Letters in Mathematical Physics (2024) 114:77
https://doi.org/10.1007/s11005-024-01824-w

Zhelobenko–Stern formulas and B_n Toda wave functions

A. Galiullin[1] · S. Khoroshkin[1,2] · M. Lyachko[1]

Received: 20 February 2024 / Revised: 10 May 2024 / Accepted: 17 May 2024 /
Published online: 5 June 2024

Abstract
Using Zhelobenko–Stern formulas for the action of the generators of orthogonal Lie algebra in corresponding Gelfand–Tsetlin basis, we derive Mellin–Barnes presentations for the wave functions of B_n Toda lattice. They are in accordance with Iorgov–Shadura formulas.

Keywords Orthogonal group · Gelfand–Tsetlin basis · Whittaker vector

Mathematics Subject Classification 20C35

1 Introduction

In the paper [4], Gerasimov, Kharchev and Lebedev applied the famous formulas [2] for the action of generators of general Lie algebra $gl(n)$ in Gelfand–Tsetlin basis of irreducible finite-dimensional representations of general linear group $GL(n,\mathbb{C})$ to obtain Mellin–Barnes presentation of the wave functions of open A_n Toda chain. Using Gelfand–Tsetlin formulas, they constructed an infinite-dimensional representation of Lie algebra $gl(n,\mathbb{C})$ in the space of meromorphic functions on $n(n-1)/2$ variables, found there two dual Whittaker vectors and realized, according to Kostant theory [11], the Toda wave function as certain matrix element in this representation. The same formulas were earlier established by Kharchev and Lebedev in the technique of Yang–Baxter formalism [10].

Besides, Toda wave functions admit another presentation, known by the name Gauss–Givental by means of integrals over spatial variables. It was found first in [5]. Gauss–Givental presentation was then derived in [3] for wave functions of Toda systems related to B_n, C_n, D_n root systems.

✉ S. Khoroshkin
skhoroshkin@gmail.com

1 National Research University Higher School of Economics, Myasnitskaya 20, Moscow, Russia 101000

2 Institute for Information Transmission Problems RAS (Kharkevich Institute), Bolshoy Karetny per. 19, Moscow, Russia 127994

Mellin transform of these formulas was computed in [9]. However, the formulas presented in [9] differ from that of [4] and are not satisfactory by several reasons. In particular, one cannot find in this presentation Sklyanin measure and thus the results of [9] cannot be used to establish the completeness and orthogonality of the wave function and develop the corresponding integral transform.

The goal of this paper is to try to fill this gap at least partially for B_n Toda system using representation theoretical tools similar to that of [4]. The only known result in this direction is the paper [7] of Iorgov and Shadura, where they constructed B_n wave function by its decomposition over related A_n Toda wave function. As well as in [10], this work was done in a framework of the Yang–Baxter formalism.

Our starting point is an analog of Gelfand–Tsetlin formulas for orthogonal groups published without a proof by Zhelobenko and Stern [12]. These formulas look much more complicated compared to [2] and we did not find numerous applications of them in the literature. However, after their check we constructed 'Gelfand–Tsetlin' infinite-dimensional representation of the orthogonal Lie algebra and found there two dual Whittaker vectors. With their help we constructed the integrals, presenting B_n wave functions in which we see all expected ingredients of Sklyanin measure. The resulting formula can be presented as an iterative procedure in two ways.

Firstly, it is an iterative procedure over the rank of orthogonal group and this is probably the most interesting result of this paper. Each step can be interpreted as an action of the raising integral operator, where kernel is itself an integral over intermediate variables. In such type of structure, we also observe in Gauss–Givental representation [3, (1.74)]. Second, we can consider two successive iterative integrals combining them in other parity. Then, intermediate step becomes precisely a degeneration of B_n Gustafson integral and can be explicitly evaluated. In this way, we arrive at Iorgov–Shadura formula.

Note the two subtle points of our construction. First, Zhelobenko–Stern formulas are written for the generators of orthogonal Lie algebras in their orthogonal realization, while Whittakker vectors refer to simple root generators. An existence of Whittaker vectors in a factorized form was not evident from the beginning. By the same reasons, action of the Cartan subalgebra in corresponding infinite-dimensional representation cannot be written, contrary to $gl(n)$, in terms of multiplications by linear functions. Fortunately, it is so for the action on Whittaker vectors.

Despite the fact that Zhelobenko–Stern formulas are written uniformly for all orthogonal Lie algebras, we succeeded to find Whittaker vectors in 'only for Lie algebras $so(2n-1)$.' More precisely, the main ingredient in the construction of Whittaker vector in 'Gelfand–Tsetlin representation' is the solution of difference equations (A.1)–(A.2). These equations describe 'degenerate' Whittaker vectors for $so(2n)$, for which one of the simple generators acts by zero, so that they are essentially Whittaker vectors for embedded $gl(n)$ Lie algebra. Restricting these vectors to $so(2n-1)$, we get 'nondegenerate' Whittaker vectors for this Lie algebra which we further use for the construction of the wave function for B_n Toda system.

2 Gelfand–Tsetlin type representation

2.1 Zhelobenko–Stern formulas

It is well known that each irreducible representation of the orthogonal groups $SO(2n+1)$ and $SO(2n)$ is parametrized by its signature, given by ordered sequences of integers or half-integers , respectively,

$$\begin{aligned} &p_1 \geq p_2 \geq \cdots p_{n-1} \geq p_n \geq 0, \qquad (SO(2n+1)) \\ &p_1 \geq p_2 \geq \cdots p_{n-1} \geq |p_n|, \qquad (SO(2n)) \end{aligned} \tag{2.1}$$

and the restriction of irreducible representation of $SO(2n+1)$ to $SO(2n)$ has simple spectrum described by all signatures $q_1, \dots q_n$, satisfying interleaving inequalities

$$p_1 \geq q_1 \geq p_2 \geq \cdots q_{n-1} \geq p_n \geq q_n \geq -p_n. \tag{2.2}$$

Analogously, the restriction of irreducible representation of $SO(2n)$ to $SO(2n-1)$ has simple spectrum described by all signatures $q_1, \dots q_n$, satisfying interleaving inequalities

$$p_1 \geq q_1 \geq p_2 \geq \cdots q_{n-1} \geq p_n. \tag{2.3}$$

This enables one to construct an orthogonal basis of irreducible representation of the orthogonal group $SO(n)$ parametrized by Gelfand–Tsetlin tableaux

$$\mathbf{p} = \begin{pmatrix} p_{n-1,1} & p_{n-1,2} & \cdots & p_{n-1,[\frac{n}{2}]} \\ & \ddots & \ddots & \vdots \\ & & p_{3,1} & p_{3,2} \\ & & & p_{2,1} \\ & & & p_{1,1} \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{n-1} \\ \vdots \\ \mathbf{p}_3 \\ \mathbf{p}_2 \\ \mathbf{p}_1 \end{pmatrix} \tag{2.4}$$

The upper row $\mathbf{p}_{n-1}$ indicates the signature of the irreducible representation of $SO(n)$ and is fixed for all tableaux parametrizing its basic vectors, the second row indicates the signature of the restriction to $SO(n-1)$, etc., and the integer p_{11} indicates the irreducible of $SO(2)$. All the numbers p_{ij} are either integers of half-integers simultaneously and should satisfy the row-by-row interleaving inequalities (2.1)–(2.2), that is

$$p_{i+1,j+1} \leq p_{i,j} \leq p_{i+1,j}, \qquad p_{2i-1,i} \leq |p_{2i,i}|, \qquad |p_{2i-1,i}| \leq p_{2i-2,i-1}$$

It is natural to shift the signatures by the corresponding half sum of positive roots of the related root system, that is, we set

$$\begin{aligned} m_{2k,j} &= p_{2k,j} + (k-j) + \frac{1}{2}, \\ m_{2k-1,j} &= p_{2k-1,j} + (k-j) \end{aligned} \tag{2.5}$$

Zhelobenko and Stern [12, Chapter II, Sect. 5.8] presented without a proof a precise expression for the matrix elements of generators of the Lie algebra $so(n)$ in the corresponding orthogonal basis.

The Lie algebra $so(n)$ is generated, as a vector space, by elements

$$I_{kj} = e_{kj} - e_{jk}, \qquad k > j$$

As a Lie algebra, it is generated by elements

$$I_{k+1,k}, \qquad k = 1, \ldots, n-1$$

with defining relations

$$\begin{aligned} &\left[I_{k+1,k}, \left[I_{k+2,k+1}, I_{k+1,k}\right]\right] = I_{k+2,k+1} \quad k = 1, \ldots, n-2 \\ &\left[I_{k+2,k+1}, \left[I_{k+1,k}, I_{k+2,k+1}\right]\right] = I_{k+1,k} \quad k = 1, \ldots, n-2 \\ &\left[I_{k+1,k}, I_{j+1,j}\right] = 0 \quad |k-j| > 1 \end{aligned} \tag{2.6}$$

After a renormalization, eliminating square roots in the coefficients and correcting misprints, their formulas look like

$$\begin{aligned} I_{2k+1,2k} = &- \sum_{j=1}^{k} \frac{\prod_{r=1}^{k-1}(m_{2k-2,r} + m_{2k-1,j} + \frac{1}{2}) \prod_{r=1}^{k}(m_{2k-1,j} - m_{2k,r} + \frac{1}{2})}{2\prod_{r\neq j}(m_{2k-1,j} - m_{2k-1,r})(m_{2k-1,j} + m_{2k-1,r} + 1)} e^{\partial_{m_{2k-1,j}}} \\ &- \sum_{j=1}^{k} \frac{\prod_{r=1}^{k-1}(m_{2k-1,j} - m_{2k-2,r} - \frac{1}{2}) \prod_{r=1}^{k}(m_{2k,r} + m_{2k-1,j} - \frac{1}{2})}{2\prod_{r\neq j}(m_{2k-1,j} - m_{2k-1,r})(m_{2k-1,j} + m_{2k-1,r} - 1)} e^{-\partial_{m_{2k-1,j}}} \end{aligned} \tag{2.7}$$

$$\begin{aligned} I_{2k+2,2k+1} = &\sum_{j=1}^{k} \frac{\prod_{r=1}^{k+1}((m_{2k,j} + \frac{1}{2})^2 - m_{2k+1,r}^2)}{2m_{2k,j}(m_{2k,j} + \frac{1}{2})\prod_{r\neq j}(m_{2k,j}^2 - m_{2k,r}^2)} e^{\partial_{m_{2k,j}}} \\ &+ \sum_{j=1}^{k} \frac{\prod_{r=1}^{k}((m_{2k,j} - \frac{1}{2})^2 - m_{2k-1,r}^2)}{2m_{2k,j}(m_{2k,j} - \frac{1}{2})\prod_{r\neq j}(m_{2k,j}^2 - m_{2k,r}^2)} e^{-\partial_{m_{2k,j}}} \\ &+ i\, \frac{\prod_{r=1}^{k} m_{2k-1,r} \prod_{r=1}^{k+1} m_{2k+1,r}}{\prod_{r=1}^{k}(m_{2k,r} + \frac{1}{2})(m_{2k,r} - \frac{1}{2})} \end{aligned} \tag{2.8}$$

Here, the operators $e^{\pm\partial m_{kj}}$ are operators of shifts of the entries of Gelfand–Tsetlin tableau: the operator $e^{\pm\partial m_{kj}}$ changes m_{kj} by $m_{kj} \pm 1$ (and, respectively, p_{kj} by $p_{kj} \pm 1$). We can extend the RHS of relations (2.7) and (2.8) to arbitrary complex parameters m_{ij} and regard them as operators acting in the space of rational functions on m_{ij}.

