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RESEARCH

FPTU recurrence within the Gardner equation

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Abstract The study of Fermi–Pasta–Ulam–Tsingou (FPUT) recurrence is examined within the framework of the Gardner equation. The evolution of harmonic waves is investigated for both positive and negative cubic nonlinearities. It is observed that harmonic waves undergo fission into solitons, which then interact with each other. For positive cubic nonlinearity, recurrence occurs periodically over time for weak and intermediate nonlinearities. However, as the dispersion becomes weaker, this phenomenon ceases to occur. Conversely, for negative cubic nonlinearity, recurrence is also observed for weak and intermediate nonlinearities, but it lacks a well-defined temporal period.

Keywords Recurrence · Fermi-Pasta-Ulmam-Tsingou · Gardner equation · Solitons

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1 Introduction

Fermi, Pasta, Ulam and Tsingou considered a model where a chain of particles with equal masses is connected by elastic springs, governed by a dynamical system with quadratic and cubic spring forces [1,2]. They initially expected the system to distribute energy evenly across all possible modes, similar to the behavior of billiards. However, their results showed significant differences in the energy levels of each mode. The energy spread to higher harmonics, but after a finite number of oscillations, the flow of energy into other modes ceased, and the dynamics reversed, causing the energy to flow back into the first mode. This energy recurrence was found to be almost complete, with only about a 2% loss of the total energy. It is like a billiard returns to a triangular shape automatically (in terms of energy). This periodic phenomenon is known as Fermi–Pasta–Ulam–Tsingou (FPUT) recurrence. In the continuum limit, the FPUT lattice is modeled by the Korteweg-de Vries (KdV) equation. Zabusky and Kruskal [3] also observed this phenomenon with a harmonic lattice within the KdV framework. They noted that the initial cosine wave splits into solitons due to nonlinearity, which then interact with each other elastically, later their interaction was classified by Lax [4]. After a critical time, the initial state is almost reconstructed through nonlinear interactions. The first recurrence closely approximated the initial state, but subsequent recurrences were "near recurrences" and not as accurate. Later, Abe and Abe [5] demonstrated that even the first recurrence is incomplete (the energy is recovered only partially) and leads to progressively more incomplete recurrences.

Surprisingly, as we decrease the dispersion, periodic behavior in the recurrence re-emerges. Figure 10 shows that recurrence occurs approximately every $t \approx 27.2$. As observed by Zabusky and Kruskal [3], the process continues, and at $t \approx 54.4$, a near-recurrence occurs, though it is not as pronounced as the first recurrence, as seen in Fig. 11. Further decreasing the dispersion eventually eliminates recurrence, although the cosine evolution remains visually similar. Figure 12 displays the pattern of the cosine wave evolution, demonstrating behavior qualitatively similar to that reported in Fig. 10. It is noteworthy that, despite the absence of recurrence in this case, there are still two specific times when the first mode recovers roughly 90% of its initial energy, see for instance Fig. 13.

Goda [6] revisited the recurrence problem for a sinusoidal initial state for the KdV equation with a different approach. The author analysed the recurrence phenomenon through energy sharing. He showed that the wave field does not recover its initial state perfeclty, but it recurs close to it. For a two-harmonic initial state it was possible to find a long but finite time, such that the energy distribution recurs nearer to the initial state than that regarded as almost recurring state in a short time. Differently from the KdV equation, Goda [7] showed that the recurrence property holds for Regularized Long Wave model (RLW) only if the nonlinearity and dispersion are perfectly balanced (same order). Nonetheless, the energy of the RLW solution is shared only among the lower modes of the system (no thermalization). There are extensive studies devoted to the study of recurrence within different frameworks; readers are referred to the works of Watanabe et al. [8] for the dissipative KdV equation, Yoshimura and Watanabe [9] for the Kawahara equation and others [10–14].

In this work, we investigate the recurrence phenomenon in the Gardner equation with both positive and negative cubic nonlinearities. The Gardner equation is prominently featured in the context of internal waves [15–22] and plasma physics [23–25]. This equation is known to support a broader class of solutions compared to the KdV equation. These solutions include cnoidal waves, solitons of both polarities, and breathers [26]. The Gardner equation has recently been employed to study soliton interactions with external forces [18,27], a topic that has also been explored in the context of various dispersive equations [28–31].

