# AN APPLICATION OF THE NONLINEAR RECIPROCAL TRANSFORMATIONS IN THE THEORY OF DR HIERARCHIES

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Abstract. We prove that the DR hierarchy corresponding to the family of F-cohomological field theories without unit considered in a previous work of the first author together with D. Gubarevich can be "trivialized", i.e. reduced to two copies of the KdV hierarchy, using a simple nonlinear reciprocal transformation. This gives the first manifestation of a role of nonlinear reciprocal transformation in the theory of integrable systems associated to the moduli spaces of stable curves, beyond the dispersionless limit.

## 1. Introduction

There are various ways to produce integrable hierarchies of evolutionary PDEs using the geometry of the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable algebraic curves of genus $g$ with $n$ marked points. The central role here is played by the notion of a *cohomological field theory (CohFT)* introduced by Kontsevich and Manin [KM94]. CohFTs are systems of cohomology classes on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ that are compatible with natural maps between the moduli spaces. The notion of a CohFT involves also a vector space called the *phase space*, a bilinear form on it called the *metric*, and a special vector in the phase space called the *unit*.

One way to produce an integrable hierarchy from a CohFT was proposed by Dubrovin and Zhang [DZ01] (for homogeneous semisimple CohFTs) and then generalized in [BPS12] (for general semisimple CohFTs). It is understood now [BS24] that a generalization of the Dubrovin–Zhang hierarchy exists for an object that is more general than a CohFT, for a so-called *F-CohFT*, introduced in [BR21], where the is no metric and there are less requirements regarding the compatibility with natural maps between the moduli spaces. All these more general hierarchies will be also called the *Dubrovin–Zhang (DZ) hierarchies*. Note, however, that the polynomiality property of the DZ hierarchy is proved only for semisimple CohFTs [BPS12] and in some concrete examples [BR21, Bur23].

The DZ hierarchies include many important hierarchies from mathematical physics, for example, the Gelfand–Dickey hierarchies, Toda hierarchies of various types, the ILW hierarchy, the Drinfeld–Sokolov hierarchies, the discrete KdV hierarchy. Certain subclasses in the class of DZ hierarchies can be conjecturally described independently of the geometry, using only the language of integrable systems [DZ01, DLYZ16, LWZ21].

There is another way to produce an integrable hierarchy starting from a CohFT, which was proposed by the first author in [Bur15], the resulting hierarchy was called the *DR hierarchy*. The DR hierarchy is polynomial by construction, and it is endowed with a remarkably rich algebraic structure, which can be described very explicitly. It is understood now that the DR hierarchy can be associated to an arbitrary F-CohFT [BR21] (a systematic study is presented in [ABLR21]), and it is again polynomial by construction, while the polynomiality of the DZ hierarchy for an arbitrary F-CohFT is an open problem. Conjecturally, the DR and DZ hierarchies are Miura equivalent: for CohFTs it was formulated in [Bur15], then in a more precise form in [BDGR18], and currently the most general version of the conjecture is formulated in [BS24].

Note that the DR hierarchy can be associated to a more general object than an F-CohFT, to an *F-CohFT without unit*: the first explicit examples were computed in [BG23]. A construction of a DZ hierarchy associated to an F-CohFT without unit is not developed yet.

*Date*: November 6, 2024.

Having a construction of a class of integrable hierarchies, it is important to understand whether some of them are related by changes of variables or other transformations. A lot of work was done regarding the action of Miura transformations on the DZ and DR hierarchies. However, there is another type of transformations, the so-called *reciprocal transformations* (see e.g. [CF89, CFA89] and the references therein about the literature on reciprocal transformations), whose role in the relation to the DZ and DR hierarchies is less studied. The dispersionless parts of these two hierarchies are hierarchies of hydrodynamic type, and the reciprocal transformations of such systems are well studied (see [XZ06] and the references at the end of page 1 in [LSV24]). However, the reciprocal transformations of dispersive deformations of the hierarchies of hydrodynamic type are much less studied (see e.g. [LZ11, LWZ23, LSV24]), and regarding the DZ or DR hierarchies (and more generally the hierarchies controlling CohFT-type correlators) there are only works [Ale21, LWZ23, YZ24], where the authors consider the simplest possible reciprocal transformations, the so-called *linear* ones.

In our paper, we give the first application of the *nonlinear* reciprocal transformations in the theory of DR hierarchies. We consider the 3-parameter family of F-CohFTs without unit of rank 2 from the paper [BG23] and the associated DR hierarchy. In [BG23], the authors computed explicitly the primary flows of the DR hierarchy. In our paper, we give an explicit description of all the flows of the DR hierarchy: we prove that after the composition of a Miura transformation and a nonlinear reciprocal transformation the two dependent variables of the hierarchy become splitted and the resulting flows can be simply described in terms of the flows of the KdV hierarchy (see Theorem 3.3).

Note that in the theory of DR and DZ hierarchies the KdV hierarchy is considered as the simplest possible hierarchy, because it corresponds to the trivial CohFT where all the classes are just units in the cohomology. The fact that the DR or DZ hierarchy corresponding to some CohFT or F-CohFT with nontrivial $R$-matrix can be "trivialized", i.e. reduced to the KdV hierarchy, using linear reciprocal transformations was observed in several papers, see e.g. [Ale21, Corollary 3.1] and [YZ24, Corollary 2]. As far as we know, our paper gives the first example where this trivialization is obtained using a nonlinear reciprocal transformation. So we believe that our result shows the importance of the role of nonlinear reciprocal transformations in the theory of DR and DZ hierarchies, which should be clarified in the future research.

**Notations and conventions.**

- We use the standard convention of sum over repeated Greek indices.
- When it doesn't lead to a confusion, we use the symbol $*$ to indicate any value, in the appropriate range, of a sub- or superscript.
- For a topological space $X$, we denote by $H^i(X)$ the cohomology groups with the coefficients in $\mathbb{C}$. Let $H^{\mathrm{even}}(X) := \bigoplus_{i\geq 0} H^{2i}(X)$.

**Acknowledgements.** The work of A. B. is an output of a research project implemented as part of the Basic Research Program at the National Research University Higher School of Economics (HSE University).

## 2. A family of F-CohFTs without unit of rank 2 and the associated DR hierarchy

Here we recall the construction of the DR hierarchy associated to an F-CohFT without unit and the main result from [BG23].

**Definition 2.1.** An *F-cohomological field theory without unit* (F-CohFT without unit) is a system of linear maps

$$c_{g,n+1}\colon V^*\otimes V^{\otimes n}\to H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n+1}),\quad 2g-1+n>0,$$

where $V$ is an arbitrary finite dimensional vector space, such that the following axioms are satisfied.

