizers under too general conditions can not be expected.

We start with a very special case, that is when μ_1 and μ_2 both coincide with the Lebesgue measure. We prove that functional $\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}$ is bounded from above and maximizers exist. Note that the existence of maximizers, and their characterisation, follow from the functional Blaschke-Santaló inequality. On the other hand we include our proof as it might be of independent interest and, jointly with the subsequent results of this paper, provides an alternative proof of the functional Blaschke-Santaló inequality.

Theorem 5.1. *Let $\mathcal{BS} = \mathcal{BS}_{1,1,1,1}$ be the classical Blaschke-Santaló functional. There exists $\Phi \in \mathcal{C}$ such that*

$$\mathcal{BS}(\Phi) = M := \sup_{\Psi \in \mathcal{C}} \mathcal{BS}(\Psi).$$

Making appropriate linear change of variables and normalization, one can assume that the measure μ with density $e^{-\Phi}$ (with respect to the Lebesgue measure) is an isotropic probability measure and Φ^ satisfies the inequality*

$$\Phi^*(y) \geq \frac{1}{2n} \sum_{i=1}^n |y_i| - c(n) \tag{17}$$

The proof of the previous result will be given in the sequel of this section. Our next result concerns the homogeneous case.

Theorem 5.2. *Assume ρ_1 is s -homogeneous and ρ_2 is t -homogeneous for $s, t > -n$ such that*

$$\frac{\alpha}{\beta} = \frac{n+s}{n+t}.$$

Then $\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}$ attains a maximum on the set of 1-symmetric functions.

The following corollary can be deduced.

Corollary 5.3. *If V is a p -homogeneous convex function then the functional $\mathcal{BS}_{p,V}$ attains a maximum on the set of 1-symmetric functions.*

The proof of the last two results will be given in the sequel of this section. Note that Corollary 5.3 does not say anything about the precise form of maximizers.

Our next result concerns the existence of maximizers of $\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}$, when ρ_1 and ρ_2 are homogeneous, and radially symmetric, and it is an application of our results on symmetrization, and in particular of Proposition 4.12. In order to apply the latter result, we would need to assume that ρ_2 is log-concave. As the unique log-concave and homogeneous functions defined on \mathbb{R} are (positive) constant functions, we assume that

$$\rho_2 \equiv 1,$$

that is, μ_2 is the Lebesgue measure. Next, we choose

$$\rho_1 = \rho_2(x) = \varphi(|x|)$$

with $\varphi: (0, +\infty) \rightarrow \mathbb{R}$ defined by

$$\varphi(r) = \frac{1}{r^\gamma}, \quad 0 \leq \gamma < n.$$

In particular the measure with density ρ_1 is locally finite (which can be seen using polar coordinates).

Theorem 5.4. Let $\gamma \in (0, n)$ and let $\rho: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ be defined by $\rho(x) = \frac{1}{|x|^\gamma}$. Let $\alpha, \beta > 0$ be such that $\alpha = \beta \frac{n-\gamma}{n}$. Then the functional defined by

$$\mathcal{BS}_{\alpha, \beta, \rho, 1}(\Phi) = \left(\int e^{-\alpha\Phi} \rho dx \right)^{\frac{1}{\alpha}} \left(\int e^{-\beta\Phi^*} dy \right)^{\frac{1}{\beta}},$$

admits a radially symmetric maximizer.

Remark 5.5. Theorem 5.4 is a particular case of Theorem A. However, the proof given below is of independent interest, because it is based only on symmetrization techniques and does not use the Prékopa–Leindler theorem.

Proof of Theorem 5.4. By Proposition 4.12 (see also the proof of the following Theorem 5.6) we may assume that there exists a maximizing sequence Φ_k , $k \in \mathbb{N}$, such that Φ_k is radially symmetric for every k . This means that

$$\sup_{\mathcal{C}} \mathcal{BS}_{\alpha, \beta, \rho, 1} = \sup_{\mathcal{C}_{1, s}} \mathcal{BS}_{\alpha, \beta, \rho, 1}.$$

The proof can be completed applying Theorem 5.2. □

Using the results on symmetrization established in Section 4, we prove that under natural geometric assumptions (symmetry, monotonicity and log-concavity) on densities any symmetric $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}$ functional having maximizers must have, in particular, symmetric maximizers.

Theorem 5.6. Let us assume that $\rho_1, \rho_2: \mathbb{R}^n \rightarrow \mathbb{R}$ satisfy the following assumptions:

(R1) ρ_1 is of the form

$$\rho = \rho(x) = \varphi(|x|)$$

where $\varphi: [0, +\infty) \rightarrow [0, +\infty)$ is decreasing;

(R2) ρ_2 is radially symmetric and log-concave.

If the functional $\mathcal{BS}_{\rho_1, \rho_2}$ admits a maximizer in some subset of \mathcal{C} , invariant under symmetrizations with respect to hyperplanes through the origin, then it admits a radially symmetric maximizer.

Proof. It is well known (see [38]) that for every $\Phi \in \mathcal{C}$ there exists a sequence of hyperplanes H_k , $k \in \mathbb{N}$, such that the sequence of functions ϕ_k defined recursively as follows:

$$\Phi_1 = \Phi, \quad \Phi_{k+1} = (\Phi_k)_{H_k} \quad \forall k \geq 1,$$

is contained in \mathcal{C} and epi-converges to a radially symmetric function $\Phi_\infty \in \mathcal{C}$. In view of this fact, of Proposition 4.12 and of the continuity of $\mathcal{BS}_{\rho_1, \rho_2}$ we conclude the proof. □

Theorem 5.7. Assume that ρ_1 and ρ_2 verify assumptions

(U1) ρ_1 is unconditional, and for every coordinate hyperplane H , with unit normal e , for every $x' \in H$, the function $\varphi: [0, +\infty) \rightarrow \mathbb{R}$ defined by

$$\varphi(t) = \rho_1(x' + te)$$

is decreasing;

(U2) ρ_2 is unconditional and log-concave.

If the functional $\mathcal{BS}_{\rho_1, \rho_2}$ admits a maximizer in some subset of \mathcal{C} , invariant under symmetrizations with respect to coordinate hyperplanes, then it admits an unconditional maximizer.

Proof. Let H_i , $i = 1, \dots, n$ be the coordinate hyperplanes in \mathbb{R}^n . Given $\Phi \in \mathcal{C}$, the function

$$\Phi_u = (\dots((\Phi_{H_1})_{H_2})\dots)_{H_n}$$

belongs to \mathcal{C} and it is unconditional. The proof is completed using Proposition 4.13. □

5.1 Proof of Theorem 5.1

We recall that a probability measure μ on \mathbb{R}^n is said to be isotropic if

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 d\mu = 1$$

for every unit vector θ . Also we recall that in this subsection \mathcal{BS} is the classical Blaschke–Santaló functional:

$$\mathcal{BS}(\Phi) = \int_{\mathbb{R}^n} e^{-\Phi} dx \int_{\mathbb{R}^n} e^{-\Phi^*} dy.$$

We start with a remark. Let $\Phi \in \mathcal{C}$. Without changing the value of $\mathcal{BS}(\Phi)$, we can assume that the measure μ with density $e^{-\Phi}$ is an isotropic probability measure. This is possible because, by Remark 2.2, \mathcal{BS} is invariant with respect to transformations of the form $\Phi(x) \rightarrow \Phi(Ax) + b$, where $b \in \mathbb{R}$ and $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an invertible linear transformation. So, we first reduce to a probability density adding a suitable constant, and then, taking a linear image, we make the measure isotropic.

Lemma 5.8. *Let $\Phi \in \text{Conv}(\mathbb{R}^n)$ be coercive and assume that the measure μ with density $e^{-\Phi}$ is an isotropic probability measure. There exists a constant $c = c(n) > 0$ depending on n such that*

$$\Phi(x) \leq c(n)$$

for every $x \in B = \{x : |x| \leq \frac{1}{2}\}$.

Proof. As Φ is convex on B , there exists $x_0 \in \partial B$ such that

$$m = \min_{x \in B} e^{-\Phi(x)} = e^{-\max_{x \in B} \Phi(x)} = e^{-\Phi(x_0)}.$$

Hence $x_0 = \frac{\theta}{2}$, for some $\theta \in \mathbb{S}^{n-1}$. Let $L = \{x \in \mathbb{R}^n : |\langle x, \theta \rangle| \leq \frac{1}{2}\}$. Clearly, $e^{-\Phi} \leq m$ on $\mathbb{R}^n \setminus L$. Since

$$1 = \int_L \langle x, \theta \rangle^2 d\mu + \int_{\mathbb{R}^n \setminus L} \langle x, \theta \rangle^2 d\mu \leq \frac{1}{4} + \int_{\mathbb{R}^n \setminus L} \langle x, \theta \rangle^2 d\mu,$$

one has

$$\begin{aligned} \frac{3}{4} &\leq \int_{\mathbb{R}^n \setminus L} \langle x, \theta \rangle^2 d\mu \leq \int_{\mathbb{R}^n \setminus L} |x|^2 d\mu \leq \left(\int_{\mathbb{R}^n \setminus L} |x|^{n+5} d\mu \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n \setminus L} \frac{1}{|x|^{n+1}} d\mu \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\mathbb{R}^n} |x|^{n+5} d\mu \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n \setminus L} \frac{dx}{|x|^{n+1}} \right)^{\frac{1}{2}} \sqrt{m}. \end{aligned}$$

Using the well-known moment equivalence result for log-concave measures (see Theorem 3.5.11 in [1]) and the isotropicity of μ , one gets that $\int_{\mathbb{R}^n} |x|^{n+5} d\mu$ is bounded by a number depending on n . Clearly, $\int_{\mathbb{R}^n \setminus L} \frac{dx}{|x|^{n+1}} = C(n) < \infty$. Thus we get $m \geq c(n) > 0$. This completes the proof. \square

Corollary 5.9. *Let $\Phi \in \text{Conv}(\mathbb{R}^n)$ be coercive, and assume that the measure with density $e^{-\Phi}$ is an isotropic probability measure. Then Φ^* satisfies the inequality*

$$\Phi^*(y) \geq \frac{1}{2n} \sum_{i=1}^n |y_i| - c(n)$$

for every $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

Proof. Let $\{e_i : i = 1, \dots, n\}$ be the standard orthonormal basis in \mathbb{R}^n . Apply inequality

$$\Phi^*(y) \geq \langle x, y \rangle - \Phi(x)$$

to $x = \pm \frac{1}{2} e_i$. The previous lemma implies $\Phi^*(y) \geq \frac{1}{2} |y_i| - c(n)$. The result immediately follows. \square

Proof of Theorem 5.1. Let Φ_k , $k \in \mathbb{N}$, be a sequence of coercive functions in $\text{Conv}(\mathbb{R}^n)$ such that

$$\lim_{k \rightarrow +\infty} \mathcal{BS}(\Phi_k) = \sup\{\mathcal{BS}(U) : U \in \text{Conv}(\mathbb{R}^n; \mathbb{R})\}.$$

As already remarked, we may assume that for every $k \in \mathbb{N}$, the measure μ_k with density $e^{-\Phi_k}$ is a probability measure and it is isotropic.

By Chebyshev inequality, we have

$$\int_{|x|>R} e^{-\Phi_k} dx \leq \frac{1}{R^2} \int_{|x|>R} |x|^2 e^{-\Phi_k} dx$$

for every $R > 0$. By the same argument used in the proof of Lemma 5.8, we obtain from the last inequality that the sequence of measures μ_k is tight, and then Prokhorov theorem can be applied (see, for instance, [8]). Therefore μ_k admits a subsequence which is weakly convergent to a probability measure μ .

As μ_k is log-concave and isotropic for every k , it is easy to see that μ is log-concave and isotropic, as well. By a well-known theorem of Borell (see [10]), the log-concavity of μ implies that it is absolutely continuous with respect to the Lebesgue measure, and its density is of the form $e^{-\Phi}$, where $\Phi \in \text{Conv}(\mathbb{R}^n)$. As μ is a probability measure, V is coercive; moreover, by Lemma 5.8, $\Phi(x) < \infty$ for every x such that $|x| \leq \frac{1}{2}$.

The weak convergence of μ_k to μ implies that

$$\lim \Phi_k(x) = \Phi(x), \quad \forall x \in \mathbb{R}^n \setminus \partial \text{dom}(\Phi).$$

By [59, Theorem 7.17], Φ_k epi-converges to Φ . By Proposition 3.5,

$$\mathcal{BS}(\Phi) = \lim_{k \rightarrow \infty} \mathcal{BS}(\Phi_k) = M.$$

Note that Φ^* verifies (17), so that, in particular, $M < +\infty$. □

Remark 5.10. *Theorem 5.1 is used in the proof of the classical Blaschke–Santaló inequality by transportation method without symmetrization arguments (see Remark 7.12).*

5.2 Proof of Theorem 5.2 and Corollary 5.3

Proof of Theorem 5.2. Write $M = \sup_{\Phi \in \mathcal{C}_{1s}} \mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi)$, and choose a sequence $\{\Phi_k\} \subseteq \mathcal{C}_{1s}$ such that $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi_k) \rightarrow M$.

First observe that for every $\Phi \in \mathcal{C}$ and every $\lambda \in \mathbb{R}$ we have $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi + \lambda) = \mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi)$, so we may assume without loss of generality that $\Phi_k(0) = 0$ for all k . This means in particular that $\Phi_k \geq 0$. Next, for $\Phi \in \mathcal{C}$ and $\lambda > 0$ we set $(H_\lambda \Phi)(x) = \Phi(\lambda x)$, and recall that $(H_\lambda \Phi)^* = H_{1/\lambda} \Phi^*$. It follows that

$$\begin{aligned} \mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(H_\lambda \Phi) &= \left(\int_{\mathbb{R}^n} e^{-\alpha \Phi(\lambda x)} \rho_1(x) dx \right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}^n} e^{-\beta \Phi^*(y/\lambda)} \rho_2(y) dy \right)^{\frac{1}{\beta}} \\ &= \left(\frac{1}{\lambda^n} \int_{\mathbb{R}^n} e^{-\alpha \Phi(z)} \rho_1\left(\frac{z}{\lambda}\right) dz \right)^{\frac{1}{\alpha}} \left(\lambda^n \int_{\mathbb{R}^n} e^{-\beta \Phi^*(w)} \rho_2(\lambda w) dw \right)^{\frac{1}{\beta}} \\ &= \left(\frac{1}{\lambda^{n+s}} \int_{\mathbb{R}^n} e^{-\alpha \Phi(z)} \rho_1(z) dz \right)^{\frac{1}{\alpha}} \left(\lambda^{n+t} \int_{\mathbb{R}^n} e^{-\beta \Phi^*(w)} \rho_2(w) dw \right)^{\frac{1}{\beta}} \\ &= \lambda^{\frac{n+t}{\beta} - \frac{n+s}{\alpha}} \mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi) = \mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi). \end{aligned}$$

For every k the set $[\Phi_k \leq 1]$ is a 1-symmetric convex body. We observe that for arbitrary 1-symmetric body L the corresponding John ellipsoid E is a ball. Indeed, if T is a linear transformation satisfying $T(L) = L$, then $T(E)$ is the John ellipsoid as well. By uniqueness of E one has $T(E) = E$. Hence E is 1-symmetric. This means that E is a ball. Therefore by replacing Φ_k with $H_\lambda \Phi_k$ for a suitable $\lambda > 0$ we may assume that $[\Phi_k \leq 1]$ is in John position. In particular $B_2^n \subseteq [\Phi_k \leq 1] \subseteq \sqrt{n} B_2^n$.

By passing to a subsequence we may assume without loss of generality that $\{\Phi_k\}$ epi-converges to a lower semi-continuous convex function $\Phi : \mathbb{R}^n \rightarrow [-\infty, \infty]$ (such a converging subsequence always exists). Clearly $\min \Phi = \Phi(0) = 0$ and Φ is 1-symmetric, so in order to prove that $\Phi \in \mathcal{C}_{1s}$ it is enough to show that $\text{int}(\text{dom}(\Phi)) \neq \emptyset$ and that $\lim_{|x| \rightarrow \infty} \Phi(x) = \infty$.

We know that $[\Phi_k \leq 1] \rightarrow [\Phi \leq 1]$ in the Hausdorff sense (see the proof of Lemma 5 of [19]), so in particular $B_2^n \subseteq [\Phi \leq 1] \subseteq \sqrt{n}B_2^n$. This immediately shows that $\text{int}(\text{dom}(\Phi)) \supseteq \text{int}(B_2^n) \neq \emptyset$. On the other hand for all $x \in \mathbb{R}^n$ with $|x| \geq \sqrt{n}$ we have

$$1 \leq \Phi\left(\frac{\sqrt{n}x}{|x|}\right) = \Phi\left(\frac{\sqrt{n}}{|x|}x + \left(1 - \frac{\sqrt{n}}{|x|}\right)0\right) \leq \frac{\sqrt{n}}{|x|}\Phi(x) + \left(1 - \frac{\sqrt{n}}{|x|}\right)\Phi(0) = \frac{\sqrt{n}}{|x|}\Phi(x),$$

or $\Phi(x) \geq \frac{|x|}{\sqrt{n}}$. This shows that $\lim_{|x| \rightarrow \infty} \Phi(x) = \infty$ and so $\Phi \in \mathcal{C}$.