To the best of our knowledge, the recurrence problem involving a single harmonic has never been addressed in the literature. It is important to mention that the evolution of long sine waves has also been explored within the Gardner framework [32,33]. However, recurrence was not investigated in these studies. The authors primarily focused on solitonic regimes, whereas our approach diverges by concentrating on a different regime. To study the recurrence phenomenon in the Gardner equation, we use a cosine wave as the initial condition and track the evolution of the Fourier mode energies over time. Our findings reveal that for both strong and intermediate dispersion, the Gardner equation with positive cubic nonlinearity exhibits the recurrence phenomenon. In contrast, with negative cubic nonlinearity, although the initial mode occasionally regains its initial energy under strong and intermediate dispersion, it does not demonstrate periodic behavior over time.

The article is structured as follows: Sect. 2 introduces the Gardner equation and outlines the numerical methods employed in our study. Section 3 presents the asymptotic results. Section 4 presents the numerical results and final conclusions in Sect. 5.

2 The Gardner equation

In our research, we investigate solitary wave interactions by focusing on the Gardner equation in its canonical form

$$u_t + uu_x + \beta u^2 u_x + \mu u_{xxx} = 0. \qquad (1)$$

Within this equation, the variable u represents the wave field at a specific position x and time t. The parameter μ controls the dispersion regime and β the cubic nonlinearity sign ($\beta = \pm 1$).

The FPUT recurrence problem is first analyzed analytically for the simple case of two harmonic interactions. However, this analytical approach is not applicable when considering a single-harmonic solution. For the latter case, we solve Eq. (1) numerically using an initial data given by a single harmonic. This is accomplished with a Fourier pseudospectral method combined with an integrating factor. The computational domain for the simulation is a periodic interval $[-L, L]$, discretized with an equidistant grid consist-

