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Baxter Operators in Ruijsenaars Hyperbolic System I: Commutativity of Q-Operators

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Abstract. We introduce Baxter Q-operators for the quantum Ruijsenaars hyperbolic system. We prove that they represent a commuting family of integral operators and also commute with Macdonald difference operators, which are gauge equivalent to the Ruijsenaars Hamiltonians of the quantum system. The proof of commutativity of the Baxter operators uses a hypergeometric identity on rational functions that generalize Ruijsenaars kernel identities.

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Birkhäuser

1. Introduction

1.1. Ruijsenaars System

In this paper, we develop the theory of Baxter operators for relativistic hyperbolic Ruijsenaars system [20]. This model is parametrized by three positive constants ω_1, ω_2 ("periods") and g (coupling constant), subject to the relation

$$0 < g < \omega_1 + \omega_2. \tag{1.1}$$

The dual coupling constant

$$g^* = \omega_1 + \omega_2 - g \tag{1.2}$$

is used as well everywhere.

The Ruijsenaars system is governed by commuting symmetric difference operators

$$H_r(\boldsymbol{x}_n, g|\boldsymbol{\omega}) = \sum_{\substack{I\subset[n] \\ |I|=r}} \prod_{\substack{i\in I \\ j\notin I}} \frac{\operatorname{sh}^{\frac{1}{2}} \frac{\pi}{\omega_2}(x_i - x_j - \imath g)}{\operatorname{sh}^{\frac{1}{2}} \frac{\pi}{\omega_2}(x_i - x_j)} \cdot T_{I,x}^{-\imath\omega_1} \cdot \prod_{\substack{i\in I \\ j\notin I}} \frac{\operatorname{sh}^{\frac{1}{2}} \frac{\pi}{\omega_2}(x_i - x_j + \imath g)}{\operatorname{sh}^{\frac{1}{2}} \frac{\pi}{\omega_2}(x_i - x_j)} \tag{1.3}$$

acting on meromorphic functions of n complex variables analytic in the strip

$$|\operatorname{Im} x_i| < \omega_1 + \varepsilon, \qquad \varepsilon > 0.$$

Here and in what follows we denote tuples of n variables as

$$\boldsymbol{x}_n = (x_1, \dots, x_n). \tag{1.4}$$

The sum in (1.3) is taken over all subsets

$$I \subset [n] = \{1, \dots, n\} \tag{1.5}$$

of cardinality r. By $T^a_{x_i}$ we denote shift operators

$$T^a_{x_i} := e^{a\partial_{x_i}}, \qquad (T^a_{x_i} f)(x_1, \dots, x_i, \dots, x_n) = f(x_1, \dots, x_i + a, \dots, x_n), \tag{1.6}$$

with $T^a_{I,x}$ being their product

$$T^a_{I,x} = \prod_{i\in I} e^{a\partial_{x_i}} \tag{1.7}$$

for any subset $I \subset [n]$.

The Ruijsenaars operators (1.3) are closely related to Macdonald operators

$$M_r(\boldsymbol{x}_n; g|\boldsymbol{\omega}) = \sum_{\substack{I\subset[n] \\ |I|=r}} \prod_{\substack{i\in I \\ j\notin I}} \frac{\operatorname{sh}\frac{\pi}{\omega_2}(x_i - x_j - \imath g)}{\operatorname{sh}\frac{\pi}{\omega_2}(x_i - x_j)} \cdot T_{I,x}^{-\imath\omega_1}. \tag{1.8}$$

Namely, denote by $\mu(z|\boldsymbol{\omega})$ the function

$$\mu(z|\boldsymbol{\omega}) = S_2(\imath z|\boldsymbol{\omega}) S_2(-\imath z + g^*|\boldsymbol{\omega}) \tag{1.9}$$

and by $\mu(\boldsymbol{x}_n|\boldsymbol{\omega})$ the product

$$\mu(\boldsymbol{x}_n|\boldsymbol{\omega}) = \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \mu(x_i - x_j). \tag{1.10}$$

Here $S_2(z|\boldsymbol{\omega})$ is the double sine function, its definition and key properties are given in Appendix A. Then,

$$\sqrt{\mu(\boldsymbol{x}_n|\boldsymbol{\omega})}\, M_r(\boldsymbol{x}_n; g|\boldsymbol{\omega})\, \frac{1}{\sqrt{\mu(\boldsymbol{x}_n|\boldsymbol{\omega})}} = H_r(\boldsymbol{x}_n, g|\boldsymbol{\omega}). \tag{1.11}$$

Note that the function $\mu(\boldsymbol{x}_n|\boldsymbol{\omega})$ is non-negative (assuming real constants g, ω_1, ω_2), since

$$\begin{aligned} \mu(x|\boldsymbol{\omega})\mu(-x|\boldsymbol{\omega}) &= S_2(\imath x|\boldsymbol{\omega}) S_2(-\imath x|\boldsymbol{\omega}) S_2(\imath x + g^*|\boldsymbol{\omega}) S_2(-\imath x + g^*|\boldsymbol{\omega}) \\ &= \left| S_2(\imath x|\boldsymbol{\omega}) S_2(\imath x + g^*|\boldsymbol{\omega})\right|^2. \end{aligned} \tag{1.12}$$

Ruijsenaars operators are symmetric with respect to the pairing

$$(\varphi, \psi) = \int_{\mathbb{R}^n} \varphi(\boldsymbol{x}_n) \bar{\psi}(\boldsymbol{x}_n) d\boldsymbol{x}_n,$$

while the Macdonald operators are symmetric with respect to the pairing

$$(\varphi, \psi) = \int_{\mathbb{R}^n} \varphi(\boldsymbol{x}_n) \bar{\psi}(\boldsymbol{x}_n) \mu(\boldsymbol{x}_n) d\boldsymbol{x}_n. \tag{1.13}$$

1.2. Kernel Function and Kernel Identities

Unlike the original Ruijsenaars' setting, we do not suppose that the periods ω_1, ω_2 and the coupling constant g are real positive. Instead, we assume everywhere that all of them are complex numbers with positive real parts

$$\operatorname{Re}\omega_1 > 0, \qquad \operatorname{Re}\omega_2 > 0, \qquad \operatorname{Re} g > 0. \tag{1.14}$$

We also require the condition

$$0 < \operatorname{Re} g < \operatorname{Re}\omega_1 + \operatorname{Re}\omega_2. \tag{1.15}$$

Further we usually fix the periods $\boldsymbol{\omega} = (\omega_1, \omega_2)$ and for brevity skip them in the notation, for example we use the symbol $S_2(z)$ for the double sine function

$$S_2(z) := S_2(z|\boldsymbol{\omega}).$$

Denote by $K(z)$ the following function of a complex variable

$$K(z) = S_2^{-1}\left(\imath z + \frac{g^*}{2}\right) S_2^{-1}\left(-\imath z + \frac{g^*}{2}\right). \tag{1.16}$$

In terms of the Ruijsenaars hyperbolic Gamma function

$$G(z) = S_2\left(\imath z + \frac{\omega_1 + \omega_2}{2}\right) \tag{1.17}$$

using reflection formula for the double sine function (A.6) it can be written as

$$K(z) = G\left(z - \frac{\imath g}{2}\right) G^{-1}\left(z + \frac{\imath g}{2}\right). \tag{1.18}$$

Also let

$$\boldsymbol{z}_n = (z_1, \dots, z_n), \qquad \boldsymbol{y}_n = (y_1, \dots, y_n), \qquad z_i, y_i \in \mathbb{C}$$

be two tuples of n complex variables. The Ruijsenaars **kernel function** $K(\boldsymbol{z}_n, \boldsymbol{y}_n)$ is defined as a product

$$K(\boldsymbol{z}_n, \boldsymbol{y}_n) = \prod_{i,j=1}^{n} K(z_i - y_j). \tag{1.19}$$

The kernel function $K(\boldsymbol{z}_n, \boldsymbol{y}_n)$ satisfies the relations

$$\left(M_r(\boldsymbol{z}_n; g) - M_r(-\boldsymbol{y}_n; g)\right) K(\boldsymbol{z}_n, \boldsymbol{y}_n) = 0, \qquad r = 1, \dots, n, \tag{1.20}$$

that is $K(\boldsymbol{z}_n, \boldsymbol{y})$ is a zero value eigenfunction for commuting difference operators

$$M_r(\boldsymbol{z}_n; g) - M_r(-\boldsymbol{y}_n; g),$$

see [22]. The relations (1.20) are the corollary of the trigonometric version of **kernel function identity** [22], valid for any tuples $\boldsymbol{z}_n$ and $\boldsymbol{y}_n$ of n complex variables and arbitrary parameter α:

$$\begin{aligned} &\sum_{\substack{I \subset [n] \\ |I| = r}} \prod_{i \in I} \left(\prod_{j \in [n] \setminus I} \frac{\sin(z_i - z_j - \alpha)}{\sin(z_i - z_j)} \prod_{a=1}^{n} \frac{\sin(z_i - y_a + \alpha)}{\sin(z_i - y_a)} \right) \\ &\quad = \sum_{\substack{A \subset [n] \\ |A| = r}} \prod_{a \in A} \left(\prod_{b \in [n] \setminus A} \frac{\sin(y_a - y_b + \alpha)}{\sin(y_a - y_b)} \prod_{i=1}^{n} \frac{\sin(z_i - y_a + \alpha)}{\sin(z_i - y_a)} \right). \end{aligned} \tag{1.21}$$

In the following we also use the kernel function with the second argument being a tuple of $n-1$ complex variables

$$K(\boldsymbol{z}_n, \boldsymbol{y}_{n-1}) = \prod_{i=1}^{n} \prod_{j=1}^{n-1} K(z_i - y_j). \tag{1.22}$$

1.3. Baxter Q-Operators

Let us introduce the family of Baxter Q-operators $Q_n(\lambda)$ parameterized by $\lambda \in \mathbb{C}$ as the integral operators

$$\left(Q_n(\lambda) f\right)(\boldsymbol{z}_n) = \int_{\mathbb{R}^n} Q(\boldsymbol{z}_n, \boldsymbol{y}_n; \lambda) f(\boldsymbol{y}_n) d\boldsymbol{y}_n \tag{1.23}$$

with the kernel

$$Q(\boldsymbol{z}_n, \boldsymbol{y}_n; \lambda) = e^{2\pi\imath\lambda(\underline{\boldsymbol{z}}_n - \underline{\boldsymbol{y}}_n)} K(\boldsymbol{z}_n, \boldsymbol{y}_n) \mu(\boldsymbol{y}_n), \qquad z_j, y_j \in \mathbb{R}. \tag{1.24}$$

Here and in what follows we denote the sum of tuple components as

$$\underline{\boldsymbol{z}}_n = z_1 + \ldots + z_n.$$

The Q-operator maps functions of n real variables to functions of n real variables.

The following theorem is a simple consequence of kernel function identities (1.21); its proof is given in Sect. 2.

Theorem 1. *Under the condition*

$$0 < \operatorname{Re} g < \operatorname{Re} \omega_2 \tag{1.25}$$

the operators $Q_n(\lambda)$ *commute with Macdonald operators* $M_r(\boldsymbol{z}_n; g|\boldsymbol{\omega})$

$$M_r(\boldsymbol{z}_n; g)\, Q_n(\lambda) = Q_n(\lambda)\, M_r(\boldsymbol{z}_n; g), \qquad r = 1, \dots, n. \tag{1.26}$$

Assume in addition to (1.14), (1.15) that

$$\nu_g = \operatorname{Re} \frac{g}{\omega_1 \omega_2} > 0. \tag{1.27}$$

Due to the bounds for the functions $K(y)$ and $\mu(y)$ (B.3) proven in Appendix B, the product of two Q-operators

$$Q_n(\lambda)\, Q_n(\rho)$$

is a well-defined integral operator with the kernel $Q_n(\boldsymbol{z}_n, \boldsymbol{w}_n; \lambda, \rho)$ given by absolutely convergent integral

$$Q_n(\boldsymbol{z}_n, \boldsymbol{w}_n; \lambda, \rho) = \int_{\mathbb{R}^n} Q(\boldsymbol{z}_n, \boldsymbol{y}_n; \lambda) Q(\boldsymbol{y}_n, \boldsymbol{w}_n; \rho) d\boldsymbol{y}_n \tag{1.28}$$

and the domain that consists of fast decreasing functions $f(\boldsymbol{w}_n)$, see Proposition 5 and remark after it in Appendix B. The main result of this paper is the commutativity of Baxter Q-operators, its proof is given in Sects. 3 and 4.

Theorem 2. *Under the conditions (1.14), (1.15), (1.27) Baxter operators commute*

$$Q_n(\lambda)\, Q_n(\rho) = Q_n(\rho)\, Q_n(\lambda). \tag{1.29}$$

The kernels of the operators in both sides of (1.29) are analytic functions of λ, ρ *in the strip*

$$|\operatorname{Im}(\lambda - \rho)| < \operatorname{Re} \frac{g}{\omega_1 \omega_2}. \tag{1.30}$$

Remark. Both sides of the relation (1.29) depend analytically on all the parameters $\lambda, \rho, g, \boldsymbol{\omega}$ in the region of absolute convergence of the integrals.

As it was observed for other integrable systems (see lectures [23] for review), the classical analog of the Baxter Q-operator is a special canonical transformation called Backlund transformation. In the paper [16] V. Kuznetsov and E. Sklyanin proposed a general scheme that relates the kernel of a Q-operator and generating function of the corresponding Backlund transformation. In the work [7] M. Hallnäs and S. Ruijsenaars showed that in the certain classical limit the kernel and measure functions $K(\boldsymbol{z}_n, \boldsymbol{y}_n), \mu(\boldsymbol{y}_n)$ contained in the Q-operator kernel (1.24) give rise to the Backlund transformation for the classical Ruijsenaars system. In this context, the Q-operator we defined is a quantum counterpart of this transformation.

We also note that for the case of two particles $n = 2$ (and real constants ω_1, ω_2, g) such an operator first implicitly appeared in the work [11]. Moreover, the commutativity of Q-operators in this particular case also follows from the results of [11].

1.4. Hypergeometric Identities

The proof of Theorem 2 consists of residue calculation of the integrals (1.28). This includes the proof of cancellation of higher-order poles and the equality of sums of ordinary poles. The latter is equivalent to certain identity for basic hypergeometric series, which resembles duality transformation theorem for multiple hypergeometric series by Y. Kajihara and M. Noumi, see [14].

Let q and t be formal variables. Denote by $(z;q)_k$ the q-analog of the Pochhammer symbol,

$$(z;q)_k = (1-z)(1-qz)\cdots(1-q^{k-1}z). \tag{1.31}$$

Let $\boldsymbol{u} = (u_1,\ldots,u_n)$ and $\boldsymbol{v} = (v_1,\ldots,v_n)$ be two tuples of n variables.

Theorem 3. *For any integer K, we have the following equality of rational functions*

$$\sum_{|\boldsymbol{k}|=K} \prod_{i=1}^{n} \frac{(qt;q)_{k_i}}{(q;q)_{k_i}} \times \prod_{\substack{i,j=1\\ i\neq j}}^{n} \frac{(t^{-1}q^{-k_j}u_i/u_j;q)_{k_i}}{(q^{-k_j}u_i/u_j;q)_{k_i}} \times \prod_{a,j=1}^{n} \frac{(tu_j/v_a;q)_{k_j}}{(u_j/v_a;q)_{k_j}}$$
$$= \sum_{|\boldsymbol{k}|=K} \prod_{a=1}^{n} \frac{(qt;q)_{k_a}}{(q;q)_{k_a}} \times \prod_{\substack{a,b=1\\ a\neq b}}^{n} \frac{(t^{-1}q^{-k_a}v_a/v_b;q)_{k_b}}{(q^{-k_a}v_a/v_b;q)_{k_b}} \times \prod_{a,j=1}^{n} \frac{(tu_j/v_a;q)_{k_a}}{(u_j/v_a;q)_{k_a}}. \tag{1.32}$$

Here the sum on both sides of the equality is taken over n tuples of non-negative integers with total sum equal to K

$$\boldsymbol{k} = (k_1,\ldots,k_n), \qquad k_i \geq 0, \qquad k_1+\ldots+k_n = K. \tag{1.33}$$

Note that the kernel function identity (1.21) is a particular limit of the hypergeometric identity (1.32), see [4] for details.

After our work was completed, O. Warnaar and H. Rosengren communicated to us that an elliptic analog of this identity was proven by different methods in the papers [17, Corollary 4.3], [6, eq. (6.7)].

1.5. Further Results

Denote by $\Lambda_n(\lambda)$ the integral operator

$$\left(\Lambda_n(\lambda)f\right)(\boldsymbol{x}_n) = d_{n-1}(g)\int_{\mathbb{R}^{n-1}} \Lambda(\boldsymbol{x}_n,\boldsymbol{y}_{n-1};\lambda) f(\boldsymbol{y}_{n-1}) d\boldsymbol{y}_{n-1} \tag{1.34}$$

with the kernel

$$\Lambda(\boldsymbol{x}_n,\boldsymbol{y}_{n-1};\lambda) = e^{2\pi\imath\lambda(\underline{\boldsymbol{x}}_n-\underline{\boldsymbol{y}}_{n-1})} K(\boldsymbol{x}_n,\boldsymbol{y}_{n-1})\mu(\boldsymbol{y}_{n-1}) \tag{1.35}$$

and the constant

$$d_{n-1}(g) = \frac{1}{(n-1)!}\left[\sqrt{\omega_1\omega_2}\, S_2(g)\right]^{-n+1}. \tag{1.36}$$

The operator $\Lambda_n(\lambda)$ maps functions of $n-1$ real variables to functions of n real variables. M. Hallnäs and S. Ruijsenaars [8] proved that for real periods $\boldsymbol{\omega}$ under the condition

$$0 < \operatorname{Re} g < \omega_2 \tag{1.37}$$

the function

$$\Psi_{\boldsymbol{\lambda}_n}(\boldsymbol{x}_n; g|\boldsymbol{\omega}) = \Lambda_n(\lambda_n)\,\Lambda_{n-1}(\lambda_{n-1})\,\cdots\,\Lambda_2(\lambda_2)\,e^{2\pi\imath\lambda_1 x_1} \tag{1.38}$$

is given by absolutely convergent integral and represents the joint eigenfunction of Macdonald operators

$$M_r(\boldsymbol{x}_n; g)\Psi_{\boldsymbol{\lambda}_n}(\boldsymbol{x}_n; g) = e_r\big(e^{2\pi\lambda_1\omega_1}, \dots, e^{2\pi\lambda_n\omega_1}\big)\,\Psi_{\boldsymbol{\lambda}_n}(\boldsymbol{x}_n; g), \qquad r = 1, \dots, n. \tag{1.39}$$

Here $e_r(z_1, \dots, z_n)$ is r-th elementary symmetric function,

$$e_r(z_1, \dots, z_n) = \sum_{1\le i_1<i_2<\ldots<i_r\le n} z_{i_1}\cdots z_{i_r}.$$

In the next paper [3], we show that the operators (1.35) can be obtained in the certain limit from Baxter Q-operators, so that the commutativity of Q-operators imply commutation relations between Λ-operators and Q-operators, and between Λ-operators themselves. These relations allow to derive important properties of the eigenfunction. In particular, we show that the eigenfunction (1.38)

1. enjoys duality property

$$\Psi_{\boldsymbol{\lambda}_n}(\boldsymbol{x}_n; g|\boldsymbol{\omega}) = \Psi_{\boldsymbol{x}_n}(\boldsymbol{\lambda}_n, \hat{g}^*|\hat{\boldsymbol{\omega}}), \tag{1.40}$$

and consequently admits another iterative integral representation given by Mellin–Barnes type of integrals over spectral parameters λ_j. Here we denoted

$$\hat{a} = \frac{a}{\omega_1\omega_2} \tag{1.41}$$

for any $a \in \mathbb{C}$, so that

$$\hat{\boldsymbol{\omega}} = \left(\frac{1}{\omega_2}, \frac{1}{\omega_1}\right), \qquad \hat{g} = \frac{g}{\omega_1\omega_2}, \qquad \hat{g}^* = \hat{\omega}_1 + \hat{\omega}_2 - \hat{g} = \frac{g^*}{\omega_1\omega_2}; \tag{1.42}$$

2. is symmetric function of the coordinates x_j, as well as of the spectral variables λ_j;
3. is an eigenfunction of the Baxter Q-operator with the eigenvalue

$$\prod_{j=1}^{n}\hat{K}(\lambda-\lambda_j) = \prod_{j=1}^{n} S_2^{-1}\Big(\imath(\lambda-\lambda_j) + \frac{\hat{g}}{2}\Big|\hat{\boldsymbol{\omega}}\Big)\, S_2^{-1}\Big(-\imath(\lambda-\lambda_j) + \frac{\hat{g}}{2}\Big|\hat{\boldsymbol{\omega}}\Big); \tag{1.43}$$

4. is a solution of bispectral problem for Macdonald operators $M_r(\boldsymbol{x}_n; g|\boldsymbol{\omega})$ and $M_s(\boldsymbol{\lambda}_n; \hat{g}^*|\hat{\boldsymbol{\omega}})$:

$$\begin{aligned} M_r(\boldsymbol{x}_n; g|\boldsymbol{\omega})\Psi_{\boldsymbol{\lambda}_n}(\boldsymbol{x}_n; g) &= e_r\big(e^{2\pi\lambda_1\omega_1}, \dots, e^{2\pi\lambda_n\omega_1}\big)\Psi_{\boldsymbol{\lambda}_n}(\boldsymbol{x}_n; g), \\ M_s(\boldsymbol{\lambda}_n; \hat{g}^*|\hat{\boldsymbol{\omega}})\Psi_{\boldsymbol{\lambda}_n}(\boldsymbol{x}_n; g) &= e_s\big(e^{\frac{2\pi x_1}{\omega_2}}, \dots, e^{\frac{2\pi x_n}{\omega_2}}\big)\Psi_{\boldsymbol{\lambda}_n}(\boldsymbol{x}_n; g), \end{aligned} \tag{1.44}$$

if $\operatorname{Re} g < \operatorname{Re}\omega_2$ and $\operatorname{Re}\hat{g}^* < \operatorname{Re}\hat{\omega}_2$.

For the hyperbolic Calogero–Sutherland model, which represents a non-relativistic limit of the Ruijsenaars system, the latter result was established in [12] and [13].

2. Baxter and Macdonald Operators Commute

Theorem 1 follows from the kernel function identity and from the invariance of the measure $\mu(\boldsymbol{x}_n)d\boldsymbol{x}_n$ with respect to the Macdonald operators.

We present here two proofs of this theorem. Both work for complex periods ω_1, ω_2 and coupling constant g. The first prove is direct analytical. The second one is its short algebraic reformulation. It exploits symmetry properties of the Macdonald operators with respect to symmetric bilinear pairing (2.22) in a way analogous to [18, Chapter VI, §9]. We describe here both proofs since the first one allows to visualize the appearing restriction on the coupling constant g, while the second outlines the responsible algebraic properties.