Now the arguments of Proposition 3.5 show that $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}$ is continuous on \mathcal{C} with respect to epi-convergence, so $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}(\Phi) = M$ as claimed. \square

Proof of Corollary 5.3. We have $\mathcal{BS}_{p, V} = \mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}$ where $\alpha = 1$, $\beta = \frac{1}{p-1}$, $\rho_1 = 1$ and $\rho_2 = \det D^2V^*$.

Since V is p -homogeneous we know that V^* is p^* -homogeneous, with $\frac{1}{p} + \frac{1}{p^*} = 1$. Therefore D^2V^* is $(p^* - 2)$ -homogeneous and ρ_2 is homogeneous of degree

$$t = n(p^* - 2) = \frac{n}{p-1} - n.$$

Of course ρ_1 is homogeneous of degree $s = 0$. Since

$$\frac{n+s}{n+t} = \frac{n}{n/(p-1)} = p-1 = \frac{\alpha}{\beta}$$

the previous theorem applies and a maximizer exists in \mathcal{C}_{1s} . \square

5.3 Inequalities for radially symmetric measures

In this subsection we compute the radially symmetric maximizer of the functional appearing in Theorem 5.4. Note that the result is a particular case of Theorem A, which be proved at the end of this section by reduction to the unconditional case.

We assume that $\beta \geq \alpha > 0$ and set $\lambda = \frac{\alpha+\beta}{\beta}$. One can easily verify that convex potential

$$U(y) = \frac{1}{\lambda}|x|^\lambda = \frac{\beta}{\alpha+\beta}|x|^{\frac{\alpha+\beta}{\beta}}$$

pushes forward the Lebesgue measure onto the measure with density

$$\det D^2U = \frac{\lambda-1}{|x|^\gamma} = \frac{\alpha}{\beta}|x|^{-\gamma}.$$

We remind the reader that $U^*(y) = \frac{1}{\lambda^*}|y|^{\lambda^*}$, where $\lambda^* = \frac{\alpha+\beta}{\alpha}$. According to Theorem 5.4, the functional

$$\begin{aligned} \left(\int e^{-\alpha\Phi(\nabla U^*(y))} dy\right)^{\frac{1}{\alpha}} \left(\int e^{-\beta\Phi^*(y)} dy\right)^{\frac{1}{\beta}} &= \left(\int e^{-\alpha\Phi(|y|^{\frac{\beta}{\alpha}-1}y)} dy\right)^{\frac{1}{\alpha}} \left(\int e^{-\beta\Phi^*(y)} dy\right)^{\frac{1}{\beta}} \\ &= \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\alpha}} \left(\int \frac{e^{-\alpha\Phi(x)}}{|x|^\gamma} dx\right)^{\frac{1}{\alpha}} \left(\int e^{-\beta\Phi^*(y)} dy\right)^{\frac{1}{\beta}} \end{aligned}$$

admits a radially symmetric maximizer, which is λ -homogeneous, by Theorem 7.9 (note that we need the assumption $\alpha \leq \beta$ to ensure that the weight $|x|^{-\gamma}$ is decreasing). Thus the choice $\Phi = U$ is optimal and we get the following result.

Corollary 5.11. *Let $\beta \geq \alpha \geq 0$. Then for every convex and even function Φ the following inequality holds:*

$$\left(\int e^{-\alpha\Phi(|y|^{\frac{\beta}{\alpha}-1}y)} dy \right)^{\frac{1}{\alpha}} \left(\int e^{-\beta\Phi^*(y)} dy \right)^{\frac{1}{\beta}} \leq \left(\int e^{-\frac{\alpha\beta}{\alpha+\beta}|y|^{1+\frac{\beta}{\alpha}}} dy \right)^{\frac{1}{\alpha}+\frac{1}{\beta}}.$$

The equivalent weighted version is:

$$\left(\int \frac{e^{-\alpha\Phi(x)}}{|x|^\gamma} dx \right)^{\frac{1}{\alpha}} \left(\int e^{-\beta\Phi^*(y)} dy \right)^{\frac{1}{\beta}} \leq \left(\int \frac{e^{-\frac{\alpha\beta}{\alpha+\beta}|x|^{\frac{\alpha+\beta}{\beta}}}}{|x|^\gamma} dx \right)^{\frac{1}{\alpha}} \left(\int e^{-\frac{\alpha\beta}{\alpha+\beta}|y|^{\frac{\alpha+\beta}{\alpha}}} dy \right)^{\frac{1}{\beta}}, \quad \gamma = n \left(1 - \frac{\alpha}{\beta} \right).$$

Taking, in particular, $p \geq 2$

$$\alpha = \frac{1}{p-1}, \quad \beta = 1$$

(by homogeneity the general case can be reduced to this situation), one gets the following result

$$\left(\int e^{-\frac{1}{p-1}\Phi(|x|^{p-2}x)} dx \right)^{p-1} \left(\int e^{-\Phi^*(y)} dy \right) \leq \left(\int e^{-\frac{1}{p}|x|^p} dx \right)^p. \quad (18)$$

Remark 5.12. *One may ask whether (18) holds also for $1 < p < 2$. We will see that this is not true. Indeed, Proposition 7.7 implies that if it is the case, then the probability measure $\mu = Ce^{-\frac{1}{p}|x|^p} dx$ must satisfy the strong Brascamp–Lieb inequality with constant $1 - \frac{1}{p}$. As we will see in Section 8, this is not true for $p < 2$.*

Remark 5.13. *The following inequality of Blaschke–Santaló type, with a radially symmetric maximizer, has been proved by Fradelizi and Meyer in [29]. Given a decreasing function ρ on \mathbb{R}_+ and positive even functions f, g satisfying*

$$f(x)g(y) \leq \rho^2(\langle x, y \rangle),$$

for all x, y such that $\langle x, y \rangle \geq 0$, one has

$$\int f dx \int g dy \leq \left(\int \rho(|x|^2) dx \right)^2. \quad (19)$$

Though in this paper we do not analyse relations between our result and inequality of Fradelizi and Meyer, we observe that the strong Brascamp–Lieb inequality deduced from (18) (see Proposition 7.7) is weaker than an infinitesimal version of (19). The latter coincides with an improvement of the Brascamp–Lieb inequality obtained by Cordero-Erausquin and Rotem in [23]. The relation between (19) and the result of Cordero-Erausquin and Rotem was noticed in [28] (see Theorem 4.1).

5.4 Examples of non-existence of maximizers

Example 5.14. *Let $\rho_1 = 1$ and ρ_2 be 0-homogeneous. Let $M = \max_{y \in \mathbb{S}^{n-1}} \rho_2(y)$ and assume that the restriction of ρ_2 to \mathbb{S}^{n-1} admits exactly two maximum points: y_0 and $-y_0$. For simplicity, we may assume that $y_0 = e_1$. Then for every admissible Φ*

$$\int e^{-\Phi} dx \int e^{-\Phi^*} \rho_2 dy < M \int e^{-\Phi} dx \int e^{-\Phi^*} dy < M(2\pi)^n.$$

Let $\Phi_\varepsilon = \frac{1}{2}x_1^2 + \frac{\varepsilon}{2} \sum_{i=2}^n x_i^2$. Then $\int e^{-\Phi_\varepsilon} dx = (2\pi)^{\frac{n}{2}} \varepsilon^{-\frac{n-1}{2}}$ and

$$\begin{aligned} \int e^{-\Phi_\varepsilon} dx \int e^{-\Phi_\varepsilon^*} \rho_2 dy &= (2\pi)^{\frac{n}{2}} \varepsilon^{-\frac{n-1}{2}} \int e^{-\frac{1}{2}y_1^2 - \frac{1}{2\varepsilon} \sum_{i=2}^n y_i^2} \rho_2 dy \\ &= (2\pi)^{\frac{n}{2}} \int e^{-\frac{1}{2}|t|^2} \rho_2(t_1, \varepsilon t_2, \dots, \varepsilon t_n) dt. \end{aligned}$$

Clearly

$$\lim_{\varepsilon \rightarrow 0} \int e^{-\Phi_\varepsilon} dx \int e^{-\Phi_\varepsilon^*} \rho_2 dy = (2\pi)^n M.$$

Thus we get that

$$\sup \int e^{-\Phi} dx \int e^{-\Phi^*} \rho_2 dy = M,$$

but

$$\int e^{-\Phi} dx \int e^{-\Phi^*} \rho_2 dy$$

does not attain the supremum.

The following example is related to Theorem 5.2.

Example 5.15. Without the compatibility condition $\frac{\alpha}{\beta} = \frac{n+s}{n+t}$ the functional $\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}$ clearly is not bounded in general, even in dimension $n = 1$ (or equivalently, in dimension n even if we assume all densities and functions are not only 1-symmetric but even rotation invariant). This is obvious from the proof in the previous example, but as a concrete example consider

$$\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(\Phi) = \int e^{-\Phi} dx \cdot \int e^{-\Phi^*} x^2 dx$$

(i.e. $\alpha = \beta = 1$, $\rho_1 = 1$, $\rho_2 = x^2$, $s = 0$ and $t = 2$). Then $\frac{\alpha}{\beta} = 1 \neq \frac{1+0}{1+2} = \frac{n+s}{n+t}$. And indeed, choosing e.g. $\Phi_\lambda(x) = \lambda \frac{x^2}{2}$ we see that

$$\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(\Phi_\lambda) = \int e^{-\lambda \frac{x^2}{2}} dx \cdot \int e^{-\frac{1}{\lambda} \frac{x^2}{2}} x^2 dx = 2\pi\lambda \xrightarrow{\lambda \rightarrow \infty} \infty.$$

Example 5.16. If ρ_1 and ρ_2 are not assumed to be homogeneous then it is possible for $\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}$ to be bounded and still not attain a maximum. For instance, in the example above replace ρ_2 with $\rho_2(x) = e^{-\frac{1}{2}x^2}$. Then

$$\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(\Phi_\lambda) = \int e^{-\lambda \frac{x^2}{2}} dx \cdot \int e^{-\frac{1}{\lambda} \frac{x^2}{2}} e^{-\frac{x^2}{2}} dx = \frac{2\pi}{\sqrt{\lambda+1}} \xrightarrow{\lambda \rightarrow 0} 2\pi.$$

On the other hand, since $\rho_2(x) < 1$ for all $x \neq 0$ we have, for all $\Phi \in \mathcal{C}$,

$$\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(\Phi) = \int e^{-\Phi} dx \cdot \int e^{-\Phi^*} e^{-\frac{x^2}{2}} dx < \int e^{-\Phi} dx \cdot \int e^{-\Phi^*} dx \leq 2\pi.$$

This shows that $\sup_{\Phi \in \mathcal{C}} \mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(\Phi) = 2\pi$, but this supremum is not attained.

5.5 Blaschke–Santaló inequality for unconditional functions : Theorem A(2)

In this section we derive Blaschke–Santaló inequality for unconditional functions from the Prékopa–Leindler inequality. The arguments go back to [29].

For $x, y \in \mathbb{R}_+^n$ and for $s, t \in \mathbb{R}$ define

$$x^s y^t = (x_1^s y_1^t, \dots, x_n^s y_n^t).$$

Lemma 5.17. Let V be a function, which is twice continuously differentiable and p -homogeneous on $(0, \infty)^n$, $p \geq 1$. Then the following properties are equivalent:

1. for every $a, b \in (0, \infty)^n$

$$\langle a, \nabla V(b) \rangle \geq \langle a^{\frac{1}{p}} b^{\frac{p-1}{p}}, \nabla V(a^{\frac{1}{p}} b^{\frac{p-1}{p}}) \rangle = pV(a^{\frac{1}{p}} b^{\frac{p-1}{p}}).$$

2. the following function is concave:

$$x \rightarrow V(x^{\frac{1}{p}}).$$

Proof. Assume that 1. holds. Let $a = b + \varepsilon c$. One has:

$$\langle b + \varepsilon c, \nabla V(b) \rangle \geq pV\left((b + \varepsilon c)^{\frac{1}{p}} b^{\frac{p-1}{p}}\right).$$

Expanding in ε , one gets $(b + \varepsilon c)^{\frac{1}{p}} b^{\frac{p-1}{p}} = b + \frac{\varepsilon}{p} c + \frac{(1-p)\varepsilon^2}{2p^2} c^2 b^{-1} + o(\varepsilon^2)$ and

$$pV\left((b + \varepsilon c)^{\frac{1}{p}} b^{\frac{p-1}{p}}\right) = pV(b) + \varepsilon \langle c, \nabla V(b) \rangle + \frac{(1-p)\varepsilon^2}{2p^2} \langle \text{diag}\left(\frac{V_{x_i}(b)}{b_i}\right) c, c \rangle + \frac{\varepsilon^2}{2p^2} \langle (D^2V)(b) c, c \rangle + o(\varepsilon^2).$$

Finally, we obtain that V must satisfy

$$D^2V(x) \leq (p-1) \text{diag}\left(\frac{V_{x_i}(x)}{x_i}\right). \quad (20)$$

To see the equivalence to concavity of $V_p(x) = V(x^{\frac{1}{p}})$, we note that

$$\begin{aligned} D^2V_p(x) &= \frac{1}{p^2} \text{diag}\left(x_i^{\frac{1-p}{p}}\right) D^2V(x^{\frac{1}{p}}) \text{diag}\left(x_i^{\frac{1-p}{p}}\right) + \frac{1-p}{p^2} \text{diag}\left(x_i^{\frac{1-2p}{p}} V_{x_i}(x^{\frac{1}{p}})\right) \\ &= \frac{1}{p^2} \text{diag}\left(x_i^{\frac{1-p}{p}}\right) \left[D^2V(x^{\frac{1}{p}}) + (1-p) \text{diag}\left(\frac{V_{x_i}(x^{\frac{1}{p}})}{x_i^{1/p}}\right) \right] \text{diag}\left(x_i^{\frac{1-p}{p}}\right). \end{aligned}$$

Thus we get that $D^2V_p \leq 0$ if and only if (20) holds.

Let us assume 2. Note that function $f(a) = V(a^{\frac{1}{p}} b^{p-1})$ is a composition of V_p with diagonal linear mapping $\text{diag}(b_i^{\frac{p-1}{p}})$, hence f is concave. In particular, $g(t) = f(b + t(a-b))$ is concave and consequently it satisfies $g(1) \leq g(0) + g'(0)$. One can easily compute $g(1) = f(a) = V(a^{\frac{1}{p}} b^{\frac{p-1}{p}})$, $g(0) = f(b) = V(b)$,

$$g'(0) = \langle a - b, \nabla f(b) \rangle = \frac{1}{p} \langle a - b, \nabla V(b) \rangle = \frac{1}{p} \langle a, \nabla V(b) \rangle - V(b).$$

This completes the proof. \square

Example 5.18. Let $V = \frac{1}{p}|x|_r^p$. Then V satisfies assumptions of the previous lemma if $r \leq p$.

Theorem 5.19. Let Φ, V be unconditional functions. Let, in addition, V satisfy the following assumptions:

1. V is p -homogeneous for some $p > 1$;
2. the function $x \rightarrow V(x^{\frac{1}{p}})$ is concave.

Then

$$\left(\int_{\mathbb{R}^n} e^{-\Phi} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} e^{-\frac{1}{p-1} \Phi^*(\nabla V)} dx \right)^{\frac{p-1}{p}} \leq \int_{\mathbb{R}^n} e^{-V} dx.$$

If, in addition, V is convex, then the inequality is sharp and $\Phi = V$ is the maximum point.

Proof. Using change of variables $x = e^r$ and Prékopa–Leindler inequality, one gets

$$\begin{aligned} & \int_{\mathbb{R}^n} e^{-\Phi(x)} dx \left[\int_{\mathbb{R}^n} e^{-\frac{1}{p-1} \Phi^*(\nabla V(x))} dx \right]^{p-1} = 2^{np} \int_{\mathbb{R}_+^n} e^{-\Phi(x)} dx \left[\int_{\mathbb{R}_+^n} e^{-\frac{1}{p-1} \Phi^*(\nabla V(x))} dx \right]^{p-1} \\ &= 2^{np} \int_{\mathbb{R}^n} e^{-\Phi(e^t)} e^{\sum_{i=1}^n t_i} dt \left[\int_{\mathbb{R}^n} e^{-\frac{1}{p-1} \Phi^*(\nabla V(e^s))} e^{\sum_{i=1}^n s_i} ds \right]^{p-1} \\ &\leq 2^{np} \left[\int_{\mathbb{R}^n} e^{-\inf_{r=\frac{t+(p-1)s}{p}} \left(\frac{\Phi(e^t) + \Phi^*(\nabla V(e^s))}{p} - \frac{\sum_{i=1}^n t_i + (p-1)s_i}{p} \right)} dr \right]^p. \end{aligned}$$

Apply the assumptions on V and the previous lemma:

$$\begin{aligned}
\Phi(e^t) + \Phi^*(\nabla V(e^s)) &\geq \langle e^t, \nabla V(e^s) \rangle \\
&\geq \langle e^{\frac{t+(p-1)s}{p}}, \nabla V(e^{\frac{t+(p-1)s}{p}}) \rangle \\
&= \langle e^r, \nabla V(e^r) \rangle \\
&= pV(e^r).
\end{aligned}$$

Thus

$$\int_{\mathbb{R}^n} e^{-\Phi(x)} dx \left[\int_{\mathbb{R}^n} e^{-\frac{1}{p-1}\Phi(\nabla V(x))} dx \right]^{p-1} \leq 2^{np} \left[\int_{\mathbb{R}^n} e^{-V(e^r) + \sum_{i=1}^n r_i} dr \right]^p = \left[\int_{\mathbb{R}^n} e^{-V(x)} dx \right]^p.$$

□

Corollary 5.20. *Let $p > 1, r > 1, r \leq p$ and $V = \frac{1}{p}|x|^p$. Then the Blaschke–Santaló inequality*

$$\left(\int e^{-\Phi} dx \right)^{\frac{1}{p}} \left(\int e^{-\frac{1}{p-1}\Phi^*(\nabla V(y))} dy \right)^{1-\frac{1}{p}} \leq \int e^{-V} dx$$

holds on the set of unconditional functions.