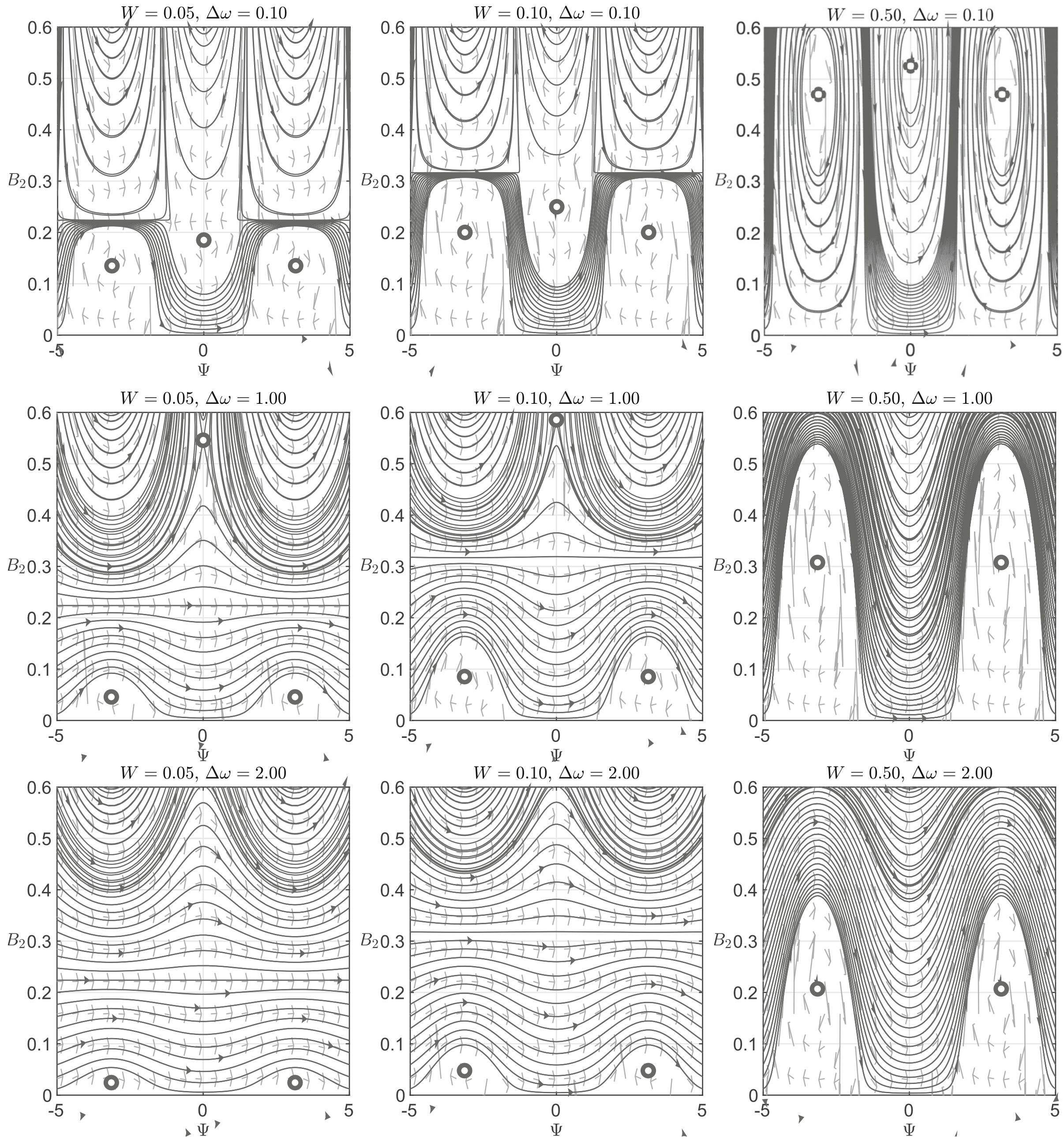


Fig. 1 Different phase portraits of the dynamical system (11) for different values of W and $\Delta\omega$

ing of N points. This grid configuration facilitates precise approximation of spatial derivatives, as discussed in [34]. For the temporal evolution of the equation, we employ the classical fourth-order Runge–Kutta method with discrete time steps of size Δt. Typical simulations use parameter values such as $L = \pi$, $N = 2^{10}$, and $\Delta t = 10^{-4}$. The initial data is always taken as

$$u(x, 0) = A_0 \cos(x), \tag{2}$$

Here, A_0 represents the wave amplitude, while the dispersion parameter μ is varied. In what follows, we fix $A_0 = 1$. The reason for this choice is that for small values of A_0, the problem is approximately linear,