I. Direct proof. Let $\Phi(\boldsymbol{z}_n)$ be a function of n complex variables $\boldsymbol{z}_n = (z_1, \ldots, z_n)$ analytic in a strip

$$\Pi_\varepsilon : \qquad -\varepsilon - \operatorname{Re}\omega_1 < \operatorname{Im} z_i < \operatorname{Re}\omega_1 + \varepsilon. \tag{2.1}$$

We are going to prove the equality

$$M_r(\boldsymbol{z}_n; g)\, Q_n(\lambda)\, \Phi(\boldsymbol{z}_n) = Q_n(\lambda)\, M_r(\boldsymbol{z}_n; g)\, \Phi(\boldsymbol{z}_n), \qquad r = 1, \ldots, n. \tag{2.2}$$

Explicitly it looks as

$$\begin{aligned} &M_r(\boldsymbol{z}_n; g) \int_{\mathbb{R}^n} \mu(\boldsymbol{y}_n)\, K(\boldsymbol{z}_n, \boldsymbol{y}_n)\, e^{2\pi\imath\lambda(\underline{\boldsymbol{z}}_n - \underline{\boldsymbol{y}}_n)}\, \Phi(\boldsymbol{y}_n)\, d\boldsymbol{y}_n \\ &\quad = \int_{\mathbb{R}^n} \mu(\boldsymbol{y}_n)\, K(\boldsymbol{z}_n, \boldsymbol{y}_n)\, e^{2\pi\imath\lambda(\underline{\boldsymbol{z}}_n - \underline{\boldsymbol{y}}_n)}\, M_r(\boldsymbol{y}_n; g)\, \Phi(\boldsymbol{y}_n)\, d\boldsymbol{y}_n. \end{aligned} \tag{2.3}$$

Here we assume convergence of the corresponding integrals and $\boldsymbol{z}_n \in \mathbb{R}^n$. Note the important property of the integration contour $\mathbb{R}^n$ in the integral (2.3): it separates two series of poles of the kernel function

$$\imath y_i = \imath z_j + \frac{g^*}{2} + m\omega_1 + k\omega_2 \qquad \text{and} \qquad \imath y_i = \imath z_k - \frac{g^*}{2} - m\omega_1 - k\omega_2, \qquad m, k \geq 0, \tag{2.4}$$

and two series of poles of the measure function

$$\imath y_i = \imath y_j + g + m\omega_1 + k\omega_2 \qquad \text{and} \qquad \imath y_i = \imath y_k - g - m\omega_1 - k\omega_2, \qquad m, k \geq 0, \tag{2.5}$$

see (A.11), (A.12) for the poles and zeros of the double sine function.

The left hand side of (2.2) looks as

$$M_r(\boldsymbol{z}_n; g)\, Q_n(\lambda)\, \Phi(\boldsymbol{z}_n) = \sum_{\substack{I \subset [n] \\ |I| = r}} \prod_{\substack{i \in I \\ j \in [n] \setminus I}} \frac{\operatorname{sh} \frac{\pi}{\omega_2}(z_i - z_j - \imath g)}{\operatorname{sh} \frac{\pi}{\omega_2}(z_i - z_j)} \cdot T_{I,z}^{-\imath\omega_1} Q_n(\lambda) \Phi(\boldsymbol{z}_n), \tag{2.6}$$

where the shift operator $T_{I,z}^{-\imath\omega_1}$ is defined in (1.7). Consider the summand corresponding to subset $I = \{i_1, i_2, \ldots, i_r\}$. Denote this summand by J_I:

$$J_I = \frac{\operatorname{sh}\frac{\pi}{\omega_2}\,(z_i - z_j - \imath g)}{\operatorname{sh}\frac{\pi}{\omega_2}\,(z_i - z_j)} \cdot T_{I,z}^{-\imath\omega_1} \int\limits_{\mathbb{R}^n} \mu(\boldsymbol{y}_n) K(\boldsymbol{z}_n, \boldsymbol{y}_n) e^{2\pi\imath\lambda(\underline{\boldsymbol{z}}_n - \underline{\boldsymbol{y}}_n)}\, \Phi(\boldsymbol{y}_n)\, d\boldsymbol{y}_n. \tag{2.7}$$

Shifts act non-trivially on the kernel $K(\boldsymbol{z}_n, \boldsymbol{y}_n)$ and exponent $e^{2\pi\imath\lambda(\underline{\boldsymbol{z}}_n - \underline{\boldsymbol{y}}_n)}$. By (1.16), (1.19) we have

$$\begin{aligned} \prod_{i\in I} T_{z_i}^{-\imath\omega_1} K(\boldsymbol{z}_n, \boldsymbol{y}_n) &= \prod_{i\in I} \prod_{a=1}^{n} \frac{\operatorname{sh}\frac{\pi}{\omega_2}(z_i - y_a - \imath\frac{g^*}{2})}{\operatorname{sh}\frac{\pi}{\omega_2}(z_i - y_a - \imath\frac{g^*}{2} - \imath g)} K(\boldsymbol{z}_n, \boldsymbol{y}_n), \\ \prod_{i\in I} T_{z_i}^{-\imath\omega_1} e^{2\pi\imath\lambda(\underline{\boldsymbol{z}}_n - \underline{\boldsymbol{y}}_n)} &= e^{2\pi r\lambda\omega_1} \cdot e^{2\pi\imath\lambda(\underline{\boldsymbol{z}}_n - \underline{\boldsymbol{y}}_n)}, \end{aligned} \tag{2.8}$$

where in the first formula we have transformed the right hand side to the form similar to (1.21).

The operator $T_{z_i}^{-\imath\omega_1}$ shifts z_i and we have to shift the integration contour in (2.7), so that the conditions (2.4) are satisfied with the replacement of $\imath z_i$ by $\imath z_i + \omega_1$. That is the shifted contour should separate set of poles

$$\begin{aligned} \imath y_a &= \imath z_j + \frac{g^*}{2} + m\omega_1 + k\omega_2, && m, k \geq 0, \quad j \notin I, \\ \imath y_a &= \imath z_j + \frac{g^*}{2} + (m+1)\omega_1 + k\omega_2, && m, k \geq 0, \quad j \in I, \end{aligned} \tag{2.9}$$

from

$$\begin{aligned} \imath y_a &= \imath z_j - \frac{g^*}{2} - m\omega_1 - k\omega_2, && m, k \geq 0, \quad j \notin I, \\ \imath y_a &= \imath z_j - \frac{g^*}{2} - (m-1)\omega_1 - k\omega_2, && m, k \geq 0, \quad j \in I, \end{aligned} \tag{2.10}$$

and also separate two series of poles (2.5) of the measure functions. For this, we can use the contour

$$C\colon \ \operatorname{Im} y_a = -c, \qquad -\operatorname{Re}\frac{g^*}{2} + \operatorname{Re}\omega_1 < c < \operatorname{Re}\frac{g^*}{2}, \qquad a = 1, \ldots, n \tag{2.11}$$

which exists provided

$$\operatorname{Re} g^* > \operatorname{Re}\omega_1, \qquad \text{or equivalently} \qquad \operatorname{Re} g < \operatorname{Re}\omega_2. \tag{2.12}$$

Since the contour C does not depend on a set I, we can permute integration and summation procedures, so that

$$\begin{aligned} M_r(\boldsymbol{z}_n; g)\, Q_n(\lambda)\, \Phi(\boldsymbol{z}_n) = \sum_{\substack{I\subset[n] \\ |I|=r}} J_I = e^{2\pi r\lambda\omega_1} \\ \times \int\limits_{C} S_r(\boldsymbol{z}_n, \boldsymbol{y}_n) \mu(\boldsymbol{y}_n) K(\boldsymbol{z}_n, \boldsymbol{y}_n) e^{2\pi\imath\lambda(\underline{\boldsymbol{z}}_n - \underline{\boldsymbol{y}}_n)}\, \Phi(\boldsymbol{y}_n)\, d\boldsymbol{y}_n, \end{aligned} \tag{2.13}$$

where

$$S_r(\boldsymbol{z}_n,\boldsymbol{y}_n) = \sum_{\substack{I\subset[n]\\|I|=r}}\prod_{i\in I}\left(\prod_{j\in[n]\setminus I}\frac{\operatorname{sh}\frac{\pi}{\omega_2}\,(z_i-z_j-\imath g)}{\operatorname{sh}\frac{\pi}{\omega_2}\,(z_i-z_j)}\prod_{a=1}^{n}\frac{\operatorname{sh}\frac{\pi}{\omega_2}(z_i-y_a-\imath\frac{g^*}{2})}{\operatorname{sh}\frac{\pi}{\omega_2}(z_i-y_a-\imath\frac{g^*}{2}-\imath g)}\right)$$

Define similar sum

$$\tilde{S}_r(\boldsymbol{y}_n,\boldsymbol{z}_n) = \sum_{\substack{A\subset[n]\\|A|=r}}\prod_{a\in A}\left(\prod_{b\in[n]\setminus A}\frac{\operatorname{sh}\frac{\pi}{\omega_2}\,(y_a-y_b+\imath g)}{\operatorname{sh}\frac{\pi}{\omega_2}\,(y_a-y_b)}\prod_{i=1}^{n}\frac{\operatorname{sh}\frac{\pi}{\omega_2}(z_i-y_a-\imath\frac{g^*}{2})}{\operatorname{sh}\frac{\pi}{\omega_2}(z_i-y_a-\imath\frac{g^*}{2}-\imath g)}\right)$$

One can see that the sum $S_r(\boldsymbol{z}_n,\boldsymbol{y}_n)$ is obtained from the left hand side of the kernel function identity (1.21) by the change of variables

$$z_k\to\frac{\imath\pi}{\omega_2}z_k,\qquad y_a\to\frac{\imath\pi}{\omega_2}\Big(y_a+\imath\frac{g^*}{2}+\imath g\Big),\qquad \alpha\to\frac{\imath\pi}{\omega_2}\imath g \tag{2.14}$$

and there is the same correspondence between $\tilde{S}_r(\boldsymbol{y}_n,\boldsymbol{z}_n)$ and the right hand side of (1.21). It implies the equality

$$S_r(\boldsymbol{z}_n,\boldsymbol{y}_n)=\tilde{S}_r(\boldsymbol{y}_n,\boldsymbol{z}_n). \tag{2.15}$$

Thus, we rewrite (2.13) as

$$M_r(\boldsymbol{z}_n;g)\,Q_n(\lambda)\,\Phi(\boldsymbol{z}_n) = e^{2\pi r\lambda\omega_1}\int_C \tilde{S}_r(\boldsymbol{y}_n,\boldsymbol{z}_n,)\mu(\boldsymbol{y}_n)K(\boldsymbol{z}_n,\boldsymbol{y}_n)e^{2\pi\imath\lambda(\underline{\boldsymbol{z}}_n-\underline{\boldsymbol{y}}_n)}\,\Phi(\boldsymbol{y}_n)\,d\boldsymbol{y}_n \tag{2.16}$$

and apply to each occurring summand the same procedure in opposite direction. Namely, for any subset $A\subset[n]$ of cardinality r in the integral

$$J'_A=\int_C\prod_{a\in A}\left(\prod_{b\in[n]\setminus A}\frac{\operatorname{sh}\frac{\pi}{\omega_2}\,(y_a-y_b+\imath g)}{\operatorname{sh}\frac{\pi}{\omega_2}\,(y_a-y_b)}\prod_{i=1}^{n}\frac{\operatorname{sh}\frac{\pi}{\omega_2}(z_i-y_a-\imath\frac{g^*}{2})}{\operatorname{sh}\frac{\pi}{\omega_2}(z_i-y_a-\imath\frac{g^*}{2}-\imath g)}\right)\times e^{2\pi\imath\lambda(\underline{\boldsymbol{z}}_n-\underline{\boldsymbol{y}}_n)}\mu(\boldsymbol{y}_n)K(\boldsymbol{z}_n,\boldsymbol{y}_n)\,\Phi(\boldsymbol{y}_n)\,d\boldsymbol{y}_n \tag{2.17}$$

we perform the change of integration variables

$$y_a\to y_a-\imath\omega_1,\qquad a\in A. \tag{2.18}$$

We have

$$\prod_{a\in A}T^{-\imath\omega_1}_{y_a}K(\boldsymbol{z}_n,\boldsymbol{y}_n)=\prod_{a\in A}\prod_{i=1}^{n}\frac{\operatorname{sh}\frac{\pi}{\omega_2}(y_a-z_i-\imath\frac{g^*}{2})}{\operatorname{sh}\frac{\pi}{\omega_2}(y_a-z_i-\imath\frac{g^*}{2}-\imath g)}K(\boldsymbol{z}_n,\boldsymbol{y}_n),$$

$$\prod_{a\in A}T^{-\imath\omega_1}_{y_a}\mu(\boldsymbol{y}_n)=\prod_{\substack{a\in A\\ b\in[n]\setminus A}}\frac{\operatorname{sh}\frac{\pi}{\omega_2}(y_a-y_b-\imath\omega_1)}{\operatorname{sh}\frac{\pi}{\omega_2}(y_a-y_b)}\cdot\frac{(-1)\operatorname{sh}\frac{\pi}{\omega_2}(y_a-y_b-\imath g)}{\operatorname{sh}\frac{\pi}{\omega_2}(y_a-y_b-\imath g^*)}\,\mu(\boldsymbol{y}_n),$$

$$\prod_{a\in A} T^{-\imath\omega_1}_{y_a} e^{2\pi\imath\lambda(\underline{z}_n-\underline{y}_n)} = e^{-2\pi r\lambda\omega_1}\cdot e^{2\pi\imath\lambda(\underline{z}_n-\underline{y}_n)}. \tag{2.19}$$

Using (2.19) and the relations

$$\begin{aligned}&\prod_{a\in A} T^{-\imath\omega_1}_{y_a}\prod_{a\in A}\left(\prod_{b\in[n]\setminus A}\frac{\mathrm{sh}\frac{\pi}{\omega_2}\left(y_a-y_b+\imath g\right)}{\mathrm{sh}\frac{\pi}{\omega_2}\left(y_a-y_b\right)}\prod_{i=1}^{n}\frac{\mathrm{sh}\frac{\pi}{\omega_2}(z_i-y_a-\imath\frac{g^*}{2})}{\mathrm{sh}\frac{\pi}{\omega_2}(z_i-y_a-\imath\frac{g^*}{2}-\imath g)}\right)\\&\quad=\prod_{a\in A}\left(\prod_{b\in[n]\setminus A}\frac{(-1)\mathrm{sh}\frac{\pi}{\omega_2}\left(y_a-y_b-\imath g^*\right)}{\mathrm{sh}\frac{\pi}{\omega_2}\left(y_a-y_b-\imath\omega_1\right)}\prod_{i=1}^{n}\frac{\mathrm{sh}\frac{\pi}{\omega_2}(z_i-y_a+\imath\frac{g^*}{2}+\imath g)}{\mathrm{sh}\frac{\pi}{\omega_2}(z_i-y_a+\imath\frac{g^*}{2})}\right)\end{aligned} \tag{2.20}$$

we see that

$$\begin{aligned}J'_A=&\int_{\tilde C} d\boldsymbol{y}_n\,\mu(\boldsymbol{y}_n)K(\boldsymbol{z}_n,\boldsymbol{y}_n)\,e^{2\pi\imath\lambda(\underline{z}_n-\underline{y}_n)}\\&\times\prod_{\substack{a\in A\\ b\in[n]\setminus A}}\frac{\mathrm{sh}\frac{\pi}{\omega_2}\left(y_a-y_b-\imath g\right)}{\mathrm{sh}\frac{\pi}{\omega_2}\left(y_a-y_b\right)}\prod_{a\in A}T^{-\imath\omega_1}_{y_a}\Phi(\boldsymbol{y}_n),\end{aligned} \tag{2.21}$$

where the contour $\tilde C$ is the deformation of the contour C according to the change of variables (2.18). In the assumption $\mathrm{Re}\,g^* > \mathrm{Re}\,\omega_1$ we may choose again $\tilde C = \mathbb{R}^n$ provided the conditions (2.5) on separation of the poles of the measure are not spoiled during the move of the contour. Note that zeros of the measure $\mu(\boldsymbol{y}_n)$ cancel the poles of the hyperbolic sine functions in the last line of (2.21).

On the other hand, zeros of the sine functions

$$\mathrm{sh}\frac{\pi}{\omega_2}\left(y_a-y_b-\imath g\right)$$

cancel poles

$$y_b = y_a - \imath g - \imath p\omega_2, \qquad p\geq 0$$

of the measure function, so that the first pole which we can meet during the move of the contour is

$$y_b = y_a - \imath g - \imath\omega_1$$

and its shift does not touch the real plane. Then, we can deform the contour $\tilde C$ to its original position $\mathbb{R}^n$. Summing up (2.21) with the integration contour replaced by $\mathbb{R}^n$ we arrive at the statement of Theorem 1. □

II. Algebraic version. In the space of functions $\varphi(\boldsymbol{z}_n)$ analytical in the strip Π_ε (2.1) and satisfying the bound

$$\varphi(\boldsymbol{z}_n) = O(|\boldsymbol{z}_n|^{-1}), \qquad \mathrm{Re}\,z\to\infty, \qquad z\in\Pi_\varepsilon$$

introduce the symmetric bilinear pairing

$$(\varphi,\psi) = \int_{\mathbb{R}^n}\varphi(\boldsymbol{y}_n)\psi(-\boldsymbol{y}_n)\mu(\boldsymbol{y}_n)d\boldsymbol{y}_n. \tag{2.22}$$

Denote by τ_y the operator that changes the sign of argument in a function

$$\tau_y \varphi(\boldsymbol{y}_n) = \varphi(-\boldsymbol{y}_n).$$

Then, we can rewrite this pairing as

$$(\varphi, \psi) = \int_{\mathbb{R}^n} \varphi(\boldsymbol{y}_n) \tau_y[\psi(\boldsymbol{y}_n)] \mu(\boldsymbol{y}_n) d\boldsymbol{y}_n. \tag{2.23}$$

The eigenvalue property (1.20) of the Ruijsenaars kernel function $K(\boldsymbol{z}_n, \boldsymbol{y}_n)$ can be written as

$$M_r(\boldsymbol{z}_n; g) K(\boldsymbol{z}_n, \boldsymbol{y}_n) = \tau_y M_r(\boldsymbol{y}_n; g) \tau_y K(\boldsymbol{z}_n, \boldsymbol{y}_n). \tag{2.24}$$

However, if we want to use the relation (2.24) for the operator with a kernel containing $K(\boldsymbol{z}_n, \boldsymbol{y}_n)$, we should impose the condition (2.12) in order to have correctly defined shift operators. Macdonald operators are symmetric with respect to the pairing (2.6) (compare with [18, Chapter VI, §9, eq.(9.4)])

$$\left(M_r(\boldsymbol{y}_n; g)\, \varphi(\boldsymbol{y}_n), \psi(\boldsymbol{y}_n)\right) = \left(M_r(\boldsymbol{y}_n; g)\, \psi(\boldsymbol{y}_n), \varphi(\boldsymbol{y}_n)\right). \tag{2.25}$$

Then, the left hand side of the relation (2.2) can be written as

$$\begin{aligned} &M_r(\boldsymbol{z}_n; g)\big(K(\boldsymbol{z}_n, \boldsymbol{y}_n)\, e^{2\pi\imath\lambda \underline{\boldsymbol{z}}_n},\ e^{2\pi\imath\lambda\underline{\boldsymbol{y}}_n}\, \tau_y\, \Phi(\boldsymbol{y}_n)\big) \\ &\quad = e^{2\pi r\lambda\omega_1}\, e^{2\pi\imath\lambda\underline{\boldsymbol{z}}_n} \big(M_r(\boldsymbol{z}_n; g) K(\boldsymbol{z}_n, \boldsymbol{y}_n),\ e^{2\pi\imath\lambda\underline{\boldsymbol{y}}_n}\, \tau_y\, \Phi(\boldsymbol{y}_n)\big). \end{aligned} \tag{2.26}$$

Using (2.24) we rewrite (2.26) as

$$\begin{aligned} &e^{2\pi r\lambda\omega_1} e^{2\pi\imath\lambda\underline{\boldsymbol{z}}_n} \big(\tau_y M_r(\boldsymbol{y}_n; g) \tau_y K(\boldsymbol{z}_n, \boldsymbol{y}_n),\ e^{2\pi\imath\lambda\underline{\boldsymbol{y}}_n}\, \tau_y\, \Phi(\boldsymbol{y}_n)\big) \\ &\quad = e^{2\pi r\lambda\omega_1} e^{2\pi\imath\lambda\underline{\boldsymbol{z}}_n} \big(M_r(\boldsymbol{y}_n; g) \tau_y K(\boldsymbol{z}_n, \boldsymbol{y}_n),\ e^{-2\pi\imath\lambda\underline{\boldsymbol{y}}_n}\, \Phi(\boldsymbol{y}_n)\big). \end{aligned} \tag{2.27}$$

Next applying (2.25) we have

$$\begin{aligned} &e^{2\pi r\lambda\omega_1} e^{2\pi\imath\lambda\underline{\boldsymbol{z}}_n} \big(\tau_y K(\boldsymbol{z}_n, \boldsymbol{y}_n),\ M_r(\boldsymbol{y}_n; g) e^{-2\pi\imath\lambda\underline{\boldsymbol{y}}_n}\, \Phi(\boldsymbol{y}_n)\big) \\ &\quad = e^{2\pi\imath\lambda\underline{\boldsymbol{z}}_n} \big(\tau_y K(\boldsymbol{z}_n, \boldsymbol{y}_n),\ e^{-2\pi\imath\lambda\underline{\boldsymbol{y}}_n} M_r(\boldsymbol{y}_n; g) \Phi(\boldsymbol{y}_n)\big) \\ &\quad = e^{2\pi\imath\lambda\underline{\boldsymbol{z}}_n} \big(K(\boldsymbol{z}_n, \boldsymbol{y}_n),\ e^{2\pi\imath\lambda\underline{\boldsymbol{y}}_n} \tau_y\, M_r(\boldsymbol{y}_n; g) \Phi(\boldsymbol{y}_n)\big). \end{aligned} \tag{2.28}$$

The last line of (2.28) coincides with the right hand side of (2.2). □

3. Commutativity of Baxter Operators

The commutativity of Q-operators

$$Q_n(\rho)\, Q_n(\rho') = Q_n(\rho')\, Q_n(\rho) \tag{3.1}$$

follows from the equality of the kernels

$$Q_n(\boldsymbol{z}_n, \boldsymbol{x}_n; \rho, \rho') = Q_n(\boldsymbol{z}_n, \boldsymbol{x}_n; \rho', \rho) \tag{3.2}$$

of their products

$$Q_n(\boldsymbol{z}_n, \boldsymbol{x}_n; \rho, \rho') = \int_{\mathbb{R}^n} d\boldsymbol{y}_n\, Q(\boldsymbol{z}_n, \boldsymbol{y}_n; \rho) Q(\boldsymbol{y}_n, \boldsymbol{x}_n; \rho'), \qquad z_j, x_j \in \mathbb{R}. \tag{3.3}$$

One can further note that the variables $\boldsymbol{z}_n$ and $\boldsymbol{x}_n$ enter the equality (3.2) in a similar way. Thus, we combine them into a common $2n$ array

$$\boldsymbol{z}_{2n} = \{z_1, \ldots, z_n, x_1, \ldots, x_n\}$$

and set

$$Q_n(\boldsymbol{z}_{2n}; \lambda) = \int_{\mathbb{R}^n} e^{2\pi\imath\lambda \underline{\boldsymbol{y}}_n} \prod_{a=1}^{2n} \prod_{i=1}^{n} K(z_a - y_i)\mu(\boldsymbol{y}_n) d\boldsymbol{y}_n. \tag{3.4}$$

Then, the equality (3.2) takes the form of the following integral identity

$$Q_n(\boldsymbol{z}_{2n}; \lambda) = e^{2\pi\imath\lambda \underline{\boldsymbol{z}}_{2n}} Q_n(\boldsymbol{z}_{2n}; -\lambda), \tag{3.5}$$

where we put

$$\lambda = \rho' - \rho.$$

Under the condition (1.27), both integrals in (3.5) absolutely converge uniformly on compact subsets of the parameters, see Proposition 5 in Appendix B. Thus, both sides of the equality (3.5) are analytic functions of all the parameters therein. Having in mind these analyticity properties we prove the equality (3.5) step by step following the plan below.