5.6 Theorem A: the general case

In this section we prove some sufficient conditions for V to be the maximizer of $\mathcal{BS}_{p,V}$. Our proof is based on symmetrization arguments and the result of the previous section on maximization of $\mathcal{BS}_{p,V}$ on the set of unconditional functions.

Theorem 5.21. *Let $p > 1$ and V be an even convex function satisfying the following assumptions:*

- V is p -homogeneous;
- V is unconditional and the function

$$x = (x_1, \dots, x_n) \mapsto V(x_1^{\frac{1}{p}}, \dots, x_n^{\frac{1}{p}})$$

is concave in \mathbb{R}_+^n ;

- *for every coordinate hyperplane H , with unit normal e , for every $x' \in H$, the function $\varphi: [0, +\infty) \rightarrow \mathbb{R}$ defined by*

$$\varphi(t) = \det D^2 V^*(x' + te)$$

is decreasing.

Then inequality (5) holds for any convex even function Φ .

Proof. Let Φ be even convex function. We observe that

$$\mathcal{BS}_{p,V}(\Phi) \leq \mathcal{BS}_{p,V}((\Phi)_{H_k})$$

for every $1 \leq k \leq n$, where $(\Phi)_{H_k}$ is symmetrization of Φ with respect to the hyperplane $\{x_k = 0\}$. This follows from Proposition 4.13. Applying consecutively the symmetrizations H_1, \dots, H_n to Φ , one obtains an unconditional function $\tilde{\Phi}$ such that

$$\mathcal{BS}_{p,V}(\Phi) \leq \mathcal{BS}_{p,V}(\tilde{\Phi}).$$

On the other hand, inequality (5) holds for unconditional functions, according to Corollary 5.20; this completes the proof. □

Theorem 5.22. *Let $p \geq r \geq 2$ and $V = \frac{1}{p}|x|_r^p$. Then the generalized Blaschke–Santaló inequality*

$$\left(\int e^{-\Phi} dx \right)^{\frac{1}{p}} \left(\int e^{-\frac{1}{p-1}\Phi^*(\nabla V(y))} dy \right)^{1-\frac{1}{p}} \leq \int e^{-V} dx$$

holds on the set of even functions.

Proof. It is sufficient to check that $\det D^2V^*$ satisfies assumption (U1) of Proposition 4.13. Indeed, one has

$$V^*(x) = \frac{1}{q}|x|_{r^*}^q,$$

where $q = \frac{p}{p-1}$, $r^* = \frac{r}{r-1}$. Next we compute (for the sake of simplicity let $x_i > 0$):

$$\nabla V^*(x) = |x|_{r^*}^{q-r^*} (x_i^{r^*-1}),$$

and

$$\begin{aligned} D^2V^*(x) &= (q - r^*)|x|_{r^*}^{q-2r^*} (x_i^{r^*-1}x_j^{r^*-1}) + (r^* - 1)|x|_{r^*}^{q-r^*} x_i^{r^*-2}\delta_{ij} \\ &= |x|_{r^*}^{q-2r^*} \left[(q - r^*)x_i^{r^*-1}x_j^{r^*-1} + (r^* - 1)|x|_{r^*}^{r^*} x_i^{r^*-2}\delta_{ij} \right]. \end{aligned}$$

We set $\Lambda = |x|_{r^*}^{\frac{r^*}{2}} \text{diag}(x_i^{\frac{r^*}{2}-1})$. Then

$$D^2V^* = |x|_{r^*}^{q-2r^*} \Lambda \left[(r^* - 1)\text{Id} + (q - r^*)a \otimes a \right] \Lambda,$$

where $a = \frac{1}{|x|_{r^*}^{\frac{r^*}{2}}}(x_i^{\frac{r^*}{2}})$. Thus one has (for all $x = (x_1, \dots, x_n)$, $x_i \in \mathbb{R} \setminus \{0\}$)

$$\det D^2V^* = (q - 1)(r^* - 1)^{n-1} |x|_{r^*}^{n(q-r^*)} \prod_{i=1}^n |x_i|^{r^*-2}.$$

We obtain that the measure D^2V^* satisfies assumption (U2) provided $q \leq r^*$ and $r^* \leq 2$. Equivalently, $r \geq 2$ and $p \geq r$. \square

6 The geometric approach: an equivalence between functional inequalities and inequalities about convex bodies

In this section we will see that for a given p -homogeneous convex function V , the inequality

$$\int_{\mathbb{R}^n} e^{-\Phi(x)} dx \cdot \left(\int_{\mathbb{R}^n} e^{-\frac{1}{p-1}\Phi^*(\nabla V)} dx \right)^{p-1} \leq \left(\int_{\mathbb{R}^n} e^{-V} dx \right)^p$$

for arbitrary convex functions Φ , is equivalent to the following geometric inequality

$$|K| \cdot |\nabla V^*(K^\circ)|^{p-1} \leq \left| \left\{ V \leq \frac{1}{p} \right\} \right|^p.$$

In particular, in the next subsection, we prove Proposition 1.3.

6.1 The reduction of the functional inequality to a geometric one

For $p \geq 1$, consider $V = |x|_M^p/p$, where M is a symmetric convex body, and $|\cdot|_M$ is the associated Minkowski functional. Then V is p -homogeneous and all of its level sets are homothetic to M . For a set K in \mathbb{R}^n we shall use the notation

$$\nabla V(K) = \{\nabla V(x) : x \in K\}.$$

Note that the volume of the set $\nabla V(K)$ is given by

$$|\nabla V(K)| = \int_K \det(\nabla^2 V(x)) dx.$$

Proposition 6.1. *Fix a symmetric convex body M and let $V = |x|_M^p/p$. Suppose that for any symmetric convex body K , one has:*

$$|K| \cdot |\nabla V^*(K^\circ)|^{p-1} \leq \left| \left\{ V \leq \frac{1}{p} \right\} \right|^p = |M|^p,$$

with equality when $K = M$. Then for any even strictly convex $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} e^{-\Phi(x)} dx \cdot \left(\int_{\mathbb{R}^n} e^{-\frac{1}{p-1}\Phi^*(\nabla V)} dx \right)^{p-1} \leq \left(\int_{\mathbb{R}^n} e^{-V} dx \right)^p.$$

Proof. We follow the scheme of Artstein-Avidan, Klartag and Milman [2] and Keith Ball [4]. Note (in view of the definition of the Legendre transform) that for any $x \in \{\Phi^*(\nabla V) \leq s\}$ and any $y \in \{\Phi(y) \leq t\}$ one has $\langle \nabla V, y \rangle \leq s + t$; therefore

$$\nabla V(\{\Phi^*(\nabla V) \leq s\}) \subset (s+t)\{\Phi \leq t\}^\circ.$$

By the $\frac{1}{p-1}$ -homogeneity of ∇V^* and the relation $\nabla V = (\nabla V^*)^{-1}$, the above is equivalent to

$$\{\Phi^*(\nabla V) \leq s\} \subset \nabla V^*((s+t)\{\Phi \leq t\}^\circ) = (s+t)^{\frac{1}{p-1}} \nabla V^*(\{\Phi \leq t\}^\circ). \quad (21)$$

Using the ‘‘layer-cake’’ representation, we write

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-\Phi(x)} dx \cdot \left(\int_{\mathbb{R}^n} e^{-\frac{1}{p-1}\Phi^*(\nabla V)} dx \right)^{p-1} &= \\ \int_0^\infty e^{-t} |\{\Phi \leq t\}| dt \cdot \left(\int_0^\infty e^{-s} |\{\Phi^*(\nabla V) \leq s(p-1)\}| ds \right)^{p-1}. \end{aligned} \quad (22)$$

Consider the functions

$$f(t) = e^{-t} |\{\Phi \leq t\}|, \quad g(s) = e^{-s} |\{\Phi^*(\nabla V) \leq s(p-1)\}| \quad \text{and} \quad h(\tau) = e^{-\tau} |\{V \leq \tau\}|.$$

Letting $K_t = \{\Phi \leq t\}$, by (21) and the assumption of the Proposition, we get

$$\begin{aligned} |K_t|^{\frac{1}{p}} \cdot |\{\Phi^*(\nabla V) \leq s(p-1)\}|^{\frac{p-1}{p}} &\leq (s(p-1) + t)^{\frac{n}{p}} |K_t|^{\frac{1}{p}} \cdot |\nabla V^*(K_t^\circ)|^{\frac{p-1}{p}} \\ &\leq (s(p-1) + t)^{\frac{n}{p}} \left| \left\{ V \leq \frac{1}{p} \right\} \right| \\ &= \left| \left\{ V \leq \frac{s(p-1) + t}{p} \right\} \right|. \end{aligned}$$

Therefore,

$$h\left(\frac{1}{p}t + \frac{p-1}{p}s\right) \geq f(t)^{\frac{1}{p}} g(s)^{\frac{p-1}{p}},$$

and the conclusion follows by (22) combined with the Prékopa-Leindler inequality. \square

Remark 6.2. Note that the assumption of proposition 6.1 is equivalent to the inequality

$$|K| \cdot \left(\int_{K^\circ} \det D^2 V^*(x) dx \right)^{p-1} \leq \left| \left\{ V \leq \frac{1}{p} \right\} \right|^p.$$

Let us finally prove that inequality (6) follows from our generalized weighted functional Blaschke–Santaló inequality.

Lemma 6.3. Let Φ be a convex p -homogeneous function: $\Phi = |x|_K^p$. Here K is a convex symmetric set. Then

$$\int_{\mathbb{R}^n} e^{-\Phi(x)} dx \cdot \left(\int_{\mathbb{R}^n} e^{-\frac{1}{p-1}\Phi^*(\nabla V(x))} dx \right)^{p-1} = c(n, p) |K| \cdot |\nabla V^*(K^\circ)|^{p-1}$$

for some constant $c(n, p)$ depending only on n and p .

Proof. The proof is based on direct computations. First we apply polar coordinates:

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-\Phi} dx &= \int_{\mathbb{R}^n} e^{-|x|_K^p} dx = \int_0^\infty \left(\int_{\mathbb{S}^{n-1}} e^{-r^p |y|_K^p} \sigma(dy) \right) r^{n-1} dr = \int_{\mathbb{S}^{n-1}} \left(\int_0^\infty e^{-r^p |y|_K^p} r^{n-1} dr \right) \sigma(dy) \\ &= \int_{\mathbb{S}^{n-1}} \frac{1}{|y|_K^n} \left(\int_0^\infty e^{-s^p} s^{n-1} ds \right) \sigma(dy) = n \text{Vol}(K) \cdot \int_0^\infty e^{-s^p} s^{n-1} ds. \end{aligned}$$

As one may check:

$$\Phi^*(y) = \frac{p^{1-q}}{q} |y|_{K^\circ}^q.$$

Hence

$$\frac{1}{p-1} \Phi^*(\nabla V(x)) = \frac{p^{1-q}}{p} |\nabla V(x)|_{K^\circ}^q.$$

Applying definition of the Minkowski functional and homogeneity of V , one gets

$$\begin{aligned} |\nabla V(x)|_{K^\circ} &= \inf\{t : \nabla V(x) \in tK^\circ\} = \inf\{t : x \in \nabla V^*(tK^\circ)\} = \inf\{t : x \in t^{\frac{1}{p-1}} \nabla V^*(K^\circ)\} \\ &= \inf\{s^{p-1} : x \in s \nabla V^*(K^\circ)\} = |x|_{\nabla V^*(K^\circ)}^{p-1}. \end{aligned}$$

Thus we get

$$\frac{1}{p-1} \Phi^*(\nabla V(x)) = p^{-q} |x|_{\nabla V^*(K^\circ)}^{q(p-1)} = p^{-q} |x|_{\nabla V^*(K^\circ)}^p.$$

Applying polar coordinates again we deduce:

$$\int_{\mathbb{R}^n} e^{-\frac{1}{p-1}\Phi^*(\nabla V(x))} dx = \int_{\mathbb{R}^n} e^{-p^{-q} |x|_{\nabla V^*(K^\circ)}^p} dx = n \text{Vol}(\nabla V^*(K^\circ)) \cdot p^{\frac{nq}{p}} \int_0^\infty e^{-s^p} s^{n-1} ds.$$

Finally

$$\left(\int e^{-\frac{1}{p-1}\Phi^*(\nabla V(x))} dx \right)^{p-1} = \left(n \text{Vol}(\nabla V^*(K^\circ)) \right)^{p-1} \cdot p^n \left(\int_0^\infty e^{-s^p} s^{n-1} ds \right)^{p-1}.$$

□

Corollary 6.4. Let V be a p -homogeneous convex symmetric function. Inequality (5) holds for arbitrary convex symmetric Φ if and only if inequality (6) holds for arbitrary symmetric convex body K .

Proof. Implication (6) \implies (5) was proved in Proposition 6.1. To prove (5) \implies (6) let us take a symmetric convex body K and define $\Phi = |x|_K^p$. One has

$$\mathcal{BS}_{p,V}(\Phi) \leq \mathcal{BS}_{p,V}(V) = \mathcal{BS}_{p,V}(\alpha V),$$

where α is arbitrary positive constant. Note that in the last equality we used the invariance of $\mathcal{BS}_{p,V}$ with respect to homotheties and the homogeneity of V . Applying Lemma 6.3 one gets

$$|K| \cdot |\nabla V^*(K^\circ)|^{p-1} \leq |K_\alpha| \cdot |\nabla V^*(K_\alpha^\circ)|^{p-1}$$

where $K_\alpha = \{V \leq \frac{1}{\alpha}\}$. The result follows from the observation (the proof is left to the reader as an exercise) that

$$\nabla V^*(K_p^\circ) = K_p.$$

□

6.2 The case of rotation-invariant measures revisited

In this subsection we show that Theorem A in the case of rotationally invariant measures follows from the classical Blaschke–Santaló inequality.

Suppose $V = \frac{|x|^p}{p}$ and $V^* = \frac{|x|^q}{q}$, with p and q conjugate to each other. Then

$$\nabla^2 V^*(x) = |x|^{q-2} \text{Id} + (q-2)|x|^{q-4} x \otimes x,$$

and thus $\det(\nabla^2 V^*(x)) = (q-1)|x|^{n(q-2)}$. Therefore, the condition of Proposition 6.1 in the case $V(x) = \frac{|x|^p}{p}$ is: for any symmetric convex K and for $p, q \geq 1$, with $\frac{1}{p} + \frac{1}{q} = 1$,

$$(q-1)^{p-1} |K| \cdot \left(\int_{K^\circ} |x|^{n(q-2)} \right)^{p-1} \leq |B_2^n|^p,$$

which becomes, in view of the fact that $q = \frac{p}{p-1}$:

$$(q-1)^{\frac{1}{q-1}} |K| \cdot \left(\int_{K^\circ} |x|^{n(q-2)} \right)^{\frac{1}{q-1}} \leq |B_2^n|^{\frac{q}{q-1}}.$$

Therefore, the rotationally invariant case of Theorem A follows immediately from Proposition 6.1 and the following result.

Proposition 6.5. *For any symmetric convex K and any $q \in (1, 2]$, we have*

$$(q-1)^{\frac{1}{q-1}} |K| \cdot \left(\int_{K^\circ} |x|^{n(q-2)} \right)^{\frac{1}{q-1}} \leq |B_2^n|^{\frac{q}{q-1}}.$$

Remark 6.6. *Note that the condition $p \geq 2$ in (1) of Theorem A corresponds to the assumption $q \in [1, 2]$, since p and q are conjugate.*

We show that Proposition 6.5 follows immediately from the classical Blaschke–Santaló inequality.