2.2 Representation in meromorphic functions

Following [4], we renormalize the variables

$$m_{kj} = \frac{\nu_{kj}}{ic} \tag{2.9}$$

in order to have an additional scaling variable in the representation. Then, we have

$$\begin{aligned} I_{2k+1,2k} = &-\frac{1}{ic}\sum_{j=1}^{k}\frac{\prod_{r=1}^{k-1}(\nu_{2k-2,r}+\nu_{2k-1,j}+\frac{ic}{2})\prod_{r=1}^{k}(\nu_{2k-1,j}-\nu_{2k,r}+\frac{ic}{2})}{2\prod_{r\neq j}(\nu_{2k-1,j}-\nu_{2k-1,r})(\nu_{2k-1,j}+\nu_{2k-1,r}+ic)}e^{ic\partial_{\nu_{2k-1,j}}} \\ &-\frac{1}{ic}\sum_{j=1}^{k}\frac{\prod_{r=1}^{k-1}(\nu_{2k-1,j}-\nu_{2k-2,r}-\frac{ic}{2})\prod_{r=1}^{k}(\nu_{2k,r}+\nu_{2k-1,j}-\frac{ic}{2})}{2\prod_{r\neq j}(\nu_{2k-1,j}-\nu_{2k-1,r})(\nu_{2k-1,j}+\nu_{2k-1,r}-ic)}e^{-ic\partial_{\nu_{2k-1,j}}} \end{aligned} \tag{2.10}$$

$$\begin{aligned} iI_{2k+2,2k+1} = &\frac{1}{c}\sum_{j=1}^{k}\frac{\prod_{r=1}^{k+1}((\nu_{2k,j}+\frac{ic}{2})^2-\nu_{2k+1,r}^2)}{2\nu_{2k,j}(\nu_{2k,j}+\frac{ic}{2})\prod_{r\neq j}(\nu_{2k,j}^2-\nu_{2k,r}^2)}e^{ic\partial_{\nu_{2k,j}}} \\ &+\frac{1}{c}\sum_{j=1}^{k}\frac{\prod_{r=1}^{k}((\nu_{2k,j}-\frac{ic}{2})^2-\nu_{2k-1,r}^2)}{2\nu_{2k,j}(\nu_{2k,j}-\frac{ic}{2})\prod_{r\neq j}(\nu_{2k,j}^2-\nu_{2k,r}^2)}e^{-ic\partial_{\nu_{2k,j}}} \\ &-\frac{1}{ic}\frac{\prod_{r=1}^{k}\nu_{2k-1,r}\prod_{r=1}^{k+1}\nu_{2k+1,r}}{\prod_{r=1}^{k}(\nu_{2k,r}+\frac{ic}{2})(\nu_{2k,r}-\frac{ic}{2})} \end{aligned} \tag{2.11}$$

Proposition 1 *The operators (2.7) and (2.8) satisfy the defining relations (2.6) of the generators of orthogonal Lie algebras $so(n)$, $n \geq 2$*

Surely, this statement follows from its validity in finite-dimensional representations, since the relations are then satisfied on sufficiently many integer points. However, since the proof of the formulas is missing in [12], we checked the defining relation (2.6) directly.

For a fixed n, the relations (2.10)–(2.11) can be interpreted as an infinite-dimensional representation M_n of Lie algebra $so(n+1)$ in the space of meromorphic functions over $\nu_{k,j}$, $k \leq n-1$ with poles at

$$\nu_{2k+1,j}-\nu_{2k+1,r}, \quad \nu_{2k+1,j}+\nu_{2k+1,r}\pm ic, \quad \nu_{2k,j}, \quad \nu_{2k,j}\pm\frac{ic}{2}, \quad \nu_{2k,j}\pm\nu_{2k,r}$$

The variables $\boldsymbol{\nu}_n = \{\nu_{n,1}, \dots \nu_{n,[\frac{n-1}{2}]}\}$ are not touched by the Lie algebra generators and can be regarded as parameters of submodules $M_{\boldsymbol{\nu}_n}$ of M_n

Proposition 2 *The center of $SO(2n+1)$ acts by multiplication on symmetric polynomials in $\nu_{2n,k}^2$. The center of $SO(2n)$ acts by multiplication of polynomials on $\nu_{2n-1,k}^2$, symmetric with respect to the permutations of the variables, and by powers of the monomial $\nu_{2n-1,1}\nu_{2n-,2}\cdots\nu_{2n-1,n}$.*

This follows from Harish–Chandra isomorphism, see, e.g., [1, Sect. 7.4].

Define the following automorphism of the space of meromorphic functions on ν_{kl} and c:

$$\tau(\nu_{2k+1,j}) = \nu_{2k+1,j}, \qquad \tau(\nu_{2k,j}) = -\nu_{2k,j}, \qquad \tau(c) = -c \tag{2.12}$$

Lemma 1 *We have the relations*

$$\tau I_{k,k+1} = -I_{k,k+1}\tau \tag{2.13}$$

3 Whittaker vectors

3.1 Two chains of groups

Zhelobenko–Stern construction of the Gelfand–Tsetlin basis for orthogonal groups uses the chain of embeddings

$$\mathbf{i}_n : SO(N) \hookrightarrow SO(N+1) \tag{3.1}$$

where the compact group $SO(N)$ is embedded into the compact group $SO(N+1)$ as the stabilizer of the vector e_{N+1} so that the generators $I_{kj}, k, j \leq N$ of the Lie algebra $so(N)$ are identified with the corresponding generators of the Lie algebra $so(N+1)$.

However, for the construction of Whittaker vectors in the related infinite-dimensional representations of $so(N)$ in meromorphic functions we pass to another, noncompact real form $SO(N, J)$ of the group $SO(N, \mathbb{C})$ and use the chain of the corresponding Lie algebras compatible with the natural chain of Lie group $SO(N)$. Here,

$$J = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ 0 & 0 & ... & 1 & 0 \\ & & ... & & \\ 0 & 1 & ... & 0 & 0 \\ 1 & 0 & ... & 0 & 0 \end{pmatrix}$$

The Lie algebra $so(N)$ is generated by elements I_{kj}, with the relation $I_{jk} = -I_{kj}$, so that the elements I_{kj} with $k > j$ are chosen as a linear basis of Lie algebra $so(N)$. The Lie algebra $so(N, J)$, acting in the space with the basis $f_1, \ldots, f_N$, is generated by the elements

$$F_{kj} = f_{kj} - f_{\hat{j}\hat{k}}, \qquad \text{where} \qquad f_{kj}(f_l) = \delta_{jl} f_k \qquad \text{and} \qquad \hat{k} = N + 1 - k$$

with the relation $F_{\hat{j},\hat{k}} = -F_{k,j}$ so that the elements F_{kj} with $k + j \leq N$ can be chosen as a linear basis of the Lie algebra $so(N, J)$. The elements F_{kj} for $k < j$ form a positive nilpotent subalgebra, and the elements F_{kk} form a Cartan subalgebra.

The chain of embedding

$$\mathbf{j}_n : SO(N, J) \hookrightarrow SO(N+1, J) \tag{3.2}$$

is different. The group $SO(2n, J)$ is the stabilizer of the vector f_{n+1} in the group $SO(2n+1, J)$, while the group $SO(2n-1, J)$ is the stabilizer of the element $f_{n-1} + (-1)^{n-1} f_n$ in $SO(2n, J)$. Let us describe the maps

$$\mathbf{s}_N : so(N) \to so(N, J) \tag{3.3}$$

of **complex** Lie algebras, which intertwine the embeddings (3.1) and (3.2). On the level of bases of the vector space $\mathbb{C}^{2n}$, the map $\mathbf{s}_{2n}$ corresponds to the transformation of initial orthogonal basis $e_1, \dots, e_{2n}$ of $\mathbb{C}^{2n}$ to the defining basis $f_1, \dots, f_{2n}$ of the form J,

$$(f_1, \dots, f_n, f_{n+1}, \dots, f_{2n}), \qquad (f_i, f_{\hat{k}}) = \delta_{ik}$$

by the relation

$$f_j = i^j \cdot \frac{ie_{2j-1} + e_{2j}}{\sqrt{2}}, \quad f_{\hat{j}} = i^{-j} \cdot \frac{-ie_{2j-1} + e_{2j}}{\sqrt{2}}, \quad j = 1, , \dots, n \tag{3.4}$$

For the group $SO(2n+1)$, we transform the initial orthogonal basis $e_1, \dots, e_{2n+1}$ of $\mathbb{C}^{2n+1}$ to the defining basis $f_1, \dots, f_{2n+1}$ of the form J,

$$(f_1, \dots, f_n, f_{n+1}, \dots, f_{2n+1}), \qquad (f_i, f_{\hat{j}}) = \delta_{ij}$$

by the relation

$$f_j = i^j \cdot \frac{ie_{2j-1} + e_{2j}}{\sqrt{2}}, \qquad f_{\hat{j}} = i^{-j} \cdot \frac{-ie_{2j-1} + e_{2j}}{\sqrt{2}}, \qquad j = 1, \dots, n, \quad f_{n+1} = e_{n+1}$$

Correspondingly, the transformation formula from Lie algebra elements I_{kj} to F_{kj} that are given by conjugation of the matrix (I_{kj}) be means of the corresponding transition matrix. In particular, we have the following expressions for the generators of Lie algebra $so(2n, J)$:

$$\begin{aligned}
F_{j,j+1} &= \frac{1}{2}(I_{2j+1,2j} - I_{2j+2,2j-1}) + \frac{i}{2}(I_{2j+2,2j} + I_{2j+1,2j-1}), \\
F_{j+1,j} &= \frac{1}{2}(I_{2j+2,2j-1} - I_{2j+1,2j}) + \frac{i}{2}(I_{2j+2,2j} + I_{2j+1,2j-1},) \\
F_{j,2n-j} &= (-1)^j \Big(\frac{1}{2}(-I_{2j+1,2j} - I_{2j+2,2j-1}) + \frac{i}{2}(I_{2j+2,2j} - I_{2j+1,2j-1}) \Big), \\
F_{2n-j,j} &= (-1)^j \Big(\frac{1}{2}(I_{2j+2,2j-1} + I_{2j+1,2j}) + \frac{i}{2}(I_{2j+2,2j} - I_{2j+1,2j-1}) \Big),
\end{aligned} \tag{3.5}$$

$$F_{j,j} = -iI_{2j,2j-1} \tag{3.6}$$

Here, $j = 1, \ldots, n-1$. Besides, instead of the use of the last simple root generator $F^{2n-1}_{n-1,n}$ of the Lie algebra $so(2n-1, J)$ it is convenient to use its image

$$\mathbf{j}_{2n-1}(F^{2n-1}_{n-1,n}) = \frac{1}{\sqrt{2}}\big(F^{2n}_{n-1,n} - (-1)^{n-1}F^{2n}_{n-1,n+1}\big)$$

$$\mathbf{j}_{2n-1}(F^{2n-1}_{n,n-1}) = \frac{1}{\sqrt{2}}\big(F^{2n}_{n,n-1} - (-1)^{n-1}F^{2n}_{n+1,n-1}\big)$$

in the Lie algebra $so(2n)$.