1. We prove that for complex periods ω_1, ω_2 with $\operatorname{Re}\omega_i > 0$ and $\omega_1/\omega_2 \notin \mathbb{R}$ both integrals may be calculated by residues technique for big enough negative values of $\operatorname{Re}\lambda$. In other words, one can find a sequence of contours that in the limit encircle all poles in the corresponding half plane (for each integration variable) and such that the integrals over encircling contours tend to zero.
2. Next we assume that the real parameters z_i are generic. Under this assumption, we prove that the sum of residues over higher order poles vanishes. For this, we accumulate the vanishing properties of the integration measure $\mu(\boldsymbol{y}_n)$ into Lemma 4, which says that the sum of 4^r integrand values over the points with interchanged coefficients at the periods gives zero of order $2r$. This lemma is used to describe the result of k successive integrations computed by residues and to show that after each integration higher-order poles vanish.
3. At this stage, we are left with the sum of simple poles on both sides; each of them is a product of one-dimensional residues integrals over shifted parameters z_i. Their sums decompose into the sums of $\binom{2n}{n}$ series, depending of which parameters z_i enter the residue calculations. The equalities of corresponding series reduce to certain identities on rational functions which we prove separately. The identities generalize Ruijsenaars' kernel function identity and could be treated as certain duality transformations for multivalued basic hypergeometric series [14].
4. Finally, due to analyticity of the statement (3.5) we conclude that it is valid first for all values of the parameters z_i, $\operatorname{Re}\lambda$ and as well for $\omega_1/\omega_2 \in \mathbb{R}$ (in particular, for real values of the periods ω_1, ω_2).

3.1. Estimates of Integrals Over Encircling Contours

In the end of this subsection, we prove that the integrals in Q-commutativity identity (3.5) can be calculated by residues in the case $\omega_1/\omega_2 \notin \mathbb{R}$.

Denote by σ_i the arguments of the periods ω_i, $|\sigma_i| < \pi/2$. Since the double sine function is invariant under permutation of ω_1, ω_2, suppose for definiteness that $\sigma_1 \geq \sigma_2$. Let D_+ and D_- be the cones of poles (A.11) and zeros (A.12) of the double sine function $S_2(z|\boldsymbol{\omega})$:

$$D_+ = \{z\colon \sigma_2 < \arg z < \sigma_1\}, \qquad D_- = \{z\colon \pi + \sigma_2 < \arg z < \pi + \sigma_1\},$$
$$D = D_+ \cup D_-.$$

In the first step of our plan, we consider contour in the limit encircling big interval in the real line to a big semicircle in the corresponding half plane. This contour contains three different parts and inside each part the needed bounds are obtained in different ways. In the part of the contour close to the real plane we apply the bound given in Proposition 6 in Appendix B. In the next part of the contour which lies in the regular region $\mathbb{C} \setminus D$, we use the general statements about at most exponential growth of the functions K and μ, see (A.29), (A.33).

A subtle point is the estimate of the integrand along the part of the contour lying in “forbidden” areas $D_+ \cup D_-$ and passing between poles of the double sine function. This estimate is not possible for purely real periods or for periods whose ratio is real. Therefore, for the estimates in the area $D_+ \cup D_-$ we assume

$$\operatorname{Im} \frac{\omega_1}{\omega_2} > 0. \tag{3.6}$$

Then, we use an infinite product representation of the double sine function,

$$S_2(z|\boldsymbol{\omega}) = e^{\frac{\pi\imath}{2} B_{2,2}(z|\boldsymbol{\omega})}\varphi(z|\boldsymbol{\omega}) = e^{-\frac{\pi\imath}{2} B_{2,2}(z|\boldsymbol{\omega})}\varphi'(z|\boldsymbol{\omega}) \tag{3.7}$$

where

$$\varphi(z|\boldsymbol{\omega}) = \frac{r(z|\boldsymbol{\omega})}{s(z|\boldsymbol{\omega})} = \frac{\prod\limits_{m=0}^{\infty}\left(1 - q^{2m} e^{\frac{2\pi i z}{\omega_2}}\right)}{\prod\limits_{m=1}^{\infty}\left(1 - \tilde{q}^{2m} e^{\frac{2\pi i z}{\omega_1}}\right)}, \tag{3.8}$$

$$\varphi'(z|\boldsymbol{\omega}) = \frac{r'(z|\boldsymbol{\omega})}{s'(z|\boldsymbol{\omega})} = \frac{\prod\limits_{m=0}^{\infty}\left(1 - \tilde{q}^{2m} e^{-\frac{2\pi i z}{\omega_1}}\right)}{\prod\limits_{m=1}^{\infty}\left(1 - q^{2m} e^{-\frac{2\pi i z}{\omega_2}}\right)} \tag{3.9}$$

with

$$q = e^{\pi\imath\frac{\omega_1}{\omega_2}}, \qquad \tilde{q} = e^{-\pi\imath\frac{\omega_2}{\omega_1}} \tag{3.10}$$

and $B_{2,2}(z|\boldsymbol{\omega})$ is a particular multiple Bernoulli polynomial (A.2). For any real t_0 and ε, $0 < \varepsilon < 1$ denote by $\Pi_{1,+}(t_0, \varepsilon)$ the strip in the complex plane of the variable z, bounded from one side

$$\Pi_{1,+}(t_0, \varepsilon) = \{z = t\omega_1 + \theta\omega_2 \mid t > t_0,\ \varepsilon < \theta < 1 - \varepsilon\}. \tag{3.11}$$

Lemma 1. *The function* $\varphi(z|\boldsymbol{\omega})$ *is restricted and bounded from zero in any strip* $\Pi_{1,+}(t_0,\varepsilon)$,

$$0 < C_1 < |\varphi(z|\boldsymbol{\omega})| < C_2 \qquad \textit{for} \qquad z \in \Pi_{1,+}(t_0,\varepsilon). \tag{3.12}$$

Proof. Let $\omega_1/\omega_2 = \alpha + i\beta,\ \beta > 0$ and assume first that $t_0 > 0$. Consider the nominator of $\varphi(z|\boldsymbol{\omega})$. It could be written as

$$\begin{aligned} r(z|\boldsymbol{\omega}) &= \prod_{m\geq 0}\left(1 - q^{2m} e^{2\pi\imath\frac{t\omega_1+\theta\omega_2}{\omega_2}}\right) = \prod_{m\geq 0}\left(1 - e^{2\pi\imath\left((m+t)\frac{\omega_1}{\omega_2}+\theta\right)}\right) \\ &= \prod_{m\geq 0}\left(1 - e^{-2\pi\beta(m+t)} e^{2\pi\imath(\alpha(m+t)+\theta)}\right). \end{aligned} \tag{3.13}$$

We have for $t > 0$

$$|e^{-2\pi\beta(m+t)} e^{2\pi\imath(\alpha(m+t)+\theta)}| = e^{-2\pi\beta(m+t)} < 1. \tag{3.14}$$

Due to inequality

$$1 - |a| < |1 - a| < 1 + |a| \tag{3.15}$$

we get the following bound for the nominator $r(z|\boldsymbol{\omega})$ of $\varphi(z|\boldsymbol{\omega})$

$$\xi_\beta(t_0) < |r(z|\boldsymbol{\omega})| < \eta_\beta(t_0)$$

where

$$\xi_\beta(t_0) = \prod_{m\geq 0}\left(1 - e^{-2\pi\beta(m+t_0)}\right), \qquad \eta_\beta(t_0) = \prod_{m\geq 0}\left(1 + e^{-2\pi\beta(m+t_0)}\right).$$

Both these infinite products are converging products not equal to zero. In order to extend the desired bound for negative values of t_0, we note that this extension adds finite product of factors; each of them is bounded from zero due to the restriction on θ. Let

$$\frac{\omega_2}{\omega_1} = c - \imath d, \qquad d > 0.$$

Consider the denominator $s(z|\boldsymbol{\omega})$ of $\varphi(z|\boldsymbol{\omega})$. It looks as

$$s(z|\boldsymbol{\omega}) = \prod_{m\geq 1}\left(1 - \tilde{q}^{2m} e^{2\pi\imath\frac{z}{\omega_1}}\right) = \prod_{m\geq 1}\left(1 - e^{-2\pi d(m-\theta)} e^{2\pi\imath(-c(m-\theta)+t)}\right). \tag{3.16}$$

For $\theta < 1$, we have

$$|e^{-2\pi d(m-\theta)} e^{2\pi\imath(-c(m+\theta)-t)}| = e^{-2\pi d(m-\theta)} < 1.$$

Again, using inequality (3.15) we get the bound

$$\xi'_d(\varepsilon) < |s(z|\boldsymbol{\omega})| < \eta'_d(\varepsilon)$$

where

$$\xi'_d(\varepsilon) = \prod_{m\geq 0}\left(1 - e^{-2\pi d(m+\varepsilon)}\right), \qquad \eta'_d(\varepsilon) = \prod_{m\geq 0}\left(1 + e^{-2\pi d(m+\varepsilon)}\right) \tag{3.17}$$

are convergent infinite products. □

Remarks. Similar arguments give the two-sided bound for the values of the functions $\varphi(z|\boldsymbol{\omega})$ and $\varphi'(z|\boldsymbol{\omega})$ in generic strips.

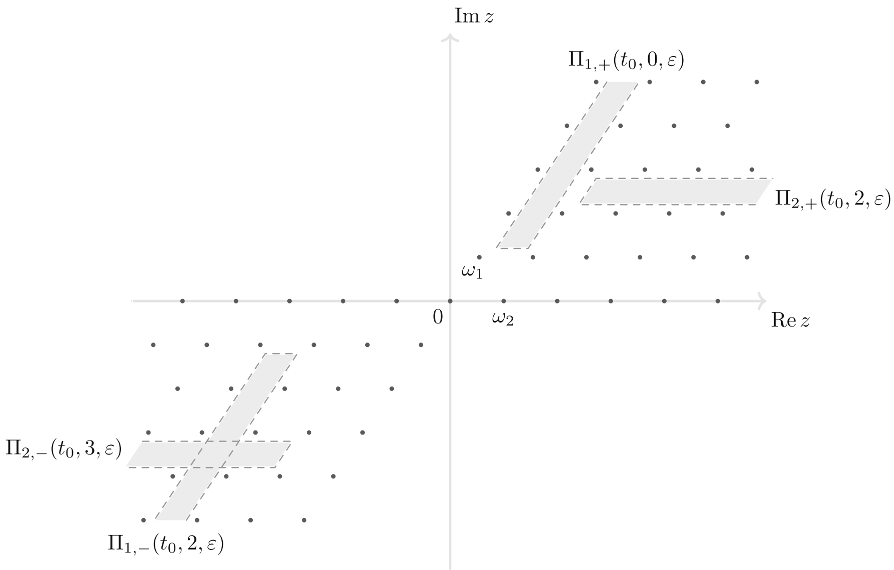


FIGURE 1. The parameter $m \in \mathbb{Z}$ controls a place of the strip $\Pi_{j,\pm}$ in the $(\mathbb{Z}\omega_1, \mathbb{Z}\omega_2)$-lattice. The value of t_0 determines where strip starts, and the parameter ε defines its thickness

1. The function $\varphi(z|\boldsymbol{\omega})$ and its inverse are restricted in strips

$$\Pi_{1,+}(t_0, m, \varepsilon) = \{z = t\omega_1 + \theta\omega_2 \mid t > t_0,\ m + \varepsilon < \theta < m + 1 - \varepsilon\}, \quad (3.18)$$

$$\Pi_{2,-}(t_0, m, \varepsilon) = \{z = -t\omega_2 - \theta\omega_1 \mid t > t_0,\ m + \varepsilon < \theta < m + 1 - \varepsilon\} \quad (3.19)$$

where $m \in \mathbb{Z},\ 0 < \varepsilon < 1$.

2. The function $\varphi'(z|\boldsymbol{\omega})$ and its inverse are restricted in any strip

$$\Pi_{2,+}(t_0, m, \varepsilon) = \{z = t\omega_2 + \theta\omega_1 \mid t > t_0,\ m + \varepsilon < \theta < m + 1 - \varepsilon\}, \quad (3.20)$$

$$\Pi_{1,-}(t_0, m, \varepsilon) = \{z = -t\omega_1 - \theta\omega_2 \mid t > t_0,\ m + \varepsilon < \theta < m + 1 - \varepsilon\} \quad (3.21)$$

where $m \in \mathbb{Z},\ 0 < \varepsilon < 1$.

Next we consider the function $\varphi(z|\boldsymbol{\omega})\varphi^{-1}(z + g|\boldsymbol{\omega})$. Assume that

$$g \notin \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2. \quad (3.22)$$

First of all note that for any $t_0 \in \mathbb{R}$ there exist $0 < \varepsilon' < \varepsilon$, $t_0' \in \mathbb{R}$ and $N_0 \in \mathbb{Z}$ such that the strip $\{z = -g + t\omega_1 + \theta\omega_2 \mid t > t_0,\ \varepsilon' < \theta < 1 - \varepsilon'\}$ is inside the strip $\Pi_{1,+}(t_0', N_0, \varepsilon)$. Then, for any $N \in \mathbb{Z}_+$ the ratio $\varphi(z|\boldsymbol{\omega})\varphi^{-1}(z + g|\boldsymbol{\omega})$ has no poles and no zeros in the strip $\Pi_{1,+}(t_0, N, \varepsilon)$ and is restricted in this strip. We now show that its bound is not more than exponential on N.

Lemma 2. *There exist real a, b and $C_1, C_2 > 0$, such that*

$$C_1 e^{aN} < \left|\varphi(z|\boldsymbol{\omega})\varphi^{-1}(z + g|\boldsymbol{\omega})\right| < C_2 e^{bN}$$

in the strip $\Pi_{1,+}(t_0, N, \varepsilon)$.

Proof. First of all note that the bounds for the product $r(z|\boldsymbol{\omega})$ do not depend on N for any strip $\Pi_{1,+}(t_0, N, \varepsilon)$ (the bound (3.14) doesn't depend on the range of θ). We just have to estimate the ratio $s(z|\boldsymbol{\omega})s^{-1}(z+g|\boldsymbol{\omega})$ in the strip $\Pi_{1,+}(t_0, N, \varepsilon)$. This product can be divided into two parts: one is an infinite convergent product

$$s_\infty(z|\boldsymbol{\omega}) = \frac{\prod\limits_{m\geq N}\left(1-\tilde{q}^{2m}e^{2\pi\imath\frac{z}{\omega_1}}\right)}{\prod\limits_{m\geq N+N_0}\left(1-\tilde{q}^{2m}e^{2\pi\imath\frac{z+g}{\omega_1}}\right)}, \tag{3.23}$$

which after the change of the product indices can be evaluated in the strip $\Pi_{1,+}(t_0, N, \varepsilon)$ independently of N

$$\frac{\xi'_d(\varepsilon)}{\eta'_d(\varepsilon)} < |s_\infty(z|\boldsymbol{\omega})| < \frac{\eta'_d(\varepsilon)}{\xi'_d(\varepsilon)}, \tag{3.24}$$

where $\xi'_d(\varepsilon)$ and $\eta'_d(\varepsilon)$ are given in (3.17); and another is a finite product

$$\begin{aligned} s_0(z|\boldsymbol{\omega}) &= \frac{\prod\limits_{1\leq m<N}\left(1-\tilde{q}^{2m}e^{2\pi\imath\frac{z}{\omega_1}}\right)}{\prod\limits_{1\leq m<N+N_0}\left(1-\tilde{q}^{2m}e^{2\pi\imath\frac{z+g}{\omega_1}}\right)} \\ &= \frac{\tilde{q}^{-N(N-1)}e^{2\pi\imath(N-1)\frac{z'}{\omega_1}}}{\tilde{q}^{-(N+N_0)(N+N_0-1)}e^{2\pi\imath(N+N_0-1)\frac{z''}{\omega_1}}}\cdot\frac{\prod\limits_{1\leq m<N}\left(\tilde{q}^{2m}e^{-2\pi\imath\frac{z'}{\omega_1}}-1\right)}{\prod\limits_{1\leq m<N+N_0}\left(\tilde{q}^{2m}e^{-2\pi\imath\frac{z''}{\omega_1}}-1\right)}, \end{aligned} \tag{3.25}$$

where both $z' = z - N\omega_2$ and $z'' = z + g - (N + N_0)\omega_2$ are now in the strip $\Pi_{1,+}(t_0, 0, \varepsilon) = \Pi_{1,+}(t_0, \varepsilon)$. The second fraction can be bounded from both sides independently of N with a help of analogous infinite product, that is

$$\frac{\xi'_d(\varepsilon)}{\eta'_d(\varepsilon)} < \left|\frac{\prod\limits_{1\leq m<N}\left(\tilde{q}^{2m}e^{-2\pi\imath\frac{z'}{\omega_1}}-1\right)}{\prod\limits_{1\leq m<N+N_0}\left(\tilde{q}^{2m}e^{-2\pi\imath\frac{z''}{\omega_1}}-1\right)}\right| < \frac{\eta'_d(\varepsilon)}{\xi'_d(\varepsilon)} \tag{3.26}$$

The estimate of the first one is also pure exponential

$$\begin{aligned} C_1e^{2\pi dN(2\varepsilon-1-N_0)} &< |\tilde{q}|^{2N_0N+N_0(N_0-1)}e^{2\pi d(N-1)(\theta'-\theta'')}e^{-2\pi dN_0} \\ &< C_2e^{2\pi dN(1-2\varepsilon-N_0)} \end{aligned} \tag{3.27}$$

This ends the proof of Lemma 2. □

Analogous statement holds for the function $\varphi'(z|\boldsymbol{\omega})$ in corresponding strips.

For each $0 < \varepsilon < 1$ and positive integers M, N denote by $\Pi_{1,+}(t_0, N, M, \varepsilon)$ the bounded open region

$$\Pi_{1,+}(t_0, M, N, \varepsilon) = \{z = t\omega_1 + \theta\omega_2 \mid t_0 < t < N,\ M + \varepsilon < \theta < M + 1 - \varepsilon\}. \tag{3.28}$$

Lemma 2, its analog for the function $\varphi'(z|\boldsymbol{\omega})$ and the relations (3.7) immediately imply the following corollary.

Corollary 1. *The ratio* $S_2(z|\boldsymbol{\omega})S_2^{-1}(z+g|\boldsymbol{\omega})$ *admits a two sided exponential bound in the region* $\Pi_{1,+}(t_0, M, N, \varepsilon)$

$$C_1 e^{a(N+M)} < |S_2(z|\boldsymbol{\omega})S_2^{-1}(z+g|\boldsymbol{\omega})| < C_2 e^{b(N+M)} \tag{3.29}$$

for some real a, b *and* $C_1, C_2 > 0$.

Remark. Analogous exponential bounds hold for the regions

$$\begin{aligned} \Pi_{2,+}(t_0, M, N, \varepsilon) =& \{z = t\omega_2 + \theta\omega_1 \mid t_0 < t < N,\ M+\varepsilon < \theta < M+1-\varepsilon\} \\ \Pi_{1,-}(t_0, M, N, \varepsilon) =& \{z = -t\omega_1 - \theta\omega_2 \mid t_0 < t < N,\ M+\varepsilon < \theta < M+1-\varepsilon\} \\ \Pi_{2,-}(t_0, M, N, \varepsilon) =& \{z = -t\omega_2 + \theta\omega_1 \mid t_0 < t < N,\ M+\varepsilon < \theta < M+1-\varepsilon\} \end{aligned} \tag{3.30}$$

The proofs are similar.

Corollary 2. *For big negative values of the real part of the parameter* λ *the integral in the left hand side of (3.5) can be computed by residues calculation, moving the integration contours to the lower half plane; and the integral in the right hand side of (3.5) can be also computed by residues calculation, moving the integration contours to the upper half plane.*

Proof. The residue calculation means that the initial straight contour in the integral in the left hand side of (3.5) is enclosed by a contour where, for instance,

$$|y_1| \gg |y_2| \gg \ldots \gg |y_n| \gg 1, \qquad \operatorname{Im} y_i < 0$$

and we argue that the integral over this enclosing contour tends to zero when the contour grows. For each variable y_i its integration contour is either lies in the regular region for all occurring function $S(\imath y_i - a_j)S^{-1}(\imath y_i + g - a_j)$ or in the cones of singularities of these functions. In the part of the contour close to the real line, we apply the bound given in Proposition 6 in Appendix B. In the next part of the contour which lies in the regular region $\mathbb{C}\backslash D$, using the general statements about at most exponential growth, see (A.29), (A.33), we suppress the integrand by fast decreasing exponent $e^{2\pi\imath\lambda \boldsymbol{y}_n}$ with sufficiently big negative values of $\operatorname{Re}\lambda$. Inside the irregular cone D we put the contour into proper regions $\Pi_{k,\pm}(t_i, M_i, N_i, \varepsilon_i)$, $k = 1, 2$ for sufficiently large M_i and N_i. In the proper regions all functions grow at most exponentially, see Corollary 1 and remark after it, therefore we suppress them by fast decreasing exponent $e^{2\pi\imath\lambda \boldsymbol{y}_n}$ as well. The same procedure for the integral in the right hand side of (3.5).