(i) The maps $c_{g,n+1}$ are equivariant with respect to the $S_n$-action permuting the $n$ copies of $V$ in $V^*\otimes V^{\otimes n}$ and the last $n$ marked points on curves from $\overline{\mathcal{M}}_{g,n+1}$, respectively.

(ii) Fixing a basis $e_1,\ldots,e_{\dim V}$ in $V$ and the dual basis $e^1,\ldots,e^{\dim V}$ in $V^*$, the following property holds:

$$\mathrm{gl}^*c_{g_1+g_2,n_1+n_2+1}(e^{\alpha_0}\otimes\otimes_{i=1}^{n_1+n_2}e_{\alpha_i})=c_{g_1,n_1+2}(e^{\alpha_0}\otimes\otimes_{i\in I}e_{\alpha_i}\otimes e_\mu)\otimes c_{g_2,n_2+1}(e^\mu\otimes\otimes_{j\in J}e_{\alpha_j})$$

for $1\le\alpha_0,\alpha_1,\ldots,\alpha_{n_1+n_2}\le\dim V$, where $I\sqcup J=\{2,\ldots,n_1+n_2+1\}$, $|I|=n_1$, $|J|=n_2$, and $\mathrm{gl}\colon\overline{\mathcal{M}}_{g_1,n_1+2}\times\overline{\mathcal{M}}_{g_2,n_2+1}\to\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2+1}$ is the corresponding gluing map. Clearly the axiom doesn't depend on the choice of a basis in $V$.

The dimension of $V$ is called the *rank* of the F-CohFT without unit.

We will use the following standard cohomology classes on $\overline{\mathcal{M}}_{g,n}$:

- The psi-class $\psi_i\in H^2(\overline{\mathcal{M}}_{g,n})$, $1\le i\le n$, is the first Chern class of the line bundle over $\overline{\mathcal{M}}_{g,n}$ formed by the cotangent lines at the $i$-th marked point of stable curves.
- The Hodge class $\lambda_j:=c_j(\mathbb{E})\in H^{2j}(\overline{\mathcal{M}}_{g,n})$, $j\ge 0$, where $\mathbb{E}$ is the rank $g$ Hodge vector bundle over $\overline{\mathcal{M}}_{g,n}$ whose fibers are the spaces of holomorphic one-forms on stable curves.
- The *double ramification (DR) cycle* $\mathrm{DR}_g(a_1,\ldots,a_n)\in H^{2g}(\overline{\mathcal{M}}_{g,n})$, $a_1,\ldots,a_n\in\mathbb{Z}$, $\sum a_i=0$, is defined as follows. There is a moduli space of projectivized stable maps to $\mathbb{CP}^1$ relative to 0 and $\infty$, with ramification profile over 0 given by the negative numbers among the $a_i$-s, ramification profile over $\infty$ given by the positive numbers among the $a_i$-s, and the zeros among the $a_i$-s correspond to additional marked points (see, e.g., [BSSZ15] for more details). This moduli space is endowed with a virtual fundamental class, which lies in homology of degree $2(2g-3+n)$. The DR cycle $\mathrm{DR}_g(a_1,\ldots,a_n)$ is the Poincaré dual to the pushforward, through the forgetful map to $\overline{\mathcal{M}}_{g,n}$, of this virtual fundamental class. The crucial property of the DR cycle is that for any cohomology class $\theta\in H^*(\overline{\mathcal{M}}_{g,n})$ the integral $\int_{\overline{\mathcal{M}}_{g,n+1}}\lambda_g\mathrm{DR}_g\left(-\sum a_i,a_1,\ldots,a_n\right)\theta$ is a homogeneous polynomial in $a_1,\ldots,a_n$ of degree $2g$ (see, e.g., [Bur15]).

Let us briefly recall main notions and notations in the formal theory of evolutionary PDEs with one spatial variable:

- We fix an integer $N\ge 1$ and consider formal variables $u^1,\ldots,u^N$. To the formal variables $u^\alpha$ we attach formal variables $u^\alpha_d$ with $d\ge 0$ and introduce the algebra of *differential polynomials* $\widehat{\mathcal{A}}_u:=\mathbb{C}[[u^*_0]][u^*_{\ge 1}][[\varepsilon]]$. We identify $u^\alpha_0=u^\alpha$ and also denote $u^\alpha_x:=u^\alpha_1$, $u^\alpha_{xx}:=u^\alpha_2$, $\ldots$. Denote by $\widehat{\mathcal{A}}_{u;d}\subset\widehat{\mathcal{A}}_u$ the homogeneous component of (differential) degree $d$, where $\deg u^\alpha_i:=i$ and $\deg\varepsilon:=-1$.
- An operator $\partial_x\colon\widehat{\mathcal{A}}_u\to\widehat{\mathcal{A}}_u$ is defined by $\partial_x:=\sum_{d\ge 0}u^\alpha_{d+1}\frac{\partial}{\partial u^\alpha_d}$.
- An operator $H\colon\widehat{\mathcal{A}}_u\to\widehat{\mathcal{A}}_u$ is called *evolutionary*, if $H$ satisfies the Leibniz rule and commutes with $\partial_x$. An operator $H$ is evolutionary if and only if it has the form $H=H_{\overline{P}}:=\sum_{n\ge 0}(\partial_x^nP^\alpha)\frac{\partial}{\partial u^\alpha_n}$ for some $\overline{P}=(P^1,\ldots,P^N)\in\widehat{\mathcal{A}}_u^N$. If the differential polynomials $P^\alpha$ satisfy the condition $P^\alpha|_{u^*_*=0}=0$, then we will write $H|_{u^*_*=0}=0$.
- We assign to an evolutionary operator $H_{\overline{P}}$ the system of *evolutionary PDEs* (with one spatial variable) $\frac{\partial u^\alpha}{\partial t}=P^\alpha$, $1\le\alpha\le N$. Two such systems are said to be *compatible* if the corresponding evolutionary operators commute.
- An element $f\in\widehat{\mathcal{A}}_u$ is called a *conservation law* for an evolutionary operator $H$ (or for the corresponding system of evolutionary PDEs) if there exists an element $R\in\widehat{\mathcal{A}}_u$ such that $H(f)=\partial_xR$.

- A *Miura transformation* is a change of variables $u^\alpha \mapsto \widetilde{u}^\alpha(u^*_*,\varepsilon)$ of the form $\widetilde{u}^\alpha(u^*_*,\varepsilon) = g^\alpha(u^*_0)+\varepsilon f^\alpha(u^*_*,\varepsilon)$, where $f^\alpha \in \widehat{\mathcal{A}}_{u;1}$ and $g^\alpha \in \mathbb{C}[[u^*_0]]$ satisfy $g^\alpha|_{u^*_0=0} = 0$ and $\det\left(\frac{\partial g^\alpha}{\partial u^\beta_0}\right)\Big|_{u^*_0=0} \neq 0$.