Note that for the Lie algebra $so(n)$, the automorphism τ represents the longest element of the Weyl group,

$$\tau F_{j,j+1} = F_{j+1,j}\tau. \tag{3.7}$$

3.2 Right and left Whittaker vectors

Recall the definition of Whittaker vectors. Let $\mathbf{g}$ be a reductive Lie algebra with Chevalley generators [8] $\{e_j, f_j, h_j|, j = 1, \ldots n\}$ of its semisimple part and M be a $\mathbf{g}$ module. Here, e_j generate the maximal nilpotent subalgebra $\mathbf{n}_+$, f_j generate the opposite maximal nilpotent subalgebra $\mathbf{n}_-$, and h_j for a basis of Cartan subalgebra $\mathbf{h}$. Vector $v \in M$ is called left Whittaker vector, if $e_j v = a_j v$, $j = 1, \ldots n$ where $a_j \in \mathbb{C}$, $a_j \neq 0$. Analogously, vector $v' \in M$ is called right Whittaker vector, if $f_j v = b_j v$, $j = 1, \ldots n$ where $b_j \in \mathbb{C}$, $b_j \neq 0$.

For further convenience, we denote by $\boldsymbol{\nu}_n$ the tuples of variables

$$\boldsymbol{\nu}_{2k} = \{\nu_{2k,1}, \ldots \nu_{2k,k}\}, \qquad \boldsymbol{\nu}_{2k-1} = \{\nu_{2k-1,1}, \ldots \nu_{2k-1,k}\} \tag{3.8}$$

and by $\hat{\boldsymbol{\nu}}_n$ the Gelfand–Tsetlin array

$$\widehat{\boldsymbol{\nu}}_n = \begin{pmatrix} \boldsymbol{\nu}_n \\ \boldsymbol{\nu}_{n-1} \\ \vdots \\ \boldsymbol{\nu}_2 \\ \boldsymbol{\nu}_1 \end{pmatrix} \tag{3.9}$$

For any two sets

$$\mathbf{X} = (x_1, \ldots, x_n), \qquad \mathbf{Y} = (y_1, \ldots, y_m)$$

set

$$s(\mathbf{X}; \mathbf{Y}) = \prod_{x\in\mathbf{X},\, y\in\mathbf{Y}} (ic)^{\frac{x+y}{ic}} \cdot \Gamma\left(\frac{x+y}{ic} + \frac{1}{2}\right) \tag{3.10}$$

With their help for any $k > 0$ define the meromorphic functions $W_k^{\pm}$ and $V_k^{\pm}$ by the relation

$$\begin{aligned} W_k^{\pm} &= e^{\mp\frac{\pi}{2c}\sum_j \nu_{2k,j}} s(\pm\boldsymbol{\nu}_{2k-1}, \boldsymbol{\nu}_{2k}) s(-\boldsymbol{\nu}_{2k}, \pm\boldsymbol{\nu}_{2k+1}), \\ V_k^{\pm} &= e^{\mp\frac{\pi}{2c}\sum_j \nu_{2k,j}} s(\pm\boldsymbol{\nu}_{2k-1}, \boldsymbol{\nu}_{2k}). \end{aligned} \tag{3.11}$$

Set

$$\begin{aligned} w_n &= e^{-\frac{\pi n}{2c}\delta_{2n}} W_1^{+} W_2^{-} \cdots W_{n-1}^{(-)^n} V_n^{(-)^{n+1}}, \\ w_n' &= e^{\frac{\pi}{c}\sum_{k=1}^{n}\delta_{2k-1}} \tau(w_n) = e^{-\frac{\pi n}{2c}\delta_{2n}} W_1^{-} W_2^{+} \cdots W_{n-1}^{(-)^{n-1}} V_n^{(-)^n} \end{aligned} \tag{3.12}$$

Here,

$$\delta_{2k} = \sum_j \nu_{2k,j}, \qquad \delta_{2k+1} = (-1)^k \sum_j \nu_{2k+1,j}.$$

Theorem 1 *The functions w_n and w_n' are left and right $SO(2n+1)$ Whittaker vectors:*

$$F_{k,k+1} w_n = \frac{(-1)^{k+1}}{ic} w_n, \qquad k < n, \qquad F_{n,n+1} w_n = \frac{(-1)^{n+1}}{\sqrt{2}ic} w_n, \tag{3.13}$$

$$F_{k+1,k} w_n' = \frac{(-1)^{k+1}}{ic} w_n', \qquad k < n, \qquad F_{n+1,n} w_n' = \frac{(-1)^{n+1}}{\sqrt{2}ic} w_n'. \tag{3.14}$$

Proof is given in Appendix A

Precise formulas for Whittaker vector and Whittaker function look better after the following change of variables

$$\gamma_{2k-1,j} = (-1)^{k+1} \nu_{2k-1,j}, \qquad \gamma_{2k,j} = \nu_{2k,j}. \tag{3.15}$$

Then,

$$\delta_k = \sum_j \gamma_{kj}, \qquad \gamma_{kj} \in \boldsymbol{\gamma}_k$$

, and the Whittaker vectors can be written as

$$\begin{aligned} w_n &= e^{-\frac{\pi n}{2c}\delta_{2n}} e^{\sum_{k=1}^{n}(-1)^k \delta_{2k}} \prod_{k=1}^{n} s(\boldsymbol{\gamma}_{2k-1}, \boldsymbol{\gamma}_{2k}) \prod_{k=1}^{n-1} s(-\boldsymbol{\gamma}_{2k}, -\boldsymbol{\gamma}_{2k+1}) \\ w_n' &= e^{-\frac{\pi n}{2c}\delta_{2n}} e^{\frac{\pi}{2c}\sum_{k=1}^{n}(-1)^{k+1}\delta_{2k}} \prod_{k=1}^{n} s(-\boldsymbol{\gamma}_{2k-1}, \boldsymbol{\gamma}_{2k}) \prod_{k=1}^{n-1} s(-\boldsymbol{\gamma}_{2k}, \boldsymbol{\gamma}_{2k+1}) \end{aligned} \tag{3.16}$$

3.3 Action of Cartan subalgebra

In Gelfand–Tsetlin representation of gl_n see [4], the Cartan subalgebra acts by multiplication by linear functions on the variables γ_{kj}. It is not so for Gelfand–Tsetlin representations of $so(n)$, which we study here. However, the Cartan subalgebra of $so(n)$ acts in a similar way on Whittaker vectors. The Cartan subalgebra of $so(n)$ is generated by the elements $F_{kk} = -iI_{2k,2k-1}$.

Proposition 3

$$F_{kk}w_n = \frac{(-1)^{k-1}}{ic}\left(\sum_{j=1}^{k}\nu_{2k-1,j} + \sum_{j=1}^{k-1}\nu_{2k-3,j} + (-1)^k(k-1)ic\right)w_n \tag{3.17}$$

$$F_{kk}w'_n = \frac{(-1)^{k-1}}{ic}\left(\sum_{j=1}^{k}\nu_{2k-1,j} + \sum_{j=1}^{k-1}\nu_{2k-3,j} - (-1)^k i(k-1)c\right)w'_n \tag{3.18}$$

In the variables γ_{kj}, the relations (3.17) look as follows:

$$F_{kk}w_n = \frac{1}{ic}\left(\sum_{j=1}^{k}\gamma_{2k-1,j} - \sum_{j=1}^{k-1}\gamma_{2k-3,j} - (k-1)ic\right)w_n \tag{3.19}$$

$$F_{kk}w'_n = \frac{1}{ic}\left(\sum_{j=1}^{k}\gamma_{2k-1,j} - \sum_{j=1}^{k-1}\gamma_{2k-3,j} + (k-1)ic\right)w'_n \tag{3.20}$$

Proof is given in Appendix B

4 Whittaker wave function

Assume that a representation V of a reductive Lie algebra contains a left Whittaker vector V, a representation V' contains a right Whittaker vector v', and there is an invariant pairing $(,)$ between V and V'. Then, it is well known [11] that the matrix coefficient

$$F(\mathbf{h}) = (v', e^{\mathbf{h}}v)$$

regarded as a function on the Cartan subalgebra $\mathbf{h}$ satisfies a system of differential equations known as eigenfunction equations for Toda system, related to the root system corresponding to $\mathbf{g}$. In this section, we study the matrix coefficient (4.4) between Whittaker vectors (3.16) and show that it is an eigenfunction of B_n quadratic Toda Hamiltonian.

4.1 Invariant pairing

The invariant pairing looks the same in both ν and γ variables.

Define the functions $\tilde{\mu}(\boldsymbol{\gamma}_{2k})$ and $\tilde{\mu}(\boldsymbol{\gamma}_{2k+1})$ by the relations

$$\begin{aligned}\tilde{\mu}(\boldsymbol{\gamma}_{2k}) &= e^{\frac{\pi}{c}\sum_j \gamma_{2k,j}} \prod_r \Gamma^{-1}\Big(\frac{2\gamma_{2k,r}}{ic}\Big)\Gamma^{-1}\Big(\frac{-2\gamma_{2k,r}}{ic}\Big) \\ &\quad \times \prod_{r\neq s} \Gamma^{-1}\Big(\frac{\gamma_{2k,r}-\gamma_{2k,s}}{ic}\Big) \prod_{r<s} \Gamma^{-1}\Big(\frac{\gamma_{2k,r}+\gamma_{2k,s}}{ic}\Big)\Gamma^{-1}\Big(\frac{-\gamma_{2k,r}-\gamma_{2k,s}}{ic}\Big), \\ \tilde{\mu}(\boldsymbol{\gamma}_{2k+1}) &= \prod_{r\neq s} \Gamma^{-1}\Big(\frac{\gamma_{2k+1,r}-\gamma_{2k+1,s}}{ic}\Big) \\ &\quad \prod_{r<s} \Gamma^{-1}\Big(1+\frac{\gamma_{2k+1,r}+\gamma_{2k+1,s}}{ic}\Big)\Gamma^{-1}\Big(1-\frac{\gamma_{2k+1,r}+\gamma_{2k+1,s}}{ic}\Big),\end{aligned} \tag{4.1}$$

that is

$$\begin{aligned}\tilde{\mu}(\boldsymbol{\gamma}_{2k}) &= e^{\frac{\pi}{c}\delta_{2k}} \prod_{r<s} \Big|\Gamma\Big(\frac{\gamma_{2k,r}-\gamma_{2k,s}}{ic}\Big)\Big|^{-2} \Big|\Gamma\Big(\frac{\gamma_{2k,r}+\gamma_{2k,s}}{ic}\Big)\Big|^{-2} \prod_r \Big|\Gamma\Big(\frac{2\gamma_{2k,r}}{ic}\Big)\Big|^{-2}, \\ \tilde{\mu}(\boldsymbol{\gamma}_{2k+1}) &= \prod_{r<s} \Big|\Gamma\Big(\frac{\gamma_{2k+1,r}-\gamma_{2k+1,s}}{ic}\Big)\Big|^{-2} \Big|\Gamma\Big(1+\frac{\gamma_{2k+1,r}+\gamma_{2k+1,s}}{ic}\Big)\Big|^{-2}\end{aligned} \tag{4.2}$$

and define a scalar product on functions in M as

$$(f,g)_{2n} = \int_{\mathbb{R}^{n^2}} \bar{f}(\widehat{\boldsymbol{\gamma}}_{2n-1}) g(\widehat{\boldsymbol{\gamma}}_{2n-1}) \tilde{\mu}(\widehat{\boldsymbol{\gamma}}_{2n-1}) d\widehat{\boldsymbol{\gamma}}_{2n-1} \tag{4.3}$$

where

$$\tilde{\mu}(\widehat{\boldsymbol{\gamma}}_n) = \prod_{k=1}^{n} \tilde{\mu}(\boldsymbol{\gamma}_k)$$

Then,

Lemma 2 *The operators $I_{2k+1,2k}$ and $i I_{2k+2,2k+1}$ are skew symmetric with respect to the pairing (4.3). In particular, operators F_{kl} are skew symmetric with respect to the pairing (4.3).*