Lemma 6.7. *For a convex body K we have*

$$\int_{K^\circ} |x|^{n(q-2)} dx = \frac{1}{(q-1)n} \int_{\mathbb{S}^{n-1}} h_K^{(1-q)n}(\theta) d\theta.$$

Proof. Using the polar coordinates, we write

$$\int_{K^\circ} |x|^{n(q-2)} dx = \int_{\mathbb{S}^{n-1}} \int_0^{\rho_{K^\circ}} t^{n-1+n(q-2)} dt,$$

and the equality follows from the fact that $h_K^{-1} = \rho_{K^\circ}$. □

Proof. Proof of Proposition 6.5] Using Lemma 6.7 combined with Hölder’s inequality, we write

$$\begin{aligned} (q-1)^{\frac{1}{q-1}}|K| \cdot \left(\int_{K^\circ} |x|^{n(q-2)} \right)^{\frac{1}{q-1}} &= (q-1)^{\frac{1}{q-1}}|K| \cdot \left(\frac{1}{(q-1)n} \int_{\mathbb{S}^{n-1}} h_K^{(1-q)n}(\theta) d\theta \right)^{\frac{1}{q-1}} \\ &\leq C(n, q)|K| \cdot \int_{\mathbb{S}^{n-1}} h_K^{-n}(\theta) d\theta \\ &= C'(n, q)|B_2^n|^2, \end{aligned}$$

where in the last step we used the Blaschke-Santaló inequality. Here $C(n, q)$ and $C'(n, q)$ are appropriate constants depending on n and p , such that equality is attained in all the inequalities above when K is B_2^n . This completes the proof of the Proposition. \square

6.3 A counterexample to the generalized Blaschke–Santaló inequality

Let

$$V(x) = \frac{1}{p}|x|_p^p = \frac{1}{p} \sum_{i=1}^n |x_i|^p.$$

Then

$$V^*(y) = \frac{1}{q}|y|_q^q = \frac{1}{q} \sum_{i=1}^n |y_i|^q$$

and

$$\det D^2 V^*(x) = (q-1)^n \prod_{i=1}^n |y_i|^{q-2} = (q-1)^n \prod_{i=1}^n |y_i|^{\frac{2-p}{p-1}}.$$

Thus Proposition 6.1 implies that the weighted Blaschke–Santaló inequality for V is equivalent to the following inequality for sets

$$(q-1)^{n(p-1)}|K| \left(\int_{K^\circ} \prod_{i=1}^n |y_i|^{\frac{2-p}{p-1}} dy \right)^{p-1} \leq |B_p^n|^p.$$

Equivalently

$$|K| \cdot \left(\int_{K^\circ} \prod_{i=1}^n |y_i|^{\frac{2-p}{p-1}} dy \right)^{p-1} \leq (p-1)^{(p-1)n} |B_p^n|^p. \quad (23)$$

Letting p tend to 1 we get:

$$|K| \cdot \sup_{y \in K^\circ} \prod_{i=1}^n |y_i| \leq \frac{2^n}{n!}. \quad (24)$$

This equality is not true in general. Indeed, the left hand side of (24) is not invariant under linear transformations. By considering $K = K_R$ to be a thin “needle” of length R pointing in the direction $(1, 1, \dots, 1)$, we see that

$$\lim_{R \rightarrow \infty} \left(|K_R| \cdot \sup_{y \in K_R^\circ} \prod_{i=1}^n |y_i| \right) = \infty.$$

We conclude that the assumption of Proposition 6.1 is false for $V(x) = \frac{|x|_p^p}{p}$ when p is close to 1.

We conclude that inequality (5) fails to hold for values of p close to 1. We will show in the last section that (5) fails to hold for $1 < p < 2$.

We show, however, that the inequality for sets in question holds when Φ is 1-symmetric and $p = 1$; this also follows from the unconditional part of Theorem A.

Proposition 6.8. *Let K be a 1-symmetric convex body. Then*

$$|K| \cdot \sup_{y \in K^\circ} \prod_{i=1}^n |y_i| \leq \frac{2^n}{n!}.$$

Proof. By the arithmetic-geometric mean inequality,

$$\sup_{y \in K^o} \prod_{i=1}^n |y_i| \leq \sup_{y \in K^o} \left(\frac{|y|_1}{n} \right)^n = \left(\frac{\sup_{y \in K^o, x \in B_\infty^n} \langle x, y \rangle}{n} \right)^n.$$

Suppose that the largest centered cube contained in K is RB_∞^n , $R > 0$. Then K^o is contained in $\frac{1}{R}B_1^n$, and therefore the above is bounded by

$$\left(\frac{\sup_{y \in B_1^n, x \in B_\infty^n} \langle x, y \rangle}{Rn} \right)^n = \frac{1}{(Rn)^n}.$$

On the other hand, since the largest centered cube that K contains is RB_∞^n , and K is 1-symmetric, we conclude that $K \subset RnB_1^n$, and therefore

$$|K| \leq (Rn)^n \frac{2^n}{n!}.$$

These facts imply the statement. \square

7 The mass transport approach to the Blaschke-Santaló-type inequalities

7.1 The Euler-Lagrange equation of \mathcal{BS}

In this subsection we derive the Euler-Lagrange equation of the Blaschke-Santaló functional. We realise that this is an equation of the Monge–Ampère type. In the following subsection we prove a kind of more precise statement: a monotonicity property of our functional, which also leads to this equation.

The following lemma is known to experts; we include it for the reader's convenience.

Lemma 7.1. *Let $V \in C^3(\mathbb{R}^n)$ be such that $D^2V(x)$ is positive definite for every x , and let $f \in C^1(\mathbb{R}^n)$ be compactly supported. Then*

$$V_\varepsilon = V + \varepsilon f.$$

Then V_ε^* can be expanded in the following way:

$$V_\varepsilon^* = V^* - \varepsilon f(\nabla V^*) + \frac{\varepsilon^2}{2} \langle D^2V^* \nabla f(\nabla V^*), \nabla f(\nabla V^*) \rangle + o(\varepsilon^2).$$

The dependence of the term $o(\varepsilon^2)$ on x in this expansion is uniform on $\nabla V(\text{supp}(f))$.

Proof. Expand V_ε^* :

$$V_\varepsilon^* = V^* + \varepsilon a + \frac{\varepsilon^2}{2} b + o(\varepsilon^2)$$

and apply relation $V_\varepsilon^*(\nabla V_\varepsilon) = \langle x, \nabla V_\varepsilon \rangle - V_\varepsilon$. In this way we obtain

$$\begin{aligned} V^*(\nabla V_\varepsilon) + \varepsilon a(\nabla V_\varepsilon) + \frac{\varepsilon^2}{2} b(\nabla V_\varepsilon) + o(\varepsilon^2) &= \langle x, \nabla V_\varepsilon \rangle - V_\varepsilon \\ &= V^*(\nabla V) + \varepsilon(\langle x, \nabla f \rangle - f). \end{aligned} \tag{25}$$

The final result follows from the expansions:

$$\begin{aligned} V^*(\nabla V_\varepsilon) &= V^*(\nabla V) + \varepsilon \langle x, \nabla f \rangle + \frac{\varepsilon^2}{2} \langle D^2V^*(\nabla V) \nabla f, \nabla f \rangle + o(\varepsilon^2) \\ a(\nabla V_\varepsilon) &= a(\nabla V) + \varepsilon \langle \nabla a(\nabla V), \nabla f \rangle + o(\varepsilon^2). \end{aligned}$$

Indeed, expanding both sides of (25) one gets

$$a(\nabla V) = -f, \quad \frac{1}{2}\langle D^2V^*(\nabla V)\nabla f, \nabla f \rangle + \langle \nabla a(\nabla V), \nabla f \rangle + \frac{1}{2}b(\nabla V) = 0.$$

Expressing a and b from these equation and changing variables one gets the statement. \square

Proposition 7.2. *Let μ_1 and μ_2 be admissible measure. Let Φ be a maximizer of $\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}$ in \mathcal{C} , and assume that*

$$\Phi \in C^2(\mathbb{R}^n) \quad \text{and} \quad D^2\Phi(x) > 0 \quad \forall x \in \mathbb{R}^n.$$

Then

$$\frac{\int_{\mathbb{R}^n} e^{-\alpha\Phi} \rho_1 dx}{\int_{\mathbb{R}^n} e^{-\alpha\Phi} \rho_1 dx} = \frac{\int_{\mathbb{R}^n} e^{-\beta\Phi^*(\nabla\Phi)} \det(D^2\Phi) \rho_2(\nabla\Phi) dy}{\int_{\mathbb{R}^n} e^{-\beta\Phi^*} \rho_2 dy}.$$

Proof. Let $\tau \in C_c^\infty(\mathbb{R}^n)$ (where the lower index c means compact support); that is, τ is a test function. For $\varepsilon > 0$ sufficinetly small in absolute value, the function

$$\Phi_\varepsilon = \Phi + \varepsilon\tau$$

belongs to \mathcal{C} (here we are using the assumption that $D^2\Phi > 0$ and that τ has compact support). Hence the function

$$\varepsilon \rightarrow \mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(\Phi_\varepsilon)$$

has a maximum for $\varepsilon = 0$. Therefore

$$\left. \frac{d}{d\varepsilon} \mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(\Phi_\varepsilon) \right|_{\varepsilon=0} = 0.$$

On the other hand

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(\Phi_\varepsilon) \right|_{\varepsilon=0} &= \left. \frac{d}{d\varepsilon} \left(\int_{\mathbb{R}^n} e^{-\alpha\Phi_\varepsilon} \rho_1 dx \right)^\frac{1}{\alpha} \right|_{\varepsilon=0} \cdot \left(\int_{\mathbb{R}^n} e^{-\beta\Phi^*} \rho_2 dx \right)^\frac{1}{\beta} + \\ &\quad \left(\int_{\mathbb{R}^n} e^{-\alpha\Phi} \rho_1 dx \right)^\frac{1}{\alpha} \cdot \left. \frac{d}{d\varepsilon} \left(\int_{\mathbb{R}^n} e^{-\beta\Phi_\varepsilon^*} \rho_2 dx \right)^\frac{1}{\beta} \right|_{\varepsilon=0} \\ &= - \int_{\mathbb{R}^n} \tau e^{-\alpha\Phi} \rho_1 dx \left(\int_{\mathbb{R}^n} e^{-\alpha\Phi_\varepsilon} \rho_1 dx \right)^\frac{1}{\alpha}-1 \left(\int_{\mathbb{R}^n} e^{-\beta\Phi^*} \rho_2 dx \right)^\frac{1}{\beta} + \\ &\quad \int_{\mathbb{R}^n} \tau(\nabla\Phi^*) e^{-\beta\Phi^*} \rho_2 dy \left(\int_{\mathbb{R}^n} e^{-\alpha\Phi_\varepsilon} \rho_1 dx \right)^\frac{1}{\alpha} \left(\int_{\mathbb{R}^n} e^{-\beta\Phi^*} \rho_2 dx \right)^\frac{1}{\beta}-1 \end{aligned}$$

where we have used Lemma 7.1. By the change of variable $\nabla\Phi^*(y) = x$ we get

$$\int_{\mathbb{R}^n} \tau(\nabla\Phi^*) e^{-\beta\Phi^*} \rho_2 dy = \int_{\mathbb{R}^n} \tau e^{-\beta\Phi^*(\nabla\Phi)} \det(D^2\Phi) \rho_2(\nabla\Phi) dx$$

We deduce that

$$\int_{\mathbb{R}^n} \tau \left[I_2 e^{-\alpha\Phi} \rho_1 - I_1 e^{-\beta\Phi^*(\nabla\Phi)} \det(D^2\Phi) \rho_2(\nabla\Phi) \right] dx = 0,$$

where

$$I_1 = \int_{\mathbb{R}^n} e^{-\alpha\Phi} \rho_1 dx, \quad I_2 = \int_{\mathbb{R}^n} e^{-\beta\Phi^*} \rho_2 dx.$$

As τ is arbitrary, the conclusion follows. \square

7.2 A comparison result for Blaschke–Santaló functional via optimal transportation

In this section we present a comparison result for the Blaschke–Santaló functional, involving optimal transportation. We will show that the functional

$$\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(\Phi) = \left(\int e^{-\alpha\Phi} \rho_1 dx \right)^{\frac{1}{\alpha}} \left(\int e^{-\beta\Phi^*} \rho_2 dy \right)^{\frac{1}{\beta}}$$

admits a remarkable monotonicity property related to optimal transportation. This property was already mentioned in [48] for $\alpha = \beta = 1$. The idea of the proof is essentially the same. However, the statement about maximum points of the Blaschke–Santaló functional that we prove here is more precise even for values $\alpha = \beta = 1$.

Let μ_1, μ_2 be non-negative Borel measures on \mathbb{R}^n , absolutely continuous with respect to the Lebesgue measure, with strictly positive densities ρ_1 and ρ_2 , respectively. Let $\Phi: \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$ be a convex function; assume that

$$\Omega := \text{int}(\text{dom}(\Phi)) \quad \text{and} \quad \Omega^* := \text{int}(\text{dom}(\Phi^*))$$

are non-empty, and

$$0 < \int_{\mathbb{R}^n} e^{-\Phi} d\mu_1, \int_{\mathbb{R}^n} e^{-\Phi^*} d\mu_2 < \infty.$$

Let $\alpha > 0, \beta > 0$ and $\nabla U: \Omega \rightarrow \Omega^*$ be the optimal transportation of the measure μ with density

$$\frac{1}{\int_{\mathbb{R}^n} e^{-\alpha\Phi} d\mu_1} e^{-\alpha\Phi} \rho_1$$

onto the measure ν with density

$$\frac{1}{\int_{\mathbb{R}^n} e^{-\beta\Phi^*} d\mu_2} e^{-\beta\Phi^*} \rho_2.$$

Let us assume that

1. $U(0) = \Phi(0)$,
2. Φ and U are lower semi-continuous,
3. $\Phi = +\infty$ on $\{U = +\infty\}$.

Remark 7.3. *It is easy to verify that there exists a unique function U satisfying assumptions 1)-3). Note that uniqueness for optimal transportation guarantees only that $T = \nabla U$ is uniquely determined μ -a.e. In particular, potentials U_i giving the same mapping T can be different outside of support μ . But in our case the support of μ is convex and U is supposed to take infinite values outside of it. This implies the uniqueness of U .*

We set

$$\nabla U(\Omega) = \{\nabla U(x) : x \in \Omega, \partial U(x) \text{ contains a unique element}\}.$$

Proposition 7.4. *In the previous assumptions*

$$\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(\Phi) \leq \mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(U). \tag{26}$$

Moreover, equality holds if and only if

$$\Phi = U \quad \text{in } \Omega,$$

and $\nabla U(\Omega)$ coincides with $\{U^* < \infty\}$ up to a set of Lebesgue measure zero.

Proof. Obviously

$$\int_{\mathbb{R}^n} e^{-\alpha\Phi} \rho_1 dx = \int_{\Omega} e^{-\alpha\Phi} \rho_1 dx, \quad \int_{\mathbb{R}^n} e^{-\beta\Phi^*} \rho_2 dy = \int_{\Omega^*} e^{-\beta\Phi^*} \rho_2 dy.$$

By convexity, Φ and Φ^* are continuous on Ω and Ω^* , respectively. By the change of variables formula (13)

$$\frac{e^{-\alpha\Phi(x)} \rho_1(x)}{\int_{\Omega} e^{-\alpha\Phi} d\mu_1} = \frac{e^{-\beta\Phi^*(\nabla U(x))} \rho_2(\nabla U)}{\int_{\Omega^*} e^{-\beta\Phi^*} d\mu_2} \det D_a^2 U(x)$$

almost everywhere in Ω .

Take the power $\frac{\alpha}{\alpha+\beta}$ of both sides of the last equality:

$$\left(\frac{\int_{\Omega^*} e^{-\beta\Phi^*} d\mu_2}{\int_{\Omega} e^{-\alpha\Phi} d\mu_1} \right)^{\frac{\alpha}{\alpha+\beta}} e^{-\frac{\alpha}{\alpha+\beta}\Phi(x)} \rho_1^{\frac{\alpha}{\alpha+\beta}}(x) = e^{-\frac{\alpha}{\alpha+\beta}\Phi^*(\nabla U(x))} \rho_2^{\frac{\alpha}{\alpha+\beta}}(\nabla U) (\det D_a^2 U(x))^{\frac{\alpha}{\alpha+\beta}}.$$

Multiply this identity by $e^{-\frac{\beta}{\alpha+\beta}\Phi} \rho_1^{\frac{\beta}{\alpha+\beta}}$:

$$\left(\frac{\int_{\Omega^*} e^{-\beta\Phi^*} d\mu_2}{\int_{\Omega} e^{-\alpha\Phi} d\mu_1} \right)^{\frac{\alpha}{\alpha+\beta}} e^{-\alpha\Phi(x)} \rho_1(x) = e^{-\frac{\alpha\beta}{\alpha+\beta}[\Phi(x)+\Phi^*(\nabla U(x))]} \rho_1^{\frac{\beta}{\alpha+\beta}} \rho_2^{\frac{\alpha}{\alpha+\beta}}(\nabla U) (\det D_a^2 U(x))^{\frac{\alpha}{\alpha+\beta}}.$$

Integrating over Ω , we obtain

$$\begin{aligned} & \left(\int_{\Omega} e^{-\alpha\Phi} \rho_1 dx \right)^{\frac{\beta}{\alpha+\beta}} \left(\int_{\Omega^*} e^{-\beta\Phi^*} \rho_2 dy \right)^{\frac{\alpha}{\alpha+\beta}} \\ &= \int_{\Omega} e^{-\frac{\alpha\beta}{\alpha+\beta}[\Phi(x)+\Phi^*(\nabla U(x))]} \rho_1^{\frac{\beta}{\alpha+\beta}} \rho_2^{\frac{\alpha}{\alpha+\beta}}(\nabla U) (\det D_a^2 U(x))^{\frac{\alpha}{\alpha+\beta}} dx. \end{aligned}$$

As

$$\Phi^*(\nabla U(x)) + \Phi(x) \geq \langle x, \nabla U(x) \rangle,$$

while

$$U^*(\nabla U(x)) + U(x) = \langle x, \nabla U(x) \rangle,$$

for every $x \in \Omega$, we get

$$\begin{aligned} \left(\int_{\Omega} e^{-\alpha\Phi} \rho_1 dx \right)^{\frac{\beta}{\alpha+\beta}} \left(\int_{\Omega^*} e^{-\beta\Phi^*} \rho_2 dy \right)^{\frac{\alpha}{\alpha+\beta}} &\leq \int_{\Omega} e^{-\frac{\alpha\beta}{\alpha+\beta}[U(x)+U^*(\nabla U(x))]} \rho_1^{\frac{\beta}{\alpha+\beta}} \rho_2^{\frac{\alpha}{\alpha+\beta}}(\nabla U) (\det D_a^2 U(x))^{\frac{\alpha}{\alpha+\beta}} dx \\ &\leq \left(\int_{\Omega} e^{-\alpha U} \rho_1 dx \right)^{\frac{\beta}{\alpha+\beta}} \left(\int_{\Omega} e^{-\beta U^*(\nabla U)} \rho_2(\nabla U) \det D_a^2 U dx \right)^{\frac{\alpha}{\alpha+\beta}} \\ &= \left(\int_{\Omega} e^{-\alpha U} \rho_1 dx \right)^{\frac{\beta}{\alpha+\beta}} \left(\int_{\nabla U(\Omega)} e^{-\beta U^*} \rho_2 dy \right)^{\frac{\alpha}{\alpha+\beta}} \\ &\leq \left(\int_{\mathbb{R}^n} e^{-\alpha U} \rho_1 dx \right)^{\frac{\beta}{\alpha+\beta}} \left(\int_{\mathbb{R}^n} e^{-\beta U^*} \rho_2 dy \right)^{\frac{\alpha}{\alpha+\beta}}, \end{aligned}$$

where we have used Hölder inequality in the second step, and the change of variable formula in the third one. This proves (26).