**Lemma 2.2.** *Consider the case $N=1$ and denote $u_n := u^1_n$. Consider a pair of compatible PDEs*

$$\begin{cases} \dfrac{\partial u}{\partial t} = P, \\ \dfrac{\partial u}{\partial s} = Q, \end{cases}$$

*where $P = uu_x + O(\varepsilon) \in \widehat{\mathcal{A}}_{u;1}$, $Q = f(u)u_x + O(\varepsilon) \in \widehat{\mathcal{A}}_{u;1}$, and $f \in \mathbb{C}[[u]]$. Then the differential polynomial $Q$ is uniquely determined by $P$ and $f$.*

*Proof.* This is a slight generalization of [BR21, Lemma 4.14], with the same proof, so we omit it. □

Consider now an arbitrary F-CohFT without unit of rank $N$ and define differential polynomials $P^\alpha_{\beta,d} \in \widehat{\mathcal{A}}_u$, $1 \le \alpha,\beta \le N$, $d \ge 0$, by

$$P^\alpha_{\beta,d} := \sum_{\substack{g,n\ge 0,\, 2g+n>0\\ k_1,\ldots,k_n\ge 0\\ \sum_{j=1}^n k_j = 2g}} \frac{\varepsilon^{2g}}{n!}\mathrm{Coef}_{(a_1)^{k_1}\ldots(a_n)^{k_n}}\left(\int_{\mathrm{DR}_g(-\sum_{j=1}^n a_j,0,a_1,\ldots,a_n)} \lambda_g\psi_2^d c_{g,n+2}(e^\alpha\otimes e_\beta\otimes \otimes_{j=1}^n e_{\alpha_j})\right)\prod_{j=1}^n u^{\alpha_j}_{k_j}.$$

The *DR hierarchy* is the following system of evolutionary PDEs:

$$\frac{\partial u^\alpha}{\partial t^\beta_d} = \partial_x P^\alpha_{\beta,d}, \qquad 1\le\alpha,\beta\le N, \quad d\ge 0. \tag{2.1}$$

All the equations of the DR hierarchy are pairwise compatible.

*Example* 2.3. Consider the trivial F-CohFT without unit given by $V=\mathbb{C}$, $e_1 = 1 \in \mathbb{C} = V$, and

$$c^{\mathrm{triv}}_{g,n+1}(e^1\otimes e_1^{\otimes n}) := 1 \in H^0(\overline{\mathcal{M}}_{g,n+1}).$$

Then the corresponding DR hierarchy is the KdV hierarchy [Bur15, Section 4.3.1] (we denote $u_d := u^1_d$ and $t_d := t^1_d$)

$$\frac{\partial u}{\partial t_d} = \partial_x P^{\mathrm{KdV}}_d,$$

where

$$P^{\mathrm{KdV}}_0 = u, \quad P^{\mathrm{KdV}}_1 = \frac{u^2}{2} + \frac{\varepsilon^2}{12}u_{xx}, \quad P^{\mathrm{KdV}}_2 = \frac{u^3}{6} + \frac{\varepsilon^2}{24}(2uu_{xx}+u_x^2) + \frac{\varepsilon^4}{240}u_{xxxx},$$

and a general formula for $P^{\mathrm{KdV}}_d$ is

$$\partial_x P^{\mathrm{KdV}}_d = \frac{\varepsilon^{2d+2}}{2(2d+1)!!}\left[\left(L^{d+\frac12}\right)_+, L\right], \quad L = \partial_x^2 + 2\varepsilon^{-2}u,$$

defining $P^{\mathrm{KdV}}_d\big|_{u_*=0} := 0$. Note that

$$P^{\mathrm{KdV}}_d = \frac{u^{d+1}}{(d+1)!} + O(\varepsilon) \in \widehat{\mathcal{A}}_{u;0}.$$

In [BG23], the authors considered the following family of F-CohFTs without unit, with phase space $V = \mathbb{C}^2$, parameterized by a vector $G = (G^1, G^2) \in \mathbb{C}^2$:

$$c^{\mathrm{triv},G}_{g,n+1}(e^{i_0} \otimes \otimes_{j=1}^n e_{i_j}) := \begin{cases} (G^{i_0})^g, & \text{if } i_0 = i_1 = \ldots = i_n, \\ 0, & \text{otherwise,} \end{cases}$$

where $e_1, e_2$ is the standard basis of $\mathbb{C}^2$. Then the authors of [BG23] applied to this F-CohFT without unit the $R$-matrix $\mathrm{Id} + R_1 z$ with

$$R_1 = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}, \quad \xi \in \mathbb{C},$$

see the details about this action in [BG23]. The resulting F-CohFT without unit is denoted by $\left((\mathrm{Id} + R_1 z)c^{\mathrm{triv},G}\right)_{g,n+1}$. In [BG23, Theorem 4.1], the authors considered the corresponding DR hierarchy and proved that after the Miura transformation

$$\widetilde{u}^1 = u^1 + \xi\frac{(u^2)^2}{2} + \frac{\varepsilon^2}{24}\partial_x^2\left(\xi G^2 u^2 + \frac{G^1}{1+\xi u^2}\right), \qquad \widetilde{u}^2 = u^2,$$

the flows $\frac{\partial}{\partial t^1_0}$ and $\frac{\partial}{\partial t^2_0}$ of the DR hierarchy become

$$\frac{\partial \widetilde{u}^1}{\partial t^1_0} = \partial_x\left[\frac{\widetilde{u}^1}{1+\xi\widetilde{u}^2}\right], \tag{2.2}$$

$$\frac{\partial \widetilde{u}^2}{\partial t^1_0} = 0, \tag{2.3}$$

$$\frac{\partial \widetilde{u}^1}{\partial t^2_0} = \xi\partial_x\left[\frac{\widetilde{u}^1\widetilde{u}^2}{1+\xi\widetilde{u}^2} - \frac{1}{2}\frac{(\widetilde{u}^1)^2}{(1+\xi\widetilde{u}^2)^2} - \frac{\varepsilon^2 G^1}{12}\left(\left(\left(\frac{\widetilde{u}^1}{1+\xi\widetilde{u}^2}\right)_x\frac{1}{1+\xi\widetilde{u}^2}\right)_x\frac{1}{1+\xi\widetilde{u}^2}\right)\right], \tag{2.4}$$

$$\frac{\partial \widetilde{u}^2}{\partial t^2_0} = \widetilde{u}^2_x, \tag{2.5}$$

and moreover

$$\frac{\partial \widetilde{u}^2}{\partial t^1_d} = 0, \qquad \frac{\partial \widetilde{u}^2}{\partial t^2_d} = \partial_x P^{\mathrm{KdV}}_d\big|_{u_n \mapsto \widetilde{u}^2_n,\, \varepsilon \mapsto \sqrt{G^2}\varepsilon}. \tag{2.6}$$

## 3. Reciprocal transformations and the main result

**3.1. Linear and nonlinear reciprocal transformations: overview.** Here, without specifying technical details, we recall the definitions of two types of reciprocal transformations.