4.2 Integral formula

Due to (3.16), the product $\bar{w'}_n w_n$ looks as

$$\begin{aligned}\bar{w'}_n w_n &= e^{-\frac{\pi n}{c}\delta_{2n}} \prod_{k=1}^{n-1} s(\boldsymbol{\gamma}_{2k-1},\boldsymbol{\gamma}_{2k})\bar{s}(-\boldsymbol{\gamma}_{2k-1},\boldsymbol{\gamma}_{2k}) s(-\boldsymbol{\gamma}_{2k},-\boldsymbol{\gamma}_{2k+1})\bar{s}(-\boldsymbol{\gamma}_{2k},\boldsymbol{\gamma}_{2k+1}) \\ &\quad \times s(\boldsymbol{\gamma}_{2n-1},\boldsymbol{\gamma}_{2n})\bar{s}(-\boldsymbol{\gamma}_{2n-1},\boldsymbol{\gamma}_{2n})\end{aligned}$$

or, in terms of Gamma functions

$$w_n \cdot \overline{w_n'} = e^{-\frac{\pi}{c}\sum\limits_{k=1}^{n-1}\delta_{2k}} c^{\frac{1}{ic}\left(\sum\limits_{k=1}^{n-1} 2k(\delta_{2k-1}-\delta_{2k+1})+2n\delta_{2n-1}\right)} \times \left(\prod_{k=1}^{n-1} \Gamma\left(\frac{\pm\boldsymbol{\gamma}_{2k}+\boldsymbol{\gamma}_{2k-1}}{ic}+\frac{1}{2}\right)\Gamma\left(\frac{\pm\boldsymbol{\gamma}_{2k}-\boldsymbol{\gamma}_{2k+1}}{ic}+\frac{1}{2}\right)\right) \cdot \Gamma\left(\frac{\pm\boldsymbol{\gamma}_{2n}+\boldsymbol{\gamma}_{2n-1}}{ic}+\frac{1}{2}\right)$$

Thus, the Whittaker function

$$\Psi_{\boldsymbol{\gamma}_{2n}} = (w_n', \, e^{-\sum_{k=1}^n x_k F_{kk}} w_n)_{2n} \tag{4.4}$$

is given by the integral

$$\Psi_{\boldsymbol{\gamma}_{2n}}(\mathbf{x}_n) = \int_{\mathbb{R}^{n^2}} \prod_{k=1}^{2n-1} \mu(\boldsymbol{\gamma}_k) d\boldsymbol{\gamma}_k \cdot c^{\frac{2}{ic}\sum_{k=1}^n \delta_{2k-1}} e^{\frac{1}{ic}\sum_{k=1}^n (\delta_{2k-3}-\delta_{2k-1}+(k-1)ic)x_k} \times \prod_{k=1}^{n-1}\prod_{r,j=1}^{k}\prod_{l=1}^{k+1} \Gamma\left(\frac{\pm\gamma_{2k,r}+\gamma_{2k-1,j}}{ic}+\frac{1}{2}\right)\Gamma\left(\frac{\pm\gamma_{2k,r}-\gamma_{2k+1,l}}{ic}+\frac{1}{2}\right) \times \prod_{r,j=1}^{n} \Gamma\left(\frac{\pm\gamma_{2n,r}+\gamma_{2n-1,j}}{ic}+\frac{1}{2}\right) \tag{4.5}$$

Here,

$$\begin{aligned} \mu(\boldsymbol{\gamma}_{2k}) &= e^{-\frac{\pi}{c}\delta_{2k}}\tilde{\mu}(\boldsymbol{\gamma}_{2k}) \\ &= \prod_{r<s}\left|\Gamma\Big(\frac{\gamma_{2k,r}-\gamma_{2k,s}}{ic}\Big)\right|^{-2}\left|\Gamma\Big(\frac{\gamma_{2k,r}+\gamma_{2k,s}}{ic}\Big)\right|^{-2}\prod_{r}\left|\Gamma\Big(\frac{2\gamma_{2k,r}}{ic}\Big)\right|^{-2}, \\ \mu(\boldsymbol{\gamma}_{2k+1}) &= \tilde{\mu}(\boldsymbol{\gamma}_{2k+1}) \\ &= \prod_{r<s}\left|\Gamma\Big(\frac{\gamma_{2k+1,r}-\gamma_{2k+1,s}}{ic}\Big)\right|^{-2}\left|\Gamma\Big(1+\frac{\gamma_{2k+1,r}+\gamma_{2k+1,s}}{ic}\Big)\right|^{-2} \end{aligned} \tag{4.6}$$

The measure functions $\mu(\boldsymbol{\gamma}_j)$ do not contain exponential factors.

The convergence of the integral (4.5) can be proved by the arguments given in [7, Appendix A]. Namely, let us complete the sequence

$$\boldsymbol{\gamma}_1 = \{\gamma_{11}\}, \boldsymbol{\gamma}_2 = \{\gamma_{21}\}, \boldsymbol{\gamma}_3 = \{\gamma_{31}, \gamma_{32}\}, \dots \boldsymbol{\gamma}_{2n} = \{\gamma_{2n,1}, \dots \gamma_{2n,n}\},$$

to the sequence

$$\boldsymbol{\gamma}'_1 = \{\gamma'_{1,-1}, \gamma'_{11}\}, \boldsymbol{\gamma}'_2 = \{\gamma'_{2,-1}, \gamma'_{2,0}, \gamma'_{21}\}, , \dots \boldsymbol{\gamma}'_{2n} = \{\gamma_{2n,-n}, \dots \gamma_{2n,n}\},$$

where

$$\gamma'_{n,k} = \gamma_{n,k}, \ \gamma'_{n,-k} = -\gamma_{n,k} \quad \text{for} \quad k > 0, \quad \text{and} \quad \gamma'_{2m,0} = 0$$

as is customary in the representation theory of orthogonal groups. Then, the inequality [7, 34] applied to this sequence is transformed to the bound

$$\begin{aligned} &\sum_{k=1}^{2n}\sum_{r,j}|\pm\gamma_{2k,r} - \gamma_{2k\pm1,j}| - \sum_{k=1}^{2n-1}\sum_{r\neq j}|\pm\gamma_{k,r} - \gamma_{k,j}| - 2\sum_{k=1}^{n-1}\sum_{j}|\gamma_{2k,j}| \\ &\geq C(\boldsymbol{\gamma}_{2n}) + \frac{2}{n}\sum_{k=1}^{2n-1}\sum_{j}|\gamma_{k,j}|, \end{aligned} \tag{4.7}$$

where the constant $C(\boldsymbol{\gamma}_{2n})$ depends on the values of $\gamma_{2n,i}$. Due to the asymptotics of the Gamma function

$$\Gamma(ix) \sim e^{-\pi|x|/2}$$

in imaginary direction, we observe that the integrand of (4.5) can be bounded by (4.7) as

$$C'(\boldsymbol{\gamma}_{2n})e^{-\frac{\pi-\varepsilon}{cn}\sum_{k=1}^{2n-1}\sum_j|\gamma_{k,j}|} \tag{4.8}$$

for any small positive $\varepsilon > 0$ and a proper positive constant $C'(\boldsymbol{\gamma}_{2n})$, which implies absolute convergence of the integral (4.5)

4.3 Toda equation

Denote by H_n^B the Toda Hamiltonian

$$H_{B_n} = \sum_{k=1}^{n}\left(-\frac{\partial^2}{\partial x_k^2} + (2n-2k+1)\frac{\partial}{\partial x_k}\right) + \sum_{k=1}^{n-1}\frac{2}{c^2}e^{x_k - x_{k+1}} + \frac{1}{c^2}e^{x_n} \tag{4.9}$$

Theorem 2 *The Whittaker function (4.5) is a wave function for B_n Toda Hamiltonian:*

$$H_{B_n}\Psi_{\boldsymbol{\gamma}_{2n}}(\mathbf{x}_n) = \left(\frac{1}{c^2}\sum_{j=1}^{n}\gamma_{2n,j}^2 + \frac{n(2n-1)(2n+1)}{12}\right)\Psi_{\boldsymbol{\gamma}_{2n}}(\mathbf{x}_n) \tag{4.10}$$

Proof is a standard game with the matrix element $G(\mathbf{x}_n)$ in representation $M_{\boldsymbol{\gamma}_{2n}}$

$$G(\mathbf{x}_n) = (w'_n, \ L_{2n+1}\, e^{-\sum_k x_k F_{kk}} w_n)_{2n} \tag{4.11}$$

where

$$L_{2n+1} = \frac{1}{2}\sum_{i,j=1}^{2n+1} I_{ij}^2 = \frac{1}{2}\sum_{i,j=1}^{2n+1} F_{ij}F_{ji}$$

is Laplace operator of $SO(2n+1)$. For this one should also know the eigenvalue of L in the representation $M_{\boldsymbol{\gamma}_{2n}}$. But it is known from the theory of highest weight representations of $so(2n+1)$. It gives us the eigenvalue

$$\sum_{j=1}^{n} m_{2n,j}^2 - (\rho, \rho)$$

where

$$\rho = \left(n - \frac{1}{2}, \ldots, \frac{1}{2}\right)$$

is a half sum of positive roots for $so(2n+1)$. Thus, $G(\mathbf{x}_n)$ (once we act by L to the left) is equal to

$$-\frac{1}{c^2}\sum_{j=1}^{n} \gamma_{2n,j}^2 - \frac{n(2n-1)(2n+1)}{12} \tag{4.12}$$

On the other hand, we can rewrite L as

$$L = \sum_{k=1}^{n} F_{kk}^2 + 2\sum_{1\le k<l\le 2n} F_{lk}F_{kl} - \sum_{k=1}^{n}(2n-2k+1)F_{kk}$$

so that

$$\begin{aligned} G(\mathbf{x}_n) = &\sum_{k=1}^{n}\left(\frac{\partial^2}{\partial x_k^2} - (2n-2k+1)\frac{\partial}{\partial x_k}\right) \\ &+ 2\sum_{1\le r<l\le 2n} (w_n',\ F_{lr}F_{rl}\, e^{-\sum_k x_k F_{kk}} w_n)_{2n} \end{aligned} \tag{4.13}$$

Due to the skew symmetry, we can act by F_{lr} on w_n' and, by (3.14) the last sum in (4.13) can be rewritten as

$$\begin{aligned} &\sum_{l=1}^{n-1} \frac{(-1)^l}{ic}(w_n',\ F_{l,l+1},\ e^{-\sum_k x_k F_{kk}} w_n)_{2n} + \frac{(-1)^n}{\sqrt{2}ic}(w_n',\ F_{n,n+1},\ e^{-\sum_k x_k F_{kk}} w_n)_{2n} \\ &= -\left(\sum_{k=1}^{n-1}\frac{1}{c^2}e^{x_k - x_{k+1}} + \frac{1}{2c^2}e^{x_n}\right)\Psi_{\boldsymbol{\gamma}_{2n}}(\mathbf{x}_n) \end{aligned}$$

Combining this with (4.13) and (4.12), we arrive to (4.10) □

Remark 1 Note that the function

$$\tilde{\Psi}_{\boldsymbol{\gamma}_{2n}}(\mathbf{x}_n) = e^{-(\rho,\mathbf{x}_n)}\Psi_{\boldsymbol{\gamma}_{2n}}(\mathbf{x}_n) = e^{-\sum\limits_{k=1}^{n}(n-k+\frac{1}{2})x_k}\Psi_{\boldsymbol{\gamma}_{2n}}(\mathbf{x}_n)$$

is the solution of more familiar spectral problem

$$\left(-\sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}+\sum_{k=1}^{n-1}\frac{2}{c^2}e^{x_k-x_{k+1}}+\frac{1}{c^2}e^{x_n}\right)\tilde{\Psi}_{\boldsymbol{\gamma}_{2n}}(\mathbf{x}_n) = \frac{1}{c^2}\left(\sum_{j=1}^{n}\gamma_{2n,j}^2\right)\tilde{\Psi}_{\boldsymbol{\gamma}_{2n}}(\mathbf{x}_n)$$