In the case $n = 1$ the Q-commutativity integral (3.4) is one-dimensional

$$Q_1(z_1, z_2; \lambda) = \int_{\mathbb{R}} \mathrm{d}y_1\, K(z_1 - y_1)K(z_2 - y_1)\, e^{2\pi\imath\lambda y_1}, \tag{3.31}$$

and the corresponding enclosing contour $C_{N,M}$ that appears when we calculate it by residues

$$\int_{\mathbb{R}} = \int_{C_{N,M}} + \sum \operatorname{Res} \tag{3.32}$$

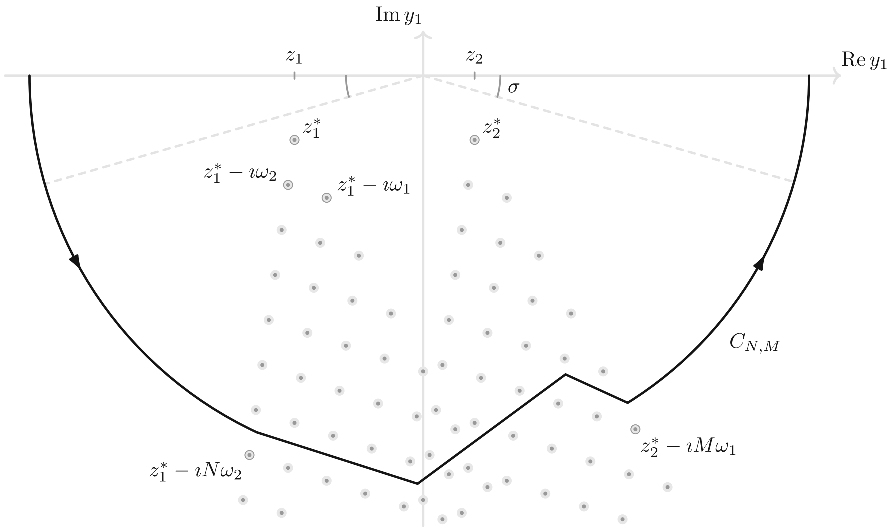


FIGURE 2. The contour in the case $n = 1$. We denoted $z_j^* = z_j - \imath g^*/2$ and for clarity circled all labeled poles

is shown in Fig. 2. There are two sequences of poles in the lower half-plane

$$y_1 = z_j - \frac{\imath g^*}{2} - \imath m^1\omega_1 - \imath m^2\omega_2, \qquad j = 1, 2, \quad m^i \geq 0. \tag{3.33}$$

Small circles around poles in Fig. 2 are restricted regions: the broken line stays away from the poles at the distance more than their radii (consequently, we have fixed exponent parameters a, b from Corollary 1). The angle σ is determined from the condition (B.16), so that we have exponentially decreasing bound near the real line given in Proposition 6. The integers N, M are chosen such that the contour $C_{N,M}$ passes right above the pole $z_1 - \imath g^*/2 - \imath N\omega_2$ from the left and the pole $z_2 - \imath g^*/2 - \imath M\omega_1$ from the right. □

3.2. Reduction to Simple Poles

3.2.1. Chains of Integrals and Double Zeros Lemma. Denote by F the integrand of the left hand side of Q-commutativity relation (3.4)

$$F(\boldsymbol{y}_n, \boldsymbol{z}_{2n}) = e^{2\pi\imath\lambda \underline{\boldsymbol{y}}_n} \prod_{a=1}^{2n}\prod_{i=1}^{n} K(y_i - z_a) \prod_{\substack{i,j=1 \\ i\neq j}}^{n} \mu(y_i - y_j). \tag{3.34}$$

We integrate this function over y_j by residues in the order of increasing indices. Let G_m be the result of m successive integrations

$$G_m(y_{m+1}, \dots, y_n, \boldsymbol{z}_{2n}) = \int_{\mathbb{R}} \mathrm{d}y_m \cdots \int_{\mathbb{R}} \mathrm{d}y_1\, F(\boldsymbol{y}_n, \boldsymbol{z}_{2n}). \tag{3.35}$$

Moving contours to the lower half-plane we meet poles of the function $K(y_i - z_a)$

$$\imath y_i = \imath z_a + \frac{g^*}{2} + m^1\omega_1 + m^2\omega_2, \qquad \imath y_i = \imath z_a - \frac{g^*}{2} - m^1\omega_1 - m^2\omega_2 \quad (3.36)$$

and of the function $\mu(y_i - y_j)$

$$\imath y_i = \imath y_j + g + m^1\omega_1 + m^2\omega_2, \qquad \imath y_i = \imath y_j - g - m^1\omega_1 - m^2\omega_2 \quad (3.37)$$

where $m^1, m^2 \geq 0$. Below we prove (see Proposition 1) that the resulting function G_m can be written solely in terms of two typical residue integrals with simple poles.

The first typical integral is $J_1(z_a|b, m^1, m^2)$. It depends on a complex parameter z_a, on the index b of the variable y_b and on the pair (m^1, m^2) of non-negative integers. It is given by one-dimensional residue

$$J_1(z_a|b, m^1, m^2) = -2\pi\imath \operatorname{Res}_{\imath y_b = \imath z_a + \frac{g^*}{2} + m^1\omega_1 + m^2\omega_2} F(\boldsymbol{y}_n, \boldsymbol{z}_{2n}). \quad (3.38)$$

Additionally, it is a function of all other parameters z_c and variables y_j different from y_b. This residue is nonzero due to the poles of functions K (3.36).

The second typical integral is k-fold residue integral $I_k(y_{i_0}|\boldsymbol{i}_k, \boldsymbol{m}_k)$. It depends on a complex valued variable y_{i_0} with $i_0 \in [n] = \{1, \ldots, n\}$, on a sequence $\boldsymbol{i}_k$ of k distinct indices

$$\boldsymbol{i}_k = (i_1, \ldots, i_k), \qquad i_a \in [n] \setminus \{i_0\}$$

corresponding to variables y_{i_a} in (3.34), and on two non-negative sequences of k integers

$$\boldsymbol{m}_k = (m_1^1, \ldots, m_k^1; m_1^2, \ldots, m_k^2), \qquad m_j^i \geq 0.$$

We define $I_k(y_{i_0}|\boldsymbol{i}_k, \boldsymbol{m}_k)$ as the following k-fold residue integral

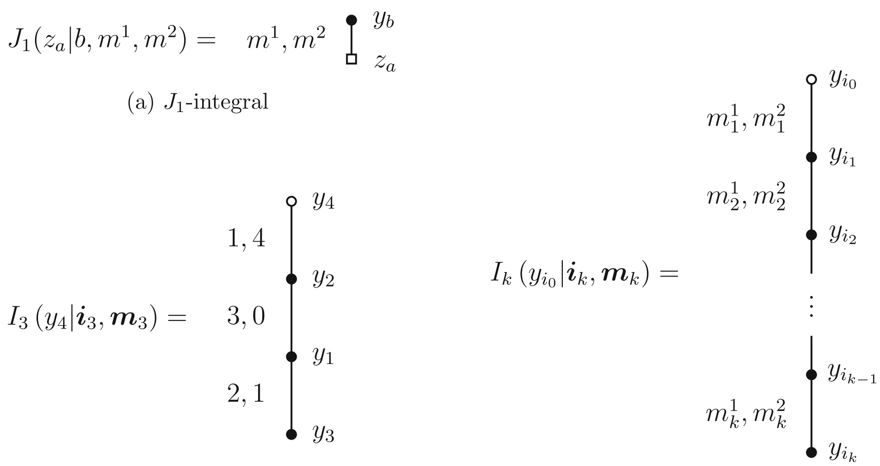
$$I_k(y_{i_0}|\boldsymbol{i}_k, \boldsymbol{m}_k) = (-2\pi\imath)^k \operatorname{Res}_{\imath y_{i_1} = \imath y_{i_0} + g + m_1^1\omega_1 + m_1^2\omega_2} \cdots \\ \cdots \operatorname{Res}_{\imath y_{i_k} = \imath y_{i_{k-1}} + g + m_k^1\omega_1 + m_k^2\omega_2} F(\boldsymbol{y}_n, \boldsymbol{z}_{2n}). \quad (3.39)$$

Additionally, it is a function of parameters z_a and all variables y_j which are not engaged in the integration procedure. It naturally refers to the point

$$\begin{aligned} \imath y_{i_1} &= \imath y_{i_0} + g + m_1^1\omega_1 + m_1^2\omega_2, \\ \imath y_{i_2} &= \imath y_{i_0} + 2g + \big(m_1^1 + m_2^1\big)\omega_1 + \big(m_1^2 + m_2^2\big)\omega_2, \\ &\vdots \\ \imath y_{i_k} &= \imath y_{i_0} + kg + \big(m_1^1 + \ldots + m_k^1\big)\omega_1 + \big(m_1^2 + \ldots + m_k^2\big)\omega_2. \end{aligned} \quad (3.40)$$

This residue is nonzero due to the poles of functions μ (3.37). Note that, although in G_m (3.35) we integrate over y_{i_a} in the order of increasing indices i_a, the multidimensional pole (3.40) is simple, so the definition (3.39) does not depend on the order of residues.

We name the integrals $J_1(z_a|b, m^1, m^2)$ and $I_k(y_{i_0}|\boldsymbol{i}_k, \boldsymbol{m}_k)$ as **chain integrals** and picture them as chains of vertices with the corresponding labels, see Fig. 3. Note that length of the line in J_1-chain is twice smaller than in



(a) J_1-integral

(b) Example: $i_0 = 4,\ \boldsymbol{i}_3 = (2, 1, 3)$, $\boldsymbol{m}_3 = (1, 3, 2; 4, 0, 1)$

(c) I_k-integral

Figure 3. Chain integrals

I_k-chains. This rule reflects difference between constants $g^*/2$ and g in residue points of chain integrals.

The chain integrals are parametrized by the cycles in the space of corresponding integration variables. It is natural to define the **direct product** of chain integrals as the integrals over direct products of corresponding cycles. More precisely, assume that all the variables y_{i_a} of the first chain integral including integration variables and the free variable are different from those of the second chain integral. Then, the direct product of these two chain integrals is defined as the integral over corresponding product of the contours. For instance, the direct product

$$I_k\left(y_{i_0}|\boldsymbol{i}_k, \boldsymbol{m}_k\right) \times I_l\left(y_{i'_0}|\boldsymbol{i}'_l, \boldsymbol{m}'_l\right)$$

of two chain integrals is defined for disjoint sets $\boldsymbol{i}_k$ and $\boldsymbol{i}'_l$ and generic parameters y_{i_0} and $y_{i'_0}$ as $k + l$ fold residue integral

$$\begin{aligned}(-2\pi\imath)^{k+l}\, & \operatorname{Res}_{\imath y_{i_1} = \imath y_{i_0} + g + m_1^1\omega_1 + m_1^2\omega_2} \cdots \operatorname{Res}_{\imath y_{i_k} = \imath y_{i_{k-1}} + g + m_k^1\omega_1 + m_k^2\omega_2} \\ & \operatorname{Res}_{\imath y_{i'_1} = \imath y_{i'_0} + g + m'^1_1\omega_1 + m'^2_1\omega_2} \cdots \\ & \operatorname{Res}_{\imath y_{i'_l} = \imath y_{i'_{l-1}} + g + m'^1_l\omega_1 + m'^2_l\omega_2} F(\boldsymbol{y}_n, \boldsymbol{z}_{2n})\end{aligned}$$

A direct product of integrals will be pictured simply by placing the corresponding chains next to each other, see Fig. 4.

The main technical result of this subsection is the following statement.

Proposition 1. *For generic $\boldsymbol{\omega}$, g and $\boldsymbol{z}_{2n}$ the function $G_m(y_{m+1}, \ldots, y_n, \boldsymbol{z}_{2n})$ defined by (3.35) is a sum of all possible direct products of chains $J_1(z_a|$*

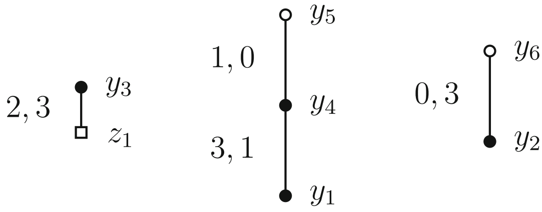


FIGURE 4. Direct product of three chain integrals

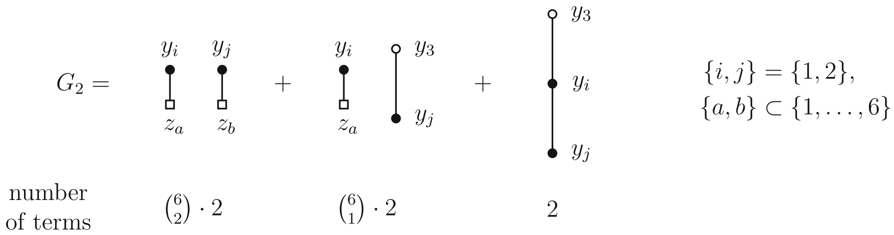


FIGURE 5. All possible direct products for $n = 3, m = 2$. The sum over all possible edge parameters and indices i, j, a, b is implied

$b, m^1, m^2)$ and $I_k\left(y_{i_0}|\boldsymbol{i}_k, \boldsymbol{m}_k\right)$, such that parameters z_a are distinct inside each summand and indices of integration variables do not exceed m.

Note that the function $G_n(\emptyset, \boldsymbol{z}_{2n})$ coincides with the integral (3.4)

$$G_n(\emptyset, \boldsymbol{z}_{2n}) = Q_n(\boldsymbol{z}_{2n}; \lambda). \tag{3.41}$$

An immediate corollary of Proposition 1 is the following observation.

Corollary 3. *The integral in the left hand side of (3.5) is the sum of all possible direct products of one-dimensional residues $J_1(z_a|b, m^1, m^2)$ with distinct z_a.*

Indeed, in this case free parameters of possible chains could not be any integration variables y_a, so we have a sum of direct products of chain integrals $J_1(z_a|b, m^1, m^2)$. The same result holds for the right hand side of (3.5).

For the proof of Proposition 1, we need the following property of zeros location of the measure function $\mu(\boldsymbol{y}_n)$. Choose a pair of variables, say y_1 and y_2. Let

$$\imath y_1 = a + \varepsilon + p^1\omega_1 + p^2\omega_2, \quad \imath y_2 = a + q^1\omega_1 + q^2\omega_2, \quad p^i, q^i \in \mathbb{Z}. \tag{3.42}$$

Set

$$m^1 = p^1 - q^1, \qquad m^2 = p^2 - q^2.$$

Let the operator τ_{12}^{1,m^1} permute p^1 and q^1 and the operator τ_{12}^{2,m^2} permute p^2 and q^2, so that

$$\begin{aligned}
&\tau_{12}^{1,m^1} F(y_1, y_2, \ldots, y_n, \boldsymbol{z}_{2n}) \\
&\quad = F(a + \varepsilon + q^1\omega_1 + p^2\omega_2, a + p^1\omega_1 + q^2\omega_2, \ldots, y_n, \boldsymbol{z}_{2n}), \\
&\tau_{12}^{2,m^2} F(y_1, y_2, \ldots, y_n, \boldsymbol{z}_{2n}) \\
&\quad = F(a + \varepsilon + p^1\omega_1 + q^2\omega_2, a + q^1\omega_1 + p^2\omega_2, \ldots, y_n, \boldsymbol{z}_{2n}).
\end{aligned}$$

In other words, the operators τ_{12}^{1,m^1} and τ_{12}^{2,m^2} are the following shift operators:

$$\tau_{12}^{1,m^1} \begin{pmatrix} \imath y_1 \\ \imath y_2 \end{pmatrix} = \begin{pmatrix} \imath y_1 - m^1\omega_1 \\ \imath y_2 + m^1\omega_1 \end{pmatrix}, \quad \tau_{12}^{2,m^2} \begin{pmatrix} \imath y_1 \\ \imath y_2 \end{pmatrix} = \begin{pmatrix} \imath y_1 - m^2\omega_2 \\ \imath y_2 + m^2\omega_2 \end{pmatrix}. \tag{3.43}$$

Lemma 3. *For generic a, variables y_j and parameters z_b*

$$(1 + \tau_{12}^{1,m^1})(1 + \tau_{12}^{2,m^2}) F(\boldsymbol{y}_n, \boldsymbol{z}_{2n}) = O(\varepsilon^2), \qquad \varepsilon \to 0. \tag{3.44}$$

In other words, the sum of four summands

$$F(\boldsymbol{y}_n, \boldsymbol{z}_{2n}) + \tau_{12}^{1,m^1} F(\boldsymbol{y}_n, \boldsymbol{z}_{2n}) + \tau_{12}^{2,m^2} F(\boldsymbol{y}_n, \boldsymbol{z}_{2n}) + \tau_{12}^{1,m^1}\tau_{12}^{2,m^2} F(\boldsymbol{y}_n, \boldsymbol{z}_{2n}) \tag{3.45}$$

has zero of the second order at the hyperplane

$$\imath(y_1 - y_2) = m^1\omega_1 + m^2\omega_2. \tag{3.46}$$

Proof of Lemma 3. Note that shift operators in (3.44) act only on y_1, y_2. Denote by A the part of function F that does not depend on y_1, y_2

$$F(\boldsymbol{y}_n, \boldsymbol{z}_{2n}) = e^{2\pi\imath\lambda(y_1+y_2)} A(y_3, \ldots, y_n, \boldsymbol{z}_{2n}) B(\boldsymbol{y}_n, \boldsymbol{z}_{2n}). \tag{3.47}$$

The exponent is invariant under the shifts. Using reflection formula

$$S_2^{-1}(z) = S_2(\omega_1 + \omega_2 - z) \tag{3.48}$$

we can write the remaining part B in the form

$$\begin{aligned}
B = {} & S_2(\imath(y_1 - y_2))\, S_2(\imath(y_2 - y_1))\, S_2(\imath(y_1 - y_2) + g^*)\, S_2(\imath(y_2 - y_1) + g^*) \\
& \times \prod_{(b,b')} S_2(\imath y_1 - b)\, S_2(\imath y_2 - b)\, S_2(b' - \imath y_1)\, S_2(b' - \imath y_2),
\end{aligned} \tag{3.49}$$

where pairs (b, b') contain all other variables y_j, z_a $(j \neq 1, 2)$ and constants. Moreover, the first two functions can be written as (A.5)

$$S_2(\imath(y_1 - y_2)) S_2(\imath(y_2 - y_1)) = -4 \sin\frac{\imath\pi}{\omega_1}(y_1 - y_2) \sin\frac{\imath\pi}{\omega_2}(y_1 - y_2). \tag{3.50}$$

For the variables y_1, y_2 at the points (3.42) we use the last formula with a factorization property (A.10)

$$S_2(z + p^1\omega_1 + p^2\omega_2) = (-1)^{p_1 p_2} \frac{S_2(z + p^1\omega_1)\, S_2(z + p^2\omega_2)}{S_2(z)}, \qquad p^j \in \mathbb{Z} \tag{3.51}$$

to separate coordinates p^j, q^j in the function B

$$B(p^1, q^1; p^2, q^2) = C\, B_1(p^1, q^1)\, B_2(p^2, q^2) \tag{3.52}$$

where C does not depend on any p^j, q^j. The signs coming from (3.51) disappear since each of them occurs an even number of times. Clearly, B_j differ only by ω_j. Therefore, it is sufficient to prove that

$$(1 + \tau_{12}^{1,m^1}) B_1(p^1, q^1) = O(\varepsilon). \tag{3.53}$$

Evaluating B_1 at $\varepsilon = 0$ we obtain a function antisymmetric with respect to p^1, q^1:

$$\begin{aligned} &B_1(p^1, q^1)\big|_{\varepsilon=0} \\ &\quad = (-1)^{p^1+q^1} \sin \frac{\pi\omega_1}{\omega_2}(p^1 - q^1)\; S_2((p^1 - q^1)\omega_1 + g^*)\, S_2((q^1 - p^1)\omega_1 + g^*) \\ &\quad \times \prod_{(b,b')} S_2(p^1\omega_1 + a - b)\, S_2(q^1\omega_1 + a - b)\, S_2(b' - a - p^1\omega_1)\, S_2(b' - a - q^1\omega_1). \end{aligned} \tag{3.54}$$

Since all functions in B_1 are analytic, the identity (3.53) follows. □

Now let $\boldsymbol{m}_k$ again denote two sequences of integers (without requiring them to be non-negative)

$$\boldsymbol{m}_k = (m_1^1, \dots m_k^1; m_1^2, \dots, m_k^2), \qquad m_j^i \in \mathbb{Z}.$$

We attach to this sequence $2k$ shift operators $\tau_{12}^{1,\boldsymbol{m}_k}$, $\tau_{12}^{2,\boldsymbol{m}_k}$, … $\tau_{2k-1,2k}^{1,\boldsymbol{m}_k}$, $\tau_{2k-1,2k}^{2,\boldsymbol{m}_k}$, so that

$$\tau_{2j-1,2j}^{1,\boldsymbol{m}_k} \begin{pmatrix} \imath y_{2j-1} \\ \imath y_{2j} \end{pmatrix} = \begin{pmatrix} \imath y_{2j-1} - m_j^1\omega_1 \\ \imath y_{2j} + m_j^1\omega_1 \end{pmatrix},$$

$$\tau_{2j-1,2j}^{2,\boldsymbol{m}_k} \begin{pmatrix} \imath y_{2j-1} \\ \imath y_{2j} \end{pmatrix} = \begin{pmatrix} \imath y_{2j-1} - m_j^2\omega_2 \\ \imath y_{2j} + m_j^2\omega_2 \end{pmatrix}.$$

Set

$$\varepsilon_1 = \imath(y_1 - y_2) - (m_1^1\omega_1 + m_1^2\omega_2), \qquad \dots$$
$$\varepsilon_k = \imath(y_{2k-1} - y_{2k}) - (m_k^1\omega_1 + m_k^2\omega_2),$$

and denote by $F_k(\boldsymbol{y}_n, \boldsymbol{z}_{2n})$ the sum of 4^k summands

$$F_k(\boldsymbol{y}_n, \boldsymbol{z}_{2n}) = \left(\prod_{j=1}^{k} \left(1 + \tau_{2j-1,2j}^{1,\boldsymbol{m}_k}\right)\left(1 + \tau_{2j-1,2j}^{2,\boldsymbol{m}_k}\right) \right) F(\boldsymbol{y}_n, \boldsymbol{z}_{2n}) \tag{3.55}$$

The following statement is a direct consequence of Lemma 3.

Lemma 4. *The function $F_k(\boldsymbol{y}_n, \boldsymbol{z}_{2n})$ has zero of order $2k$ on the intersection of hyperplanes*

$$\varepsilon_1 = \ldots = \varepsilon_k = 0. \tag{3.56}$$

Moreover, its Taylor expansion in a generic point of the plane (3.56) starts from $\varepsilon_1^2 \cdots \varepsilon_k^2$

$$F_k(\boldsymbol{y}_n, \boldsymbol{z}_{2n}) = \varepsilon_1^2 \cdots \varepsilon_k^2 \cdot H_k(\boldsymbol{y}_n, \boldsymbol{z}_{2n}), \tag{3.57}$$

where $H_k(\boldsymbol{y}_n, \boldsymbol{z}_{2n})$ *is regular at generic point of (3.56).*

Proof. Using Lemma 3 for generic values of all the variables and parameters, we have

$$\big(1+\tau^{1,\boldsymbol{m}_k}_{2j-1,2j}\big)\big(1+\tau^{2,\boldsymbol{m}_k}_{2j-1,2j}\big)F(\boldsymbol{y}_n,\boldsymbol{z}_{2n}) = \varepsilon_j^2 \cdot H_j(\boldsymbol{y}_n,\boldsymbol{z}_{2n})$$

for any $j = 1, \ldots, k$, where $H_j(\boldsymbol{y}_n, \boldsymbol{z}_{2n})$ is analytic function on the hyperplane

$$\imath(y_{2j-1} - y_{2j}) = m_j^1\omega_1 + m_j^2\omega_2$$

and in particular on the plane (3.56). Since all operators $\tau^{1,\boldsymbol{m}_k}_{2j-1,2j}$ in the product (3.55) commute, the same is true for the total expression F_k. So, F_k is analytic with respect to ε_j and has zero of the second order at $\varepsilon_j = 0$ for all j. Then its Taylor expansion in ε_j starts with the $\varepsilon_1^2 \cdots \varepsilon_k^2$ term. □

3.2.2. Induction Step: Fusion of Chain Integrals. We are ready now to prove the induction step of Proposition 1. We regard the result $G_m(y_{m+1}, \ldots, y_n, \boldsymbol{z}_{2n})$ of the first m integrations as an analytical function of parameters $y_{m+1}, \ldots, y_n$, $\boldsymbol{z}_{2n}$. Thus, during the integration over the variable y_{m+1} we can assume that all the parameters $y_{m+2}, \ldots, y_n, \boldsymbol{z}_{2n}$ are generic so that there are no singularities between different factors in each summand of $G_m(y_{m+1}, \ldots, y_n, \boldsymbol{z}_{2n})$. It means that the induction step reduces to the consideration of the fusions of chain integrals (3.38) and (3.39). Namely, we now consider one-dimensional integrals

$$\operatorname{Res}_{\imath y_{i_0} = \imath y_{j_0} + c}\ I_k\left(y_{i_0} | \boldsymbol{i}_k, \boldsymbol{m}_k\right) \times I_l\left(y_{j_0} | \boldsymbol{j}_l, \tilde{\boldsymbol{m}}_l\right), \tag{3.58}$$

$$\operatorname{Res}_{\imath y_{i_0} = \imath z_j + c}\ I_k\left(y_{i_0} | \boldsymbol{i}_k, \boldsymbol{m}_k\right) \times J_1(z_a | b, \tilde{m}^1, \tilde{m}^2), \tag{3.59}$$

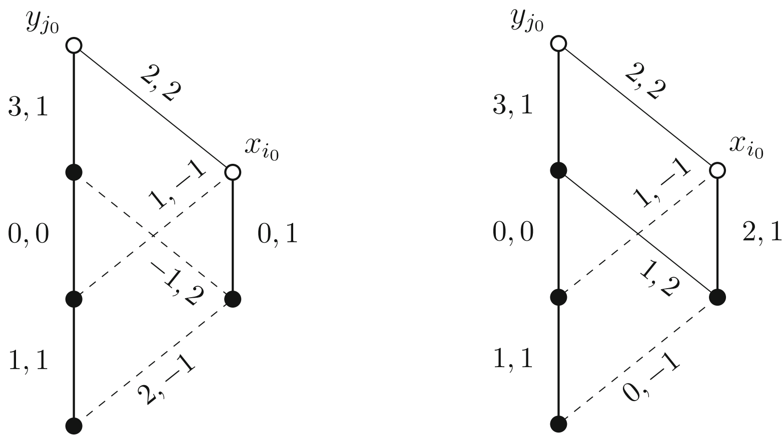
$$\operatorname{Res}_{\imath y_{i_0} = \imath z_j + c}\ I_k\left(y_{i_0} | \boldsymbol{i}_k, \boldsymbol{m}_k\right). \tag{3.60}$$

Below we prove that all such residues cancel each other, except ones that form new chain integrals I_{k+l+1} and J_1. Having in mind that in the original integral (3.4) all the parameters were real, we assume that in the first integral the variable y_{j_0} is real and in the second and third integrals the parameter z_a is also real.