Consider a compatible system of PDEs

$$\frac{\partial u^\alpha}{\partial t_i} = P^\alpha_i, \quad P^\alpha_i \in \widehat{\mathcal{A}}_u, \quad 1 \le \alpha \le N, \quad i \ge 1. \tag{3.1}$$

A *linear reciprocal transformation* is a change of the spatial variable $x$ of the following form:

$$x \mapsto y = \gamma_0 x + \sum_{i\ge1}\gamma_i t_i, \quad \gamma_0, \gamma_1, \gamma_2, \ldots \in \mathbb{C}.$$

These transformations will not be considered in this paper. Suppose now that $f \in \widehat{\mathcal{A}}_u$ is a nonconstant conservation law for our system (3.1):

$$\frac{\partial f}{\partial t_i} = \partial_x R_i, \quad R_i \in \widehat{\mathcal{A}}_u.$$

A *nonlinear reciprocal transformation* is a change of the spatial variable $x \mapsto y$ of the form

$$dy = f dx + \sum_{i\ge1} R_i dt_i.$$

In the next section, we will describe how this transformation changes the system (3.1). In particular, some conditions on our system and on $f$ will be added, so that the transformed system does not go beyond the class of systems of evolutionary PDEs that we consider.

3.2. **Nonlinear reciprocal transformations: formal definition.** Proofs of the properties of nonlinear reciprocal transformations can be found, for example, in [LZ11], however for completeness we present short proofs in the appendix.

Consider two $N$-tuples of variables $u^1,\dots,u^N$ and $v^1,\dots,v^N$, and the associated algebras of differential polynomials $\widehat{\mathcal{A}}_u$ and $\widehat{\mathcal{A}}_v$. In the algebra $\widehat{\mathcal{A}}_v$, let us denote the spatial variable by $y$, i.e. we denote $\partial_y := \sum_{n\ge 0} v^\alpha_{n+1}\frac{\partial}{\partial v^\alpha_n}$, and also $v^\alpha_y := v^\alpha_1$, $v^\alpha_{yy} := v^\alpha_2$, .... Choose a nonzero element $f\in\widehat{\mathcal{A}}_{u;0}$ such that $f|_{u^*_*=0}=0$. Then the element $1+f\in\widehat{\mathcal{A}}_{u;0}$ is invertible. Define an algebra homomorphism $\Phi_f\colon\widehat{\mathcal{A}}_v\to\widehat{\mathcal{A}}_u$ by

$$\Phi_f(P) := P|_{v^\alpha_k\mapsto((1+f)^{-1}\partial_x)^k(u^\alpha)},\qquad P\in\widehat{\mathcal{A}}_v.$$

The homomorphism $\Phi_f$ is an isomorphism. It is called a *nonlinear reciprocal transformation.* We have

$$(1+f)^{-1}\partial_x\circ\Phi_f = \Phi_f\circ\partial_y,$$

and therefore under the isomorphism $\Phi_f\colon\widehat{\mathcal{A}}_v\to\widehat{\mathcal{A}}_u$ the operators $\partial_y$ and $(1+f)^{-1}\partial_x$ become identified.

By abuse of notation, we will identify elements of $\widehat{\mathcal{A}}_v$ and their images under $\Phi_f$ in $\widehat{\mathcal{A}}_u$, as well as the operators $\partial_y$ and $(1+f)^{-1}\partial_x$. Note that under this identification evolutionary operators on $\widehat{\mathcal{A}}_v$ in general do not correspond to evolutionary operators on $\widehat{\mathcal{A}}_u$. However, suppose $f$ is a conservation law for an evolutionary operator $H$ on $\widehat{\mathcal{A}}_u$, so that $H(f)=\partial_x R$ for some $R\in\widehat{\mathcal{A}}_u$. Then $H-R\partial_y$ is an evolutionary operator on $\widehat{\mathcal{A}}_v$. Moreover, let us denote $H_1:=H$, $R_1:=R$, and suppose that $f$ is a conservation law for another evolutionary operator $H_2$ on $\widehat{\mathcal{A}}_u$, $H_2(f)=\partial_x R_2$, which commutes with $H_1$. Suppose also that $H_1|_{u^*_*=0}=H_2|_{u^*_*=0}=0$. Then the corresponding evolutionary operators $H_1-R_1\partial_y$ and $H_2-R_2\partial_y$ on $\widehat{\mathcal{A}}_v$ commute.

We know that any evolutionary operator $H$ on $\widehat{\mathcal{A}}_u$ has the form $H=H_{\overline{P}}$, where $\overline{P}=(P^1,\dots,P^N)\in\widehat{\mathcal{A}}_u^N$, and we assign to $H$ the system of PDEs

$$\frac{\partial u^\alpha}{\partial t} = P^\alpha,\quad 1\le\alpha\le N. \tag{3.2}$$

Given a conservation law $f\in\widehat{\mathcal{A}}_{u;0}$ of $H$, $H(f)=\partial_x R$, satisfying $f|_{u^*_*=0}=0$, we obtain the evolutionary operator $H-R\partial_y$ on $\widehat{\mathcal{A}}_v$, to which we assign the system of PDEs

$$\frac{\partial v^\alpha}{\partial t} = P^\alpha - R v^\alpha_y,\quad 1\le\alpha\le N. \tag{3.3}$$

We will say that the system (3.3) is obtained from the system (3.2) by the *nonlinear reciprocal transformation* given by the conservation law $f$.