4.4 Iterative procedures

1. The integral (4.5) can be formulated as an iterative integral presentation of the Whittaker wave function,

$$\begin{aligned}\Psi_{\boldsymbol{\gamma}_{2n}}(x_1,\ldots,x_n) = &\int_{\mathbb{R}^{2n-1}}\mu(\boldsymbol{\gamma}_{2n-1})d\boldsymbol{\gamma}_{2n-1}\,\mu(\boldsymbol{\gamma}_{2n-2})d\boldsymbol{\gamma}_{2n-2}\\ &\prod_{j=1}^{n-1}\prod_{l=1}^{n}\Gamma\left(\frac{\pm\gamma_{2n-2,j}-\gamma_{2n-1,l}}{ic}+\frac{1}{2}\right)\prod_{r,j=1}^{n}\Gamma\left(\frac{\pm\gamma_{2n,r}+\gamma_{2n-1,j}}{ic}+\frac{1}{2}\right)\\ &\times c^{\frac{2}{ic}\delta_{2n-1}}e^{-\frac{\delta_{2n-1}}{ic}x_n+((n-1)+\sum_{k=1}^{n-2}k)x_n}\Psi_{\boldsymbol{\gamma}_{2n-2}}(x_1-x_n,\ldots,x_{n-1}-x_n)\end{aligned} \tag{4.14}$$

or

$$\Psi_{\boldsymbol{\gamma}_{2n}}(x_1,\ldots,x_n) = \Lambda_n(x_n)\left(\Psi_{\boldsymbol{\gamma}_{2n-2}}(x_1-x_n,\ldots,x_{n-1}-x_n)\right) \tag{4.15}$$

where $\Lambda(x_n)$ is an integral operator

$$(\Lambda_n(x)f)(\boldsymbol{\gamma}_{2n}) = \int_{\mathbb{R}^{n-1}}K(\boldsymbol{\gamma}_{2n};\boldsymbol{\gamma}_{2n-2}|x)f(\boldsymbol{\gamma}_{2n-2})\mu(\boldsymbol{\gamma}_{2n-2})d\boldsymbol{\gamma}_{2n-2} \tag{4.16}$$

with the kernel

$$\begin{aligned}K(\boldsymbol{\gamma}_{2n};\boldsymbol{\gamma}_{2n-2}|x) = e^{\frac{n(n-1)}{2}x}&\int_{\mathbb{R}^n}\mu(\boldsymbol{\gamma}_{2n-1})d\boldsymbol{\gamma}_{2n-1}c^{\frac{2}{ic}\delta_{2n-1}}e^{-\frac{\delta_{2n-1}}{ic}x}\\ &\prod_{j=1}^{n-1}\prod_{l=1}^{n}\Gamma\left(\frac{\pm\gamma_{2n-2,j}-\gamma_{2n-1,l}}{ic}+\frac{1}{2}\right)\\ &\prod_{r,j=1}^{n}\Gamma\left(\frac{\pm\gamma_{2n,r}+\gamma_{2n-1,j}}{ic}+\frac{1}{2}\right)\end{aligned} \tag{4.17}$$

2. Another recurrent procedure uses the observation that the above construction of the Whittaker vectors and Whittaker functions, restricted to $SO(2n)$, produced actually GL_n Whittaker vectors and functions. It can be seen from the relation (A.1), (A.2). In this way, we arrive by using Gustafson integrals [6] to Iorgov–Shadura formula [7], which expresses the B_n Toda wave function via A_n Toda wave function and contains in total twice less integrals.

The restriction $\Psi_{\boldsymbol{\gamma}_{2n-1}}(\mathbf{x}_n)$ of the wave function (4.5) to $SO(2n)$ is given by the integral

$$\Psi_{\boldsymbol{\gamma}_{2n-1}}(\mathbf{x}_n) = \int_{\mathbb{R}^{n^2-n}} \prod_{k=1}^{2n-2} \mu(\boldsymbol{\gamma}_k) d\boldsymbol{\gamma}_k \cdot c^{\frac{2}{ic}\sum_{k=1}^{n-1}\delta_{2k-1}} e^{\frac{1}{ic}\sum_{k=1}^{n}(\delta_{2k-3}-\delta_{2k-1}+(k-1)ic)x_k} \\ \times \prod_{k=1}^{n-1}\prod_{r,j=1}^{k}\prod_{l=1}^{k+1} \Gamma\left(\frac{\pm\gamma_{2k,r}+\gamma_{2k-1,j}}{ic}+\frac{1}{2}\right)\Gamma\left(\frac{\pm\gamma_{2k,r}-\gamma_{2k+1,l}}{ic}+\frac{1}{2}\right) \tag{4.18}$$

and the functions $\Psi_{\boldsymbol{\gamma}_{2n}}(\mathbf{x}_n)$ and $\Psi_{\boldsymbol{\gamma}_{2n-1}}(\mathbf{x}_n)$ are related as follows:

$$\Psi_{\boldsymbol{\gamma}_{2n}}(\mathbf{x}_n) = \int_{\mathbb{R}^n} \mu(\boldsymbol{\gamma}_{2n-1}) d\boldsymbol{\gamma}_{2n-1} c^{\frac{2}{ic}\delta_{2n-1}} \prod_{r,j=1}^{n} \Gamma\left(\frac{\pm\gamma_{2n,r}+\gamma_{2n-1,j}}{ic}+\frac{1}{2}\right)\Psi_{\boldsymbol{\gamma}_{2n-1}}(\mathbf{x}_n) \tag{4.19}$$

For each k, the integral over $\boldsymbol{\gamma}_{2k}$ in (4.18) can be explicitly calculated by means of the degenerate B_n Gustafson integral

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\prod_{i=1}^{2n+1}\prod_{j=1}^{n}\Gamma(a_i+iz_j))\Gamma(a_i-iz_j))}{\prod_{1\le i<j\le n}\left|\Gamma(i(z_i+z_j))\Gamma(i(z_i-z_j))\right|^2\prod_{j=1}^{n}\left|\Gamma(2iz_j)\right|^2} d\mathbf{x}_n \\ = n!2^n \prod_{1\le i<j\le 2n+1}\Gamma(a_i+a_j)\ , \tag{4.20}$$

where all a_i are assumed to have a positive real part. The integral (4.20) is a limiting case

$$a_{2n+2} = \varepsilon + iL, \qquad L\to\infty, \qquad \varepsilon\to +0$$

of a general B_n Gustafson integral [6]

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\prod_{l=1}^{2n+2}\prod_{j=1}^{n}\Gamma(a_l+iz_j))\Gamma(a_l-iz_j))}{\prod_{1\le l<j\le n}\left|\Gamma(i(z_l+z_j))\Gamma(i(z_l-z_j))\right|^2\prod_{j=1}^{n}\left|\Gamma(2iz_j)\right|^2} d\mathbf{x}_n \\ = \frac{n!2^n\prod_{1\le l<j\le 2n+2}\Gamma(a_l+a_j)}{\Gamma\left(\sum_{l=1}^{2n+2}a_l\right)}.$$

Using

$$a_1 = \frac{\gamma_{2k-1,1}}{ic} + \frac{1}{2}, \ldots, a_k = \frac{\gamma_{2k-1,k}}{ic} + \frac{1}{2}, \qquad a_{k+1} = -\frac{\gamma_{2k+1,1}}{ic} + \frac{1}{2}, \ldots$$
$$a_{2k+1} = -\frac{\gamma_{2k+1,k+1}}{ic} + \frac{1}{2},$$

we get

$$\int\limits_{\mathbb{R}^k} \frac{\prod_{i=1}^{k}\left(\prod_{j=1}^{k}\Gamma\left(\frac{\pm\gamma_{2k,i}+\gamma_{2k-1,j}}{ic}+\frac{1}{2}\right)\prod_{j=1}^{k+1}\Gamma\left(\frac{\pm\gamma_{2k,i}-\gamma_{2k+1,j}}{ic}+\frac{1}{2}\right)\right)}{\prod_{r<s}\left|\Gamma\left(\frac{\gamma_{2k,r}-\gamma_{2k,s}}{ic}\right)\right|^2\left|\Gamma\left(\frac{\gamma_{2k,r}+\gamma_{2k,s}}{ic}\right)\right|^2\prod_r\left|\Gamma\left(\frac{2\gamma_{2k,r}}{ic}\right)\right|^2} d\gamma_{2k,1}\ldots d\gamma_{2k,k}$$
$$= c^k(2\pi)^k 2^k \cdot k! \cdot \prod_{r<s}\Gamma\left(\frac{\gamma_{2k-1,r}+\gamma_{2k-1,s}}{ic}+1\right)\cdot\prod_{i,j}\Gamma\left(\frac{\gamma_{2k-1,i}-\gamma_{2k+1,j}}{ic}+1\right)$$
$$\times\prod_{r<s}\Gamma\left(-\frac{\gamma_{2k+1,r}+\gamma_{2k+1,s}}{ic}+1\right) \tag{4.21}$$

and

$$\Psi_{\boldsymbol{\gamma}_{2n-1}}(\mathbf{x}_n) = d_n \cdot c^{\frac{n+1}{ic}\delta_{2n-1}} \prod_{1\le r<s\le n}\Gamma\left(1-\frac{\gamma_{2n-1,r}+\gamma_{2n-1,s}}{ic}\right)\Phi_{\boldsymbol{\gamma}_{2n-1}}(\mathbf{x}_n) \tag{4.22}$$

where

$$\Phi_{\boldsymbol{\gamma}_{2n-1}}(\mathbf{x}_n) = \int\limits_{\mathbb{R}^{n(n-1)/2}} e^{\frac{1}{ic}\sum_{k=1}^{n}(\delta_{2k-3}-\delta_{2k-1}+(k-1)ic)x_k} d\gamma_1\ldots d\gamma_{2n-3}$$
$$\times\frac{\prod_{k=1}^{n-1}\prod_{l=1}^{k}\prod_{j=1}^{k+1} c^{\frac{\gamma_{2k-1,l}-\gamma_{2k+1,j}}{ic}}\Gamma\left(\frac{\gamma_{2k-1,l}-\gamma_{2k+1,j}}{ic}+1\right)}{\prod_{k=1}^{n-1}\prod_{r<s}\left|\Gamma\left(\frac{\gamma_{2k-1,r}-\gamma_{2k-1,s}}{ic}\right)\right|^2} \tag{4.23}$$

and

$$d_n = \prod_{k=1}^{n-1} c^k(2\pi)^k 2^k \cdot k! \tag{4.24}$$

Both functions $\Psi_{\mathbf{x}_n}(\gamma_{2n-1})$ and $\Phi_{\boldsymbol{\gamma}_{2n-1}}(\mathbf{x}_n)$ are solutions of $GL(n)$ Toda equations. In order to compare the final results with Iorgov–Shadura formula [7], we perform the change of integration variables

$$\gamma_{2k-1,j} \to \gamma_{2k-1,j} - \left(n-k+\frac{1}{2}\right)ic.$$

Then,

$$\Phi_{\boldsymbol{\gamma}_{2n-1}}(\mathbf{x}_n) = e^{\sum\limits_{k=1}^{n}(n-k+\frac{1}{2})x_k} \int\limits_{C_n} e^{\frac{1}{ic}\sum\limits_{k=1}^{n}(\delta_{2k-3}-\delta_{2k-1})x_k} d\gamma_1 \dots d\gamma_{2n-3}$$
$$\times \frac{\prod_{k=1}^{n-1}\prod_{l=1}^{k}\prod_{j=1}^{k+1} c^{\frac{\gamma_{2k-1,l}-\gamma_{2k+1,j}}{ic}} \Gamma\left(\frac{\gamma_{2k-1,l}-\gamma_{2k+1,j}}{ic}\right)}{\prod_{k=1}^{n-1}\prod_{r<s}\left|\Gamma\left(\frac{\gamma_{2k-1,r}-\gamma_{2k-1,s}}{ic}\right)\right|^2} \tag{4.25}$$

and

$$\Psi_{\boldsymbol{\gamma}_{2n}}(\mathbf{x}_n) = d_n \int\limits_{C'_n} \frac{\prod_{l=1}^{n}\prod_{j=1}^{n}\Gamma\left(\frac{\gamma_{2n-1,l}+\gamma_{2n,j}}{ic}\right)\Gamma\left(\frac{\gamma_{2n-1,l}-\gamma_{2n,j}}{ic}\right)}{\prod_{r<s}\Gamma\left(\frac{\gamma_{2n-1,r}+\gamma_{2n-1,s}}{ic}\right)\prod_{r<s}\left|\Gamma\left(\frac{\gamma_{2n-1,r}-\gamma_{2n-1,s}}{ic}\right)\right|^2}$$
$$\times c^{\frac{n+1}{ic}\delta_{2n-1}} \Phi_{\boldsymbol{\gamma}_{2n-1}}(\mathbf{x}_n) d\gamma_{2n-1} \tag{4.26}$$