Assume now that

$$\mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(\Phi) = \mathcal{BS}_{\alpha,\beta,\rho_1,\rho_2}(U).$$

This is possible only if

$$\Phi^*(\nabla U(x)) + \Phi(x) = \langle x, \nabla U(x) \rangle, \tag{27}$$

on Ω . On the other hand, if $x \in \Omega$ and Φ is differentiable at x , then

$$\Phi^*(\nabla\Phi(x)) + \Phi(x) = \langle x, \nabla\Phi(x) \rangle, \quad (28)$$

and $\nabla\Phi(x)$ is the only vector for which (28) is valid. Hence

$$\nabla U(x) = \nabla\Phi(x)$$

for almost every $x \in \Omega$. Together with assumption $V(0) = \Phi(0)$ and lower semi-continuity, this clearly implies $\Phi = U$.

In addition, by the argument used in the first part of this proof, equality is possible if and only if $\nabla U(\Omega) = \{U^* < \infty\}$ up to a set of zero measure, this implies the last statement of the Theorem. \square

Theorem 7.5. *Let $\alpha, \beta > 0$ be numbers and ρ_1, ρ_2 be positive functions. Assume that Φ is a maximum point of the functional $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}$. Then $\nabla\Phi$ is the optimal transportation pushing forward measure μ onto ν , where*

$$d\mu = \frac{e^{-\alpha\Phi} \rho_1 dx}{\int e^{-\alpha\Phi} \rho_1 dx}, \quad d\nu = \frac{e^{-\beta\Phi^*} \rho_2 dy}{\int e^{-\beta\Phi^*} \rho_2 dy}.$$

In addition,

$$\nabla\Phi(\Omega) = \{\Phi^* < \infty\}$$

up to a set of Lebesgue measure zero.

Proof. The result follows immediately from Proposition 7.4. \square

Remark 7.6. *Another remarkable monotonicity property in of the Blaschke–Santaló functional in terms of a Gaussian diffusion semigroup has been recently obtained by Nakamura and Tsuji in [56]. This result provides an alternative and purely analytical proof of the classical Blaschke–Santaló inequality.*

7.3 Brascamp–Lieb type inequality for maximizers

In this subsection we derive partial differential inequality for the maximum point of $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}$ and we observe that this is an inequality of the Brascamp–Lieb type.

Proposition 7.7. *Let Φ be the maximum point of $\mathcal{BS}_{\alpha, \beta, \rho_1, \rho_2}$. Assume, in addition, that Φ is strictly convex and twice continuously differentiable inside of $\{\Phi < \infty\}$. Then the measure μ with density*

$$\frac{e^{-\alpha\Phi} \rho_1}{\int e^{-\alpha\Phi} \rho_1 dx}$$

satisfies

$$\text{Var}_\mu f \leq \frac{1}{\alpha + \beta} \int \langle (D^2\Phi)^{-1} \nabla f, \nabla f \rangle d\mu.$$

for every function $f \in C^1(\mathbb{R}^n)$.

Proof. Take f satisfying $\int f e^{-\alpha\Phi} \rho_1 dx = 0$. One has

$$\frac{1}{\alpha} \log \int e^{-\alpha\Phi_\varepsilon} \rho_1 dx + \frac{1}{\beta} \log \int e^{-\beta\Phi_\varepsilon^*} \rho_2 dy \leq \frac{1}{\alpha} \log \int e^{-\alpha\Phi} \rho_1 dx + \frac{1}{\beta} \log \int e^{-\beta\Phi^*} \rho_2 dy,$$

where $\Phi_\varepsilon = \Phi + \varepsilon f$. Lemma 7.1 implies

$$\begin{aligned} \log \int e^{-\alpha\Phi_\varepsilon} \rho_1 dx &= \log \int e^{-\alpha\Phi} \left(1 - \varepsilon\alpha f + \frac{\varepsilon^2\alpha^2}{2} f^2 + o(\varepsilon^2) \right) \rho_1 dx \\ &= \log \int e^{-\alpha\Phi} \left(1 + \frac{\varepsilon^2\alpha^2}{2} f^2 + o(\varepsilon^2) \right) \rho_1 dx \\ &= \log \int e^{-\alpha\Phi} \rho_1 dx + \frac{\varepsilon^2\alpha^2}{2} \frac{\int f^2 e^{-\alpha\Phi} \rho_1 dx}{\int e^{-\alpha\Phi} \rho_1 dx} + o(\varepsilon^2), \end{aligned}$$

$$\begin{aligned}
\log \int e^{-\beta\Phi_\varepsilon^*} \rho_2 dx &= \log \int e^{-\beta\Phi^*} e^{\beta(\varepsilon f(\nabla\Phi^*) - \frac{\varepsilon^2}{2} \langle D^2\Phi^* \nabla f(\nabla\Phi^*), \nabla f(\nabla\Phi^*) \rangle + o(\varepsilon^2))} \rho_2 dy \\
&= \log \int e^{-\beta\Phi^*} \left(1 + \varepsilon\beta f(\nabla\Phi^*) - \frac{\varepsilon^2\beta}{2} \langle D^2\Phi^* \nabla f(\nabla\Phi^*), \nabla f(\nabla\Phi^*) \rangle + \frac{\varepsilon^2\beta^2}{2} f^2(\nabla\Phi^*) + o(\varepsilon^2) \right) \rho_2 dy.
\end{aligned}$$

Since $\nabla\Phi^*$ sends the measure $\frac{e^{-\beta\Phi^*} \rho_2 dy}{\int e^{-\beta\Phi^*} \rho_2 dy}$ to the measure $\frac{e^{-\alpha\Phi} \rho_1 dx}{\int e^{-\alpha\Phi} \rho_1 dx}$, one gets

$$\int f(\nabla\Phi^*) e^{-\beta\Phi^*} \rho_2 dy = C \int f e^{-\alpha\Phi} \rho_1 dx = 0.$$

Thus

$$\begin{aligned}
\log \int e^{-\beta\Phi_\varepsilon^*} \rho_2 dy &= \log \int e^{-\beta\Phi^*} \left(1 - \frac{\varepsilon^2\beta}{2} \langle D^2\Phi^* \nabla f(\nabla\Phi^*), \nabla f(\nabla\Phi^*) \rangle + \frac{\varepsilon^2\beta^2}{2} f^2(\nabla\Phi^*) + o(\varepsilon^2) \right) \rho_2 dx \\
&= \log \int e^{-\beta\Phi^*} \rho_2 dy + \frac{\varepsilon^2}{2 \int e^{-\beta\Phi^*} \rho_2 dy} \int \left(\beta^2 f^2(\nabla\Phi^*) - \beta \langle D^2\Phi^* \nabla f(\nabla\Phi^*), \nabla f(\nabla\Phi^*) \rangle \right) e^{-\beta\Phi^*} \rho_2 dy + o(\varepsilon^2) \\
&= \log \int e^{-\beta\Phi^*} \rho_2 dy + \frac{\varepsilon^2}{2 \int e^{-\alpha\Phi} \rho_1 dx} \int \left(\beta^2 f^2 - \beta \langle (D^2\Phi)^{-1} \nabla f, \nabla f \rangle \right) e^{-\alpha\Phi} \rho_1 dx + o(\varepsilon^2).
\end{aligned}$$

Finally, one gets the relation

$$\begin{aligned}
\frac{1}{\alpha} \log \int e^{-\alpha\Phi_\varepsilon} \rho_1 dx + \frac{1}{\beta} \log \int e^{-\beta\Phi_\varepsilon^*} \rho_2 dy &= \frac{1}{\alpha} \log \int e^{-\alpha\Phi} \rho_1 dx + \frac{1}{\beta} \log \int e^{-\beta\Phi^*} \rho_2 dy \\
&\quad + \frac{\varepsilon^2}{\int e^{-\alpha\Phi} \rho_1 dx} \left(\int \left(\frac{\alpha + \beta}{2} f^2 - \frac{1}{2} \langle (D^2\Phi)^{-1} \nabla f, \nabla f \rangle \right) e^{-\alpha\Phi} \rho_1 dx \right)
\end{aligned}$$

and the claim follows. \square

Let $p > 1$ and V be a p -homogeneous convex function. The latter means, in particular,

$$V = \frac{1}{p-1} V^*(\nabla V). \quad (29)$$

Consider functional

$$\int e^{-\Phi} dx \cdot \left(\int e^{-\frac{1}{p-1}\Phi^*} \rho dy \right)^{p-1} = \int e^{-\Phi} dx \cdot \left(\int e^{-\frac{1}{p-1}\Phi^*(\nabla V)} dx \right)^{p-1},$$

where ρdy is the image of the Lebesgue measure under ∇V , in particular

$$\rho = \det D^2 V^*.$$

Note that ∇V is the optimal transportation mapping of $\frac{e^{-V} dx}{\int e^{-V} dx}$ onto $\frac{e^{-\frac{1}{p-1}V^*} \rho dy}{\int e^{-\frac{1}{p-1}V^*} \rho dy}$. This follows from (29) and the fact that ρdy is the image of Lebesgue measure under ∇V .

Thus we get that V is a natural candidate to maximize $\mathcal{BS}_{p,V}$. The expected inequality reads as:

$$\int e^{-\Phi} dx \cdot \left(\int e^{-\frac{1}{p-1}\Phi^*(\nabla V)} dx \right)^{p-1} \leq \left(\int e^{-V} dx \right)^p.$$

In particular, V satisfies the corresponding Euler–Lagrange equation. The second order condition obtained in Proposition 7.7 means that if V is a maximizer, it must satisfy the inequality.

$$\text{Var}_\mu f \leq \left(1 - \frac{1}{p} \right) \int \langle (D^2 V)^{-1} \nabla f, \nabla f \rangle d\mu,$$

where $\mu = \frac{e^{-V} dx}{\int e^{-V} dx}$.

7.4 Homogeneity of maximizers of \mathcal{BS} -functional

In this section we consider a smooth (say, $C^3(\mathbb{R}^n)$) and strictly convex function $V: \mathbb{R}^n \rightarrow \mathbb{R}$; we will refer to V as a *potential*. It will be assumed throughout that V is a p -homogeneous convex function for some fixed $p > 1$. The proof of the following properties will be left to the reader as an exercise.

Proposition 7.8. *Let $V \in C^3(\mathbb{R}^n)$ be a convex, even p -homogeneous function. Then V verifies the following properties.*

1. $\langle \nabla V(x), x \rangle = pV(x)$;
2. $(p-1)V = V^*(\nabla V)$;
3. $(p-1)\nabla V(x) = D^2V(x) \cdot x$;
4. for every vector e one has $(D^2V(x))_e \cdot x = (p-2)D^2V(x) \cdot e$

(where $(\cdot)_e$ indicates partial differentiation along e).

We consider the Blaschke-Santaló functional

$$\mathcal{BS}_{p,V}(\Phi) = \int e^{-\Phi} dx \cdot \left(\int e^{-\frac{1}{p-1}\Phi^*} \rho dy \right)^{p-1} = \int e^{-\Phi} dx \cdot \left(\int e^{-\frac{1}{p-1}\Phi^*(\nabla V)} dx \right)^{p-1},$$

where ρdy is the image of Lebesgue measure under ∇V . Note that ∇V is the optimal transportation mapping of $\frac{e^{-V} dx}{\int e^{-V} dx}$ onto $\frac{e^{-\frac{1}{p-1}V^*} \rho dy}{\int e^{-\frac{1}{p-1}V^*} \rho dy}$. This follows from (29) and the fact that ρdy is the image of Lebesgue measure under ∇V .

Theorem 7.9. *Any symmetric maximum point of the functional*

$$\mathcal{BS}_{p,V}(\Phi) = \int e^{-\Phi} dx \cdot \left(\int e^{-\frac{1}{p-1}\Phi^*(\nabla V)} dx \right)^{p-1}$$

is p -homogeneous (up to addition of a constant).

We start with some preliminary considerations. If Φ is a maximum point, Theorem 7.5 implies that:

1. $\nabla \Phi$ is the optimal transportation mapping between the measures

$$\mu = \frac{e^{-\Phi} dx}{\int_{\mathbb{R}^n} e^{-\Phi} dx} \quad \text{and} \quad \nu = \frac{e^{-\frac{1}{p-1}\Phi^*} \det D^2V^* dy}{\int_{\mathbb{R}^n} e^{-\frac{1}{p-1}\Phi^*} \det D^2V^* dy},$$

2. $\nabla \Phi(\mathbb{R}^n) = \{\Phi^* < \infty\}$.

Without loss of generality we assume

$$\Phi(0) = \Phi^*(0) = 0.$$

First we observe that Φ, Φ^* are smooth on $\mathbb{R}^n \setminus \{0\}$ provided V is smooth on $\mathbb{R}^n \setminus \{0\}$ (we can not assume that V is smooth on entire \mathbb{R}^n , because V is p -homogeneous). Indeed, to prove this we apply local Hölder estimates for solution to the Monge–Ampère equation. We refer to [41], proof of Lemma 5.2. We choose a neighbourhood U of a point $y_0 \neq 0$ which does not contain the origin. Using that $\det D^2V^*$ is smooth inside of U , we consider equation

$$C \det D^2V^* e^{-\frac{1}{p-1}\Phi^*} = e^{-\Phi(\nabla \Phi^*)} \det D^2\Phi^*,$$

where C is the corresponding normalizing constant, and apply the Forzani–Maldonado estimate (see [27]) to ensure that $\nabla \Phi^*$ is locally Hölder and then the Trudinger–Wang estimates (see [62]) to prove higher regularity. See details in [41]. Applying standard bootstrapping arguments we can conclude that Φ^* is smooth on $\Omega^* \setminus \{0\}$ and the similar statement holds for Φ .

Next we prove a lemma about behaviour of Φ near the boundary of $\{\Phi < \infty\}$.

Lemma 7.10. $\partial\Phi(x)$ is empty for every $x \in \partial\Omega$ and $\partial\Phi^*(x)$ is empty for every $y \in \partial\Omega^*$. In particular, $|\nabla\Phi(x_n)| \rightarrow \infty$ for every sequence $x_n \rightarrow x$, where $x \in \partial\Omega$.

Proof. Assume that $a \in \partial\Phi(x)$, $x \in \partial\Omega$. Then $a + tn \in \partial\Phi(x)$ for every $t \geq 0$, where n is the outer normal to $\partial\Omega$ at x . In one hand, we note that by strict convexity of Φ one has: $L_a = \{a + tn \in \partial\Phi(x), t > 0\} \subset \mathbb{R}^n \setminus \nabla\Phi(\Omega)$. In the other hand, we note that

$$\Phi^*(a + nt) = \langle x, a + nt \rangle - \Phi(x) < \infty.$$

Hence Φ^* is finite on L_a and by convexity Φ^* is finite in some neighborhood \tilde{L} of L_a . Hence there exists an open set $\tilde{L} \subset \mathbb{R}^n \setminus \nabla\Phi(\Omega)$ with the property $L \subset \Omega^*$. But this contradicts to the fact that $\nabla\Phi(\Omega) = \Omega^*$ (up to a set of measure zero), see Theorem 7.5. \square

In what follows we consider function

$$W(x) = \langle x, \nabla\Phi(x) \rangle - p\Phi(x).$$

In a sense, W “measures the homogeneity” of Φ . If Φ is p -homogeneous, then $W = 0$ according to Proposition 7.8.

We work with the operator

$$Lf = \text{Tr} \left[(D^2\Phi)^{-1} D^2 f \right] - \left\langle \nabla f, \frac{x}{p-1} - [\nabla \log \det D^2 V^*] \circ \nabla \Phi \right\rangle,$$

described in Section 2. Recall that L is symmetric with respect to μ : for every smooth g vanishing in a neighbourhood of $\partial\Omega \cup \{0\}$ one has

$$- \int Lfg d\mu = \int \langle (D^2\Phi)^{-1} \nabla f, \nabla g \rangle d\mu.$$

The following lemma is proved by direct computations (and differentiation of change of variables).