*Remark* 3.1. Let us describe how the nonlinear reciprocal transformations act on solutions of systems of PDEs. Consider a collection of pairwise commuting evolutionary operators $H_i$, $i\ge 1$, on $\widehat{\mathcal{A}}_u$, $H_i=H_{\overline{P}_i}$, and suppose that $P^\alpha_i|_{u^*_*=0}=0$. Suppose that $0\ne f\in\widehat{\mathcal{A}}_{u;0}$ satisfying $f|_{u^*_*=0}=0$ is a common conservation law for the evolutionary operators $H_i$, $H_i f=\partial_x R_i$. Consider a solution $(\mathbf{u}^1,\dots,\mathbf{u}^N)\in\mathbb{C}[[x,t_*,\varepsilon]]^N$, $\mathbf{u}^\alpha|_{x=t_*=0}=0$, of the system of PDEs

$$\frac{\partial u^\alpha}{\partial t_i} = P^\alpha_i,\quad 1\le\alpha\le N,\ i\ge 1. \tag{3.4}$$

Define a formal power series $\mathbf{y} \in \mathbb{C}[[x, t_*, \varepsilon]]$ satisfying $\mathbf{y}|_{x=t_*=0} = 0$ by the equation

$$d\mathbf{y} = \left(1 + f|_{u^\alpha_n = \partial_x^n \mathbf{u}^\alpha}\right) dx + \sum_{i\geq 1} \left(R_i|_{u^\alpha_n = \partial_x^n \mathbf{u}^\alpha}\right) dt_i.$$

The 1-form on the right-hand side is closed because $H_i(f) = \partial_x f$ and $H_i(R_j) = H_j(R_i)$. The last equality is true because $\partial_x(H_i(R_j) - H_j(R_i)) = H_i(H_j(f)) - H_j(H_i(f)) = 0$, which implies that $H_i(R_j) - H_j(R_i) \in \mathbb{C}[[\varepsilon]]$. However, since $H_i|_{u^*_*=0} = H_j|_{u^*_*=0} = 0$, we immediately obtain $H_i(R_j) - H_j(R_i) = 0$. Let $\mathbf{v}^\alpha \in \mathbb{C}[[y, t_*, \varepsilon]]$ be a unique formal power series satisfying

$$\mathbf{v}^\alpha|_{y=\mathbf{y}} = \mathbf{u}^\alpha.$$

Then $(\mathbf{v}^1, \ldots, \mathbf{v}^N)$ is a solution of the system

$$\frac{\partial v^\alpha}{\partial t_i} = P^\alpha_i - R_i v^\alpha_y, \quad 1 \leq \alpha \leq N,\ i \geq 1,$$

which is obtained from the system (3.4) by the nonlinear reciprocal transformation given by the common conservation law $f$ of the evolutionary operators $H_i$.

*Example* 3.2. Consider the KdV hierarchy

$$\frac{\partial u}{\partial t_d} = \partial_x P^{\mathrm{KdV}}_d, \quad d \geq 0.$$

We see that $u$ is a common conservation law of the flows of the KdV hierarchy. So for any $\xi \in \mathbb{C}^*$ we have the nonlinear reciprocal transformation of the KdV hierarchy given by the conservation law $\xi u$:

$$\frac{\partial v}{\partial t_d} = \underbrace{\partial_x P^{\mathrm{KdV}}_d - \xi P^{\mathrm{KdV}}_d v_y}_{=:Q^{\xi-\mathrm{KdV}}_d \in \widehat{\mathcal{A}}_v}, \quad d \geq 0.$$

Since $u^2$ is a conservation law of the KdV hierarchy, we obtain $u\partial_x P^{\mathrm{KdV}}_d \in \mathrm{Im}(\partial_x)$, which implies that $(1 + \xi u)\partial_x P^{\mathrm{KdV}}_d - \xi P^{\mathrm{KdV}}_d u_x \in \mathrm{Im}(\partial_x)$, and therefore $Q^{\xi-\mathrm{KdV}}_d \in \mathrm{Im}(\partial_y)$. Thus, the nonlinear reciprocal transformation of the KdV hierarchy given by the conservation law $\xi u$ has the form

$$\frac{\partial v}{\partial t_d} = \partial_y P^{\xi-\mathrm{KdV}}_d, \quad d \geq 0,$$

where $P^{\xi-\mathrm{KdV}}_d\Big|_{v_*=0} = 0$. For example,

$$P^{\xi-\mathrm{KdV}}_0 = v, \qquad P^{\xi-\mathrm{KdV}}_1 = \frac{v^2}{2} + \xi\frac{v^3}{6} + \frac{\varepsilon^2}{12}(1 + \xi v)^3 v_{yy}.$$

Consider again the DR hierarchy associated to the F-CohFT without unit $\left((\mathrm{Id} + R_1 z)c^{\mathrm{triv},G}\right)_{g,n+1}$ from Section 2. Since $P^2_{1,d} = 0$ and $P^2_{2,d}$ doesn't depend on $u^1_n$ for any $n \geq 0$, after any Miura transformation that doesn't change the variable $u^2$, this variable will be a conservation law for the transformed hierarchy.

**Theorem 3.3.** *Consider the F-CohFT without unit* $\left((\mathrm{Id} + R_1 z)c^{\mathrm{triv},G}\right)_{g,n+1}$*, with* $\xi \neq 0$*, and the associated DR hierarchy. Then the composition of the Miura transformation*

$$\widehat{u}^1 = \frac{1}{1 + \xi u^2}\left(u^1 + \xi\frac{(u^2)^2}{2} + \frac{\varepsilon^2}{24}\partial_x^2\left(\xi G^2 u^2 + \frac{G^1}{1 + \xi u^2}\right)\right), \qquad \widehat{u}^2 = u^2, \tag{3.5}$$

*and the nonlinear reciprocal transformation given by the conservation law* $\xi\widehat{u}^2$ *transforms the DR hierarchy to the system*

$$\frac{\partial v^1}{\partial t^1_d} = \left(\partial_x P^{\mathrm{KdV}}_d\right)\Big|_{u_l\mapsto v^1_l,\,\varepsilon\mapsto\sqrt{G^1}\varepsilon}, \qquad \frac{\partial v^2}{\partial t^1_d} = 0,$$
$$\frac{\partial v^1}{\partial t^2_d} = -\left(\xi\partial_x P^{\mathrm{KdV}}_{d+1}\right)\Big|_{u_l\mapsto v^1_l,\,\varepsilon\mapsto\sqrt{G^1}\varepsilon}, \qquad \frac{\partial v^2}{\partial t^2_d} = \left(\partial_y P^{\xi-\mathrm{KdV}}_d\right)\Big|_{v_l\mapsto v^2_l,\,\varepsilon\mapsto\sqrt{G^2}\varepsilon}.$$

## 4. Proof of Theorem 3.3

4.1. **Computation of the dispersionless part of the hierarchy.** Let us recall how to compute the dispersionless part, i.e. when $\varepsilon=0$, of the DR hierarchy associated to an arbitrary F-CohFT without unit $\{c_{g,n+1}\colon V^*\otimes V^{\otimes n}\to H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n+1})\}$. For this, define an $N$-tuple of formal power series $(F^1,\ldots,F^N)\in\mathbb{C}[[u^1,\ldots,u^N]]^N$ by

$$F^\alpha := \sum_{n\ge 2}\frac{1}{n!}\sum_{1\le\alpha_1,\ldots,\alpha_n\le N}\left(\int_{\overline{\mathcal{M}}_{0,n+1}} c_{0,n+1}(e^\alpha\otimes\otimes_{j=1}^n e_{\alpha_j})\right)\prod_{j=1}^n u^{\alpha_j}.$$