Consider the residue (3.58) first. For better convenience denote the variables y_{i_a} taking part in the integral $I_k\left(y_{i_0} | \boldsymbol{i}_k, \boldsymbol{m}_k\right)$ by letters x_{i_a}. The residue (3.58) could be nonzero only if the integration variable x_{i_0} meets the point corresponding to the pole of the measure function either

$$S(\imath(y_{j_b} - x_{i_a}) + g^*) \qquad \text{or} \qquad S(\imath(x_{i_a} - y_{j_b}) + g^*). \tag{3.61}$$

This happens when the variables $\imath(y_{j_b} - x_{i_a})$ or $\imath(x_{i_a} - y_{j_b})$ in the arguments of the functions in (3.61) equal to $g + m^1\omega_1 + m^2\omega_2$ for some non-negative integers m^1 and m^2, see (A.11). Moreover, a number of such singularities can appear together giving a multiple pole. A typical example is shown in Fig. 6. The pairs of numbers on each edge indicate corresponding integers m^1 and m^2. When one of them is negative, the corresponding edge is dashed, which means the missing of the corresponding singularity. In the case of the multiple singularity instead of single fusion integral, we consider the fusion of several summands of the function $G_m(y_{m+1}, \ldots, y_n, \boldsymbol{z}_{2n})$ which fit using of Lemma 4.



(a) First order pole: $\boldsymbol{m}_1 = (0; 1)$, $\tilde{\boldsymbol{m}}_3 = (3, 0, 1; 1, 0, 1)$

(b) Second order pole: $\boldsymbol{m}_1 = (2; 1)$, $\tilde{\boldsymbol{m}}_3 = (3, 0, 1; 1, 0, 1)$

FIGURE 6. Fusions of chains with different edge parameters. Solid lines correspond to singularities

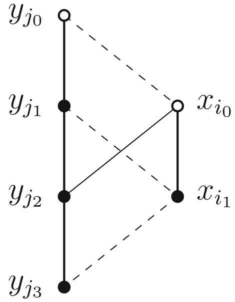


FIGURE 7. The pole (3.62) with $a = 0$, $b = 2$

We then justify that such sum either vanishes or it is given by a simple residue of the first order which we analyze further.

Consider each type of singularity (3.61) separately. In the first singularity imposed by the pole of $S(\imath(y_{j_b} - x_{i_a}) + g^*)$, we have the relation

$$\imath y_{j_b} = \imath x_{i_a} + g + p^1\omega_1 + p^2\omega_2, \qquad p^1, p^2 \geq 0. \tag{3.62}$$

In the second

$$\imath x_{i_a} = \imath y_{j_b} + g + q^1\omega_1 + q^2\omega_2, \qquad q^1, q^2 \geq 0. \tag{3.63}$$

Consider the first case (3.62), example is shown in Fig. 7.

If $a < k$, then in the chain I_k there is a variable $x_{i_{a+1}}$ such that

$$\imath x_{i_{a+1}} = \imath x_{i_a} + g + m^1_{a+1}\omega_1 + m^2_{a+1}\omega_2, \qquad m^1_{a+1}, m^2_{a+1} \geq 0, \tag{3.64}$$

and at the residue point we have the relation

$$\imath y_{j_b} = \imath x_{i_{a+1}} + n^1\omega_1 + n^2\omega_2, \qquad n^i = p^i - m^i_{a+1}. \tag{3.65}$$

This relation coincides with the hyperplane from Lemma 3, which we apply as follows. By definition (3.43) shift operators $\tau^{i,n^i}_{j_b,i_{a+1}}$ act on the variables y_{j_b}, $y_{i_{a+1}} \equiv x_{i_{a+1}}$ as

$$\tau^{1,n^1}_{j_b,i_{a+1}} \begin{pmatrix} \imath y_{j_b} \\ \imath x_{i_{a+1}} \end{pmatrix} = \begin{pmatrix} \imath y_{j_b} - n^1\omega_1 \\ \imath x_{i_{a+1}} + n^1\omega_1 \end{pmatrix}, \qquad \tau^{2,n^2}_{j_b,i_{a+1}} \begin{pmatrix} \imath y_{j_b} \\ \imath x_{i_{a+1}} \end{pmatrix} = \begin{pmatrix} \imath y_{j_b} - n^2\omega_2 \\ \imath x_{i_{a+1}} + n^2\omega_2 \end{pmatrix}. \tag{3.66}$$

In our case y_{j_b}, $x_{i_{a+1}}$ are integration variables inside the residue integral (3.58). By action of shift operators on the residue integral we assume action on its residue points, for instance

$$\tau^{1,n^1}_{j_b,i_{a+1}} \operatorname{Res}_{\imath x_{i_{a+1}} = \imath x_{i_a} + g + m^1_{a+1}\omega_1 + m^2_{a+1}\omega_2} = \operatorname{Res}_{\imath x_{i_{a+1}} = \imath x_{i_a} + g + (m^1_{a+1} - n^1)\omega_1 + m^2_{a+1}\omega_2}. \tag{3.67}$$

Then, instead of the single residue integral (3.58) consider the sum of four integrals

$$\operatorname{Res}_{\imath x_{i_0} = \imath y_{j_0} + c} \left(1 + \tau^{1,n^1}_{j_b,i_{a+1}}\right)\left(1 + \tau^{2,n^2}_{j_b,i_{a+1}}\right) I_k\left(x_{i_0} | \boldsymbol{i}_k, \boldsymbol{m}_k\right) \times I_l\left(y_{j_0} | \boldsymbol{j}_l, \tilde{\boldsymbol{m}}_l\right). \tag{3.68}$$

Here

$$\begin{aligned} c = (b - a - 1)g &+ (\tilde{m}^1_1 + \ldots + \tilde{m}^1_b - m^1_1 - \ldots - m^1_a - p^1)\omega_1 \\ &+ (\tilde{m}^2_1 + \ldots + \tilde{m}^2_b - m^2_1 - \ldots - m^2_a - p^2)\omega_2, \end{aligned} \tag{3.69}$$

where p^1 and p^2 are given by (3.62). The four residue integrals (3.68) differ by parameters on the edges. At the same time by induction assumption the function G_m (3.35) equals the sum of all possible chain integrals. In particular, it contains direct products $I_k \times I_l$ with all possible non-negative edge parameters

$$\sum_{m^i_j, \tilde{m}^i_j \geq 0} I_k\left(x_{i_0} | \boldsymbol{i}_k, \boldsymbol{m}_k\right) \times I_l\left(y_{j_0} | \boldsymbol{j}_l, \tilde{\boldsymbol{m}}_l\right). \tag{3.70}$$

Parameters on the edges in the shifted integrals from the sum (3.68) could be negative, depending on the integers $\boldsymbol{m}_k$, $\tilde{\boldsymbol{m}}_l$, p^i. However, the chain integral with at least one negative edge parameter equals zero, since, as we noted earlier, the residues in it can be taken in any order and

$$\operatorname{Res}_{\imath y_i = \imath y_j + g + h^1\omega_1 + h^2\omega_2} F(\boldsymbol{y}_n, \boldsymbol{z}_{2n}) = 0, \tag{3.71}$$

unless both $h^i \geq 0$. Thus, all of the four residue integrals in (3.68) either are contained in the sum (3.70) or equal to zero.

To apply Lemma 3 to the sum (3.68), we shift the integration variables in each integral in order to remove the dependence on edge parameters from residue points. For the chain integral $I_k(x_{i_0}|\boldsymbol{i}_k, \boldsymbol{m}_k)$ in (3.68), we define new integration variables $\tilde{x}_{i_s}$ by the following shifts

$$\begin{aligned} \imath x_{i_0} &= \imath \tilde{x}_{i_0} + c, \\ \imath x_{i_1} &= \imath \tilde{x}_{i_1} + g + m_1^1\omega_1 + m_1^2\omega_2 + c, \\ \imath x_{i_2} &= \imath \tilde{x}_{i_2} + 2g + (m_1^1 + m_2^1)\omega_1 + (m_1^2 + m_2^2)\omega_2 + c, \\ &\ \ \vdots \\ \imath x_{i_k} &= \imath \tilde{x}_{i_k} + kg + (m_1^1 + \ldots + m_k^1)\omega_1 + (m_1^2 + \ldots + m_k^2)\omega_2 + c \end{aligned} \tag{3.72}$$

and similarly for all other chain integrals in the sum (3.68). Now only the integrands depend on edge parameters. Denote two tuples

$$\boldsymbol{x}_{k+1} = (x_{i_0}, \ldots, x_{i_k}), \qquad \boldsymbol{y}_{l+1} = (y_{j_0}, \ldots, y_{j_l}) \tag{3.73}$$

with components given by the formulas (3.72) and analogous ones with y_{j_d}. Then, we rewrite (3.68) as one single residue integral

$$\begin{aligned} \operatorname{Res}_{\imath\tilde{x}_{i_0} = \imath\tilde{y}_{j_0}} \prod_{s=1}^{k} \operatorname{Res}_{\imath\tilde{x}_{i_s} = \imath\tilde{x}_{i_{s-1}}} \prod_{s=1}^{l} \operatorname{Res}_{\imath\tilde{y}_{j_s} = \imath\tilde{y}_{j_{s-1}}} \\ \big(1 + \tau^{1,n^1}_{j_b, i_{a+1}}\big)\big(1 + \tau^{2,n^2}_{j_b, i_{a+1}}\big) F(\boldsymbol{x}_{k+1}, \boldsymbol{y}_{l+1}, \ldots). \end{aligned} \tag{3.74}$$

Here by dots we mean all other variables of the integrand. Lemma 3 says that the integrand in the last formula has additional zero of the order two at the hyperplane (3.65). This double zero compensates two possible simple singularities along the hyperplanes

$$\imath y_{j_b} = \imath x_{i_a} + g + p^1\omega_1 + p^2\omega_2, \qquad \text{and} \qquad \imath y_{j_{b-1}} = \imath x_{i_{a+1}} + g + r^1\omega_1 + r^2\omega_2. \tag{3.75}$$

The same statement holds for the singularity (3.63) once $b < k$.

Note also that due to inequalities $p^i \geq 0$, $m^i_{i_{a+1}} \geq 0$, $\operatorname{Re} g \geq 0$ $\operatorname{Re}\omega_i \geq 0$, all the new points the shifted variables $\imath y_{j_b}$ and $\imath x_{i_{a+1}}$ have positive real part once the variable $\imath x_{i_a}$ does have.

Suppose there are no singularities in the residue integral (3.58) except (3.75). Then we cancel this integral applying Lemma 3 in the described way. Next assume there are other singularities besides (3.75). Among all pairs with singularities let y_{j_b}, x_{i_a} be the one with the smallest index a (upper diagonal line). The corresponding singularity is of the type either (3.62) or (3.63). Consider the first case (3.62). Denote

$$r = \min(k - a - 1, l - b), \tag{3.76}$$

see example in Fig. 8.

Then, there are $r+1$ pairs of variables $y_{j_{b+\alpha}}, x_{i_{a+1+\alpha}}$ with $\alpha \in \{0, \ldots, r\}$, for which we have relations

$$\imath y_{j_b} = \imath x_{i_{a+1}} + n^1_{a,b}\omega_1 + n^2_{a,b}\omega_2, \quad \ldots \quad \imath y_{j_{b+r}} = \imath x_{i_{a+1+r}} + n^1_{a+r,b+r}\omega_1 + n^2_{a+r,b+r}\omega_2. \tag{3.77}$$

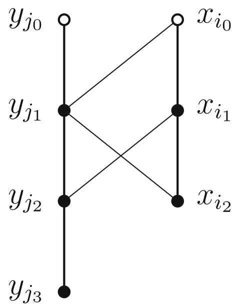


FIGURE 8. Third order pole with $a = 0$, $b = 1$ and $r = 1$

Now we apply Lemma 4 for the intersection of hyperplanes (3.77) repeating the procedure described above for Lemma 3. Let us consider the sum of 4^{r+1} integrals

$$\operatorname{Res}_{\imath x_{i_0}=\imath y_{j_0}+c} T_{kl}\, I_k\left(x_{i_0}|\boldsymbol{i}_k, \boldsymbol{m}_k\right) I_l\left(y_{j_0}|\boldsymbol{j}_l, \tilde{\boldsymbol{m}}_l\right) \tag{3.78}$$

where T_{kl} denotes the sum of shift operators

$$T_{kl} = \prod_{\alpha=0}^{r} \left(1 + \tau_{j_{b+\alpha},i_{a+1+\alpha}}^{1,n^1_{a+\alpha,b+\alpha}}\right)\left(1 + \tau_{j_{b+\alpha},i_{a+1+\alpha}}^{2,n^2_{a+\alpha,b+\alpha}}\right). \tag{3.79}$$

This operator shifts the parameters on the edges of chain integrals I_k, I_l. The arguments above show that all such integrals are either contained in the sum (3.70) or equal to zero. Therefore, we can consider the sum (3.78) instead of the single residue (3.58).

Making linear changes of variables analogous to (3.72), we may regard this sum of integrals as a residue integral

$$\operatorname{Res}_{\imath\tilde{x}_{i_0}=\imath\tilde{y}_{j_0}} \prod_{s=1}^{k} \operatorname{Res}_{\imath\tilde{x}_{i_s}=\imath\tilde{x}_{i_{s-1}}} \prod_{s=1}^{l} \operatorname{Res}_{\imath\tilde{y}_{j_s}=\imath\tilde{y}_{j_{s-1}}} T_{kl}\, F(\boldsymbol{x}_{k+1}, \boldsymbol{y}_{l+1}, \dots), \tag{3.80}$$

where $\boldsymbol{x}_{k+1}$ components are given by (3.72) and similarly for $\boldsymbol{y}_{l+1}$. By Lemma 4 the number of poles in this one-dimensional integral could exceed the number of zeros of integrand by one only in two cases.

I. We have the singularity

$$\imath y_{j_d} = \imath x_{i_k} + g + p^1\omega_1 + p^2\omega_2, \qquad p^1, p^2 \geq 0. \tag{3.81}$$

II. We have the singularity

$$\imath x_{i_d} = \imath y_{j_l} + g + q^1\omega_1 + q^2\omega_2, \qquad q^1, q^2 \geq 0. \tag{3.82}$$

Otherwise the sum (3.78) vanishes. Similarly, for the residue between pair of variables x_{i_a}, y_{j_b} of the second type (3.63) we have vanishing sum, unless the same two cases.

Consider the case **I**. Note that $d > 0$ since otherwise $\operatorname{Im} x_{i_0} > 0$. Define two new chain integrals $I_{k+l-d+1}\left(x_{i_0}|\boldsymbol{i}'_{k+l-d+1}, \boldsymbol{m}'_{k+l-d+1}\right)$ and $I_{d-1}\left(y_{j_0}|\boldsymbol{j}'_{d-1},\right.$

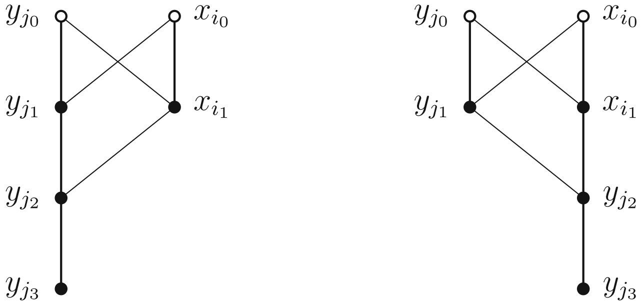


Figure 9. Singularity of the type (3.81) with $d = 2$ and the corresponding pair of new chains from the right

$\tilde{\boldsymbol{m}}'_{d-1})$ as follows

$$\begin{array}{lll} s \le k: & i'_s = i_s, & m'^i_s = m^i_s, \\ s = k+1: & i'_{k+1} = j_d, & m'^i_{k+1} = p^i, \\ s > k+1: & i'_s = j_{d-1+s-k}, & m'^i_s = \tilde{m}^i_{d-1+s-k}, \\ s < d: & j'_s = j_s, & \tilde{m}'^i_s = \tilde{m}^i_s. \end{array} \tag{3.83}$$

This definition is rather simple in terms of pictures, see example in Fig. 9. The following lemma describes cancellation mechanism for such integrals.

Lemma 5.

$$\begin{aligned} &\mathrm{Res}_{\imath x_{i_0} = \imath y_{j_0} + c}\, T_{kl}\, I_k\left(x_{i_0} | \boldsymbol{i}_k, \boldsymbol{m}_k\right) I_l\left(y_{j_0} | \boldsymbol{j}_l, \tilde{\boldsymbol{m}}_l\right) + \\ &\mathrm{Res}_{\imath x_{i_0} = \imath y_{j_0} + c}\, T_{kl}\, I_{k+l-d+1}\left(x_{i_0} | \boldsymbol{i}'_{k+l-d+1}, \boldsymbol{m}'_{k+l-d+1}\right) I_{d-1}\left(y_{j_0} | \boldsymbol{j}'_{d-1}, \tilde{\boldsymbol{m}}'_{d-1}\right) = 0, \end{aligned} \tag{3.84}$$

where the operator T_{kl} is defined in (3.79).

Proof. Indeed, by the arguments above both integrals are simple residue integrals with the same integrand and singularities. In the shifted variables $\tilde{x}_{i_s}$ and $\tilde{y}_{j_s}$, all the singularities are at the diagonals

$$\begin{array}{llll} \tilde{x}_{i_s} = \tilde{x}_{i_{s+1}}, & s = 0, \dots, k-1, & \tilde{y}_{j_s} = \tilde{y}_{j_{s+1}}, & s = 0, \dots, l-1, \\ \tilde{x}_{i_s} = \tilde{y}_{j_{s+d-k}}, & s = a, \dots, k, & \tilde{x}_{i_{s+1}} = \tilde{y}_{j_{s+d-k-1}}, & s = a, \dots, k-1, \end{array} \tag{3.85}$$

where $\tilde{y}_{j_0} = y_{j_0}$. Lemma 4 says that we may present the integrand in the form

$$\frac{\left(\tilde{x}_{i_{a+1}} - \tilde{y}_{j_{a+d-k}}\right)^2 \cdots \left(\tilde{x}_{i_k} - \tilde{y}_{j_{d-1}}\right)^2}{\prod_{s=0}^{k-1}\left(\tilde{x}_{i_s} - \tilde{x}_{i_{s+1}}\right) \prod_{s=0}^{l-1}\left(\tilde{y}_{j_s} - \tilde{y}_{j_{s+1}}\right) \prod_{s=a}^{k}\left(\tilde{x}_{i_s} - \tilde{y}_{j_{s+d-k}}\right) \prod_{s=a}^{k-1}\left(\tilde{x}_{i_{s+1}} - \tilde{y}_{j_{s+d-k-1}}\right)} H\left(\tilde{\boldsymbol{x}}_{k+1}, \tilde{\boldsymbol{y}}_{l+1}\right)$$

where $H\left(\tilde{\boldsymbol{x}}_{k+1}, \tilde{\boldsymbol{y}}_{l+1}\right)$ does not have singularities on integration contour. Replacing each factor $\left(\tilde{x}_{i_{s+1}} - \tilde{y}_{j_{s+d-k}}\right)^2$ of the nominator by

$$\left(\left(\tilde{x}_{i_s} - \tilde{x}_{i_{s-1}}\right) + \left(\tilde{x}_{i_{s-1}} - \tilde{y}_{j_{s+d-k}}\right)\right)\left(\left(\tilde{x}_{i_s} - \tilde{y}_{j_{s+d-k-1}}\right) + \left(\tilde{y}_{j_{s+d-k-1}} - \tilde{y}_{j_{s+d-k}}\right)\right)$$

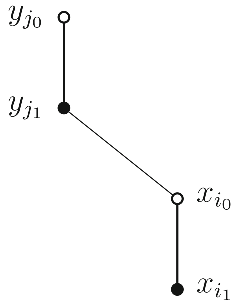


FIGURE 10. Two chain integrals give a new one

we obtain the sum of simple fractions such that the number of factors in denominator of each of them equals the number of integration. One can then also note that only one fraction

$$\frac{1}{\left(\tilde{x}_{i_k}-\tilde{y}_{j_d}\right)\prod_{s=0}^{k-1}\left(\tilde{x}_{i_s}-\tilde{x}_{i_{s+1}}\right)\prod_{s=0}^{l-1}\left(\tilde{y}_{j_s}-\tilde{y}_{j_{s+1}}\right)}H\left(\tilde{\boldsymbol{x}}_{k+1},\tilde{\boldsymbol{y}}_{l+1}\right) \tag{3.86}$$

gives non-trivial contribution to the both integrals (3.84). The corresponding integrals can be computed. The first one equals

$$H\left(\tilde{\boldsymbol{x}}_{k+1},\tilde{\boldsymbol{y}}_{l+1}\right)\big|_{\tilde{x}_{i_s}=\tilde{y}_{j_0},\, s=0,\dots,k;\ \tilde{y}_{j_s}=\tilde{y}_{j_0},\, s=1,\dots,l},$$

the second to

$$-H\left(\tilde{\boldsymbol{x}}_{k+1},\tilde{\boldsymbol{y}}_{l+1}\right)\big|_{\tilde{x}_{i_s}=\tilde{y}_{j_0},\, s=0,\dots,k;\ \tilde{y}_{j_s}=\tilde{y}_{j_0},\, s=1,\dots,l},$$

so that their sum equals zero. □

The same arguments hold for fusions of type **II** unless $d \neq 0$. For $d = 0$ the corresponding fusion of chain integrals is by definition the chain integral

$$\frac{1}{-2\pi\imath}I_{k+l+1}\left(y_{i_0}|\boldsymbol{i}'_{k+l+1},\boldsymbol{m}'_{k+l+1}\right) \tag{3.87}$$

where

$$\begin{array}{lll} s\le l: & i'_s=j_s, & {m'}^i_s=\tilde{m}^i_s,\\ s=l+1: & i'_{l+1}=i_0, & {m'}^i_{l+1}=q^i,\\ s>l+1: & i'_s=i_{s-l-1}, & {m'}^i_s=m^i_{s-l-1}. \end{array}$$

Here, the integers q^i are given by the relation (3.82). See example in Fig. 10.