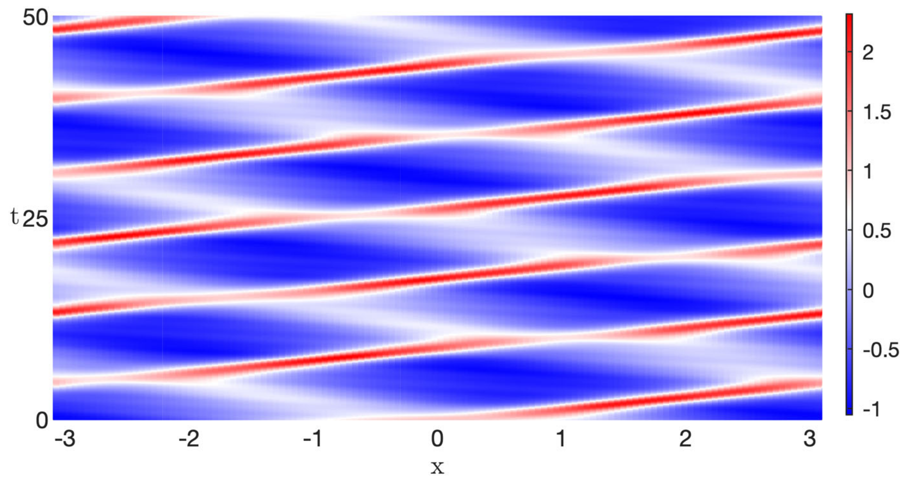


Fig. 2 The evolution of the cosine wave with $\mu = 0.1$ and $\beta = +1$

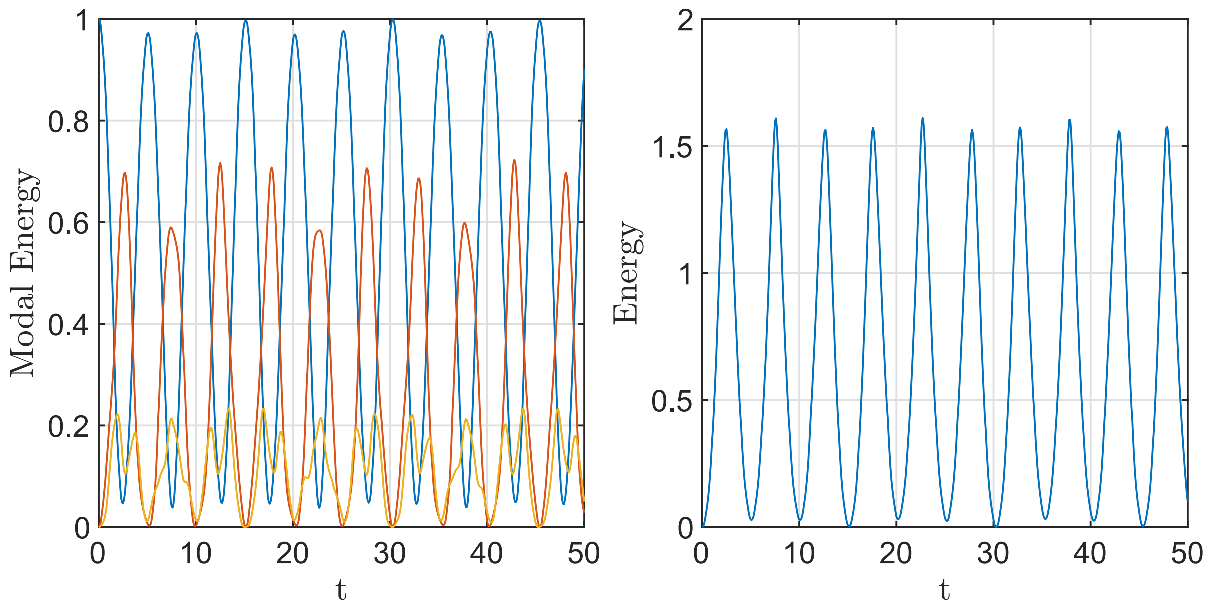


Fig. 3 The modal energy of the first three harmonic modes $k = 1, k = 2, k = 3$ in blue, red and yellow respectively and the variation of the energy defined in Eq. (13) as a function of time for $\mu = 0.1$ and $\beta = +1$

necessitating long simulation times to observe nonlinear effects. Conversely, large values of A_0 quickly lead to the appearance of higher harmonics.

3 Simple analysis of two harmonics interactions

The investigation of the recurrence phenomenon for two harmonics is motivated by the search for a two-harmonic solution to the Gardner Eq. (1). To achieve this, we follow the approach outlined by Pelinovsky and Shavratsky [19]. We begin by examining the linearized form of Eq. (1), considering a two-harmonic solution in its complex form

$$u(x,t) = A_1 \exp(i(\omega_1 t - k_1 x)) + A_2 \exp(i(\omega_2 t - 2k_1 x)). \quad (3)$$

The solution can be easily obtained from the dispersion relation. Substituting into the linearized version of (1) we obtain

$$\omega_1 = -\mu k_1^3 \text{ and } \omega_2 = -\mu(2k_1)^3. \quad (4)$$

In particular, defining $\Delta\omega = \omega_2 - \omega_1$ we have that $\Delta\omega \neq 0$.