Let $c^\alpha_{\beta\gamma}=\frac{\partial^2 F^\alpha}{\partial u^\beta\partial u^\gamma}$ and define $N\times N$ matrices $C_\gamma:=(c^\alpha_{\beta\gamma})_{1\le\alpha,\beta\le N}$. Then the $N\times N$ matrices $P^{[0]}_d:=(P^\alpha_{\beta,d}|_{\varepsilon=0})_{1\le\alpha,\beta\le N}$ are uniquely determined by the relations

$$\frac{\partial P^{[0]}_d}{\partial u^\gamma} = C_\gamma\cdot P^{[0]}_{d-1},\qquad P^{[0]}_d\Big|_{u^*=0}=0,\quad d\ge 0,\ 1\le\gamma\le N, \tag{4.1}$$

where $P^{[0]}_{-1}:=\mathrm{Id}$.

Let us return to our F-CohFT without unit $\left((\mathrm{Id}+R_1z)c^{\mathrm{triv},G}\right)_{g,n+1}$.

**Lemma 4.1.** *We have*

$$F^1(u^1,u^2)=\frac{\frac{(u^1)^2}{2}+u^1\cdot\frac{\xi(u^2)^2}{2}-\frac{\xi(u^2)^3}{24}(4+\xi u^2)}{1+\xi u^2},\qquad F^2(u^1,u^2)=\frac{(u^2)^2}{2}.$$

*Proof.* We proceed in the same way as in [BG23, proof of Theorem 4.1]: in order to compute the integrals

$$\int_{\overline{\mathcal{M}}_{0,n+1}}\left((\mathrm{Id}+R_1z)c^{\mathrm{triv},G}\right)_{0,n+1}(e^\alpha\otimes\otimes_{j=1}^n e_{\alpha_j}),$$

we express the class $\left((\mathrm{Id}+R_1z)c^{\mathrm{triv},G}\right)_{0,n+1}(e^\alpha\otimes\otimes_{j=1}^n e_{\alpha_j})$ as a sum over stable trees. All the trees that give a nontrivial contribution to the integral are depicted in Figure 1, where the first four trees contribute to $F^1(u^1,u^2)$, and the last one contributes to $F^2(u^1,u^2)$. The respective sums are

$$F^1(u^1,u^2)=\frac{(u^1)^2/2}{1+\xi u^2}+\frac{u^1\cdot\xi(u^2)^2/2}{1+\xi u^2}+\frac{\xi^2(u^2)^4/8}{1+\xi u^2}-\frac{\xi(u^2)^3}{6},\qquad F^2(u^1,u^2)=\frac{(u^2)^2}{2}.$$

□

**Lemma 4.2.** *For any* $d\ge 0$, *we have*

$$P^{[0]}_d=\begin{pmatrix}\frac{(\bar u^1)^{d+1}}{(d+1)!} & M_d\\ 0 & \frac{(\bar u^2)^{d+1}}{(d+1)!}\end{pmatrix},$$

*where*

$$\bar u^1:=\widehat{u}^1|_{\varepsilon=0}=\frac{u^1+\frac{\xi(u^2)^2}{2}}{1+\xi u^2},\quad \bar u^2:=u^2,\quad M_d:=-\frac{\xi\left((\bar u^1)^{d+2}-(d+2)\bar u^1(\bar u^2)^{d+1}+(d+1)(\bar u^2)^{d+2}\right)}{(d+2)!}.$$

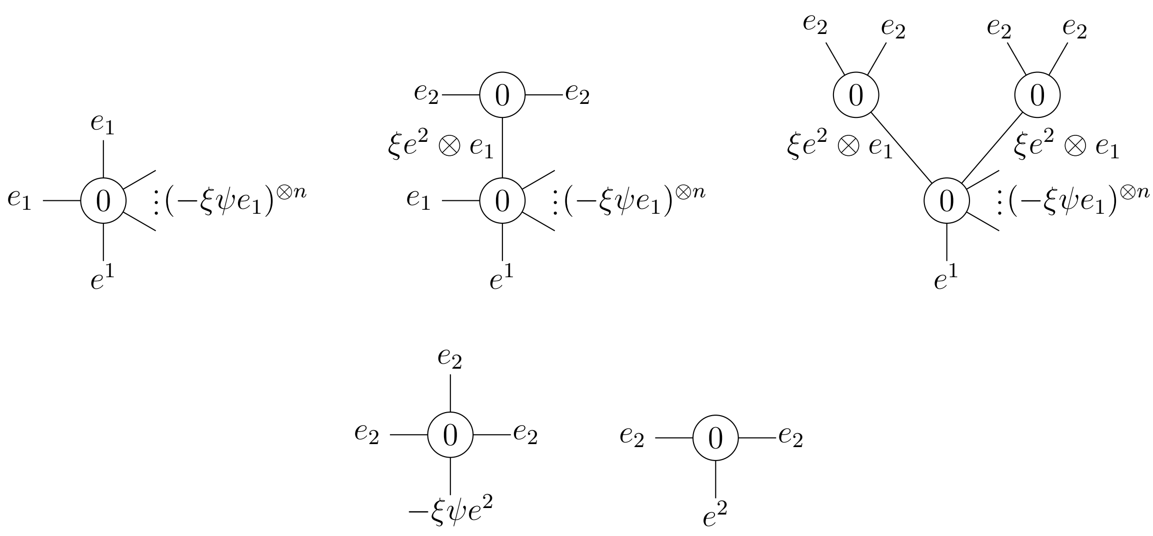


FIGURE 1. Stable trees contributing to $(F^1, F^2)$

*Proof.* We compute

$$C_1 = \begin{pmatrix} \frac{1}{1+\xi\bar{u}^2} & \frac{\xi(\bar{u}^2-\bar{u}^1)}{1+\xi\bar{u}^2} \\ 0 & 0 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} \frac{\xi(\bar{u}^2-\bar{u}^1)}{1+\xi\bar{u}^2} & \frac{\xi(\bar{u}^1-\bar{u}^2)(1+\xi\bar{u}^1)}{1+\xi\bar{u}^2} \\ 0 & 1 \end{pmatrix}. \tag{4.2}$$