Here, the contour C_n in (4.25) is a deformation of $\mathbb{R}^{n(n-1)/2}$, such that the integration over the variable γ_{2k-1} is performed in such a way that the singularity of $\Gamma\left(\frac{\gamma_{2k-1,l}-\gamma_{2k+1,j}}{ic}\right)$ is under the line of integration and the singularity of $\Gamma\left(\frac{\gamma_{2k-3,l}-\gamma_{2k-1,j}}{ic}\right)$ is above the line of integration over γ_{2k-1}. The contour C'_n in (4.26) is a deformation of $\mathbb{R}^n$ where the singularities of the nominators are under the contours of integrations over all variables.

The relations (4.25)–(4.26) are in accordance with Iorgov–Shadura formula [7, (26),(27)]. More precisely, in Iorgov–Shadura description the boundary wall corresponds to the first coordinate x_1, while we work with the boundary wall related to the last coordinate. One can observe the coincidence of formulas after the change of variables

$$x_k \to -x_{n+1-k}$$

and the following symmetry of the A_n Toda wave function:

$$\Phi_{\gamma_1,\dots,\gamma_n}(x_1,\dots x_n) = \Phi_{-\gamma_1,\dots,-\gamma_n}(-x_n,\dots,-x_1).$$

5 Examples

$\mathbf{n = 1}$. For one particle, the system and the wave function coincide with that of $sl(2)$ Toda system:

$$\Psi_{\gamma_{21}}(x_1) = \int_{\mathbb{R}} d\gamma_{11} c^{\frac{2}{ic}\gamma_{11}} e^{-\frac{1}{ic}\gamma_{11}x_1} \Gamma\left(\frac{\pm\gamma_{21}+\gamma_{11}}{ic}+\frac{1}{2}\right)$$

n = 2. The wave function for B_2 Toda system is given by fourfold integral, which can be reduced to threefold by using Gustafson integral,

$$\Psi_{\gamma_{41},\gamma_{42}}(x_1,x_2)=\int_{\mathbb{R}^4} d\gamma_{11}\frac{d\gamma_{21}}{\left|\Gamma\left(\frac{2\gamma_{21}}{ic}\right)\right|^2}\frac{d\gamma_{31}d\gamma_{32}}{\left|\Gamma\left(\frac{\gamma_{31}-\gamma_{32}}{ic}\right)\Gamma\left(1+\frac{\gamma_{31}+\gamma_{32}}{ic}\right)\right|^2}$$
$$\times c^{\frac{2}{ic}(\gamma_{11}+\gamma_{31}+\gamma_{32})}e^{\frac{1}{ic}(-\gamma_{11}x_1+(\gamma_{11}-\gamma_{31}-\gamma_{32}+ic)x_2)}$$
$$\times\Gamma\left(\frac{\pm\gamma_{21}+\gamma_{11}}{ic}+\frac{1}{2}\right)\Gamma\left(\frac{\pm\gamma_{21}-\gamma_{31}}{ic}+\frac{1}{2}\right)$$
$$\Gamma\left(\frac{\pm\gamma_{21}-\gamma_{32}}{ic}+\frac{1}{2}\right)\prod_{i,j=1}^{2}\Gamma\left(\frac{\pm\gamma_{4i}+\gamma_{3j}}{ic}+\frac{1}{2}\right)$$

that is

$$\Psi_{\gamma_{41},\gamma_{42}}(x_1,x_2)$$
$$=\int_{\mathbb{R}^3}\frac{d\gamma_{21}}{\left|\Gamma\left(\frac{2\gamma_{21}}{ic}\right)\right|^2}\frac{d\gamma_{31}d\gamma_{32}}{\left|\Gamma\left(\frac{\gamma_{31}-\gamma_{32}}{ic}\right)\Gamma\left(1+\frac{\gamma_{31}+\gamma_{32}}{ic}\right)\right|^2}c^{\frac{2}{ic}(\gamma_{31}+\gamma_{32})}e^{\frac{1}{ic}((-\gamma_{31}-\gamma_{32}+ic)x_2)}$$
$$\Gamma\left(\frac{\pm\gamma_{21}-\gamma_{31}}{ic}+\frac{1}{2}\right)\Gamma\left(\frac{\pm\gamma_{21}-\gamma_{32}}{ic}+\frac{1}{2}\right)$$
$$\prod_{i,j=1}^{2}\Gamma\left(\frac{\pm\gamma_{4i}+\gamma_{3j}}{ic}+\frac{1}{2}\right)\Psi_{\gamma_{21}}(x_1-x_2)$$

or

$$\Psi_{\gamma_{41},\gamma_{42}}(x_1,x_2)$$
$$=\int_{\mathbb{R}^2+\varepsilon}\frac{\prod_{i,j=1}^{2}\Gamma\left(\frac{\pm\gamma_{4i}+\gamma_{3j}}{ic}\right)}{\Gamma\left(\frac{\gamma_{31}+\gamma_{32}}{ic}\right)\left|\Gamma\left(\frac{\gamma_{31}-\gamma_{32}}{ic}\right)\right|^2}c^{\frac{3}{ic}(\gamma_{31}+\gamma_{32})}\Phi_{\gamma_{31},\gamma_{32}}(x_1,x_2)d\gamma_{31}d\gamma_{32}$$

where $\Phi_{\gamma_{31},\gamma_{32}}(x_1,x_2)$ is the wave function of $GL(2)$ Toda system

$$\Phi_{\gamma_{31},\gamma_{32}}(x_1,x_2)=e^{\frac{3x_1+x_2}{2}}\int_{\mathbb{R}+\varepsilon}d\gamma_{11}e^{\frac{1}{ic}(-\gamma_{11}x_1+(\gamma_{11}-\gamma_{31}-\gamma_{32})x_2)}c^{\frac{1}{ic}(2\gamma_{11}-\gamma_{31}-\gamma_{32})}$$
$$\Gamma\left(\frac{\gamma_{11}-\gamma_{31}}{ic}\right)\Gamma\left(\frac{\gamma_{11}-\gamma_{32}}{ic}\right)$$

A Calculations of Whittaker vector

In this section, we prove Theorem 1. First we note that the relation (3.14) follows from (3.13) by using the automorphism τ. Proof of the equality (3.13) reduces to check of the following equalities, where $k=1,\dots n-1$ in the relations (A.2) and (A.3)

$$(F_{k,k+1}-(-1)^k F_{k,2n-k})w_n=(I_{2k+1,2k}+iI_{2k+1,2k-1})w_n=\frac{(-1)^{k+1}}{ic}w_n \tag{A.1}$$

$$(F_{k,k+1}+(-1)^k F_{k,2n-k})w_n=(-I_{2k+2,2k-1}+iI_{2k+2,2k})w_n=\frac{(-1)^{k+1}}{ic}w_n \tag{A.2}$$

$$F_{n,n+1}w_n=\frac{1}{\sqrt{2}}(I_{2n+1,2n}+iI_{2n+1,2n-1})w_n=\frac{(-1)^{n+1}}{ic\sqrt{2}}w_n \tag{A.3}$$

Besides, the relation (A.3) is a particular case of (A.1). So we have to prove the relations (A.1) and (A.2).

The proof of (A.1)–(A.2) requires certain calculations. For their visualization, we introduce some intermediate notations. First rewrite the operators (2.10) and (2.11) as

$$I_{2k+1,2k}=\sum_{\varepsilon=\pm1}\sum_{j=1}^{k}P^{\varepsilon}_{kj},\qquad I_{2k+2,2k+1}=\sum_{j=1}^{k}Q_{kj}+\sum_{j=1}^{k}R_{kj}+T_k \tag{A.4}$$

where

$$P^{\varepsilon}_{k,j}=-\frac{1}{ic}\frac{\prod_{r=1}^{k-1}(\nu_{2k-1,j}+\varepsilon(\nu_{2k-2,r}+\frac{ic}{2}))\prod_{r=1}^{k}(\nu_{2k-1,j}-\varepsilon(\nu_{2k,r}-\frac{ic}{2}))}{2\prod_{r\neq j}(\nu_{2k-1,j}-\nu_{2k-1,r})(\nu_{2k-1,j}+\nu_{2k-1,r}+\varepsilon ic)}$$
$$e^{\varepsilon ic\partial_{\nu_{2k-1,j}}},$$

$$Q_{k,j}=\frac{1}{ic}\frac{\prod_{r=1}^{k+1}((\nu_{2k,j}+\frac{ic}{2})^2-\nu^2_{2k+1,r})}{2\nu_{2k,j}(\nu_{2k,j}+\frac{ic}{2})\prod_{r\neq j}(\nu^2_{2k,j}-\nu^2_{2k,r})}e^{ic\partial_{\nu_{2k,j}}}$$

$$R_{k,j}=\frac{1}{ic}\frac{\prod_{r=1}^{k}((\nu_{2k,j}-\frac{ic}{2})^2-\nu^2_{2k-1,r})}{2\nu_{2k,j}(\nu_{2k,j}-\frac{ic}{2})\prod_{r\neq j}(\nu^2_{2k,j}-\nu^2_{2k,r})}e^{-ic\partial_{\nu_{2k,j}}}$$

$$T_k=\frac{1}{c}\frac{\prod_{r=1}^{k}\nu_{2k-1,r}\prod_{r=1}^{k+1}\nu_{2k+1,r}}{\prod_{r=1}^{k}(\nu^2_{2k,r}+\frac{c^2}{4})}$$

Set also for $\varepsilon,\delta=\pm1$

$$J^{\varepsilon}_{k,\delta,j}=\sum_{s=1}^{k}\frac{\varepsilon ic}{\nu_{2k+\delta,j}-\varepsilon(\nu_{2k,s}+\frac{ic}{2})}Q_{k,s}+\sum_{s=1}^{k}\frac{\varepsilon ic}{\nu_{2k+\delta,j}+\varepsilon(\nu_{2k,s}-\frac{ic}{2})}$$
$$R_{k,s}+\frac{\varepsilon ic}{\nu_{2k+\delta,j}}T_k. \tag{A.5}$$

In these notation, we can present the following expressions for other generators of Lie algebra $so(n)$ needed in the equations on Whittaker vectors. They can be checked by straightforward calculations.