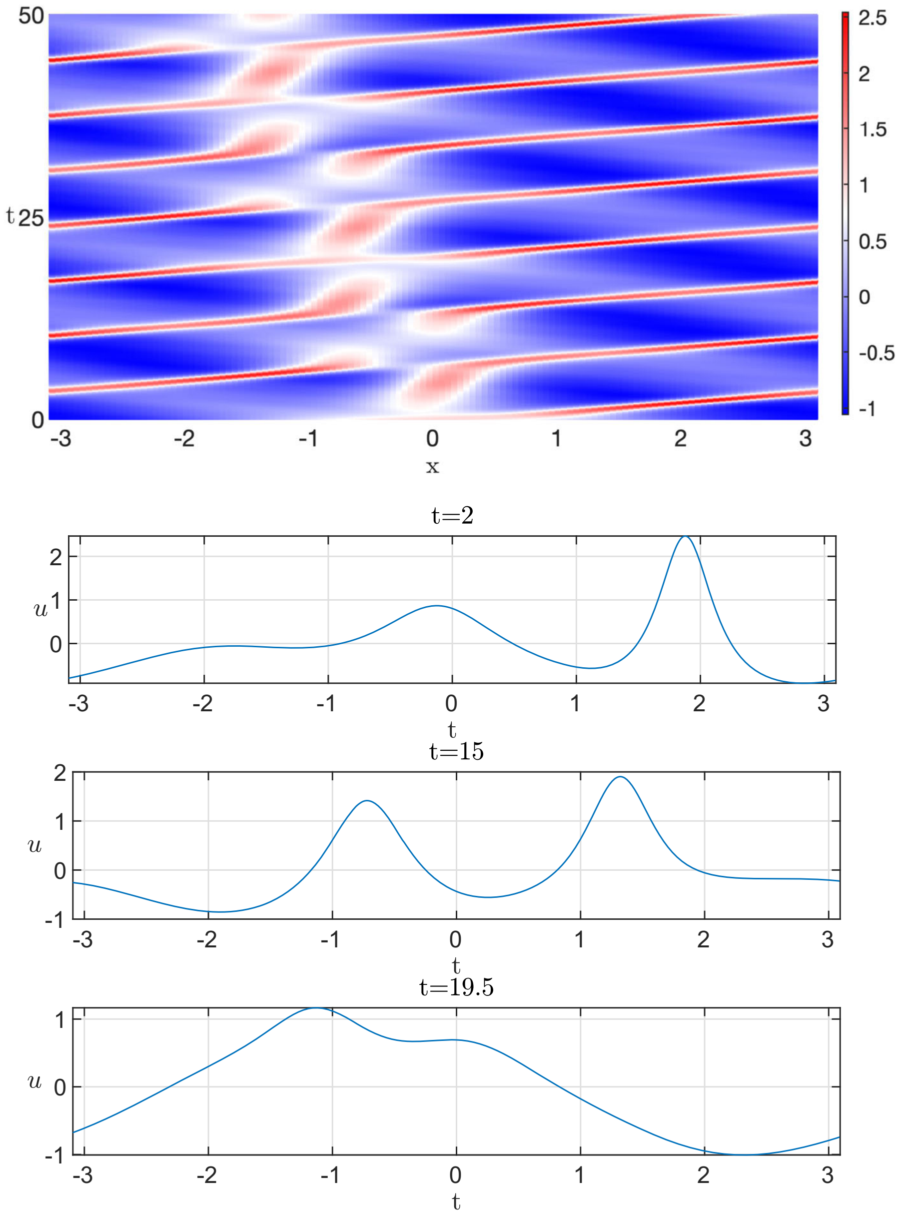


Fig. 4 The evolution of the cosine wave with $\mu = 0.05$ and $\beta = +1$ and a series of snapshots at different times

Now, for the nonlinear case, we assume that the amplitudes are time-dependent and small, therefore the cubic nonlinearity can be neglected. In other words, we seek for a two-harmonic solution of the form

$$u(x,t) = A_1(t)\exp(i\Theta) + A_2(t)\exp(i(2\Theta + \Delta\omega t)) + c.c, \quad (5)$$

where $\Theta(x,t) = \omega_1 t - k_1 x$ and $c.c$ denotes the harmonic complex conjugate. Substituting expression (5) into the Gardner Eq. (1) and collecting the terms with $e^{i\Theta}$ and $e^{2i\Theta}$ lead to the following complex dynamical system

$$\begin{aligned} \frac{dA_1}{dt} &= -\frac{ik}{2}A_1^* A_2 e^{i\Delta\omega t}, \\ \frac{dA_2}{dt} &= -\frac{ik}{2}A_1^2 e^{-i\Delta\omega t}. \end{aligned} \quad (6)$$