Then one directly checks the relations (4.1), expressing

$$\frac{\partial}{\partial u^1} = \frac{1}{1+\xi\bar{u}^2}\frac{\partial}{\partial \bar{u}^1}, \qquad \frac{\partial}{\partial u^2} = \xi\frac{\bar{u}^2-\bar{u}^1}{1+\xi\bar{u}^2}\frac{\partial}{\partial \bar{u}^1} + \frac{\partial}{\partial \bar{u}^2}.$$

□

**Proposition 4.3.** *After the composition of the Miura transformation* (3.5) *and the nonlinear reciprocal transformation given by the conservation law $\xi\widehat{u}^2$, the DR hierarchy of Theorem 3.3 has the form*

$$\frac{\partial v^1}{\partial t^1_d} = \partial_y\left(\frac{(v^1)^{d+1}}{(d+1)!}\right) + O(\varepsilon), \qquad \frac{\partial v^2}{\partial t^1_d} = 0,$$
$$\frac{\partial v^1}{\partial t^2_d} = -\xi\partial_y\left(\frac{(v^1)^{d+2}}{(d+2)!}\right) + O(\varepsilon), \qquad \frac{\partial v^2}{\partial t^2_d} = \left(\partial_y P^{\xi-\mathrm{KdV}}_d\right)\Big|_{v_l\mapsto v^2_l,\,\varepsilon\mapsto\sqrt{G^2}\varepsilon}.$$

*Proof.* This is a direct computation. We apply the Miura transformation (3.5) to the dispersionless part of the DR hierarchy, computed in Lemma 4.2, and obtain

$$\frac{\partial \widehat{u}^1}{\partial t^1_d} = \frac{1}{1+\xi\widehat{u}^2}\partial_x\left(\frac{(\widehat{u}^1)^{d+1}}{(d+1)!}\right) + O(\varepsilon), \qquad \frac{\partial \widehat{u}^2}{\partial t^1_d} = 0,$$
$$\frac{\partial \widehat{u}^1}{\partial t^2_d} = -\xi\frac{\widehat{u}^1_x\left((\widehat{u}^1)^{d+1} - (\widehat{u}^2)^{d+1}\right)}{(d+1)!(1+\xi\widehat{u}^2)} + O(\varepsilon), \qquad \frac{\partial \widehat{u}^2}{\partial t^2_d} = \partial_x P^{\mathrm{KdV}}_d\Big|_{u_n\mapsto\widehat{u}^2_n,\,\varepsilon\mapsto\sqrt{G^2}\varepsilon}.$$

One can easily see that after the nonlinear reciprocal transformation given by the conservation law $\xi\widehat{u}^2$ the hierarchy has the desired form. □

4.2. **The full hierarchy.** We are ready to finish the proof of Theorem 3.3. We know that after the composition of the Miura transformation (3.5) and the nonlinear reciprocal transformation given by the conservation law $\xi\widehat{u}^2$, the DR hierarchy has the form

$$
\begin{aligned}
\frac{\partial v^1}{\partial t^1_d} &= S_d, & \frac{\partial v^2}{\partial t^1_d} &= 0,\\
\frac{\partial v^1}{\partial t^2_d} &= T_d, & \frac{\partial v^2}{\partial t^2_d} &= \left(\partial_y P_d^{\xi-\mathrm{KdV}}\right)\Big|_{v_l\mapsto v^2_l,\,\varepsilon\mapsto\sqrt{G^2}\varepsilon},
\end{aligned}
$$

where

$$
S_d=\partial_y\left(\frac{(v^1)^{d+1}}{(d+1)!}\right)+O(\varepsilon)\in\widehat{\mathcal{A}}_{v;1},\qquad T_d=-\xi\partial_y\left(\frac{(v^1)^{d+2}}{(d+2)!}\right)+O(\varepsilon)\in\widehat{\mathcal{A}}_{v;1}.
$$

A direct computation using formulas (2.2)–(2.5) gives that

$$
S_0=v^1_y,\qquad T_0=-\xi\partial_y\left(\frac{(v^1)^2}{2}+\frac{\varepsilon^2G^1}{12}v^1_{yy}\right).
$$

Let us prove that $S_d$ and $T_d$ don't depend on the variables $v^2_n$. Indeed, the fact that the flows $\frac{\partial}{\partial t^1_0}$ and $\frac{\partial}{\partial t^1_d}$ commute gives that

$$
0=\frac{\partial}{\partial t^1_d}\frac{\partial v^1}{\partial t^1_0}-\frac{\partial}{\partial t^1_0}\frac{\partial v^1}{\partial t^1_d}=\partial_yS_d-\sum_{n\ge 0}v^1_{n+1}\frac{\partial S_d}{\partial v^1_n}=\sum_{n\ge 0}v^2_{n+1}\frac{\partial S_d}{\partial v^2_n},
$$

which implies that $\frac{\partial S_d}{\partial v^2_n}=0$ for any $n$. In the same way, the commutativity of the flows $\frac{\partial}{\partial t^1_0}$ and $\frac{\partial}{\partial t^2_d}$ gives that $\frac{\partial T_d}{\partial v^2_n}=0$ for any $n$. Since the flows of the KdV hierarchy pairwise commute, Proposition 4.3 and Lemma 2.2 imply that $S_d=\left(\partial_xP_d^{\mathrm{KdV}}\right)\big|_{u_l\mapsto v^1_l,\,\varepsilon\mapsto\sqrt{G^1}\varepsilon}$ and $T_d=-\left(\xi\partial_xP_{d+1}^{\mathrm{KdV}}\right)\big|_{u_l\mapsto v^1_l,\,\varepsilon\mapsto\sqrt{G^1}\varepsilon}$. This completes the proof of Theorem 3.3.

## Appendix A. Nonlinear reciprocal transformations

Here, for completeness, we present short proofs of the properties of nonlinear reciprocal transformations mentioned in Section 3, see Parts 1 and 2 of Proposition A.1. There are also Part 3 in Proposition A.1 and Remark A.2, which we think are of independent interest, and which as far as we know didn't appear in the literature before.