This ends the consideration of fusions of chains integrals (3.58). All of them cancel, except new chains (3.87).

Consider now the fusion (3.59)

$$\mathrm{Res}_{\imath x_{i_0}=\imath z_j+c}\, I_k\left(y_{i_0}|\boldsymbol{i}_k,\boldsymbol{m}_k\right)J_1(z_j|b,\tilde{m}^1,\tilde{m}^2). \tag{3.88}$$

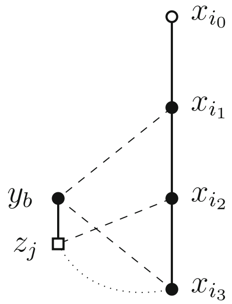


FIGURE 11. Singularities (3.89), (3.90), (3.91) are shown with dashed lines ($a = 2$). Dotted line shows a possible zero (3.92)

Here we again change symbols y to x in the chain I_k. There could appear such singularities, labeled by a fixed number a:

$$\imath x_{i_a} = \imath z_j + \frac{g^*}{2} + p^1\omega_1 + p^2\omega_2, \qquad p^1, p^2 \geq 0, \tag{3.89}$$

$$\imath x_{i_{a+1}} = \imath y_b + g + q^1\omega_1 + q^2\omega_2, \qquad q^1, q^2 \geq 0, \tag{3.90}$$

$$\imath y_b = \imath x_{i_{a-1}} + g + r^1\omega_1 + r^2\omega_2, \qquad r^1, r^2 \geq 0. \tag{3.91}$$

See example in Fig. 11. Once there is a pole (3.90), there is a zero at the point

$$\imath x_{i_{a+1}} = \imath z_j - \frac{g^*}{2} + s^1\omega_1 + s^2\omega_2, \qquad s^i = \tilde{m}^i + q^i + 1 > 0 \tag{3.92}$$

given by the function $S^{-1}(\imath x_{i_{a+1}} - \imath z_j + g^*/2)$. Once we have a singularity (3.89) or (3.91), we are in the position of Lemma 3 with respect to the variables y_b and x_{i_a} and consider instead of the residue (3.88) the corresponding sum of four integrals with shift operators, for which we have additional zero of the second order.

Analyzing the balance of poles and zeros, we conclude that the residue (3.88) could be non-trivial only when $a = k + 1$, so that the variables x_{i_a} and $x_{i_{a+1}}$ are missing and the residue (3.88) is taken along the only singularity

$$\imath y_b = \imath x_{i_k} + g + r^1\omega_1 + r^2\omega_2, \qquad r^1, r^2 \geq 0. \tag{3.93}$$

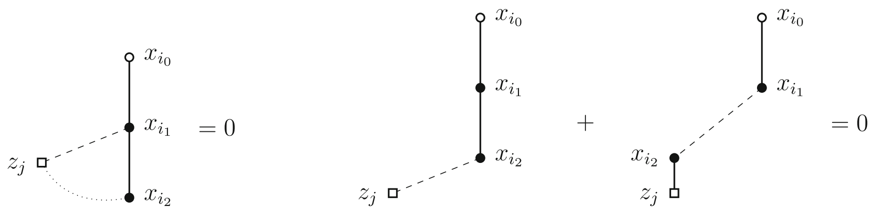
Finally, consider the residue (3.60). In the calculation

$$\operatorname{Res}_{\imath x_{i_0} = \imath z_j + c}\, I_k\left(y_{i_0} | \boldsymbol{i}_k, \boldsymbol{m}_k\right)$$

we could meet only one pole of the form (3.89) together with zero (3.92) if the variable $x_{i_{a+1}}$ exists, see Fig. 12. Thus, the only non-trivial result could be only when $k = a$ and the residue is taken along the singularity

$$\imath x_{i_k} = \imath z_j + \frac{g^*}{2} + p^1\omega_1 + p^2\omega_2, \qquad p^1, p^2 \geq 0. \tag{3.94}$$

One can note that the resulting integrals obtained in the calculations of the fusion integrals (3.59) and (3.60) coincide up to a sign and cancel each other except for the case $k = 0$ in (3.60), see Fig. 12.



(a) Pole and zero cancel each other ($a \neq k$)

(b) Residues of the type (3.59) and (3.60) cancel each other

FIGURE 12. Cancellation of residues

But for $k = 0$ this gives precisely the new chain $J_1\big(z_j|i_0, p^1, p^2\big)$. For their direct products $J_1(z_a|b, m^1, m^2) \times J_1(z_c|d, n^1, n^2)$ the indices a and c could not coincide since otherwise we can apply Lemma 3 to the variables y_b and y_d and cancel the sum of four corresponding terms.

This ends the proof of Proposition 1. □

3.3. Simple Poles Calculations

In the previous subsections, we proved that

1. Both integrals in (3.5) can be calculated by residues, moving the contours of integration either to the lower or to the upper half planes depending on the sign of $\operatorname{Re}\lambda$;
2. In residue calculations only direct products of simple poles $J_1\big(z_a|b, m^1, m^2\big)$ (3.38) with distinct z_a contribute to the integrals, see Corollary 3 to Proposition 1.

Now consider the integral $Q_n(\boldsymbol{z}_{2n};\lambda)$ in the left hand side of (3.5). By Corollary 3, it equals the sum of all possible direct products

$$J_1\big(z_{a_1}|b_1, m_1^1, m_1^2\big) \times \ldots \times J_1\big(z_{a_n}|b_n, m_n^1, m_n^2\big) \tag{3.95}$$

with distinct indices $a_1, \ldots, a_n$. These indices form a subset $I \subset [2n] = \{1, \ldots, 2n\}$ of cardinality n. Collecting the direct products with a given choice of the set I we arrive at the sum of $\binom{2n}{n}$ series

$$Q_n(\boldsymbol{z}_{2n};\lambda) = \sum_{\substack{I\subset[2n]\\|I|=n}} e^{2\pi\lambda\left(\frac{ng^*}{2}+\imath\sum\limits_{i\in I} z_i\right)} L^I(u,v) \tag{3.96}$$

over

$$u = e^{2\pi\lambda\omega_1} \qquad \text{and} \qquad v = e^{2\pi\lambda\omega_2}. \tag{3.97}$$

The series converges for sufficiently big negative $\operatorname{Re}\lambda$. More precisely, denote two sequences of non-negative integers

$$\boldsymbol{m}^1 = (m_1^1, \ldots, m_n^1), \qquad \boldsymbol{m}^2 = (m_1^2, \ldots, m_n^2) \tag{3.98}$$

and the sum of their components in a standard way

$$|\boldsymbol{m}^i| = m_1^i + \ldots + m_n^i. \tag{3.99}$$

Then, the function $L^I(u,v)$ equals

$$L^I(u,v) = n!\,(-2\pi\imath)^n \sum_{M,K\geq 0} L^I_{M,K} u^M v^K \tag{3.100}$$

where $L^I_{M,K}$ is the following sum of multiple residues

$$L^I_{M,K} = \sum_{\substack{m_i^1\geq 0:\ |\boldsymbol{m}^1|=M\\ m_i^2\geq 0:\ |\boldsymbol{m}^2|=K}} L^I_{\boldsymbol{m}^1,\boldsymbol{m}^2} \tag{3.101}$$

with

$$\begin{aligned} L^I_{\boldsymbol{m}^1,\boldsymbol{m}^2} &= \operatorname{Res}_{\imath y_1=\imath z_{i_1}+\frac{g^*}{2}+m_1^1\omega_1+m_1^2\omega_2}\cdots \\ &\cdots \operatorname{Res}_{\imath y_n=\imath z_{i_n}+\frac{g^*}{2}+m_n^1\omega_1+m_n^2\omega_2}\ \mu(\boldsymbol{y}_n)\prod_{a=1}^{2n}\prod_{j=1}^{n}K(y_j-z_a). \end{aligned} \tag{3.102}$$

In the same way, we compute the integral in the right hand side of (3.5), but moving the integration contours to the upper half plane. Again it is expressed via the sum of chain integrals with different sign due to the opposite orientation of the contours. Collecting the terms where the indices a_j of the parameters z_{a_j} belong to a given subset $J\subset[2n]$ of cardinality n we get a sum of $\binom{2n}{n}$ series

$$Q_n(\boldsymbol{z}_{2n};-\lambda) = \sum_{\substack{J\subset[2n]\\ |J|=n}} e^{-2\pi\lambda\left(-\frac{ng^*}{2}+\imath\sum\limits_{i\in J} z_i\right)} R^J(u,v) \tag{3.103}$$

over the same variables u and v defined in (3.97). Here

$$R^J(u,v) = n!\,(2\pi\imath)^n \sum_{M,K\geq 0} R^J_{M,K} u^M v^K \tag{3.104}$$

where $R^J_{M,K}$ is the following sum of multiple residues

$$R^J_{M,K} = \sum_{\substack{m_i^1\geq 0:\ |\boldsymbol{m}^1|=M\\ m_i^2\geq 0:\ |\boldsymbol{m}^2|=K}} R^J_{\boldsymbol{m}^1,\boldsymbol{m}^2} \tag{3.105}$$

with

$$\begin{aligned} R^J_{\boldsymbol{m}^1,\boldsymbol{m}^2} &= \operatorname{Res}_{\imath y_1=\imath z_{j_1}-\frac{g^*}{2}-m_1^1\omega_1-m_1^2\omega_2}\cdots \\ &\cdots \operatorname{Res}_{\imath y_n=\imath z_{j_n}-\frac{g^*}{2}-m_n^1\omega_1-m_n^2\omega_2}\ \mu(\boldsymbol{y}_n)\prod_{a=1}^{2n}\prod_{j=1}^{n}K(y_j-z_a). \end{aligned} \tag{3.106}$$

For any subset $I\subset[2n]$ of cardinality n denote by $\bar{I}$ the complement

$$\bar{I} = [2n]\setminus I$$

of I in the set $[2n]$.

Proposition 2. *For any $I \subset [2n]$, $|I| = n$, we have the equality of series*

$$L^I(u,v) = (-1)^n R^{\bar{I}}(u,v) \tag{3.107}$$

Equivalently,

$$L^I_{M,K} = (-1)^n R^{\bar{I}}_{M,K} \qquad \textit{for any} \qquad M,K \geq 0. \tag{3.108}$$

Proposition 2 immediately implies the equality (3.5) due to (3.96) and (3.103). Thus, it also implies the commutativity of Q-operators (3.1).

Proof of Proposition 2. It is clear by symmetry arguments, that it is sufficient to prove the equalities (3.108) for the set $I_0 = \{1,\ldots,n\}$. For the sake of convenience below we change the notation for variables z_a in the following way

$$\imath z_a \to z_a, \qquad a = 1,\ldots,n, \qquad \imath z_{n+i} \to x_i, \qquad i = 1,\ldots,n. \tag{3.109}$$

Using (A.14), we get the precise value of the multiple residue (3.102),

$$\begin{aligned} L^{I_0}_{\boldsymbol{m}^1,\boldsymbol{m}^2} =& \frac{(\omega_1\omega_2)^{n/2}}{(-2\pi\imath)^n} \prod_{a=1}^n \frac{(-1)^{m_a^1 m_a^2 + m_a^1 + m_a^2}\, S_2^{-1}(g^* + m_a^1\omega_1 + m_a^2\omega_2)}{\prod_{j=1}^{m_a^1} 2\sin\frac{\pi j\omega_1}{\omega_2} \prod_{l=1}^{m_a^2} 2\sin\frac{\pi l\omega_2}{\omega_1}} \\ & \times \prod_{\substack{a,b=1\\a\neq b}}^n \Big[S_2(z_a - z_b + (m_a^1 - m_b^1)\omega_1 + (m_a^2 - m_b^2)\omega_2) \\ & S_2(z_a - z_b + g^* + (m_a^1 - m_b^1)\omega_1 + (m_a^2 - m_b^2)\omega_2) \\ & S_2^{-1}(z_a - z_b + g^* + m_a^1\omega_1 + m_a^2\omega_2)\, S_2^{-1}(z_a - z_b - m_b^1\omega_1 - m_b^2\omega_2)\Big] \\ & \times \prod_{i,a=1}^n S_2^{-1}(z_a - x_i + g^* + m_a^1\omega_1 + m_a^2\omega_2) \\ & S_2^{-1}(x_i - z_a - m_a^1\omega_1 - m_a^2\omega_2). \end{aligned} \tag{3.110}$$

For a variable x and integer m,k denote by $[x|\boldsymbol{\omega}]_{m,k}$ the hyperbolic analog of the Pochhammer symbol:

$$[x|\boldsymbol{\omega}]_{m,k} = (-1)^{mk} \frac{S_2(x)}{S_2(x + m\omega_1 + k\omega_2)}. \tag{3.111}$$

For non-negative m and k

$$[x|\boldsymbol{\omega}]_{m,k} = \prod_{s=0}^{m-1} 2\sin\pi\frac{x + s\omega_1}{\omega_2} \prod_{l=0}^{k-1} 2\sin\pi\frac{x + l\omega_2}{\omega_1}. \tag{3.112}$$

In these notations, the expression (3.110) can be rewritten as follows

$$\begin{aligned} L^{I_0}_{\boldsymbol{m}^1,\boldsymbol{m}^2} =& \frac{(\omega_1\omega_2)^{n/2}}{(-2\pi\imath S_2(g^*))^n} \prod_{i,a=1}^n S_2^{-1}(z_a - x_i + g^*)\, S_2^{-1}(x_i - z_a) \\ & \times \prod_{a=1}^n \frac{[g^*|\boldsymbol{\omega}]_{m_a^1,m_a^2}}{[\omega_1 + \omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}} \end{aligned}$$

$$\times \prod_{a\neq b} \frac{[z_a - z_b + g^* + (m_a^1 - m_b^1)\omega_1 + (m_a^2 - m_b^2)\omega_2|\boldsymbol{\omega}]_{m_b^1,m_b^2}}{[z_a - z_b - m_b^1\omega_1 - m_b^2\omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}}$$
$$\times \prod_{i,a=1}^{n} \frac{[z_a - x_i + g^*|\boldsymbol{\omega}]_{m_a^1,m_a^2}}{[x_i - z_a - m_a^1\omega_1 - m_a^2\omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}}. \tag{3.113}$$

The inversion formula (A.6) implies the following symmetry of the hyperbolic Pochhammer symbol:

$$[x - m\omega_1 - k\omega_2|\boldsymbol{\omega}]_{m,k} = [\omega_1 + \omega_2 - x|\boldsymbol{\omega}]_{m,k}. \tag{3.114}$$

With its use we simplify the relation (3.113) as follows

$$L^{I_0}_{\boldsymbol{m}^1,\boldsymbol{m}^2} = \frac{(\omega_1\omega_2)^{n/2}}{(-2\pi\imath S_2(g^*))^n} \prod_{i,a=1}^{n} S_2^{-1}(z_a - x_i + g^*)\, S_2^{-1}(x_i - z_a)$$
$$\times \prod_{a=1}^{n} \frac{[g^*|\boldsymbol{\omega}]_{m_a^1,m_a^2}}{[\omega_1 + \omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}} \prod_{a\neq b} \frac{[z_a - z_b + g - m_b^1\omega_1 - m_b^2\omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}}{[z_a - z_b - m_b^1\omega_1 - m_b^2\omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}}$$
$$\times \prod_{i,a=1}^{n} \frac{[z_a - x_i + g^*|\boldsymbol{\omega}]_{m_a^1,m_a^2}}{[z_a - x_i + \omega_1 + \omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}}. \tag{3.115}$$

Analogous calculations for the multiple residue (3.106) for $J = \bar{I}_0$ give

$$R^{\bar{I}_0}_{\boldsymbol{m}^1,\boldsymbol{m}^2} = \frac{(\omega_1\omega_2)^{n/2}}{(2\pi\imath)^n} \prod_{i=1}^{n} \frac{(-1)^{m_i^1 m_i^2 + m_i^1 + m_i^2}\, S_2^{-1}(g^* + m_i^1\omega_1 + m_i^2\omega_2)}{\prod_{j=1}^{m_i^1} 2\sin\frac{\pi j\omega_1}{\omega_2} \prod_{l=1}^{m_i^2} 2\sin\frac{\pi l\omega_2}{\omega_1}}$$
$$\times \prod_{\substack{i,j=1\\ i\neq j}}^{n} \Big[S_2(x_i - x_j + (m_j^1 - m_i^1)\omega_1 + (m_j^2 - m_i^2)\omega_2)$$
$$S_2(x_i - x_j + g^* + (m_j^1 - m_i^1)\omega_1 + (m_j^2 - m_i^2)\omega_2)$$
$$S_2^{-1}(x_i - x_j + g^* + m_j^1\omega_1 + m_j^2\omega_2)\, S_2^{-1}(x_i - x_j - m_i^1\omega_1 - m_i^2\omega_2)\Big]$$
$$\times \prod_{i,a=1}^{n} S_2^{-1}(z_a - x_i + g^* + m_i^1\omega_1 + m_i^2\omega_2)$$
$$S_2^{-1}(x_i - z_a - m_i^1\omega_1 - m_i^2\omega_2), \tag{3.116}$$

so that

$$R^{\bar{I}_0}_{\boldsymbol{m}^1,\boldsymbol{m}^2} = \frac{(\omega_1\omega_2)^{n/2}}{(2\pi\imath S(g^*))^n} \prod_{i,a=1}^{n} S^{-1}(z_a - x_i + g^*) S^{-1}(x_i - z_a)$$
$$\times \prod_{i=1}^{n} \frac{[g^*|\boldsymbol{\omega}]_{m_i^1,m_i^2}}{[\omega_1 + \omega_2|\boldsymbol{\omega}]_{m_i^1,m_i^2}} \prod_{i\neq j} \frac{[x_i - x_j + g - m_i^1\omega_1 - m_i^2\omega_2|\boldsymbol{\omega}]_{m_j^1,m_j^2}}{[x_i - x_j - m_i^1\omega_1 - m_i^2\omega_2|\boldsymbol{\omega}]_{m_j^1,m_j^2}}$$
$$\times \prod_{i,a=1}^{n} \frac{[z_a - x_i + g^*|\boldsymbol{\omega}]_{m_i^1,m_i^2}}{[z_a - x_i + \omega_1 + \omega_2|\boldsymbol{\omega}]_{m_i^1,m_i^2}}. \tag{3.117}$$

Comparing (3.115) and (3.117), we see that the equality (3.108) is equivalent to the relation

$$\sum_{\substack{|\boldsymbol{m}^1|=M\\|\boldsymbol{m}^2|=K}} \prod_{a=1}^{n} \frac{[g^*|\boldsymbol{\omega}]_{m_a^1,m_a^2}}{[\omega_1+\omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}} \prod_{a\neq b} \frac{[z_a-z_b+g-m_b^1\omega_1-m_b^2\omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}}{[z_a-z_b-m_b^1\omega_1-m_b^2\omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}}$$
$$\times \prod_{i,a=1}^{n} \frac{[z_a-x_i+g^*|\boldsymbol{\omega}]_{m_a^1,m_a^2}}{[z_a-x_i+\omega_1+\omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}}$$
$$= \sum_{\substack{|\boldsymbol{m}^1|=M\\|\boldsymbol{m}^2|=K}} \prod_{i=1}^{n} \frac{[g^*|\boldsymbol{\omega}]_{m_i^1,m_i^2}}{[\omega_1+\omega_2|\boldsymbol{\omega}]_{m_i^1,m_i^2}} \prod_{i\neq j} \frac{[x_i-x_j+g-m_i^1\omega_1-m_i^2\omega_2|\boldsymbol{\omega}]_{m_j^1,m_j^2}}{[x_i-x_j-m_i^1\omega_1-m_i^2\omega_2|\boldsymbol{\omega}]_{m_j^1,m_j^2}}$$
$$\times \prod_{i,a=1}^{n} \frac{[z_a-x_i+g^*|\boldsymbol{\omega}]_{m_i^1,m_i^2}}{[z_a-x_i+\omega_1+\omega_2|\boldsymbol{\omega}]_{m_i^1,m_i^2}}. \tag{3.118}$$

Here, the sums in both sides of the relation are taken over two sequences $\boldsymbol{m}^1$ and $\boldsymbol{m}^2$ (3.98) of non-negative integers with their fixed sums equal to M and K,

$$m_i^j \geq 0, \qquad |\boldsymbol{m}^1| = \sum_{i=1}^{n} m_i^1 = M, \qquad |\boldsymbol{m}^2| = \sum_{i=1}^{n} m_i^2 = K. \tag{3.119}$$

Make the change of variables

$$x_i \mapsto x_i + \omega_1 + \omega_2. \tag{3.120}$$

Then, the relation (3.118) looks as

$$\sum_{\substack{|\boldsymbol{m}^1|=M\\|\boldsymbol{m}^2|=K}} \prod_{a=1}^{n} \frac{[\omega_1+\omega_2-g|\boldsymbol{\omega}]_{m_a^1,m_a^2}}{[\omega_1+\omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}}$$
$$\prod_{\substack{a,b=1\\a\neq b}}^{n} \frac{[z_a-z_b+g-m_b^1\omega_1-m_b^2\omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}}{[z_a-z_b-m_b^1\omega_1-m_b^2\omega_2|\boldsymbol{\omega}]_{m_a^1,m_a^2}}$$
$$\times \prod_{i,a=1}^{n} \frac{[z_a-x_i-g|\boldsymbol{\omega}]_{m_a^1,m_a^2}}{[z_a-x_i|\boldsymbol{\omega}]_{m_a^1,m_a^2}}$$
$$= \sum_{\substack{|\boldsymbol{m}^1|=M\\|\boldsymbol{m}^2|=K}} \prod_{i=1}^{n} \frac{[\omega_1+\omega_2-g|\boldsymbol{\omega}]_{m_i^1,m_i^2}}{[\omega_1+\omega_2|\boldsymbol{\omega}]_{m_i^1,m_i^2}}$$
$$\prod_{\substack{i,j=1\\i\neq j}}^{n} \frac{[x_i-x_j+g-m_i^1\omega_1-m_i^2\omega_2|\boldsymbol{\omega}]_{m_j^1,m_j^2}}{[x_i-x_j-m_i^1\omega_1-m_i^2\omega_2|\boldsymbol{\omega}]_{m_j^1,m_j^2}}$$

$$\times \prod_{i,a=1}^{n} \frac{[z_a - x_i - g|\boldsymbol{\omega}]_{m_i^1, m_i^2}}{[z_a - x_i|\boldsymbol{\omega}]_{m_i^1, m_i^2}}. \tag{3.121}$$

The factorization formula (A.10) is equivalent to the factorization of the hyperbolic Pochhammer symbol:

$$[x|\boldsymbol{\omega}]_{m,k} = [x|\omega_1]_m \cdot [x|\omega_2]_k. \tag{3.122}$$