Lemma 7.11. *The following equation holds for all $x \in \Omega \setminus \{0\}$:*

$$LW(x) = 0.$$

Proof. One has

$$\nabla W(x) = D^2\Phi(x) \cdot x + (1-p)\nabla\Phi(x).$$

For every $e \in \mathbb{R}^n$

$$D^2W \cdot e = (2-p)D^2\Phi \cdot e + (D^2\Phi)_e \cdot x,$$

(where $(\cdot)_e$ indicates partial differentiation along e). Therefore, for every $e_i, e_j \in \mathbb{R}^n$,

$$W_{e_i e_j} = \langle D^2W \cdot e_i, e_j \rangle = (2-p)\Phi_{e_i e_j} + \langle \nabla\Phi_{e_i e_j}, x \rangle = (2-p)\Phi_{e_i e_j} + \sum_{k=1}^n \Phi_{e_i e_j e_k} x_k.$$

Thus

$$LW = \sum_{k=1}^n \langle (D^2\Phi)^{-1} (D^2\Phi)_{e_k} \cdot x, e_k \rangle - \langle D^2\Phi \cdot x + (1-p)\nabla\Phi, \frac{x}{p-1} - [\nabla \log \det D^2 V^*] \circ \nabla \Phi \rangle + (2-p)n.$$

Let $e_k, k \in \{1, \dots, n\}$, be a basis of eigenvectors for $D^2\Phi(x)$ at x :

$$D^2\Phi(x)e_k = \lambda_k e_k,$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $D^2\Phi(x)$. One gets the following expression for LW :

$$LW = \sum_{k,j=1}^n \frac{\Phi_{e_k e_k e_j} \cdot x_j}{\lambda_k} - \langle D^2\Phi \cdot x + (1-p)\nabla\Phi, \frac{x}{p-1} - [\nabla \log \det D^2V^*] \circ \nabla\Phi \rangle + (2-p)n. \quad (30)$$

Taking logarithms in the transport equation for $\nabla\Phi$ we get

$$\Phi = \frac{\Phi^*(\nabla\Phi)}{p-1} - \log \det D^2\Phi - \log \det D^2V^*(\nabla\Phi) + C$$

for some constant C . Differentiating this equation along the vector field x we get

$$\nabla\Phi = \frac{1}{p-1}D^2\Phi \cdot x - \sum_{j=1}^n \text{Tr}(D^2\Phi)^{-1}(D^2\Phi)_{e_j} \cdot e_j - D^2\Phi \cdot [\nabla \log \det D^2V^*] \circ \nabla\Phi.$$

Here we have used the differentiation formula for determinants: $\partial_e \log \det D^2\Phi = \text{Tr}(D^2\Phi)^{-1}(D^2\Phi)_e$.

Whence

$$\langle \nabla\Phi, x \rangle = \frac{1}{p-1} \langle D^2\Phi \cdot x, x \rangle - \sum_{k,j=1}^n \frac{\Phi_{e_k e_k e_j} x_j}{\lambda_k} \cdot x_j - \langle D^2\Phi \cdot [\nabla \log \det D^2V^*] \circ \nabla\Phi, x \rangle.$$

Putting this expression into (30) one gets

$$\begin{aligned} LW &= \frac{1}{p-1} \langle D^2\Phi \cdot x, x \rangle - \langle \nabla\Phi, x \rangle - \langle D^2\Phi \cdot [\nabla \log \det D^2V^*] \circ \nabla\Phi, x \rangle \\ &\quad - \langle D^2\Phi \cdot x + (1-p)\nabla\Phi, \frac{x}{p-1} - [\nabla \log \det D^2V^*] \circ \nabla\Phi \rangle \\ &= (1-p) \langle \nabla\Phi, [\nabla \log \det D^2V^*] \circ \nabla\Phi \rangle + n(2-p) \\ &= (1-p) \langle x, \nabla \log \det D^2V^* \rangle \circ \nabla\Phi + n(2-p). \end{aligned}$$

The claim follows from the observation that $\det D^2V^*$ is a $n\frac{2-p}{p-1}$ -homogeneous function, hence

$$\langle x, \nabla \log \det D^2V^* \rangle = n\frac{2-p}{p-1}.$$

□

Proof of Theorem 7.9. We prove that W is constant. Set:

$$B = \{x \in \partial\Omega, \Phi(x) < \infty\},$$

$$B_\infty = \{x \in \partial\Omega, \Phi(x) = +\infty\}.$$

Note that $W = +\infty$ on B by Lemma 7.10, because $|\nabla\Phi|(x_n) \rightarrow \infty$ as $x_n \rightarrow x \in \partial\Omega$.

In what follows we consider a smooth, convex, non negative and decreasing function f , which vanishes on $[a, +\infty)$ for some a . In addition, we assume that $-f' \leq C$ for some $C > 0$. Using the condition $LW = 0$, which comes from the previous Lemma, one gets

$$Lf(W) = f''(W) \langle (D^2\Phi)^{-1} \nabla W, \nabla W \rangle.$$

Note that $f(W)$ vanishes on some neighbourhood of B .

Let $g_{N,\varepsilon}$ be a family of smooth nonnegative functions, satisfying the following assumptions: $g_{N,\varepsilon}(t) = 0$ for $t \leq \frac{\varepsilon}{2}$, $g_{N,\varepsilon}(t)$ is increasing on $[\frac{\varepsilon}{2}, \varepsilon]$, $g_{N,\varepsilon} = 1$ on $[\varepsilon, N]$, $g_{N,\varepsilon}$ is decreasing on $[N, 2N]$, $g_{N,\varepsilon} = 0$ on $[2N, \infty)$.

Note that $g_{N,\varepsilon}(\Phi)$ vanishes in neighborhoods of B_∞ and of the origin. Hence the product $f(W)g_{N,\varepsilon}(\Phi)$ vanishes on $\partial\Omega$ and has compact support not containing the origin. Thus one can integrate by parts

$$\int f''(W)g_{N,\varepsilon}(\Phi)\langle(D^2\Phi)^{-1}\nabla W, \nabla W\rangle d\mu = \int Lf(W)g_{N,\varepsilon}(\Phi)d\mu = - \int f'(W)g'_{N,\varepsilon}(\Phi)\langle(D^2\Phi)^{-1}\nabla W, \nabla\Phi\rangle d\mu.$$

Let us consider two monotone functions: $g_{N,\varepsilon}^1$ and $g_{N,\varepsilon}^2$ such that $g_{N,\varepsilon}^1 = g_{N,\varepsilon}$ on $-\infty, N]$ and $g_{N,\varepsilon}^1 = 1$ on $[N, +\infty)$; $g_{N,\varepsilon}^2 = 1$ on $-\infty, N]$ and $g_{N,\varepsilon}^2 = g_{N,\varepsilon}$ on $[N, +\infty)$. Note that $g_{N,\varepsilon}^1$ is non-decreasing and $g_{N,\varepsilon}^2$ is non-increasing. Note that

$$g'_{N,\varepsilon} = (g_{N,\varepsilon}^1)' + (g_{N,\varepsilon}^2)'.$$

In particular, the following holds

$$\begin{aligned} \int f''(W)g_{N,\varepsilon}(\Phi)\langle(D^2\Phi)^{-1}\nabla W, \nabla W\rangle d\mu &= - \int f'(W)(g_{N,\varepsilon}^1)'(\Phi)\langle(D^2\Phi)^{-1}\nabla W, \nabla\Phi\rangle d\mu \\ &\quad + \int f(W)L(g_{N,\varepsilon}^2(\Phi))d\mu. \end{aligned} \quad (31)$$

We will choose $g_{N,\varepsilon}$ in such a way that $\lim_{\varepsilon, N} g_{N,\varepsilon} = 1$ and the limit of the right-hand side of (31) is not positive.

We estimate the first term of the right-hand side of (31) Using that f is decreasing and $g_{N,\varepsilon}^1$ is increasing we observe

$$\begin{aligned} - \int f'(W)(g_{N,\varepsilon}^1)'(\Phi)\langle(D^2\Phi)^{-1}\nabla W, \nabla\Phi\rangle d\mu &= - \int f'(W)(g_{N,\varepsilon}^1)'(\Phi)\langle x, \nabla\Phi(x)\rangle d\mu \\ &\quad + (p-1) \int f'(W)(g_{N,\varepsilon}^1)'(\Phi)\langle(D^2\Phi)^{-1}\nabla\Phi, \nabla\Phi\rangle d\mu \\ &\leq - \int f'(W)(g_{N,\varepsilon}^1)'(\Phi)\langle x, \nabla\Phi(x)\rangle d\mu \\ &\leq C \int (g_{N,\varepsilon}^1)'(\Phi)\langle x, \nabla\Phi(x)\rangle d\mu. \end{aligned}$$

Using approximations one can relax smoothness assumption and suppose that $g_{N,\varepsilon}^1(t) = \frac{2}{\varepsilon}t - 1$ on $[\varepsilon/2, \varepsilon]$ for all N . Thus

$$- \int f'(W)(g_{N,\varepsilon}^1)'(\Phi)\langle(D^2\Phi)^{-1}\nabla W, \nabla\Phi\rangle d\mu \leq \frac{2C}{\varepsilon \int e^{-\Phi} dx} \int_{\{\Phi \leq \varepsilon\}} \langle x, \nabla\Phi(x)\rangle e^{-\Phi} dx.$$

Then we estimate

$$\begin{aligned} \frac{1}{\varepsilon} \int_{\{\Phi \leq \varepsilon\}} \langle x, \nabla\Phi(x)\rangle e^{-\Phi} dx &\leq \frac{1}{\varepsilon} \int_{\{\Phi \leq \varepsilon\}} \langle x, \nabla\Phi(x)\rangle dx = \frac{1}{\varepsilon} \left(-n \int_{\{\Phi \leq \varepsilon\}} \Phi(x) dx + \int_{\Phi=\varepsilon} \Phi(x, \nu) d\mathcal{H}^{n-1} \right) \\ &\leq \int_{\Phi=\varepsilon} \langle x, \nu \rangle d\mathcal{H}^{n-1} = n|\{\Phi \leq \varepsilon\}|. \end{aligned}$$

Thus we get

$$\lim_{\varepsilon \rightarrow 0} \left(- \int f'(W)(g_{N,\varepsilon}^1)'(\Phi)\langle(D^2\Phi)^{-1}\nabla W, \nabla\Phi\rangle d\mu \right) \leq 0.$$

Let us analyse the second term

$$\int f(W)L(g_{N,\varepsilon}^2(\Phi))d\mu$$

of the right-hand side of (31).

The function $g_{N,\varepsilon}^2$ will not depend on ε and we will write

$$g_N = g_{N,\varepsilon}^2.$$

Recall that g_N is supposed to be non-increasing and satisfying

$$0 \leq g_N \leq 1, \quad g_N|_{(-\infty, N]} = 1, \quad g_N|_{[2N, +\infty)} = 0.$$

We fix a decreasing smooth function ψ satisfying $\psi(t) = 1$ for $t \leq 0$ and $\psi(t) = 0$ for $t \geq 1$. Then we set

$$g_N(t) = \psi\left(\frac{t}{N} - 1\right)$$

for $t \geq N$.

We observe that for some $C > 0$ one has

$$|g'_N(t)| \leq \frac{C}{N}, \quad |g''_N(t)| \leq \frac{C}{N^2} e^{-\frac{t}{N}}.$$

One has

$$Lg_N(\Phi) = g'_N(\Phi) \left(n(p-1) - \frac{1}{p-1} \langle \nabla \Phi(x), x \rangle \right) + g''_N(\Phi) \langle (D^2 \Phi)^{-1} \nabla \Phi, \nabla \Phi \rangle := I + II.$$

On the set $\{x : f(W(x)) > 0\}$, one has $\langle x, \nabla \Phi(x) \rangle \leq a + p\Phi(x)$. Finally, we obtain

$$I = f(W) |g'_N(\Phi)| \left| n(p-1) - \frac{1}{p-1} \langle \nabla \Phi(x), x \rangle \right| \leq \frac{c_1}{N} (1 + |W|) \leq \frac{c_2}{N} (1 + \Phi).$$

for some $c_1, c_2 > 0$. Since $\Phi \in L^1(\mu)$ we immediately conclude that $\int f(W) I e^{-\Phi} dx \rightarrow 0$ as $N \rightarrow \infty$.

Next we use the bound $|g''_N(t)| \leq \frac{C}{N^2} e^{-\frac{t}{N}}$:

$$f(W) |g''_N(\Phi)| \leq \frac{c}{N^2} e^{-\frac{\langle x, \nabla \Phi(x) \rangle}{pN}}$$

and

$$\begin{aligned} \int f(W) II e^{-\Phi} dx &= \int f(W) |g''_N(\Phi)| \langle (D^2 \Phi)^{-1} \nabla \Phi, \nabla \Phi \rangle e^{-\Phi} dx \\ &\leq \frac{c}{N^2} \int \langle (D^2 \Phi)^{-1} \nabla \Phi, \nabla \Phi \rangle e^{-\frac{\langle x, \nabla \Phi(x) \rangle}{pN}} e^{-\Phi(x)} dx \\ &= \frac{\int e^{-\Phi} dx}{\int e^{-\frac{\Phi^*}{p-1}} dy} \frac{c}{N^2} \int \langle D^2 \Phi^* y, y \rangle e^{-\frac{\langle y, \nabla \Phi^*(y) \rangle}{pN}} e^{-\frac{\Phi^*(y)}{p-1}} dy. \end{aligned}$$

Finally, we get that for some constant d

$$\int f(W) |g''_N(\Phi)| \langle (D^2 \Phi)^{-1} \nabla \Phi, \nabla \Phi \rangle e^{-\Phi} dx \leq -\frac{d}{N} \int \langle \nabla e^{-\frac{\langle y, \nabla \Phi^*(y) \rangle}{pN}}, y \rangle e^{-\frac{\Phi^*(y)}{p-1}} dy.$$

Integrating by parts we get

$$\begin{aligned} \int f(W) |g''_N(\Phi)| \langle (D^2 \Phi)^{-1} \nabla \Phi, \nabla \Phi \rangle e^{-\Phi} dx &\leq -\frac{d}{N} \int_{\Omega^*} \langle \nabla e^{-\frac{\langle y, \nabla \Phi^*(y) \rangle}{pN}}, y \rangle e^{-\frac{\Phi^*(y)}{p-1}} dy \\ &= \frac{d}{N} \left(\int_{\Omega^*} \left(n - \frac{\langle \nabla \Phi^*(y), y \rangle}{p-1} \right) e^{-\frac{\langle y, \nabla \Phi^*(y) \rangle}{pN}} e^{-\frac{\Phi^*(y)}{p-1}} dy - \frac{d}{N} \int_{\partial \Omega^*} \langle \nu, y \rangle e^{-\frac{\langle y, \nabla \Phi^*(y) \rangle}{pN}} e^{-\frac{\Phi^*(y)}{p-1}} d\mathcal{H}_{n-1} \right). \end{aligned}$$

Note that by Proposition 7.10 (applied to Φ^*) one has $\langle y, \nabla \Phi^*(y) \rangle = +\infty$ on Ω^* , hence

$$\int_{\partial \Omega^*} \langle \nu, y \rangle e^{-\frac{\langle y, \nabla \Phi^*(y) \rangle}{pN}} e^{-\frac{\Phi^*(y)}{p-1}} d\text{vol}_{n-1} = 0.$$

Then we use that $\langle \nabla \Phi^*(y), y \rangle \geq 0$ (because Φ^* is even) and obtain

$$\int f(W) |g_N''(\Phi)| \langle (D^2\Phi)^{-1} \nabla \Phi, \nabla \Phi \rangle e^{-\Phi} dx \leq \frac{dn}{N} \int e^{-\frac{\Phi^*(y)}{p-1}} dy \rightarrow 0.$$

Finally

$$\int f''(W) \langle (D^2\Phi)^{-1} \nabla W, \nabla W \rangle d\mu = \lim_N \int f''(W) g_N(\Phi) \langle (D^2\Phi)^{-1} \nabla W, \nabla W \rangle d\mu \leq 0.$$

From this we get $\nabla W = 0$. The proof is complete. \square

Remark 7.12. *Theorem 7.9 provides an alternative proof of the classical Blaschke–Santaló inequality (4) without application of symmetrization arguments. Indeed, according to Theorem 5.1 there exists a maximizer Φ of the classical functional \mathcal{BS} . According to Theorems 7.5 and 7.9, Φ is a 2-homogeneous solution to the corresponding Monge–Ampère equation. Then following the arguments from [16] one can prove that Φ is a quadratic function. This establishes inequality (4).*

Remark 7.13. *Theorem 7.9 can be used to establish the precise form of maximizers in (5) in the rotationally invariant case (see Subsection 5.3). Unfortunately, we do not know any other examples of closed-form solutions apart of the radially symmetric cases, where Theorem 7.9 can be used. We show in the following subsection that homogeneity of the maximizers allows to reduce the problem to L^q -Minkowski problem, which is in general ill-posed.*

7.5 Relations to L^q -Minkowski problems

Lemma 7.14. *Let $\Phi: \mathbb{R}^n \rightarrow \mathbb{R}$ be α -homogeneous and convex, for some $\alpha \geq 1$. Then*

- $\Phi \geq 0$ in \mathbb{R}^n ;
- $\phi = \Phi^{1/\alpha}$ is a 1-homogeneous convex function, i.e. is the support function of a convex body (containing the origin).

Proof. Let u be a unit vector. The function $\Phi_u: \mathbb{R} \rightarrow \mathbb{R}$ defined by $\Phi_u(t) = \Phi(tu)$ is a α -homogeneous convex function on the real line. Hence it must be of the form ct^α , for some $c \geq 0$. This proves that Φ is non-negative.