**Proposition A.1.** *Suppose $0\ne f\in\widehat{\mathcal{A}}_{u;0}$ satisfying $f|_{u^*_*=0}=0$ is a conservation law for an evolutionary operator $H$ on $\widehat{\mathcal{A}}_u$, so that $H(f)=\partial_xR$ for some $R\in\widehat{\mathcal{A}}_u$. Consider the associated nonlinear reciprocal transformation $\Phi_f\colon\widehat{\mathcal{A}}_v\to\widehat{\mathcal{A}}_u$.*

1. *$H-R\partial_y$ is an evolutionary operator on $\widehat{\mathcal{A}}_v$.*
2. *Denote $H_1:=H$, $R_1:=R$, and suppose that $f$ is a conservation law for another evolutionary operator $H_2$ on $\widehat{\mathcal{A}}_u$, $H_2(f)=\partial_xR_2$, which commutes with $H_1$. Suppose also that $H_1|_{u^*_*=0}=H_2|_{u^*_*=0}=0$. Then the evolutionary operators $H_1-R_1\partial_y$ and $H_2-R_2\partial_y$ on $\widehat{\mathcal{A}}_v$ commute.*
3. *The map $g\mapsto\frac{g}{1+f}$ gives a one-to-one correspondence between the conservation laws of the operator $H$ on $\widehat{\mathcal{A}}_u$ and the conservation laws of the operator $H-R\partial_y$ on $\widehat{\mathcal{A}}_v$. Moreover, if $H(g)=\partial_xR_g$, then $(H-R\partial_y)\left(\frac{g}{1+f}\right)=\partial_y\left(R_g-\frac{gR}{1+f}\right)$.*

*Proof. 1.* The Leibniz rule holds for $H - R\partial_y$ by linearity, so we only need to check the equality $[H - R\partial_y, \partial_y] = 0$. Indeed,

$$[H - R\partial_y, \partial_y] = [H, \partial_y] - [R\partial_y, \partial_y] = \left[H, \frac{1}{1+f}\partial_x\right] - [R\partial_y, \partial_y] =$$
$$= H\left(\frac{1}{1+f}\right)\partial_x + (\partial_y R)\partial_y = -\frac{H(f)}{(1+f)^2}\partial_x + \frac{\partial_x R}{(1+f)^2}\partial_x = 0.$$

*2.* We observe that $H_2(R_1) - H_1(R_2) = 0$. Indeed, $\partial_x(H_2(R_1) - H_1(R_2)) = H_2H_1(f) - H_1H_2(f) = 0$, which implies that $H_2(R_1) - H_1(R_2) \in \mathbb{C}[[\varepsilon]]$. However, since $H_1|_{u^*_*=0} = H_2|_{u^*_*=0} = 0$, we immediately obtain $H_2(R_1) - H_1(R_2) = 0$. We further compute

$$[H_1 - R_1\partial_y, H_2 - R_2\partial_y] = [H_1, H_2] - [H_1, R_2\partial_y] - [R_1\partial_y, H_2] + [R_1\partial_y, R_2\partial_y] =$$
$$= -\left[H_1, \frac{R_2}{1+f}\partial_x\right] - \left[\frac{R_1}{1+f}\partial_x, H_2\right] + [R_1\partial_y, R_2\partial_y],$$
$$-\left[H_1, \frac{R_2}{1+f}\partial_x\right] = \left(-\frac{H_1(R_2)}{1+f} + \frac{R_2 \cdot H_1(f)}{(1+f)^2}\right)\partial_x = \left(-\frac{H_1(R_2)}{1+f} + \frac{R_2 \cdot \partial_x R_1}{(1+f)^2}\right)\partial_x,$$
$$-\left[\frac{R_1}{1+f}\partial_x, H_2\right] = \left(\frac{H_2(R_1)}{1+f} - \frac{R_1 \cdot H_2(f)}{(1+f)^2}\right)\partial_x = \left(\frac{H_2(R_1)}{1+f} - \frac{R_1 \cdot \partial_x R_2}{(1+f)^2}\right)\partial_x,$$
$$[R_1\partial_y, R_2\partial_y] = (R_1 \cdot \partial_y R_2 - R_2 \cdot \partial_y R_1)\,\partial_y = \frac{1}{(1+f)^2}(R_1 \cdot \partial_x R_2 - R_2 \cdot \partial_x R_1)\,\partial_x.$$

We see that all terms in the resulting sum cancel out.

*3.* If $H(g) = \partial_x R_g$, then we have

$$(H - R\partial_y)\left(\frac{g}{1+f}\right) = \frac{H(g)}{1+f} - \frac{g \cdot H(f)}{(1+f)^2} - \frac{R \cdot \partial_y g}{1+f} + \frac{gR \cdot \partial_y f}{(1+f)^2} =$$
$$= \partial_y R_g - \frac{g \cdot \partial_y R}{1+f} - \frac{R \cdot \partial_y g}{1+f} + \frac{gR \cdot \partial_y f}{(1+f)^2} =$$
$$= \partial_y\left(R_g - \frac{gR}{1+f}\right).$$

Conversely, if $(H - R\partial_y)\widetilde{g} = \partial_y R_{\widetilde{g}}$, then in the same way we check that $H\left((1+f)\widetilde{g}\right) = \partial_x\left(R_{\widetilde{g}} + \widetilde{g}R\right)$. ☐

*Remark* A.2. We see that the nonlinear reciprocal transformations give an action of the set $\mathrm{CL}(H) := \{f \in \widehat{\mathcal{A}}_{u;0} | H(f) \in \mathrm{Im}(\partial_x),\ f|_{u^*_*=0} = 0\}$ on the evolutionary operator $H$: given $f \in \mathrm{CL}(H)$, we transform the operator $H$ to the operator $H - R_f\partial_y$, where $R_f$ is given by $H(f) = \partial_x R_f$ with $R_f|_{u^*_*=0} = 0$. Since, by Part 3 of the proposition, there is a one-to-one correspondence between the conservation laws of the operator $H$ and the conservation laws of the operator $H - R_f\partial_y$, we can further act on $H - R_f\partial_y$ by any $g \in \mathrm{CL}(H)$, meaning that we act on $H - R_f\partial_y$ by $\frac{g}{1+f} \in \mathrm{CL}(H - R_f\partial_y)$. Let us compute the composition of two such transformations. Indeed, for this we consider another $N$-tuple of formal variables $w^1, \ldots, w^N$ and the nonlinear reciprocal transformation $\Phi_{\frac{g}{1+f}} : \widehat{\mathcal{A}}_w \to \widehat{\mathcal{A}}_v$. Let us denote the spatial variable in the algebra $\widehat{\mathcal{A}}_w$ by $z$. Acting by the conservation law $\frac{g}{1+f} \in \mathrm{CL}(H - R_f\partial_y)$ on the operator $H - R_f\partial_y$, we obtain the evolutionary operator $H - R_f\partial_y - \left(R_g - \frac{gR_f}{1+f}\right)\partial_z$ on $\widehat{\mathcal{A}}_w$, and a direct computation shows that it is equal to the operator $H - (R_f + R_g)\partial_z$. So the result is the same as the result of the action on $H$ by $f + g \in \mathrm{CL}(H)$. Thus, the constructed action of the set $\mathrm{CL}(H)$ on the evolutionary operator $H$ is actually a group action with the respect to the standard additive group structure on $\mathrm{CL}(H)$.

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