Lemma 3 *We have the following relations*

$$\begin{aligned} I_{2k+1,2k-1} &= \sum_{\varepsilon=\pm1}\sum_{j=1}^{k} P^{\varepsilon}_{kj} J^{\varepsilon}_{k-1,1,j} \qquad k=1,\ldots,n \\ I_{2k+2,2k} &= -\sum_{\varepsilon=\pm1}\sum_{j=1}^{k} P^{\varepsilon}_{kj} J^{\varepsilon}_{k,-1,j} \qquad k=1,\ldots,n-1 \\ I_{2k+2,2k-1} &= -\sum_{\varepsilon=\pm1}\sum_{j=1}^{k} P^{\varepsilon}_{kj} J^{\varepsilon}_{k,-1,j} J^{\varepsilon}_{k-1,1,j} \qquad k=1,\ldots,n-1 \end{aligned} \tag{A.6}$$

For more brevity in further formulas, we denote

$$\theta_k = (-1)^k, \qquad \theta_{k+1} = (-1)^{k+1}.$$

The following statement is one of the important steps in the proof of Theorem 1.

Lemma 4 *For any $k = 0,\ldots,n-1$ ($\delta = 1$ when $k = 0$), we have the relations*

$$J^{\theta_k}_{k,\delta,j} w_n = i w_n \tag{A.7}$$

$$J^{\theta_{k+1}}_{k,\delta,j} w_n = i\left(1 - \frac{\prod_{r=1}^{k+1}(\nu_{2k+\delta,j}+\nu_{2k+1,r})\prod_{r=1}^{k}(\nu_{2k+\delta,j}+\nu_{2k-1,r})}{\nu_{2k+\delta,j}\prod_{r=1}^{k}((\nu_{2k+\delta,j}+(-1)^k\frac{ic}{2})^2-\nu_{2k,r}^2)}\right) w_n \qquad \delta=\pm1 \tag{A.8}$$

Proof Note that the operator $J^{\theta_k}_{k,\delta,j}$ contains only shifts of variables $\nu_{2k,s}$, and thus only the factor $W_k^{\theta_{k+1}}$ can change. Let us check the equality (A.7). We then have

$$\begin{aligned} J^{\theta_k}_{k,\delta,j} w_n &= \Bigg(\sum_{s=1}^{k} \frac{\theta_k ic}{\nu_{2k+\delta,j}-\theta_k(\nu_{2k,s}+\frac{ic}{2})} Q_{k,s} + \sum_{s=1}^{k}\frac{\theta_k ic}{\nu_{2k+\delta,j}+\theta_k(\nu_{2k,s}-\frac{ic}{2})} R_{k,s} + \frac{\theta_k ic}{\nu_{2k+\delta,j}} T_k\Bigg) w_n \\ &= \Bigg(\sum_{s=1}^{k}\frac{\theta_k ic}{\nu_{2k+\delta,j}-\theta_k(\nu_{2k,s}+\frac{ic}{2})}\cdot\frac{1}{ic}\cdot\frac{\prod_{r=1}^{k+1}((\nu_{2k,s}+\frac{ic}{2})^2-\nu_{2k+1,r}^2)}{2\nu_{2k,s}(\nu_{2k,s}+\frac{ic}{2})\prod_{r\neq s}(\nu_{2k,s}^2-\nu_{2k,r}^2)} \\ &\quad \times i^{\theta_k}\times\frac{\prod_{r=1}^{k}(\nu_{2k,s}-\theta_k\nu_{2k-1,r}+\frac{ic}{2})}{\prod_{r=1}^{k+1}(-\nu_{2k,s}-\theta_k\nu_{2k+1,r}-\frac{ic}{2})} + \sum_{s=1}^{k}\frac{\theta_k ic}{\nu_{2k+\delta,j}+\theta_k(\nu_{2k,s}-\frac{ic}{2})}\cdot\frac{1}{ic} \\ &\quad \times\frac{\prod_{r=1}^{k+1}((\nu_{2k,s}-\frac{ic}{2})^2-\nu_{2k+1,r}^2)}{2\nu_{2k,s}(\nu_{2k,s}-\frac{ic}{2})\prod_{r\neq s}(\nu_{2k,s}^2-\nu_{2k,r}^2)}\cdot i^{-\theta_k}\times\frac{\prod_{r=1}^{k+1}(-\nu_{2k,s}-\theta_k\nu_{2k+1,r}+\frac{ic}{2})}{\prod_{r=1}^{k}(\nu_{2k,s}-\theta_k\nu_{2k-1,r}-\frac{ic}{2})} \\ &\quad + i\,\frac{\prod_{r=1}^{k}(-\theta_k\nu_{2k-1,r})\prod_{r=1}^{k+1}(-\theta_k\nu_{2k+1,r})}{(-\theta_k\nu_{2k+\delta,j})\prod_{r=1}^{k}(-\frac{ic}{2}-\nu_{2k,r})(-\frac{ic}{2}+\nu_{2k,r})}\Bigg) w_n = i w_n \end{aligned}$$

The last line is obtained by the use of the following well-known identity:

$$\sum_{i=1}^{m} \frac{\prod_{j=1}^{m-1}(x_i - y_j)}{\prod_{r\neq i}(x_i - x_r)} = 1 \tag{A.9}$$

where for indeterminates x_i we choose $2k+2$ variables $\{\pm\nu_{2k,s}, -\frac{ic}{2}, \theta_k\nu_{2k+\delta} - \frac{ic}{2}\}$, and for indeterminates y_i we choose $\{\theta_k\nu_{2k-1} - \frac{ic}{2}, \theta_k\nu_{2k+1} - \frac{ic}{2}\}$.

Let us check the equality (A.8). We have

$$J^{\theta_{k+1}}_{k,\delta,j} w_n = \Bigg(\sum_{s=1}^{k} \frac{\theta_{k+1} ic}{\nu_{2k+\delta,j} - \theta_{k+1}(\nu_{2k,s} + \frac{ic}{2})} Q_{k,s} + \sum_{s=1}^{k} \frac{\theta_{k+1} ic}{\nu_{2k+\delta,j} + \theta_{k+1}(\nu_{2k,s} - \frac{ic}{2})} R_{k,s} + \frac{\theta_{k+1} ic}{\nu_{2k+\delta,j}} T_k \Bigg) w_n$$

$$= \Bigg(\sum_{s=1}^{k} \frac{\theta_{k+1} ic}{\nu_{2k+\delta,j} - \theta_{k+1}(\nu_{2k,s} + \frac{ic}{2})} \cdot \frac{1}{ic} \cdot \frac{\prod_{r=1}^{k+1}((\nu_{2k,s} + \frac{ic}{2})^2 - \nu^2_{2k+1,r})}{2\nu_{2k,s}(\nu_{2k,s} + \frac{ic}{2}) \prod_{r\neq s}(\nu^2_{2k,s} - \nu^2_{2k,r})}$$

$$\times i^{\theta_{k+1}} \times \frac{\prod_{r=1}^{k}(\nu_{2k,s} - \theta_k\nu_{2k-1,r} + \frac{ic}{2})}{\prod_{r=1}^{k+1}(-\nu_{2k,s} - \theta_k\nu_{2k+1,r} - \frac{ic}{2})}$$

$$- \sum_{s=1}^{k} \frac{\theta_{k+1} ic}{\nu_{2k+\delta,j} + \theta_{k+1}(\nu_{2k,s} - \frac{ic}{2})} \cdot \frac{1}{ic} \cdot \frac{\prod_{r=1}^{k+1}((\nu_{2k,s} - \frac{ic}{2})^2 - \nu^2_{2k+1,r})}{2\nu_{2k,s}(\nu_{2k,s} - \frac{ic}{2}) \prod_{r\neq s}(\nu^2_{2k,s} - \nu^2_{2k,r})}$$

$$\times i^{-\theta_{k+1}} \cdot \frac{\prod_{r=1}^{k+1}(-\nu_{2k,s} - \theta_k\nu_{2k+1,r} + \frac{ic}{2})}{\prod_{r=1}^{k}(\nu_{2k,s} - \theta_k\nu_{2k-1,r} - \frac{ic}{2})}$$

$$+ i \frac{\prod_{r=1}^{k}(-\theta_k\nu_{2k-1,r}) \prod_{r=1}^{k+1}(-\theta_k\nu_{2k+1,r})}{(-\theta_{k+1}\nu_{2k+\delta}) \prod_{r=1}^{k}(-\frac{ic}{2} - \nu_{2k,r})(-\frac{ic}{2} + \nu_{2k,r})} \Bigg) w_n$$

$$= i\Bigg(1 - \frac{\prod_{r=1}^{k+1}(\nu_{2k+\delta,j} + \nu_{2k+1,r}) \prod_{r=1}^{k}(\nu_{2k+\delta,j} + \nu_{2k-1,r})}{\nu_{2k+\delta,j} \prod_{r=1}^{k}((\nu_{2k+\delta,j} + \theta_k\frac{ic}{2})^2 - \nu^2_{2k,r})}\Bigg) w_n$$

The last line of the equality is obtained after simplifications of the ratios by use of the relation ($A.9$), where for indeterminates x_i we substitute x variables $\{\pm\nu_{2k,s}, -\frac{ic}{2}, \theta_{k+1}\nu_{2k+\delta} - \frac{ic}{2}\}$, and for y_i we substitute $\{\theta_k\nu_{2k-1} - \frac{ic}{2}, \theta_k\nu_{2k+1} - \frac{ic}{2}\}$.

Proof of (A.1) According to Lemmas 3 and 4, for any $k = 1, \ldots, n$ we have:

$$(I_{2k+1,2k} + iI_{2k+1,2k-1})w_n = \sum_{\varepsilon=\pm1} \sum_{j=1}^{k} P^{\varepsilon}_{kj}(1 + iJ^{\varepsilon}_{k-1,1,j})w_n$$

$$= \sum_{j=1}^{k} P^{\theta_k}_{kj} \frac{\prod_{r=1}^{k}(\nu_{2k-1,j} + \nu_{2k-1,r}) \prod_{r=1}^{k-1}(\nu_{2k-1,j} + \nu_{2k-3,r})}{\nu_{2k-1,j} \prod_{r=1}^{k-1}((\nu_{2k-1,j} + \theta_{k-1}\frac{ic}{2})^2 - \nu^2_{2k-2,r})} w_n \tag{A.10}$$

$$= -\frac{1}{ic}\sum_{j=1}^{k}\frac{\prod_{r=1}^{k-1}(\nu_{2k-1,j}+\theta_k(\nu_{2k-2,r}+\frac{ic}{2}))\prod_{r=1}^{k}(\nu_{2k-1,j}-\theta_k(\nu_{2k,r}-\frac{ic}{2}))}{2\prod_{r\neq j}(\nu_{2k-1,j}-\nu_{2k-1,r})(\nu_{2k-1,j}+\nu_{2k-1,r}+\theta_k ic)}$$

$$e^{\theta_k ic\partial_{\nu_{2k-1,j}}}$$

$$\times 2\frac{\prod_{r\neq j}^{k}(\nu_{2k-1,j}+\nu_{2k-1,r})\prod_{r=1}^{k-1}(\nu_{2k-1,j}+\nu_{2k-3,r})}{\prod_{r=1}^{k-1}((\nu_{2k-1,j}+\theta_{k-1}\frac{ic}{2})^2-\nu_{2k-2,r}^2)}w_n$$

$$= -\frac{1}{ic}\sum_{j=1}^{k}\frac{\prod_{r=1}^{k-1}(\nu_{2k-1,j}+\nu_{2k-3,r}+\theta_k ic)}{\prod_{r\neq j}(\nu_{2k-1,j}-\nu_{2k-1,r})}$$

$$\cdot\frac{\prod_{r=1}^{k}\left(\nu_{2k-1,j}+\theta_{k+1}\nu_{2k,r}+\theta_k\frac{ic}{2}\right)}{\prod_{r=1}^{k-1}\left(\nu_{2k-1,j}+\theta_{k+1}\nu_{2k-2,r}+\theta_k\frac{ic}{2}\right)}e^{\theta_k ic\partial_{\nu_{2k-1,j}}}w_n$$