Let $\phi = \Phi^{1/\alpha}$. By convexity, for every $s \geq 0$

$$\{\Phi \leq s\}$$

is convex. This proves that for every $\tau \geq 0$, the set

$$\{\phi \leq \tau\}$$

is convex. This implies that

$$\phi((1-t)x_0 + tx_1) \leq \min\{\phi(x_0), \phi(x_1)\} \tag{32}$$

for every $x_0, x_1 \in \mathbb{R}^n$, for every $t \in [0, 1]$. Let $x_0, x_1 \in \mathbb{R}^n$ and let $t \in [0, 1]$. Set

$$\bar{x}_0 = \frac{x_0}{\phi(x_0)}, \quad \bar{x}_1 = \frac{x_1}{\phi(x_1)}, \quad \bar{t} = \frac{t\phi(x_1)}{(1-t)\phi(x_0) + t\phi(x_1)}.$$

Applying (32) to \bar{x}_0 , \bar{x}_1 and \bar{t} , and using the fact that ϕ is 1-homogeneous, one gets:

$$\phi((1-t)x_0 + tx_1) \leq (1-t)\phi(x_0) + t\phi(x_1),$$

that is, ϕ is convex. \square

The following computational result should be known to the experts, we give the proof for completeness of the picture.

Let us fix a point $\nu \in \mathbb{S}^{n-1}$. We create a local coordinates system $\theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$ in a neighbourhood $U_0 \subset \mathbb{S}^{n-1}$ of ν as described below. Fix any orthogonal frame (e_1, \dots, e_{n-1}) in the hyperplane $L = \{\theta : \theta \perp \nu\}$ orthogonal to ν . Let $\theta = (\theta_1, \dots, \theta_{n-1})$ be the corresponding coordinate system with orthonormal frame (e_1, \dots, e_{n-1}) in L and

$$\nu(\theta) = \frac{\nu + \theta}{\sqrt{1 + \theta^2}}.$$

This $\theta \rightarrow \nu(\theta)$ is a parametrization of a neighborhood of $\theta \in \mathbb{S}^{n-1}$. In particular, it is easy to check the following relations: at the point ν one has

$$\partial_{e_i} e_i = -\nu, \quad \partial_{e_i} e_j = 0, \quad i \neq j.$$

This implies, in particular, that the Levi-Civita connection Γ_{jk}^i of \mathbb{S}^{n-1} vanishes at ν and, in particular, the spherical Hessian $\nabla_{\mathbb{S}^{n-1}}^2$ of a function $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$ at ν coincides with the matrix $\partial_{\theta_i \theta_j} f$.

Next we consider the polar coordinate system (r, θ) on the cone $C_0 = \mathbb{R}_+ \times U_0$:

$$C_0 \ni x = r \cdot \nu(\theta).$$

Lemma 7.15. *Let Φ be a function defined on the neighbourhood of $x_0 = r_0 \cdot \nu$. The Euclidean Hessian of Φ has the following representation in the frame $(\nu, e_1, e_2, \dots, e_{n-1})$ ant the point x_0 :*

$$D^2\Phi = \begin{bmatrix} \Phi_{rr} & \frac{\Phi_{r\theta_1} - \frac{\Phi_{\theta_1}}{r^2}}{r} & \dots & \frac{\Phi_{r\theta_{n-1}} - \frac{\Phi_{\theta_{n-1}}}{r^2}}{r} \\ \frac{\Phi_{r\theta_1} - \frac{\Phi_{\theta_1}}{r^2}}{r} & a_{1,1} & \dots & a_{1,n-1} \\ \vdots & \vdots & a_{i,j} & \vdots \\ \frac{\Phi_{r\theta_{n-1}} - \frac{\Phi_{\theta_{n-1}}}{r^2}}{r} & a_{n-1,1} & \dots & a_{n-1,n-1} \end{bmatrix},$$

where

$$A = (a_{i,j})_{(n-1) \times (n-1)} = \frac{\Phi_r}{r} \delta_{ij} + \frac{1}{r^2} \nabla_{\mathbb{S}^{n-1}}^2 \Phi.$$

and $\nabla_{\mathbb{S}^{n-1}}^2 \Phi$ is the spherical Hessian of the function $\theta \rightarrow \Phi(r_0 \cdot \theta)$.

Proof. To prove the Lemma we perform the following computations at $x = x_0 = r_0 \cdot \nu$:

$$\partial_{rr}^2 \Phi(x) = \partial_r (\langle \nabla \Phi(x), \nu \rangle) = \partial_\nu \langle \nabla \Phi(x), \nu \rangle = \langle D^2 \Phi(x) \nu, \nu \rangle + \langle \nabla \Phi(x), \partial_\nu \nu \rangle = \langle D^2 \Phi(x) \nu, \nu \rangle,$$

$$\begin{aligned} \partial_{r\theta_i}^2 \Phi(x) &= \partial_r (\partial_{\theta_i} \Phi(x)) = \partial_r (r \cdot \partial_{e_i} \Phi(x)) = \partial_{e_i} \Phi(x) + r \partial_r (\partial_{e_i} \Phi(x)) = \partial_{e_i} \Phi(x) + r \partial_\nu \langle e_i, \nabla \Phi(x) \rangle \\ &= \partial_{e_i} \Phi(x) + r \langle \partial_\nu e_i, \nabla \Phi(x) \rangle + r \langle e_i, D^2 \Phi(x) \cdot \nu \rangle = \partial_{e_i} \Phi(x) + r \langle e_i, D^2 \Phi(x) \cdot \nu \rangle \\ &= \frac{\partial_{\theta_i} \Phi(x)}{r} + r \langle e_i, D^2 \Phi(x) \cdot \nu \rangle, \end{aligned}$$

$$\begin{aligned} \partial_{\theta_i \theta_j}^2 \Phi(x) &= \partial_{\theta_i} (\partial_{\theta_j} \Phi(x)) = r \partial_{e_i} (r \partial_{e_j} \Phi(x)) = r^2 \partial_{e_i} (\langle e_j, \nabla \Phi(x) \rangle) = r^2 (\langle \partial_{e_i} e_j, \nabla \Phi(x) \rangle + \langle e_j, D^2 \Phi(x) \cdot e_i \rangle) \\ &= r^2 \left(-\frac{\partial_\nu \Phi(x)}{r} \delta_{ij} + \langle e_j, D^2 \Phi(x) \cdot e_i \rangle \right) = -r \partial_r \Phi(x) + r^2 \langle e_j, D^2 \Phi(x) \cdot e_i \rangle. \end{aligned}$$

Using these formulas one can easily get the desired expression for $D^2\Phi$. We remind the reader that $\nabla_{\mathbb{S}^{n-1}}^2 \Phi = (\partial_{\theta_i \theta_j}^2 \Phi)$. \square

Corollary 7.16. *Let $\Phi = r^\alpha \phi^\alpha$ be a α -homogeneous convex function, where ϕ is the restriction of $\Phi^{\frac{1}{\alpha}}$ onto \mathbb{S}^{n-1} . Then*

$$\det D^2 \Phi = (\alpha - 1) \alpha^n r^{n(\alpha-2)} \phi^{(\alpha-1)n+1} \det(\phi \delta_{ij} + \nabla_{\mathbb{S}^{n-1}}^2 \phi).$$

Proof. Apply previous Lemma. One has

$$\begin{aligned} \Phi_{rr} &= \alpha(\alpha - 1) r^{\alpha-2} \phi^\alpha \\ \frac{\Phi_{r\theta_i}}{r} - \frac{\Phi_{\theta_i}}{r^2} &= \alpha(\alpha - 1) r^{\alpha-2} \phi^{\alpha-1} \phi_{\theta_i} \\ A &= r^{\alpha-2} \left(\alpha \phi^\alpha \delta_{ij} + \alpha \phi^{\alpha-1} \nabla_{\mathbb{S}^{n-1}}^2 \phi + \alpha(\alpha - 1) \phi^{\alpha-2} \nabla_{\mathbb{S}^{n-1}} \phi \oplus \nabla_{\mathbb{S}^{n-1}} \phi \right). \end{aligned}$$

Assume that the frame (e_1, \dots, e_{n-1}) is chosen in such a way that the matrix $\phi \delta_{ij} + \nabla_{\mathbb{S}^{n-1}}^2 \phi$ is diagonal with eigenvalues λ_i . Then $D^2 \Phi$ takes the form $D^2 \Phi = \alpha r^{\alpha-2} \phi^{\alpha-1} C$, where

$$C = \begin{bmatrix} (\alpha - 1)\phi & (\alpha - 1)\phi_{\theta_1} & \cdots & (\alpha - 1)\phi_{\theta_{n-1}} \\ (\alpha - 1)\phi_{\theta_1} & b_{1,1} & \cdots & b_{1,n-1} \\ \vdots & \vdots & b_{i,j} & \vdots \\ (\alpha - 1)\phi_{\theta_{n-1}} & b_{n-1,1} & \cdots & b_{n-1,n-1} \end{bmatrix},$$

and

$$b_{i,j} = \lambda_i \cdot \delta_{ij} + \frac{\alpha - 1}{\phi} \phi_{\theta_i} \phi_{\theta_j}.$$

Elementary computations give $\det C = (\alpha - 1) \phi \prod_{i=1}^{n-1} \lambda_i = (\alpha - 1) \phi \det(\phi \delta_{ij} + \nabla_{\mathbb{S}^{n-1}}^2 \phi)$. This completes the proof. \square

Now let $p > 1$, and assume that the potential V is p -homogeneous. In particular, V and V^* have the following forms

$$V(x) = |x|^p v^p \left(\frac{x}{|x|} \right), \quad V^*(x) = |x|^{p^*} \tilde{v}^{p^*} \left(\frac{x}{|x|} \right),$$

where

$$p^* = \frac{p}{p-1}.$$

According to Theorem 7.9, any symmetric maximum point Φ of the functional $\mathcal{BS}_{p,V}$ is p -homogeneous. Thus the similar representation holds

$$\Phi(x) = |x|^p \phi^p \left(\frac{x}{|x|} \right), \quad \Phi^*(x) = |x|^{p^*} \tilde{\phi}^{p^*} \left(\frac{x}{|x|} \right).$$

Applying the change of variables formula

$$\frac{1}{\int e^{-\frac{\Phi^*}{p-1}} dy} e^{-\frac{\Phi^*}{p-1}} \det D^2 V^* = \frac{1}{\int e^{-\Phi} dx} e^{-\Phi(\nabla \Phi^*)} \det D^2 \Phi^*,$$

the relation $\Phi = \frac{1}{p-1} \Phi^*(\nabla \Phi)$ and Corollary 7.16, we get the following result.

Theorem 7.17. *The following equation holds*

$$\tilde{\phi}^{(p^*-1)n+1} \det(\tilde{\phi} \delta_{ij} + \nabla_{\mathbb{S}^{n-1}}^2 \tilde{\phi}) = C \tilde{v}^{(p^*-1)n+1} \det(\tilde{v} \delta_{ij} + \nabla_{\mathbb{S}^{n-1}}^2 \tilde{v}), \quad (33)$$

where $C = \frac{\int e^{-\frac{\Phi^*}{p-1}} dy}{\int e^{-\Phi} dx}$.

The above Theorem establishes that any maximizer of $\mathcal{BS}_{p,V}$ is a solution to a corresponding L^q -Minkowski problem. This fact gives in a sense more precise information about relation between the functional and the set versions of the problem (see Proposition 1.3).

In particular, uniqueness of solution to (33) (for fixed C, \tilde{v} and unknown $\tilde{\phi}$) would imply an affirmative answer to Question 1.2. Unfortunately, it is known that in general equation (33) has many (possibly, infinitely many) solutions for those values of p which are of interest for us (see [18], [33], [36], [57]).

Remark 7.18. *Note that a variational problem related to equation (9) is known in the literature about L^q -Minkowski problem. Usually it is stated in the following form: maximize the functional*

$$\int_{\mathbb{S}^{n-1}} h^q f dx$$

with constraint $|K_h| = 1$, where h is the support function of K_h . Then the solution satisfies equation (9). This problem is equivalent to our maximization problem (6) for sets.

8 Strong Brascamp-Lieb inequalities

In this section we study the following strengthening of $\mathcal{BS}_{p,V}$ on the set of even functions:

$$\text{Var}_\mu f \leq \lambda \int \langle (D^2V)^{-1} \nabla f, \nabla f \rangle d\mu, \quad (34)$$

with $\lambda < 1$. Here $\mu = \frac{e^{-V(x)} dx}{\int e^{-V(x)} dx}$. As we have seen in Subsection 7.3, the maximizers of the generalized Blaschke–Santaló functional satisfy (34) with $\lambda = 1 - \frac{1}{p}$. We estimate the best value of λ in (34) for log-concave measures with potential of the form $V = c|x|_q^p$. We prove that in general (34) fails to hold with $\lambda = 1 - \frac{1}{p}$. This proves, in particular, that V is not always maximizer for $\mathcal{BS}_{p,V}$.

8.1 Powers of l_q -norms

In this subsection we study strong Brascamp–Lieb inequality for measure

$$\mu = \frac{e^{-\frac{|x|_q^p}{p}} dx}{\int e^{-\frac{|x|_q^p}{p}} dx} = \frac{e^{-\frac{1}{p}(\sum_{i=1}^n |x_i|^q)^{\frac{p}{q}}} dx}{\int e^{-\frac{|x|_q^p}{p}} dx}.$$

All the functions below are assumed to be even.

Remark 8.1. *Due to homogeneity invariance the constant λ in (34) remains the same for $V = c|x|_q^p$ with any $c > 0$.*

At the first step we do the following change of variables:

$$x_i = \text{sign}(y_i) |y_i|^{\frac{2}{q}}.$$

The reader can easily verify that the image of μ under the mapping $x \rightarrow y(x)$ coincides with

$$\nu = \frac{1}{C} e^{-\frac{1}{p}|y|^{\frac{2p}{q}}} \left(\prod_{i=1}^n |y_i| \right)^{\frac{2}{q}-1} dy.$$

The measure ν can be represented in polar coordinates as follows:

$$\nu = \gamma(dr)m(d\theta),$$

where

$$\gamma = \frac{e^{-\frac{1}{p}r^{\frac{2p}{q}}} r^{\frac{2n}{q}-1} dr}{\int_0^\infty e^{-\frac{1}{p}r^{\frac{2p}{q}}} r^{\frac{2n}{q}-1} dr}$$

and m is a probability measure on \mathbb{S}^{n-1} which has the form

$$m = \frac{|y_1 \cdots y_n|^{\frac{2}{q}-1} \cdot \sigma}{\int_{\mathbb{S}^{n-1}} |y_1 \cdots y_n|^{\frac{2}{q}-1} d\sigma},$$

where σ is the normalized probability surface measure on \mathbb{S}^{n-1} .

We are interested in the best estimate of λ in the strong Brascamb–Lieb inequality for μ

$$\text{Var}_\mu f \leq \lambda \int \langle (D^2V)^{-1} \nabla f, \nabla f \rangle d\mu, \quad (35)$$

where $V = \frac{1}{p} (\sum_{i=1}^n |x_i|^q)^{\frac{p}{q}}$.

Remark 8.2. Note that the best value of λ we can hope for is $1 - \frac{1}{p}$. Indeed, let V be p -homogeneous and convex. One can easily prove that $f = \langle \nabla V(x), x \rangle$ satisfies equality $\text{Var}_\mu f = \lambda \int \langle (D^2V)^{-1} \nabla f, \nabla f \rangle d\mu$, with $\lambda = 1 - \frac{1}{p}$.

Indeed, one has $f = pV$, $\nabla f = p\nabla V$ and $\langle (D^2V)^{-1} \nabla f, \nabla f \rangle = p^2 \langle (D^2V)^{-1} \nabla V, \nabla V \rangle = \frac{p^2}{(p-1)} \langle x, \nabla V(x) \rangle$. Integrating by parts one gets

$$\int f d\mu = \int \langle \nabla V(x), x \rangle d\mu = n,$$

$$\int f^2 d\mu = p \int V(x) \langle x, \nabla V(x) \rangle d\mu = np \int V d\mu + p \int \langle x, \nabla V(x) \rangle d\mu = (n+p) \int \langle x, \nabla V(x) \rangle d\mu = n(n+p).$$

Thus $\text{Var}_\mu f = np$. On the other hand $\int \langle (D^2V)^{-1} \nabla f, \nabla f \rangle d\mu = \frac{p^2}{p-1} \int \langle x, \nabla V(x) \rangle d\mu = \frac{np^2}{p-1}$. This proves the claim.

One has $\nabla V = (\sum_{i=1}^n |x_i|^q)^{\frac{p}{q}-1} a$,

$$\begin{aligned} D^2V(x) &= \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{p}{q}-1} \left[(q-1) \text{diag}(|x_i|^{q-2}) + (p-q) \frac{a \oplus a}{\sum_{i=1}^n |x_i|^q} \right] \\ &= \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{p}{q}-1} \text{diag}(|x_i|^{\frac{q}{2}-1}) \left[(q-1)I + (p-q) \frac{b \oplus b}{\sum_{i=1}^n |x_i|^q} \right] \text{diag}(|x_i|^{\frac{q}{2}-1}) \end{aligned}$$

where $a = (\text{sign}(x_i)|x_i|^{q-1})$ and $b = (\text{sign}(x_i)|x_i|^{\frac{q}{2}})$.