Here

$$[x|\omega_1]_m = \frac{S_2(x)}{S_2(x + m\omega_1)}, \qquad [x|\omega_2]_k = \frac{S_2(x)}{S_2(x + k\omega_2)}. \tag{3.123}$$

For non-negative m and k

$$[x|\omega_1]_m = \prod_{i=0}^{m-1} 2\sin\pi\frac{x + i\omega_1}{\omega_2}, \qquad [x|\omega_2]_k = \prod_{j=0}^{k-1} 2\sin\pi\frac{x + j\omega_2}{\omega_1}. \tag{3.124}$$

By using (3.122) and canceling the appearing sings, we can factorize each ratio in (3.121) into the product over periods:

$$\frac{[\omega_1 + \omega_2 - g|\boldsymbol{\omega}]_{m_a^1, m_a^2}}{[\omega_1 + \omega_2|\boldsymbol{\omega}]_{m_a^1, m_a^2}} = \frac{[\omega_1 - g|\omega_1]_{m_a^1}}{[\omega_1|\omega_1]_{m_a^1}} \times \frac{[\omega_2 - g|\omega_2]_{m_a^1}}{[\omega_2|\omega_1]_{m_a^2}},$$

$$\frac{[z_a - z_b + g - m_b^1\omega_1 - m_b^2\omega_2|\boldsymbol{\omega}]_{m_a^1, m_a^2}}{[z_a - z_b - m_b^1\omega_1 - m_b^2\omega_2|\boldsymbol{\omega}]_{m_a^1, m_a^2}} = \frac{[z_a - z_b + g - m_b^1\omega_1|\omega_1]_{m_a^1}}{[z_a - z_b - m_b^1\omega_1|\omega_1]_{m_a^1}}$$

$$\times \frac{[z_a - z_b + g - m_b^2\omega_2|\omega_2]_{m_a^2}}{[z_a - z_b - m_b^2\omega_2|\omega_2]_{m_a^2}},$$

$$\frac{[z_a - x_i - g|\boldsymbol{\omega}]_{m_a^1, m_a^2}}{[z_a - x_i|\boldsymbol{\omega}]_{m_a^1, m_a^2}} = \frac{[z_a - x_i - g|\omega_1]_{m_a^1}}{[z_a - x_i|\omega_1]_{m_a^1}} \times \frac{[z_a - x_i - g|\omega_2]_{m_a^2}}{[z_a - x_i|\omega_2]_{m_a^2}}.$$

Thus, the relation (3.121) decouples into two independent identities

$$\sum_{|\boldsymbol{m}^1| = M} \prod_{a=1}^{n} \frac{[\omega_1 - g|\omega_1]_{m_a^1}}{[\omega_1|\omega_1]_{m_a^1}} \prod_{\substack{a,b=1 \\ a \neq b}}^{n} \frac{[z_a - z_b + g - m_b^1\omega_1|\omega_1]_{m_a^1}}{[z_a - z_b - m_b^1\omega_1|\omega_1]_{m_a^1}}$$
$$\prod_{i,a=1}^{n} \frac{[z_a - x_i - g|\omega_1]_{m_a^1}}{[z_a - x_i|\omega_1]_{m_a^1}}$$
$$= \sum_{|\boldsymbol{m}^1| = M} \prod_{i=1}^{n} \frac{[\omega_1 - g|\omega_1]_{m_i^1}}{[\omega_1|\omega_1]_{m_i^1}} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{[x_i - x_j + g - m_i^1\omega_1|\omega_1]_{m_j^1}}{[x_i - x_j - m_i^1\omega_1|\omega_1]_{m_j^1}}$$
$$\prod_{i,a=1}^{n} \frac{[z_a - x_i - g|\omega_1]_{m_i^1}}{[z_a - x_i|\omega_1]_{m_i^1}} \tag{3.125}$$

and

$$\sum_{|\boldsymbol{m}^2| = K} \prod_{a=1}^{n} \frac{[\omega_2 - g|\omega_2]_{m_a^2}}{[\omega_2|\omega_2]_{m_a^2}} \prod_{\substack{a,b=1 \\ a \neq b}}^{n} \frac{[z_a - z_b + g - m_b^2\omega_2|\omega_2]_{m_a^2}}{[z_a - z_b - m_b^2\omega_2|\omega_2]_{m_a^2}}$$

$$\prod_{i,a=1}^{n} \frac{[z_a - x_i - g|\omega_2]_{m_a^2}}{[z_a - x_i|\omega_2]_{m_a^2}}$$
$$= \sum_{|\boldsymbol{m}^2|=K} \prod_{i=1}^{n} \frac{[\omega_2 - g|\omega_2]_{m_i^2}}{[\omega_2|\omega_2]_{m_i^2}} \prod_{\substack{i,j=1\\ i\neq j}}^{n} \frac{[x_i - x_j + g - m_i^2\omega_2|\omega_2]_{m_j^2}}{[x_i - x_j - m_i^2\omega_2|\omega_2]_{m_j^2}}$$
$$\prod_{i,a=1}^{n} \frac{[z_a - x_i - g|\omega_2]_{m_i^2}}{[z_a - x_i|\omega_2]_{m_i^2}}. \tag{3.126}$$

These are precisely hypergeometric identities (1.32) written in additive form. Their proof is given in the next section. Using it we complete the proof of Proposition 2 and of the main statement of commutativity of Baxter Q-operators. □

4. Proof of Hypergeometric Identities

The relations (3.125) and (3.126) are equivalent modulo the interchange of the periods. We choose (3.126). Rewrite it in the common multiplicative notations of basic hypergeometry. Set

$$q = e^{\frac{2\pi\imath\omega_2}{\omega_1}}, \qquad t = e^{\frac{-2\pi\imath g}{\omega_1}}, \qquad u_i = e^{\frac{2\pi\imath z_i}{\omega_1}}, \qquad v_a = e^{\frac{2\pi\imath x_a}{\omega_1}}. \tag{4.1}$$

Denote by $(z;q)_k$ and $[z;q]_k$ non-symmetric and symmetric q-analogs of Pochhammer symbols,

$$\begin{aligned}(z;q)_k &= (1-z)(1-qz)\cdots(1-q^{k-1}z),\\ [z;q]_k &= (z^{1/2}-z^{-1/2})(q^{1/2}z^{1/2}-q^{-1/2}z^{-1/2})\\ &\qquad \cdots(q^{(k-1)/2}z^{1/2}-q^{(-k+1)/2}z^{-1/2}).\end{aligned} \tag{4.2}$$

Then, (3.126) becomes

$$\sum_{|\boldsymbol{k}|=K} \prod_{i=1}^{n} \frac{[qt;q]_{k_i}}{[q;q]_{k_i}} \times \prod_{\substack{i,j=1\\ i\neq j}}^{n} \frac{[t^{-1}q^{-k_j}u_i/u_j;q]_{k_i}}{[q^{-k_j}u_i/u_j;q]_{k_i}} \times \prod_{a,j=1}^{n} \frac{[tu_j/v_a;q]_{k_j}}{[u_j/v_a;q]_{k_j}}$$
$$= \sum_{|\boldsymbol{k}|=K} \prod_{a=1}^{n} \frac{[qt;q]_{k_a}}{[q;q]_{k_a}} \times \prod_{\substack{a,b=1\\ a\neq b}}^{n} \frac{[t^{-1}q^{-k_a}v_a/v_b;q]_{k_b}}{[q^{-k_a}v_a/v_b;q]_{k_b}} \times \prod_{a,j=1}^{n} \frac{[tu_j/v_a;q]_{k_a}}{[u_j/v_a;q]_{k_a}} \tag{4.3}$$

in terms of symmetric q-Pochhammers. Here, the sum in both sides of the equality is taken over n-tuples of non-negative integers with total sum equal to K

$$\boldsymbol{k} = (k_1, \ldots, k_n), \qquad k_i \geq 0, \qquad k_1 + \ldots + k_n = K.$$

It has the same form in terms of traditional non-symmetric q-Pochhammer symbols:

$$\sum_{|\boldsymbol{k}_n|=K} \prod_{i=1}^{n} \frac{(qt;q)_{k_i}}{(q;q)_{k_i}} \times \prod_{\substack{i,j=1\\ i\neq j}}^{n} \frac{(t^{-1}q^{-k_j}u_i/u_j;q)_{k_i}}{(q^{-k_j}u_i/u_j;q)_{k_i}} \times \prod_{a,j=1}^{n} \frac{(tu_j/v_a;q)_{k_j}}{(u_j/v_a;q)_{k_j}}$$
$$= \sum_{|\boldsymbol{k}_n|=K} \prod_{a=1}^{n} \frac{(qt;q)_{k_a}}{(q;q)_{k_a}} \times \prod_{\substack{a,b=1\\ a\neq b}}^{n} \frac{(t^{-1}q^{-k_a}v_a/v_b;q)_{k_b}}{(q^{-k_a}v_a/v_b;q)_{k_b}} \times \prod_{a,j=1}^{n} \frac{(tu_j/v_a;q)_{k_a}}{(u_j/v_a;q)_{k_a}}. \tag{4.4}$$

However, it is more convenient for us to prove the symmetric version of identity (4.3).

The proof follows the standard line of complex analysis: in a rather tricky way we check that the difference of the left and right hand sides has zero residues at all possible simple poles. Thus, both sides are the Laurent polynomials symmetric over the variables u_i and over the variables v_j. Then, the asymptotic analysis of these polynomials shows that their difference is actually equal to zero.

The crucial step—calculation of the residues of both sides of the equality—divides into two parts. First we show that each side is regular at the diagonals $u_i = q^p u_j$ and $v_a = q^s v_b$ between the variables of the same group, see Lemma 6. In this calculation, we actually observe the canceling of terms grouped in corresponding pairs. Then, we show that residues at mixed diagonals $u_i = q^p v_a$ vanish. This is done by induction, using the non-trivial relation between such residues stated in Lemma 7. Below we give a brief proof of both lemmas; all technical details are presented in our paper [4].

It is not difficult to verify that all the poles in (4.3) are simple. Consider the left hand side of (4.3) as the function of u_1 and calculate the residue of this function at the point

$$u_1 = u_2 q^p, \qquad p \in \mathbb{Z}. \tag{4.5}$$

For each $\boldsymbol{k}$, $\sum_{j=1}^{n} k_j = K$ denote by $\mathrm{U}_{\boldsymbol{k}} = \mathrm{U}_{\boldsymbol{k}}(\boldsymbol{u};\boldsymbol{v})$ the corresponding summand of the left hand side of (4.3), and by $\mathrm{V}_{\boldsymbol{k}} = \mathrm{V}_{\boldsymbol{k}}(\boldsymbol{u};\boldsymbol{v})$ the corresponding summand of the right hand side of (4.3),

$$\mathrm{U}_{\boldsymbol{k}} = \prod_{i=1}^{n} \frac{[qt;q]_{k_i}}{[q;q]_{k_i}} \times \prod_{\substack{i,j=1\\ i\neq j}}^{n} \frac{[t^{-1}q^{-k_j}u_i/u_j;q]_{k_i}}{[q^{-k_j}u_i/u_j;q]_{k_i}} \times \prod_{a,j=1}^{n} \frac{[tu_j/v_a;q]_{k_j}}{[u_j/v_a;q]_{k_j}},$$

$$\mathrm{V}_{\boldsymbol{k}} = \prod_{a=1}^{n} \frac{[qt;q]_{k_a}}{[q;q]_{k_a}} \times \prod_{\substack{a,b=1\\ a\neq b}}^{n} \frac{[t^{-1}q^{-k_a}v_a/v_b;q]_{k_b}}{[q^{-k_a}v_a/v_b;q]_{k_b}} \times \prod_{a,j=1}^{n} \frac{[tu_j/v_a;q]_{k_a}}{[u_j/v_a;q]_{k_a}}.$$

The summands $\mathrm{U}_{\boldsymbol{k}}$, which contribute to the residue at the point (4.5), are divided into two groups. The denominators of the terms $\mathrm{U}_{\boldsymbol{k}}$ from the first group $\boldsymbol{k} \in I_p$ contain Pochhammer symbol

$$[q^{-k_2}u_1/u_2;q]_{k_1}$$

which vanishes at the point (4.5). It happens when

$$k_2 - k_1 + 1 \le p \le k_2,$$

so that

$$I_p = \{\boldsymbol{k},\ |\boldsymbol{k}| = K\colon k_1 \ge k_2 + 1 - p,\ \ k_2 \ge p\}.$$

The denominators of the terms $\mathrm{U}_{\boldsymbol{l}}$ in the second group $\boldsymbol{l} \in II_p$ contain Pochhammer

$$[q^{-l_1} u_2/u_1; q]_{l_2}$$

which vanishes at the point (4.5). It happens when

$$-l_1 \le p \le l_2 - l_1 - 1,$$

so that

$$II_p = \{\boldsymbol{l},\ |\boldsymbol{l}| = K\colon l_1 \ge -p, l_2 \ge l_1 + 1 + p\}.$$

Define the maps of sets $\varphi_p : I_p \to II_p$ and $\psi_p : II_p \to I_p$ by the same formulas

$$\begin{aligned} &\phi_p\colon I_p \to II_p \qquad \phi_p(k_1, k_2, \boldsymbol{k}') = (k_2 - p, k_1 + p, \boldsymbol{k}'),\\ &\psi_p\colon II_p \to I_p \qquad \psi_p(k_1, k_2, \boldsymbol{k}') = (k_2 - p, k_1 + p, \boldsymbol{k}') \end{aligned}$$

where $\boldsymbol{k}' = (k_3, \dots, k_n)$.

Lemma 6. *1. Maps ϕ_p and ψ_p establish bijections between the sets I_p and II_p;*

2. For any $\boldsymbol{k} \in I_p$

$$\operatorname{Res}_{u_1=u_2q^p} \mathrm{U}_{\boldsymbol{k}}(\boldsymbol{u};\boldsymbol{v}) + \operatorname{Res}_{u_1=u_2q^p} \mathrm{U}_{\phi_p(\boldsymbol{k})}(\boldsymbol{u};\boldsymbol{v}) = 0, \tag{4.6}$$

$$\operatorname{Res}_{v_2=v_1q^p} \mathrm{V}_{\boldsymbol{k}}(\boldsymbol{u};\boldsymbol{v}) + \operatorname{Res}_{v_2=v_1q^p} \mathrm{V}_{\phi_p(\boldsymbol{k})}(\boldsymbol{u};\boldsymbol{v}) = 0. \tag{4.7}$$

Proof of Lemma 6. The first part is purely combinatorial and can be checked directly. Let us prove the second part.

Note first that each summand $\mathrm{U}_{\boldsymbol{k}}(\boldsymbol{u};\boldsymbol{v})$ of the left hand side of (4.3) has the following structure

$$\mathrm{U}_{\boldsymbol{k}}(\boldsymbol{u};\boldsymbol{v}) = \frac{\mathcal{U}_{\boldsymbol{k}}(\boldsymbol{u};\boldsymbol{v};t)}{\mathcal{U}_{\boldsymbol{k}}(\boldsymbol{u};\boldsymbol{v};1)} \tag{4.8}$$

where

$$\mathcal{U}_{\boldsymbol{k}}(\boldsymbol{u};\boldsymbol{v};t) = \prod_{i=1}^{n} [qt;q]_{k_i} \times \prod_{\substack{i,j=1\\ i\ne j}}^{n} [t^{-1}q^{-k_j} u_i/u_j; q]_{k_i} \times \prod_{a,j=1}^{n} [tu_j/v_a;q]_{k_j}. \tag{4.9}$$

The following identity

$$\begin{aligned} &\mathcal{U}_{k_1,k_2,\boldsymbol{k}'}(\boldsymbol{u};\boldsymbol{v};t)|_{u_1=q^pu_2} = \mathcal{U}_{k_2-p,k_1+p,\boldsymbol{k}'}(\boldsymbol{u};\boldsymbol{v};t)|_{u_1=q^pu_2},\\ &\boldsymbol{k} = (k_1, k_2, \boldsymbol{k}') \in I_p \end{aligned} \tag{4.10}$$

valid for any $\boldsymbol{k} = (k_1, k_2, \boldsymbol{k}') \in I_p$ is established with a help of an explicit bijection between linear factors of the products in both sides of equality (4.10). Then, this equality implies the statement (4.6) about zero sum of the residues. Indeed, the relation (4.10) establishes a bijection between all nonzero factors of the denominators $\mathcal{U}_{k_1,k_2,\boldsymbol{k}'}(\boldsymbol{u};\boldsymbol{v};1)|_{u_1=q^pu_2}$ and $\mathcal{U}_{k_2-p,k_1+p,\boldsymbol{k}'}(\boldsymbol{u};\boldsymbol{v};1)$

and the equality of their products. Factors in denominators of $\mathrm{U}_{k_1,k_2,\boldsymbol{k}'}(\boldsymbol{u};\boldsymbol{v})$ and $\mathrm{U}_{k_2-p,k_1+p,\boldsymbol{k}'}(\boldsymbol{u};\boldsymbol{v})$ which tend to zero when u_1 tends to $q^p u_2$ are

$$q^{-p/2}u_1/u_2 - q^{p/2}u_2/u_1 \qquad \text{and} \qquad q^{p/2}u_2/u_1 - q^{-p/2}u_1/u_2. \tag{4.11}$$

They give inputs into residues, which just differ by sign. Thus, we arrive at (4.6). For the proof of (4.7), we note that the involution

$$\tau\colon u_i \mapsto v_i^{-1}, \qquad v_i \mapsto u_i^{-1} \tag{4.12}$$

exchanges each $\mathrm{U}_{\boldsymbol{k}}$ with $\mathrm{V}_{\boldsymbol{k}}$, as well as the left and right hand sides of (4.3). □

Corollary 4. *Both sides of (4.3) have no poles of the form* $u_i = q^p u_j$ *and* $v_a = q^p v_b$.

For any non-negative integer p denote by $\varphi_p(\boldsymbol{u};\boldsymbol{v})$ the following rational function of $\boldsymbol{u} = (u_1,\ldots,u_n)$ and $\boldsymbol{v} = (v_1,\ldots,v_n)$:

$$\varphi_p(\boldsymbol{u};\boldsymbol{v}) = (-1)^p \frac{[tq;q]_{2p}}{[q;q]_p[q;q]_{p-1}} \prod_{j=2}^{n} \frac{[tu_j/v_1;q]_p}{[u_1/u_j;q]_p} \prod_{b=2}^{n} \frac{[tu_1/v_b;q]_p}{[v_b/v_1;q]_p} \tag{4.13}$$

Lemma 7. *For any* $1 \le p \le k_1$ *and* $\boldsymbol{k}' \in \mathbb{Z}_{\ge 0}^{n-1}$

$$\mathrm{Res}_{v_1=q^{p-1}u_1} \frac{1}{v_1} \mathrm{V}_{k_1,\boldsymbol{k}'}(\boldsymbol{u};\boldsymbol{v}) = \varphi_p(\boldsymbol{u};\boldsymbol{v}) \times \mathrm{V}_{k_1-p,\boldsymbol{k}'}(qv_1,\boldsymbol{u}';q^{-1}u_1,\boldsymbol{v}'), \tag{4.14}$$

$$\mathrm{Res}_{v_1=q^{p-1}u_1} \frac{1}{v_1} \mathrm{U}_{k_1,\boldsymbol{k}'}(\boldsymbol{u};\boldsymbol{v}) = \varphi_p(\boldsymbol{u};\boldsymbol{v}) \times \mathrm{U}_{k_1-p,\boldsymbol{k}'}(qv_1,\boldsymbol{u}';q^{-1}u_1,\boldsymbol{v}'). \tag{4.15}$$

Proof of Lemma 7. It is a direct computation which uses the following properties of q-Pochhammer symbols:

$$[q^p u;q]_m \times [u]_n = [q^p u;q]_{n-p} \times [u;q]_{m+p}, \tag{4.16}$$

$$[qu;q]_m \times [q^{-(m+p)}u^{-1};q]_n = (-1)^p [qu;q]_{m+p} \times [q^{-m}u^{-1};q]_{n-p} \tag{4.17}$$

which are valid for any u and integer m,n,p. Here we assume that

$$[z;q]_{-n} = (q^{1/2}z^{1/2} - q^{-1/2}z^{-1/2})^{-1} \cdots (q^{n/2}z^{1/2} - q^{-n/2}z^{-1/2})^{-1}, \qquad n > 0. \tag{4.18}$$

For more technical details, see [4]. □

Proof of Theorem 3. We are ready now to prove the equality (4.3) and thus Theorem 3 by induction over K. Denote the difference of the left and right hand sides of (4.3) by $W_K(\boldsymbol{u};\boldsymbol{v})$. Assume that $W_K(\boldsymbol{u},\boldsymbol{v}) = 0$ for all $K < N$ and any m-tuples of variables $\boldsymbol{u} = (u_1,\ldots,u_m)$, $\boldsymbol{v} = (v_1,\ldots,v_m)$ for arbitrary m. Summing up the difference of (4.14) and (4.15) over all $\boldsymbol{k}$ with $|\boldsymbol{k}| = K$, we get the relation

$$\mathrm{Res}_{v_1=q^{p-1}u_1} \frac{1}{v_1} W_K(\boldsymbol{u};\boldsymbol{v}) = \varphi_p(\boldsymbol{u};\boldsymbol{v}) \times W_{K-p}(\boldsymbol{u}^*,\boldsymbol{v}^*), \tag{4.19}$$

where

$$\boldsymbol{u}^* = (qv_1, \boldsymbol{u}'), \qquad \boldsymbol{v}^* = (q^{-1}u_1, \boldsymbol{v}'). \tag{4.20}$$

By the induction assumption the right hand side of (4.19) equals zero. Taking in mind the symmetricity of $W_K(\boldsymbol{u};\boldsymbol{v})$ with respect to permutation of u_i and of v_j we conclude that it has no poles at all. Since $W_K(\boldsymbol{u};\boldsymbol{v})$ is a homogeneous rational function of the variables u_i and v_j of total degree zero, it is equal to a constant, which could depend on q and t. To compute this constant, we consider the behavior of this function in asymptotic zone

$$u_1 \ll u_2 \ll \ldots \ll u_n \ll v_n \ll v_{n-1} \ll \ldots \ll v_1. \tag{4.21}$$

Here both sides of (4.3) tend to

$$\sum_{|\boldsymbol{k}_n|=K} \prod_{i=1}^{n} \frac{[qt;q]_{k_i}}{[q;q]_{k_i}} \times t^{\frac{1}{2}\left((n-1)k_1+(n-3)k_2+\ldots+(3-n)k_{n-1}+(1-n)k_n\right)} \times t^{-\frac{nK}{2}}. \tag{4.22}$$

Therefore, $W_K(\boldsymbol{u};\boldsymbol{v})$ tends to zero in this asymptotic zone and so equals zero identically. This completes the induction step, the proof of the identity (4.3) and of Theorem 3. □

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Appendix

A. Double Gamma and Sine Functions

The Barnes double Gamma function $\Gamma_2(z|\boldsymbol{\omega})$ [1] is defined by the relation

$$\Gamma_2(z|\boldsymbol{\omega}) = \exp\left(\frac{\partial}{\partial s}\zeta_2(s,z|\boldsymbol{\omega})\right)\Big|_{s=0},$$

where $\zeta_2(s,z|\boldsymbol{\omega})$ is the analytical continuation of the series

$$\zeta_2(s,z|\boldsymbol{\omega}) = \sum_{n_1,n_2\geq 0}(z+n_1\omega_1+n_2\omega_2)^{-s}, \qquad \operatorname{Re} s > 2$$

which under assumptions (1.14) and $\operatorname{Re} z > 0$ can be presented by the integral

$$\zeta_2(s,z|\boldsymbol{\omega}) = \Gamma(1-s)\int_C \frac{e^{-zt}(-t)^s}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}\frac{dt}{2\pi\imath t}$$

over the Hankel contour C enclosing the ray $\{t \geq 0\}$ counterclockwise. Under the same assumptions analogous integral presentation of $\ln\Gamma_2(z|\boldsymbol{\omega})$ looks as follows

$$\ln\Gamma_2(z|\boldsymbol{\omega}) = \frac{\gamma}{2}B_{2,2}(z|\boldsymbol{\omega}) + \int_C \frac{e^{-zt}\ln(-t)}{(1-e^{-\omega_1 t})(1-e^{-\omega_2 t})}\frac{dt}{2\pi\imath t}. \tag{A.1}$$

Here

$$B_{2,2}(z|\boldsymbol{\omega}) = \frac{z^2}{\omega_1\omega_2} - \frac{\omega_1+\omega_2}{\omega_1\omega_2}z + \frac{\omega_1^2+3\omega_1\omega_2+\omega_2^2}{6\omega_1\omega_2} \tag{A.2}$$

is a particular multiple Bernoulli polynomial, γ is the Euler constant.