In the product, presenting the function w_n, only the factors $W_{k-1}^{\theta_k}$ and $W_k^{\theta_{k+1}}$ depend on the variables $\nu_{2k-1,j}$. Thus, using (3.11), functional relations on the Euler Gamma function and (A.9), we get

$$(I_{2k+1,2k}+iI_{2k+1,2k-1})w_n = \frac{\theta_{k+1}}{ic}\sum_{j=1}^{k}\frac{\prod_{r=1}^{k-1}(\nu_{2k-1,j}+\nu_{2k-3,r}+\theta_k ic)}{\prod_{r\neq j}(\nu_{2k-1,j}-\nu_{2k-1,r})}w_n = \frac{\theta_{k+1}}{ic}w_n \tag{A.11}$$

□

Proof of (A.2) Again, according to Lemmas 3 and 4, for any $k = 1,\dots,n$ we have:

$$(-I_{2k+2,2k-1}+iI_{2k+2,2k})w_n = \sum_{\varepsilon=\pm 1}\sum_{j=1}^{k}P_{kj}^{\varepsilon}J_{k,-1,j}^{\varepsilon}(J_{k-1,1,j}^{\varepsilon}-i)w_n$$

$$= -i\sum_{j=1}^{k}P_{kj}^{\theta_k}J_{k,-1,j}^{\theta_k}\frac{\prod_{r=1}^{k}(\nu_{2k-1,j}+\nu_{2k-1,r})\prod_{r=1}^{k-1}(\nu_{2k-1,j}+\nu_{2k-3,r})}{\nu_{2k-1,j}\prod_{r=1}^{k-1}((\nu_{2k-1,j}+\theta_{k-1}\frac{ic}{2})^2-\nu_{2k-2,r}^2)}w_n. \tag{A.12}$$

Since the operator $J_{k,-1,j}^{\theta_k}$ contains shifts only of the variables $\nu_{2k,s}$, we can rewrite the result as

$$-i\sum_{j=1}^{k}P_{kj}^{\theta_k}\frac{\prod_{r=1}^{k}(\nu_{2k-1,j}+\nu_{2k-1,r})\prod_{r=1}^{k-1}(\nu_{2k-1,j}+\nu_{2k-3,r})}{\nu_{2k-1,j}\prod_{r=1}^{k-1}((\nu_{2k-1,j}+\theta_{k-1}\frac{ic}{2})^2-\nu_{2k-2,r}^2)}J_{k,-1,j}^{\theta_k}w_n \tag{A.13}$$

Now Lemma 4 says that

$$J_{k,-1,j}^{\theta_k}w_n = iw_n.$$

Thus, $(-I_{2k+2,2k-1}+iI_{2k+2,2k})w_n$ equals to

$$=\sum_{j=1}^{k} P_{kj}^{\theta_k}\frac{\prod_{r=1}^{k}(\nu_{2k-1,j}+\nu_{2k-1,r})\prod_{r=1}^{k-1}(\nu_{2k-1,j}+\nu_{2k-3,r})}{\nu_{2k-1,j}\prod_{r=1}^{k-1}((\nu_{2k-1,j}+\theta_{k-1}\frac{ic}{2})^2-\nu_{2k-2,r}^2)}w_n=\frac{\theta_{k+1}}{ic}w_n \tag{A.14}$$

The latter equality was proved during the derivation of (A.11) from (A.10).

□

This ends the proof of Theorem 1. □

Remark 2 Analyzing the proof of Theorem 1, we see that the derivations over variables $\nu_{2k-1,j}$ enter the game only in the last stage of calculations. Moreover, we can freely add to Whittaker vectors factors of the form

$$e^{i\frac{\alpha_j}{c}\delta_{2j-1}},\qquad \text{where}\qquad \delta_{2j-1}=\sum_i \nu_{2j-1,i}=(-1)^{j+1}\sum_i \gamma_{2j-1,i}$$

where α is arbitrary real number. This does not affect to the convergence of integrals and does not change the action of Cartan subalgebra. In Toda equation, we earn thus arbitrary positive constants $c_j=e^{\alpha_j}$ at exponentials $e^{x_{j-1}-x_j}$ which can be equivalently obtained by successive shifts of the variables x_j.

B Action of Cartan subalgebra

It is sufficient to calculate $F_{k,k}w_n$, and then use the automorphism τ.

$$\begin{aligned}-iI_{2k,2k-1}w_n=-\Bigg(&\frac{1}{c}\sum_{j=1}^{k-1}\frac{\prod_{r=1}^{k}((\nu_{2k-2,j}+\frac{ic}{2})^2-\nu_{2k-1,r}^2)}{2\nu_{2k-2,j}(\nu_{2k-2,j}+\frac{ic}{2})\prod_{r\neq j}(\nu_{2k-2,j}^2-\nu_{2k-2,r}^2)}\\
&i^{\theta_{k-1}}\frac{\prod_{r=1}^{k-1}(\nu_{2k-2,j}-\theta_{k-1}\nu_{2k-3,r}+\frac{ic}{2})}{\prod_{r=1}^{k}(-\nu_{2k-2,j}-\theta_{k-1}\nu_{2k-1,r}-\frac{ic}{2})}\\
&+\frac{1}{c}\sum_{j=1}^{k-1}\frac{\prod_{r=1}^{k-1}((\nu_{2k-2,j}-\frac{ic}{2})^2-\nu_{2k-3,r}^2)}{2\nu_{2k-2,j}(\nu_{2k-2,j}-\frac{ic}{2})\prod_{r\neq j}(\nu_{2k-2,j}^2-\nu_{2k-2,r}^2)}\\
&i^{-\theta_{k-1}}\frac{\prod_{r=1}^{k}(-\nu_{2k-2,j}-\theta_{k-1}\nu_{2k-1,r}+\frac{ic}{2})}{\prod_{r=1}^{k-1}(\nu_{2k-2,j}-\theta_{k-1}\nu_{2k-3,r}-\frac{ic}{2})}\\
&-\frac{1}{ic}\frac{\prod_{r=1}^{k-1}\nu_{2k-3,r}\prod_{r=1}^{k}\nu_{2k-1,r}}{\prod_{r=1}^{k-1}(\nu_{2k-2,r}^2+\frac{c^2}{4})}\Bigg)w_n\end{aligned} \tag{B. 1}$$

$$
\begin{aligned}
&= -\frac{1}{ic}\Bigg(\sum_{j=1}^{k-1}\frac{\prod_{r=1}^{k}(\nu_{2k-2,j}-\theta_{k-1}\nu_{2k-1,r}+\frac{ic}{2})\prod_{r=1}^{k-1}(\nu_{2k-2,j}-\theta_{k-1}\nu_{2k-3,r}+\frac{ic}{2})}{2\nu_{2k-2,j}(\nu_{2k-2,j}+\frac{ic}{2})\prod_{r\neq j}(\nu_{2k-2,j}^2-\nu_{2k-2,r}^2)}\\
&\quad+\sum_{j=1}^{k-1}\frac{\prod_{r=1}^{k-1}(-\nu_{2k-2,j}-\theta_{k-1}\nu_{2k-3,r}+\frac{ic}{2})\prod_{r=1}^{k}(-\nu_{2k-2,j}-\theta_{k-1}\nu_{2k-1,r}+\frac{ic}{2})}{2\nu_{2k-2,j}(\nu_{2k-2,j}-\frac{ic}{2})\prod_{r\neq j}(\nu_{2k-2,j}^2-\nu_{2k-2,r}^2)}\\
&\quad+\frac{\prod_{r=1}^{k-1}(-\theta_{k-1}\nu_{2k-3,r})\prod_{r=1}^{k}(-\theta_{k-1}\nu_{2k-1,r})}{\prod_{r=1}^{k-1}(-\frac{ic}{2}-\nu_{2k-2,r})(-\frac{ic}{2}+\nu_{2k-2,r})}\Bigg)w_n\\
&= \frac{\theta_{k-1}}{ic}\left(\sum_{j=1}^{k}\nu_{2k-1,j}+\sum_{j=1}^{k-1}\nu_{2k-3,j}-\theta_{k-1}(k-1)ic\right). \qquad \text{(B. 2)}
\end{aligned}
$$

Here, we again use the identity (A.9), where for x_i we substitute $2k-1$ variables $\{\pm\nu_{2k-2,s}, -\frac{ic}{2}\}$, a and for y_i we use $\{\theta_{k-1}\nu_{2k-1}-\frac{ic}{2}, \theta_{k-1}\nu_{2k-3}-\frac{ic}{2}\}$. □

Acknowledgements The authors thank N.Belousov, S.Derkachov and S.Kharchev for their interest in the work and for fruitful discussions. They are also grateful to the referee for useful comments. The work of S. Kh. is an output of a research project implemented as part of the Basic Research Program at the National Research University Higher School of Economics (HSE University).

Data availability statement Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

References

1. Dixmier, J.: Algebres enveloppantes, vol. 192. Gauthier-Villars, Paris (1974)
2. Gelfand, I.M., Tsetlin, M.L.: Finite-dimensional representations of the group of unimodular matrices. DAN SSSR **51**, 825–828 (1950). ((**in Russian**))
3. Gerasimov, A.A., Lebedev, D.R., Oblezin, S.V.: New integral representations of Whittaker functions for classical Lie groups. Russian Math. Surv. **67**(1), 1 (2012)
4. Gerasimov, A., Kharchev, S., Lebedev, D.: Scattering method: the open Toda chain and the hyperbolic Sutherland. Int. Math. Res. Not. **2004**(17), 823–854 (2004)
5. Givental, A.: Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture. Topics in singularity theory, American Mathematical Society Transl. Ser. 2, 180, American Mathematical Society, Providence, RI, 103–115 (1997)
6. Gustafson, R.A.: Some q-beta and Mellin–Barnes integrals on compact lie groups and lie algebras. Trans. Am. Math. Soc. **341**(1), 69–119 (1994)
7. Iorgov, N.Z., Shadura, V.N.: Wave functions of the Toda chain with boundary interaction. Theor. Math. Phys. **142**, 289–305 (2005)
8. Kac, V.G.: Infinite-dimensional Lie Algebras. Cambridge University Press, Cambridge (1990)
9. Kharchev, S., Khoroshkin, S.: Mellin–Barnes presentations for Whittaker wave functions. Adv. Math. **375**, 107368 (2020)
10. Kharchev, S., Lebedev, D.: Eigenfunctions of $GL(N, R)$ Toda chain: Mellin–Barnes representation. J. Exp. Theor. Phys. Lett. **71**, 235–238 (2000)
11. Kostant, B.: Quantization and representation theory, In: Representation Theory of Lie Groups, Proceedings of Symposium, Oxford, 1977, pp. 287–317, London Math. Soc. Lecture Notes series, 34, Cambridge (1979)

12. Zhelobenko, D.P., Stern, A.I.: Representation of Lie groups (in Russian), Science (1983)