Let us make the change of variables. Apply inequality (35) to $f = g(\text{sign}(x_i)|x_i|^{\frac{q}{2}})$. One has

$$\text{Var}_\mu f = \text{Var}_\nu g$$

and

$$\langle (D^2V)^{-1} \nabla f, \nabla f \rangle = \frac{q^2}{4|y|^{2(\frac{p}{q}-1)}} \left\langle \left[(q-1)I + (p-q) \frac{y \oplus y}{|y|^2} \right]^{-1} \nabla g(y), \nabla g(y) \right\rangle.$$

Thus we get that (35) is equivalent to the following inequality:

$$\text{Var}_\nu g \leq \frac{\lambda q^2}{4(q-1)} \int \frac{1}{|y|^{2(\frac{p}{q}-1)}} \left\langle \left(I + \left(\frac{p-q}{q-1} \right) \frac{y \oplus y}{|y|^2} \right)^{-1} \nabla g(y), \nabla g(y) \right\rangle d\nu.$$

In what follows we denote by $\nabla_{\mathbb{S}^{n-1}}g$ the projection of ∇g onto \mathbb{S}^{n-1} :

$$\nabla_{\mathbb{S}^{n-1}}g(y) = g - \langle \nabla g(y), \frac{y}{r} \rangle \frac{y}{r}.$$

Note that

$$\nabla_{\theta}g = \frac{\nabla_{\mathbb{S}^{n-1}}g}{r}.$$

Thus $\nabla g = g_r \cdot \frac{y}{r} + \nabla_{\mathbb{S}^{n-1}}g$, where $|y| = r$. We get that in polar coordinates the last inequality looks like

$$\text{Var}_{\nu}g \leq \frac{\lambda q^2}{4(q-1)} \int \int \frac{1}{r^{2(\frac{p}{q}-1)}} \left(\frac{q-1}{p-1} g_r^2 + \frac{|\nabla_{\theta}g|^2}{r^2} \right) \gamma(dr) m(d\theta). \quad (36)$$

Let $g^r(\theta) = \int_0^{\infty} g(r, \theta) d\gamma$. One has

$$\text{Var}_{\nu}g = \text{Var}_{\nu}(g - g^r(\theta)) + \text{Var}_{\nu}(g^r(\theta)).$$

To estimate the first term we apply the following one-dimensional Poincaré-type inequality and the Fubini theorem.

$$\text{Var}_{e^{-u}dr}(g) \leq \int_0^{+\infty} \frac{(g')^2}{u'' + \frac{u'}{r}} e^{-u} dr. \quad (37)$$

Inequality (37) is the 1-dimensional case of the result obtained by Cordero-Erausquin and Rotem (see Theorem 3 in [23]). Note that one can extend measures and functions symmetrically to get an equivalent inequality on \mathbb{R} , and then apply the result from [23] for $n = 1$.

In particular, one gets

$$\text{Var}_{\nu}(g - g^r(\theta)) \leq \frac{q^2}{4p} \int \frac{g_r^2}{r^{2(\frac{p}{q}-1)}} d\nu.$$

To estimate the second term we apply Poncaré inequality for measure m for even functions with the best constant C_m

$$\begin{aligned} \text{Var}_{\nu}(g^r(\theta)) &= \text{Var}_m(g^r(\theta)) \leq C_m \int |\nabla_{\theta}g^r(\theta)|^2 dm \\ &= C_m \int r^2 \left| \int \nabla_{\mathbb{S}^{n-1}}g d\gamma \right|^2 dm \leq C_m \int r^{\frac{2p}{q}} d\gamma \int \frac{|\nabla_{\mathbb{S}^{n-1}}g|^2}{r^{2(\frac{p}{q}-1)}} d\nu. \end{aligned}$$

Next we compute

$$\int r^{\frac{2p}{q}} d\gamma = \frac{\int_0^{\infty} e^{-\frac{1}{p}r^{\frac{2p}{q}}} r^{(\frac{2n}{q}-1) + \frac{2p}{q}} dr}{\int_0^{\infty} e^{-\frac{1}{p}r^{\frac{2p}{q}}} r^{(\frac{2n}{q}-1)} dr} = -\frac{q}{2} \frac{\int_0^{\infty} (e^{-\frac{1}{p}r^{\frac{2p}{q}}})' r^{\frac{2n}{q}} dr}{\int_0^{\infty} e^{-\frac{1}{p}r^{\frac{2p}{q}}} r^{(\frac{2n}{q}-1)} dr} = n.$$

Finally, we get that every even g satisfies

$$\text{Var}_{\nu}g \leq \frac{q^2}{4p} \int \frac{g_r^2}{r^{2(\frac{p}{q}-1)}} d\nu + nC_m \int \frac{|\nabla_{\mathbb{S}^{n-1}}g|^2}{r^{2(\frac{p}{q}-1)}} d\nu.$$

Thus comparing this result with (36) we obtain get the following

Proposition 8.3. *Assume that*

$$\lambda \geq 1 - \frac{1}{p}, \quad C_m \leq \frac{\lambda q^2}{4n(q-1)}.$$

Then inequality (35) holds on the set of even functions.

Thus we have reduced our problem to the following question: what is the best Poincaré inequality on the set of even functions for the measure m ?

The associated weighted Laplacian for m has the form

$$Lf = \Delta_{\mathbb{S}^{n-1}} f + \left(\frac{2}{q} - 1\right) \langle \omega, \nabla_{\mathbb{S}^{n-1}} f \rangle,$$

where $\Delta_{\mathbb{S}^{n-1}}, \nabla_{\mathbb{S}^{n-1}}$ are the spherical Laplacian and gradient,

$$\omega = \left(\frac{1}{x_1}, \dots, \frac{1}{x_n}\right).$$

Thus we have to find the first non-zero eigenvalue on the domain of even functions for operator L .

Remark 8.4. *Making appropriate change of variables one can show that for $n = 2$ equation $Lf = -\lambda f$ can be reduced to the so-called Legendre equation and the corresponding eigenvalue functions are known as Legendre functions. In general, they are not elementary,*

For the sake of simplicity the computations below are done on $\{x_i > 0\}$. Recall that Euclidean and spherical Laplacians are related by

$$\Delta = \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-1}}.$$

Using this representation, we get immediately

$$\begin{aligned} Lx_i &= -\left(\frac{2n}{q} - 1\right)x_i + \frac{2-q}{qx_i} \\ Lx_i^2 &= -\frac{4n}{q}\left(x_i^2 - \frac{1}{n}\right) \\ L(x_i x_j) &= -\frac{4n}{q}x_i x_j + \left(\frac{2}{q} - 1\right)\left(\frac{x_i}{x_j} + \frac{x_j}{x_i}\right), \quad i \neq j. \end{aligned}$$

Lemma 8.5. *Let $i \neq j$. Then*

$$L[(x_i x_j)^N] = -4N \left[\frac{n}{q} + N - 1\right] (x_i x_j)^N + N \left[N + \frac{2}{q} - 2\right] (x_i x_j)^{N-1} (x_i^2 + x_j^2).$$

In particular

$$L(x_i^2 x_j^2)^{1-\frac{1}{q}} = -8 \left(1 - \frac{1}{q}\right) \left(1 + \frac{n-2}{q}\right) (x_i^2 x_j^2)^{1-\frac{1}{q}}.$$

Proof. One has

$$L[(x_i x_j)^N] = N(x_i x_j)^{N-1} L(x_i x_j) + N(N-1)(x_i x_j)^{N-2} |\nabla_{\mathbb{S}^{n-1}}(x_i x_j)|^2.$$

One can easily verify:

$$\nabla_{\mathbb{S}^{n-1}}(x_i x_j) = x_j e_i + x_i e_j - 2x_i x_j \cdot x$$

and

$$|\nabla_{\mathbb{S}^{n-1}}(x_i x_j)|^2 = x_i^2 + x_j^2 - 4x_i^2 x_j^2.$$

Thus

$$\begin{aligned} L[(x_i x_j)^N] &= N(x_i x_j)^{N-1} \left[-\frac{4n}{q} x_i x_j + \left(\frac{2}{q} - 1\right) \left(\frac{x_i}{x_j} + \frac{x_j}{x_i}\right) \right] + N(N-1)(x_i x_j)^{N-2} [x_i^2 + x_j^2 - 4x_i^2 x_j^2] \\ &= -N(x_i x_j)^N \left[\frac{4n}{q} + 4(N-1) \right] + N(x_i x_j)^{N-1} \left[\left(\frac{2}{q} - 1\right) + N - 1 \right] (x_i^2 + x_j^2). \end{aligned}$$

This completes the proof. □

We observe that L preserves even and unconditional functions. Using this observation and the above computations, we obtain the following corollary.

Corollary 8.6. *The following functions are eigenfunctions of L :*

- $$x_i^2 - \frac{1}{n}$$

with eigenvalue $-\frac{4n}{q}$.

- $$|x_i x_j|^{2\left(1-\frac{1}{q}\right)}$$

with eigenvalue $-8\left(1-\frac{1}{q}\right)\left(1+\frac{n-2}{q}\right)$.

Theorem 8.7. *The best constant C_m of measure m in the Poincarè inequality on the set of even functions satisfies*

$$C_m = \max\left(\frac{q}{4n}, \frac{q^2}{8(q-1)(n+q-2)}\right).$$

Proof. Given even function f we represent it as follows

$$f = \sum_{a \in \{0,1\}^n} f_a, \tag{38}$$

where every function $f_a(x_1, \dots, x_n)$ is even in x_i if $a_i = 0$ and odd in x_i if $a_i = 1$. For instance, if all a_i are zero, then f_a is unconditional. Note that if $a = (a_1, \dots, a_n)$ contains odd amount of 1, then $f_a = 0$, because f is even.

To obtain this representation we use the operators

$$\sigma_i(x) = (x_1, \dots, -x_i, x_n)$$

and

$$T_i^+ f = \frac{f(x) + f(\sigma_i(x))}{2}, \quad T_i^- f = \frac{f(x) - f(\sigma_i(x))}{2}.$$

Note that $f(x) = T_i^+ f + T_i^- f$, where $T_i^+ f$ is even in x_i and $T_i^- f$ is odd in x_i . Consequently applying the operators

$$T_1^\pm, T_2^\pm, \dots, T_n^\pm,$$

we obtain representation (38), where

$$f_a = T_1^{b_1} \dots T_n^{b_n} f.$$

Here $b_i = 1$, if $a_i = 1$ and $b_i = -1$ if $a_i = 0$.

Next we note that

$$\text{Var}_m f = \sum_{a \in \{0,1\}^n} \text{Var}_m f_a.$$

This is because $f_a f_b$ is odd at least in one variable for $a \neq b$, hence $\int f_a f_b dm = 0$, because measure m is unconditional.

Similarly

$$\int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} f|^2 dm = \sum_{a \in \{0,1\}^n} \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} f_a|^2 dm.$$

Indeed, let $a \neq b$. There exists j such that f_a (say) is even in x_j and f_b is odd in x_j . Then $\partial_{x_i} f_a \cdot \partial_{x_i} f_b$ is odd in x_j for all i . Indeed, if $i \neq j$, then $\partial_{x_i} f_a$ is even in x_j and $\partial_{x_i} f_b$ is odd in x_j . If $i = j$, then $\partial_{x_i} f_a$ is odd in x_j and $\partial_{x_i} f_b$ is even in x_j . Finally, $\int \partial_{x_i} f_a \cdot \partial_{x_i} f_b d\mu = 0$ and

$$\int \langle \nabla f_a, \nabla f_b \rangle d\mu = \sum_{i=1}^n \int \partial_{x_i} f_a \cdot \partial_{x_i} f_b d\mu = 0.$$

Thus we have reduced the statement to the case of f_a for arbitrary f, a .
For an unconditional function f_0 we have

$$\text{Var}_m f_0 \leq \frac{q}{4n} \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} f_0|^2 dm. \quad (39)$$

Indeed, Theorem 5.19 and Proposition 1.7 imply inequality (34) for unconditional functions for $q > 1$ with $\lambda = 1 - \frac{1}{q}$. Then we deduce (39) applying (34) to homogeneous functions (see computations in the next subsection).

Let a contain a non-zero amount of 1. We have shown above that this is an even number. For simplicity let us assume that $a_1 = a_2 = 1$. Then $f_a = 0$ on the sets $\{x_1 = 0\}, \{x_2 = 0\}$. Using this observation and identity $Lg = -\lambda g$, where $g = |x_i x_j|^{2(1-\frac{1}{q})}$, $\lambda = -8\left(1 - \frac{1}{q}\right)\left(1 + \frac{n-2}{q}\right)$ one gets

$$\begin{aligned} \lambda \text{Var}_m f_a &= \lambda \int_{\mathbb{S}^{n-1}} f_a^2 dm = - \int_{\mathbb{S}^{n-1}} f_a^2 \frac{Lg}{g} dm = 2 \int_{\mathbb{S}^{n-1}} \frac{\langle \nabla_{\mathbb{S}^{n-1}} g, \nabla_{\mathbb{S}^{n-1}} f_a \rangle}{g} f_a dm - \int_{\mathbb{S}^{n-1}} f_a^2 \frac{|\nabla_{\mathbb{S}^{n-1}} g|^2}{g^2} dm \\ &\leq \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} f_a|^2 dm. \end{aligned}$$

This completes the proof. □

Theorem 8.8. *Assume that*

$$\lambda \geq \max\left(1 - \frac{1}{p}, 1 - \frac{1}{q}, \frac{1}{2(1 + \frac{q-2}{n})}\right).$$

Then inequality (35) holds on the set of even functions.

In particular, if $p \leq q$, then one can take

$$\lambda = 1 - \frac{1}{q}, \text{ if } q \geq 2,$$

$$\lambda = \frac{1}{2(1 + \frac{q-2}{n})}, \text{ if } q \leq 2.$$

Inequality (35) is sharp and holds with $\lambda = 1 - \frac{1}{p}$ if

$$p \geq q, \quad p \geq \frac{2(n+q-2)}{n+2(q-2)} = 2 - \frac{2(q-2)}{n+2(q-2)}.$$

Proof. The estimate of λ follows from Proposition 8.3 and Theorem 8.7. The sharpness result follows from the observation that the value $1 - \frac{1}{p}$ in inequality $\text{Var}_\mu f \leq (1 - \frac{1}{p}) \int \langle (D^2 V)^{-1} \nabla f, \nabla f \rangle d\mu$, where V is p -homogeneous can not be improved, because $\text{Var}_\mu f = (1 - \frac{1}{p}) \int \langle (D^2 V)^{-1} \nabla f, \nabla f \rangle d\mu$ for $f = \langle \nabla V(x), x \rangle$. □

8.2 Counterexamples to the strong Brascamp–Lieb inequality

Theorem 8.8 gives, in particular, sharp inequalities $\text{Var}_\mu f \leq (1 - \frac{1}{p}) \int \langle (D^2 V)^{-1} \nabla f, \nabla f \rangle d\mu$ for

$$V = \frac{1}{p} |x|_q^p$$

if

$$p \geq q \geq 2$$

and

$$q < 2, p \geq 2 - \frac{2(q-2)}{n+2(q-2)}.$$

Note that in both cases the best constant in the inequality (34) is strictly bigger than $\frac{1}{2}$ except the Gaussian case $p = q = 2$.

Unfortunately, we can not claim that the values of λ in other cases are optimal. However, we are able to answer the following natural questions:

1. Is it true that inequality (34) holds with $p = q < 2$ and $\lambda = 1 - \frac{1}{q}$?
2. Is it true that inequality (34) holds with $p = 2$, $\lambda = \frac{1}{2}$, and some $q \neq 2$?

The answers to both questions are negative. Indeed, in the first case the inequality in question is equivalent (see (36)) to

$$\text{Var}_\nu g \leq \frac{q}{4} \int \int \left(g_r^2 + \frac{|\nabla_\theta g|^2}{r^2} \right) \gamma(dr) m(d\theta)$$

with

$$\gamma = \frac{e^{-\frac{1}{q}r^2} r^{\frac{2n}{q}-1} dr}{\int_0^\infty e^{-\frac{1}{q}r^2} r^{\frac{2n}{q}-1} dr}, \quad m = \frac{|y_1 \cdots y_n|^{\frac{2}{q}-1} \cdot \sigma}{\int_{\mathbb{S}^{n-1}} |y_1 \cdots y_n|^{\frac{2}{q}-1} d\sigma}.$$

Apply this inequality to $g = r^2 \omega(\theta)$ with $\int \omega dm = 0$. One gets

$$\begin{aligned} \text{Var}_\nu g &= \int r^4 d\gamma \cdot \int \omega^2 dm \\ \frac{q}{4} \int \int \left(g_r^2 + \frac{|\nabla_\theta g|^2}{r^2} \right) \gamma(dr) m(d\theta) &= \frac{q}{4} \int r^2 d\gamma \cdot \left(4 \int \omega^2 dm + \int |\nabla_\theta \omega|^2 dm \right) \end{aligned}$$

Applying integration by parts, we get

$$\int r^4 d\gamma = \frac{q}{2} \left(2 + \frac{2n}{q} \right) \int r^2 d\gamma.$$

After rearrangement of the terms we get inequality $\text{Var}_m \omega \leq \frac{q}{4n} \int |\nabla_\theta \omega|^2 dm$ for arbitrary symmetric ω . But this contradicts Theorem 8.7, because the best constant in the Poincarè inequality is $\frac{q^2}{8(q-1)(n+q-2)} > \frac{q}{4n}$.

To prove that inequality

$$\text{Var}_\mu f \leq \frac{1}{2} \int \langle (D^2 V)^{-1} \nabla f, \nabla f \rangle d\mu,$$

where $p = 2$, does not hold for all values of q except 2 we use the same arguments: apply the corresponding equivalent inequality (36) to function f with appropriate degree of homogeneity (this is $\frac{4}{q}$) and show that it contradicts the sharp estimate obtained in Theorem 8.7. We omit the computations here.

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