The double sine function $S_2(z) := S_2(z|\boldsymbol{\omega})$, see [15] and references therein, is then defined as

$$S_2(z|\boldsymbol{\omega}) = \Gamma_2(\omega_1+\omega_2-z|\boldsymbol{\omega})\Gamma_2^{-1}(z|\boldsymbol{\omega}). \tag{A.3}$$

It satisfies functional relations

$$\frac{S_2(z)}{S_2(z+\omega_1)} = 2\sin\frac{\pi z}{\omega_2}, \qquad \frac{S_2(z)}{S_2(z+\omega_2)} = 2\sin\frac{\pi z}{\omega_1} \tag{A.4}$$

and inversion relation

$$S_2(z)S_2(-z) = -4\sin\frac{\pi z}{\omega_1}\sin\frac{\pi z}{\omega_2}, \tag{A.5}$$

or equivalently

$$S_2(z)S_2(\omega_1+\omega_2-z) = 1. \tag{A.6}$$

The double sine function is a homogeneous function of all its arguments

$$S_2(\gamma z|\gamma\omega_1,\gamma\omega_2) = S_2(z|\omega_1,\omega_2), \qquad \gamma\in(0,\infty) \tag{A.7}$$

and is invariant under permutation of periods

$$S_2(z|\omega_1,\omega_2) = S_2(z|\omega_2,\omega_1). \tag{A.8}$$

The relation (A.4) has a useful corollary

$$\frac{S_2(z)}{S_2(z+m\omega_1+k\omega_2)} = (-1)^{mk}\prod_{j=0}^{m-1} 2\sin\frac{\pi}{\omega_2}(z+j\omega_1)\prod_{j=0}^{k-1} 2\sin\frac{\pi}{\omega_1}(z+j\omega_2),$$
$$\frac{S_2(z-m\omega_1-k\omega_2)}{S_2(z)} = (-1)^{mk}\prod_{j=1}^{m} 2\sin\frac{\pi}{\omega_2}(z-j\omega_1)\prod_{j=1}^{k} 2\sin\frac{\pi}{\omega_1}(z-j\omega_2) \tag{A.9}$$

that holds for $m,k\geq 0$. The latter relations also imply the following factorization formula

$$S_2(z)S_2(z+m\omega_1+k\omega_2) = (-1)^{mk}S_2(z+m\omega_1)S_2(z+k\omega_2) \tag{A.10}$$

for $m,k\in\mathbb{Z}$. The function $S_2(z)$ is a meromorphic function of z with poles at

$$z_{m,k} = m\omega_1+k\omega_2, \qquad m,k\geq 1 \tag{A.11}$$

and zeros at

$$z_{-m,-k} = -m\omega_1-k\omega_2, \qquad m,k\geq 0. \tag{A.12}$$

For $\omega_1/\omega_2\notin\mathbb{Q}$ all poles and zeros are simple. The residues of $S_2(z)$ and $S_2^{-1}(z)$ at these points are

$$\operatorname*{Res}_{z=z_{m,k}} S_2(z) = \frac{\sqrt{\omega_1\omega_2}}{2\pi}\frac{(-1)^{mk}}{\prod_{s=1}^{m-1} 2\sin\frac{\pi s\omega_1}{\omega_2}\prod_{l=1}^{k-1} 2\sin\frac{\pi l\omega_2}{\omega_1}}, \tag{A.13}$$

$$\operatorname*{Res}_{z=z_{-m,-k}} S_2^{-1}(z) = \frac{\sqrt{\omega_1\omega_2}}{2\pi}\frac{(-1)^{mk+m+k}}{\prod_{s=1}^{m} 2\sin\frac{\pi s\omega_1}{\omega_2}\prod_{l=1}^{k} 2\sin\frac{\pi l\omega_2}{\omega_1}}. \tag{A.14}$$

The integral representation for the logarithm of double sine function

$$\ln S_2(z) = \int_0^\infty \frac{dt}{2t}\left(\frac{\operatorname{sh}\left[(2z-\omega_1-\omega_2)t\right]}{\operatorname{sh}(\omega_1 t)\operatorname{sh}(\omega_2 t)} - \frac{2z-\omega_1-\omega_2}{\omega_1\omega_2 t}\right) \tag{A.15}$$

holds true for $\operatorname{Re} z\in(0,\operatorname{Re}(\omega_1+\omega_2))$.

The double dine function also can be written in terms of Ruijsenaars hyperbolic Gamma function $G(z|\boldsymbol{\omega})$ [21]

$$G(z|\boldsymbol{\omega}) = S_2\Big(\imath z+\frac{\omega_1+\omega_2}{2}\,\Big|\,\boldsymbol{\omega}\Big) \tag{A.16}$$

or Faddeev quantum dilogarithm $\gamma(z|\boldsymbol{\omega})$ [5]

$$\gamma(z|\boldsymbol{\omega}) = S_2\Big(-\imath z+\frac{\omega_1+\omega_2}{2}\,\Big|\,\boldsymbol{\omega}\Big)\exp\Big(\frac{\imath\pi}{2\omega_1\omega_2}\Big[z^2+\frac{\omega_1^2+\omega_2^2}{12}\Big]\Big). \tag{A.17}$$

Both functions $G(z|\boldsymbol{\omega})$ and $\gamma(z|\boldsymbol{\omega})$ were investigated independently.

In what follows, we use the same notations as in Sect. 3.1. Denote by σ_i the arguments of the periods ω_i, $|\sigma_i|<\pi/2$. Since the double sine function is invariant under permutation of ω_1,ω_2, suppose for definiteness that $\sigma_1\geq\sigma_2$.

Let D_+ and D_- be the cones of poles and zeros of the double sine function $S_2(z|\boldsymbol{\omega})$:

$$D_+ = \{z\colon \sigma_2 < \arg z < \sigma_1\}, \qquad D_- = \{z\colon \pi+\sigma_2 < \arg z < \pi+\sigma_1\},$$
$$D = D_+ \cup D_-.$$

Denote by $d(z, D_+)$ and $d(z, D_-)$ the distances between a point z and the cones $D_\pm$. Then, the Barnes' Stirling formula for the logarithm of the double Gamma function, see [1, §§85–86], with error term suggested by E. Rains [19, Theorem 2.6] looks as

$$\ln \Gamma_2(z|\boldsymbol{\omega}) = -\frac{1}{2}B_{2,2}(z|\boldsymbol{\omega}) \ln z + \frac{3}{4\omega_1\omega_2}\, z^2 - \frac{\omega_1+\omega_2}{2\omega_1\omega_2}\, z + O\Big(d^{-1}(z, D_-)\Big)\,. \tag{A.18}$$

Here $z \in \mathbb{C} \setminus D_-$. Moreover, the estimates for the error term given in [2, §57] are uniform on compact subsets of parameters $\boldsymbol{\omega}$ separated from zero. Then, for $z \in \mathbb{C}\backslash(D_+ \cup D_-)$

$$\begin{aligned} \ln S_2(z|\boldsymbol{\omega}) &= \ln \Gamma_2(\omega_1+\omega_2 - z|\omega_1,\omega_2) - \ln \Gamma_2(z|\omega_1,\omega_2) \\ &= \pm\frac{\pi\imath}{2}B_{2,2}(z|\boldsymbol{\omega}) + O\Big(d^{-1}(z, D)\Big). \end{aligned} \tag{A.19}$$

where the sign $+$ is taken for z in the upper half plane and the sign $-$ for z in the lower half plane (and not in D). Finally, in the same notations,

$$\ln \frac{S_2(z|\boldsymbol{\omega})}{S_2(z+g|\boldsymbol{\omega})} = \mp\pi\imath \frac{g}{\omega_1\omega_2}\left(z - \frac{g^*}{2}\right) + O\Big(d^{-1}(z, D)\Big). \tag{A.20}$$

Equivalently, for $z \in \mathbb{C} \setminus D$

$$\frac{S_2(z|\boldsymbol{\omega})}{S_2(z+g|\boldsymbol{\omega})} = e^{\mp\pi\imath \frac{g}{\omega_1\omega_2}\left(z-\frac{g^*}{2}\right)}\Big(1 + O\Big(d^{-1}(z, D)\Big)\Big). \tag{A.21}$$

Using the asymptotics (A.21), we can derive the following bounds which we use for the study of integrals convergence throughout the paper.

Let $K \subset \mathbb{C}$ be a closed subset of a complex plane satisfying the following conditions:

1. K is inside the domain of analyticity of $S_2(z|\boldsymbol{\omega})S_2^{-1}(z+g|\boldsymbol{\omega})$;
2. There exists $R > 0$ and $\rho > |g|$ such that $K \cap \{|z| > R\}$ does not intersect with D and
$$d(K \cap \{|z| > R\}, D) \geq \rho. \tag{A.22}$$

Proposition 3. *Under the conditions* 1 *and* 2 *above, we have a bound*

$$\big|S_2(z|\boldsymbol{\omega})S_2^{-1}(z+g|\boldsymbol{\omega})\big| < Ce^{\mp \mathrm{Re}\, \frac{\pi\imath g z}{\omega_1\omega_2}}, \qquad z \in K. \tag{A.23}$$

The constant C can be stated uniform as the parameters g, ω_1, ω_2 range in a compact domain separated from zero values of periods.

Proof. Due to (A.21) and condition 2, there exist $R_1 > R$ and C_1 such that

$$\big|S_2(z|\boldsymbol{\omega})S_2^{-1}(z+g|\boldsymbol{\omega})\big| < C_1e^{\mp \mathrm{Re}\, \frac{\pi\imath g z}{\omega_1\omega_2}}, \qquad z \in K,\ |z| > R_1. \tag{A.24}$$

On the other hand, the set

$$K \cap \{|z| \leq R_1\} \tag{A.25}$$

is compact and belongs to the region of analyticity of the function $S_2(z|\boldsymbol{\omega})$ $S_2^{-1}(z+g|\boldsymbol{\omega})$. Thus, this function is bounded on the set (A.25),

$$\left|S_2(z|\boldsymbol{\omega})S_2^{-1}(z+g|\boldsymbol{\omega})\right| < C_2, \qquad z \in K, \qquad |z| < R_1. \tag{A.26}$$

At the same time both real functions $e^{\mp\mathrm{Re}\,\frac{\pi\imath g z}{\omega_1\omega_2}}$ are analytic and positive on the compact set (A.25). Thus, they are bounded from below on this set

$$e^{\mp\mathrm{Re}\,\frac{\pi\imath g z}{\omega_1\omega_2}} > C_3 > 0. \tag{A.27}$$

Combining (A.26) and (A.27), we conclude that there exists a positive constant C_4 such that

$$\left|S_2(z|\boldsymbol{\omega})S_2^{-1}(z+g|\boldsymbol{\omega})\right| < C_4 e^{\mp\mathrm{Re}\,\frac{\pi\imath g z}{\omega_1\omega_2}}, \qquad z \in K,\ |z| \le R_1. \tag{A.28}$$

Combining (A.24) and (A.28) we arrive at the proof of Proposition 3. □

There are two straightforward corollaries of Proposition 3. First, since $|\mathrm{Re}\,z| < |z|$, (A.23) implies that the function $S_2(z|\boldsymbol{\omega})S_2^{-1}(z+g|\boldsymbol{\omega})$ grows at most exponentially

$$\left|S_2(z|\boldsymbol{\omega})S_2^{-1}(z+g|\boldsymbol{\omega})\right| < Ce^{a|z|}, \qquad z \in K. \tag{A.29}$$

Second, assume that K is contained in a strip $|\mathrm{Re}\,z| < b$ for some $b > 0$. Then, (A.23) implies the bound

$$\left|S_2(z|\boldsymbol{\omega})S_2^{-1}(z+g|\boldsymbol{\omega})\right| < \tilde{C} e^{\mathrm{Re}\,\frac{\pi g}{\omega_1\omega_2}|y|}, \qquad z = x+iy \in K. \tag{A.30}$$

The same statement holds for the inverse ratio. Namely, Let $K' \subset \mathbb{C}$ be a closed subset of a complex plane satisfying the following conditions:

1′. K' is inside the domain of analyticity of $S_2^{-1}(z|\boldsymbol{\omega})S_2(z+g|\boldsymbol{\omega})$;
2′. There exists $R' > 0$ and $\rho' > |g|$ such that $K' \cap \{|z| > R'\}$ does not intersect with D and

$$d(K' \cap \{|z| > R'\}, D) \ge \rho'. \tag{A.31}$$

Proposition 4. *Under the conditions* 1′ *and* 2′ *above, we have a bound*

$$\left|S_2^{-1}(z|\boldsymbol{\omega})S_2(z+g|\boldsymbol{\omega})\right| < C' e^{\pm\mathrm{Re}\,\frac{\pi\imath g z}{\omega_1\omega_2}}. \tag{A.32}$$

In particular, the function $S_2^{-1}(z|\boldsymbol{\omega})S_2(z+g|\boldsymbol{\omega})$ grows at most exponentially

$$\left|S_2^{-1}(z|\boldsymbol{\omega})S_2(z+g|\boldsymbol{\omega})\right| < C' e^{a|z|}, \qquad z \in K. \tag{A.33}$$

If K' is contained in a strip $|\mathrm{Re}\,z| < b'$ for some $b' > 0$, then

$$\left|S_2^{-1}(z|\boldsymbol{\omega})S_2(z+g|\boldsymbol{\omega})\right| < \tilde{C}' e^{-\mathrm{Re}\,\frac{\pi g}{\omega_1\omega_2}|y|}, \qquad z = x+iy \in K'. \tag{A.34}$$

B. Bounds for Integrals

Both functions $\mu(z)$ and $K(z)$ can be presented as the ratios of double sine functions that appear in Propositions 3 and 4

$$\mu(z) = S_2(\imath z) S_2^{-1}(\imath z + g),$$
$$K(z) = S_2\left(\imath z + \frac{\omega_1+\omega_2}{2} + \frac{g}{2}\right) S_2^{-1}\left(\imath z + \frac{\omega_1+\omega_2}{2} - \frac{g}{2}\right). \tag{B.1}$$

The conditions (1.14) and (1.15) imply that both functions have a strip of analyticity which include the real line of the parameter z. For brevity, we also denote by ν_g the constant in the assumption (1.27)

$$\nu_g = \operatorname{Re} \frac{g}{\omega_1\omega_2} > 0. \tag{B.2}$$

Then, by (A.30) and (A.34) we have

$$|K(y)| < C e^{-\pi\nu_g|y|}, \qquad |\mu(y)| < C e^{\pi\nu_g|y|}, \qquad y \in \mathbb{R}, \tag{B.3}$$

where C is a positive constant uniform for a compact subset of parameters ω_1, ω_2 and g preserving the conditions above. Assume also the condition

$$|\operatorname{Im}\lambda| \le \delta < \nu_g \tag{B.4}$$

with some positive δ.

Proposition 5. *The integral (3.4) corresponding to the kernel of Q-operators product*

$$Q_n(\boldsymbol{z}_{2n};\lambda) = \int_{\mathbb{R}^n} d\boldsymbol{y}_n \prod_{\substack{i,j=1\\ i\neq j}}^{n} \mu(y_i - y_j) \prod_{a=1}^{2n}\prod_{i=1}^{n} K(y_i - z_a)\, e^{2\pi\imath\lambda \underline{\boldsymbol{y}}_n} \tag{B.5}$$

converges uniformly with respect to the parameters λ, z_a, $\boldsymbol{\omega}$, g, while the parameters z_a, $\boldsymbol{\omega}$, g range over compact sets preserving the conditions (1.14), (1.15) and (B.2) and the parameter λ varies satisfying the condition (B.4)

Proof. Denote integrand by F. Using (B.3), we arrive at the following bound

$$|F| \le C \exp\left(\pi\nu_g \sum_{\substack{i,j=1\\ i\neq j}}^{n} |y_i - y_j| - \pi\nu_g \sum_{a=1}^{2n}\sum_{i=1}^{n} |y_i - z_a| + \operatorname{Re}\left(2\pi\imath\lambda \sum_{i=1}^{n} y_i\right)\right), \tag{B.6}$$

where constant C depends on $g, \boldsymbol{\omega}$. Using for the first two sums inequalities

$$|y_i - y_j| \le |y_i| + |y_j|, \qquad |y_i - z_a| \ge |y_i| - |z_a| \tag{B.7}$$

together with $|z_a| \le M$ (since all z_a vary over compact set) and for the last sum inequality

$$\left|\sum_{i=1}^{n} y_i\right| \le \sum_{i=1}^{n} |y_i|$$

we arrive at

$$|F| \leq C \exp\left(2\pi\nu_g n^2 M + 2\pi(|\operatorname{Im}\lambda| - \nu_g)\sum_{i=1}^{n}|y_i|\right). \tag{B.8}$$

Since λ satisfies (B.4), the bound (B.8) implies the statement of the proposition. □

Remark. As it can be seen from the bound (B.6), the second inequality from the line (B.7) and the bound on the measure function

$$|\mu(\boldsymbol{z}_n)| \leq C \exp\left(\pi\nu_g \sum_{\substack{i,j=1 \\ i\neq j}}^{n} |z_i - z_j|\right) \leq C \exp\left(2\pi\nu_g(n-1)\sum_{i=1}^{n}|z_i|\right) \tag{B.9}$$

the product of two Q-operators $Q_n(\lambda)Q_n(\rho)$ is well defined on fast decreasing functions $f(\boldsymbol{z}_n)$ bounded as

$$|f(\boldsymbol{z}_n)| \leq C \exp\left(-\pi\nu_g\left[3n - 2 + \frac{2|\operatorname{Im}\rho|}{\nu_g} + \varepsilon\right]\sum_{i=1}^{n}|z_i|\right) \tag{B.10}$$

with any $\varepsilon > 0$. In the case $\operatorname{Im}\rho = 0$, the bound does not depend on the Q-operators parameters

$$|f(\boldsymbol{z}_n)| \leq C \exp\left(-\pi\nu_g(3n - 2 + \varepsilon)\sum_{i=1}^{n}|z_i|\right). \tag{B.11}$$

Besides the integral (3.4) over the real plane in Sect. 3.1 we consider the iterated integral with the same kernel over big semicircles. The study of its convergence and vanishing in the limit splits into three parts: the behavior near real plane where the integrand should rapidly vanish; the total exponential bound of the integrand

$$\tilde{F} = \mu(\boldsymbol{y}_n)\prod_{a=1}^{2n}\prod_{i=1}^{n}K(y_i - z_a) \tag{B.12}$$

with the exponent $e^{2\pi\imath\lambda\underline{\boldsymbol{y}}_n}$ in the regular domain $\mathbb{C}\setminus D$; and the exponential bound of the integrand (B.12) in the irregular domain D. The second part is performed by using inequalities (B.7) and exponential bounds (A.29) and (A.33). The third part follows from the same inequalities (B.7) and the results of Sect. 3.1.

Finally, for the first part we need a bound similar to (B.8) but for the arguments in a cone around a real line. This can be done for the parameter λ with negative real part, so that

$$2\pi\lambda = -R + \imath\theta, \qquad R > 0, \qquad |\theta| \leq 2\pi\delta < 2\pi\nu_g, \tag{B.13}$$

and the integration variables on the cone around a real line with negative imaginary parts

$$y_i = \bar{y}_i(1 \pm \imath\operatorname{tg}\varphi_i), \qquad \bar{y}_i \in \mathbb{R}, \qquad \pm\bar{y}_i\operatorname{tg}\varphi_i < 0, \qquad 0 < \varphi_i < \sigma. \tag{B.14}$$

Here σ is the angle of the cone. The sign + (or −) corresponds to $\bar{y}_i < 0$ (or $\bar{y}_i > 0$). Denote also

$$\frac{g}{\omega_1\omega_2} = \nu_g(1 + \imath \mathrm{tg}\varphi_g), \qquad \alpha = |\mathrm{tg}\varphi_g \mathrm{tg}\sigma|. \tag{B.15}$$

Suppose the following inequality is satisfied

$$2\pi\nu_g(1 - (2n-1)\alpha) - |\theta| > \varepsilon \tag{B.16}$$

for some $\varepsilon > 0$. For the fixed parameters g, θ this inequality tells us, how small should be σ, that is how narrow should be the cone around a real line, in order to have the following bound.

Proposition 6. *Under the conditions (B.13), (B.14), (B.16) we have the bound*

$$|F| \leq C \, \exp\left(-\varepsilon \sum_{i=1}^{n} |\bar{y}_i|\right). \tag{B.17}$$

Proof. For the variables on the cone (B.14) use the bound (A.23) for measure function together with (B.15)

$$|\mu(y_i - y_j)| \leq C \exp\left(\left|\mathrm{Re}\, \frac{\pi g}{\omega_1\omega_2}(y_i - y_j)\right|\right) \leq C \exp\Big(\pi\nu_g(1+\alpha)(|\bar{y}_i| + |\bar{y}_j|)\Big). \tag{B.18}$$

In the same spirit we use the bound (A.32) for kernel function assuming big enough values of $|y_i|$ (compared to z_a) and, as before, $|z_a| \leq M$

$$|K(y_i - z_a)| \leq C' \exp\left(\mp \mathrm{Re}\, \frac{\pi \imath g}{\omega_1\omega_2}\left[\imath(y_i - z_a) + \frac{g^*}{2}\right]\right) \leq \tilde{C}' \exp\Big(-\pi\nu_g(1-\alpha)|\bar{y}_i|\Big). \tag{B.19}$$

Therefore, for the whole integrand we have the bound

$$|F| \leq C \, \exp\left(\pi\nu_g(1+\alpha) \sum_{\substack{i,j=1 \\ i\neq j}}^{n} (|\bar{y}_i| + |\bar{y}_j|) - \pi\nu_g(1-\alpha) \sum_{a=1}^{2n}\sum_{i=1}^{n} |\bar{y}_i| + |\theta| \left|\sum_{i=1}^{n} \bar{y}_i\right|\right), \tag{B.20}$$

which implies

$$|F| \leq C \, \exp\left(\Big[2\pi\nu_g((2n-1)\alpha - 1) + |\theta|\Big] \sum_{i=1}^{n} |\bar{y}_i|\right). \tag{B.21}$$

Then, the proposition follows from the condition (B.16). □

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