It is convenient to write the amplitudes in real form. To this end, we write

$$A_n(t) = B_n(t)\exp(i\varphi_n(t)), \ \text{for } n = 1, 2. \quad (7)$$

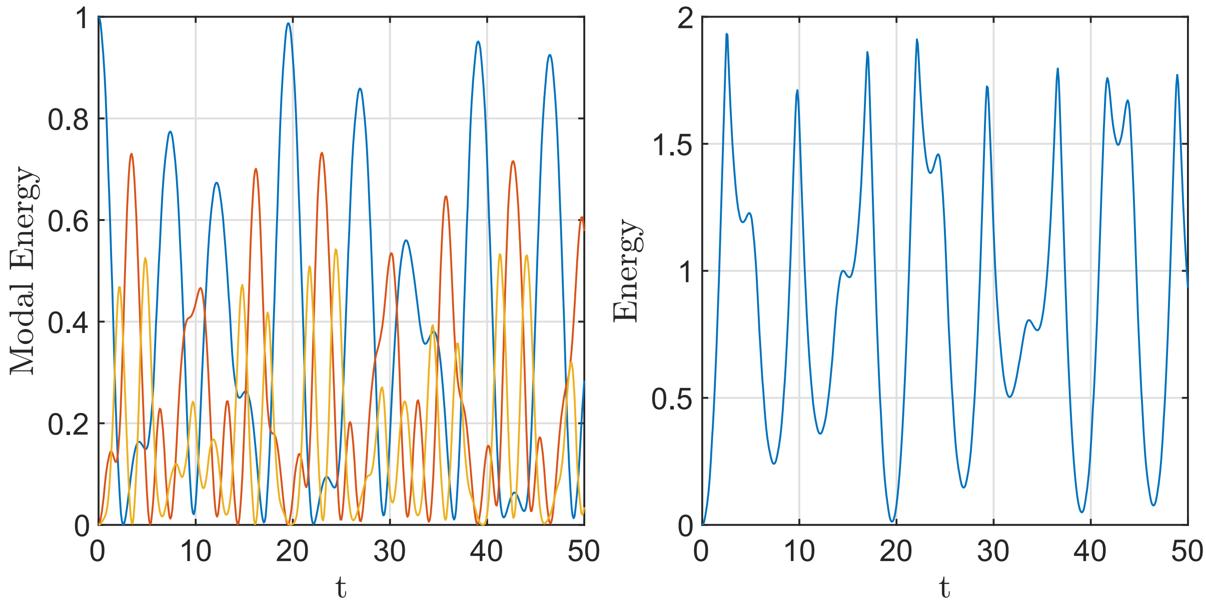


Fig. 5 The modal energy of the first three harmonic modes $k = 1, k = 2, k = 3$ in blue, red and yellow respectively and the variation of the energy defined in Eq. (13) as a function of time for $\mu = 0.05$ and $\beta = +1$

Consequently, we can write the dynamical system (6) in terms of real quantities as

$$\begin{aligned}\frac{dB_1}{dt} &= -\frac{k_1}{2}B_1B_2\sin(\varphi_2 - 2\varphi_1 + \Delta\omega t),\\ \frac{dB_2}{dt} &= -k_1B_1^2\sin 2(\varphi_1 - 2\varphi_2 - \Delta\omega t),\\ \frac{d\varphi_1}{dt} &= -\frac{k_1}{2}B_2\cos(\varphi_2 - 2\varphi_1 + \Delta\omega t),\\ \frac{d\varphi_2}{dt} &= k_1\frac{B_1^2}{B_2}\cos(\varphi_1 - \varphi_2 - \Delta\omega t).\end{aligned} \tag{8}$$

Defining $\Psi = \varphi_2 - 2\varphi_1 + \Delta\omega t$ and using it in the dynamical system (8) yields

$$\begin{aligned}\frac{dB_1}{dt} &= -\frac{k_1}{2}B_1B_2\sin(\Psi),\\ \frac{dB_2}{dt} &= k_1B_1^2\sin(\Psi),\\ \frac{d\Psi}{dt} &= k_1\left(\frac{B_1^2}{B_2} - B_2\right)\cos(\Psi) + \Delta\omega.\end{aligned} \tag{9}$$

A trivial integral of system (9) is given by

$$B_1^2 + B_2^2 = W, \tag{10}$$

where W is constant. Consequently, we can derive a new dynamical system that involves only B_2 and Ψ, namely

$$\begin{aligned}\frac{dB_2}{dt} &= k_1(W - B_2^2)\sin(\Psi),\\ \frac{d\Psi}{dt} &= k_1\frac{W - 2B_2^2}{B_2}\cos(\Psi) + \Delta\omega.\end{aligned} \tag{11}$$

This dynamical system can be rescaled by making $B_2 \to B_2/k_1$, $\Psi \to \Psi/k_1$ and $\Delta\omega \to \Delta\omega/k_1$, which yields

$$\begin{aligned}\frac{dB_2}{dt} &= (W - B_2^2)\sin(\Psi),\\ \frac{d\Psi}{dt} &= \frac{W - 2B_2^2}{B_2}\cos(\Psi) + \Delta\omega.\end{aligned} \tag{12}$$

The phase portrait of (12) is depicted in Fig. 1 for different values of W and $\Delta\omega$. The center equilibrium points correspond to traveling waves, indicating perfect recurrence, while the closed trajectories represent the recurrence phenomenon.

4 Results with one harmonic

The study of the recurrence is carried out using the energy of the harmonics instead of the wave shape itself. This idea was first considered the work of Goda [6] while investigating the recurrence phenomenon

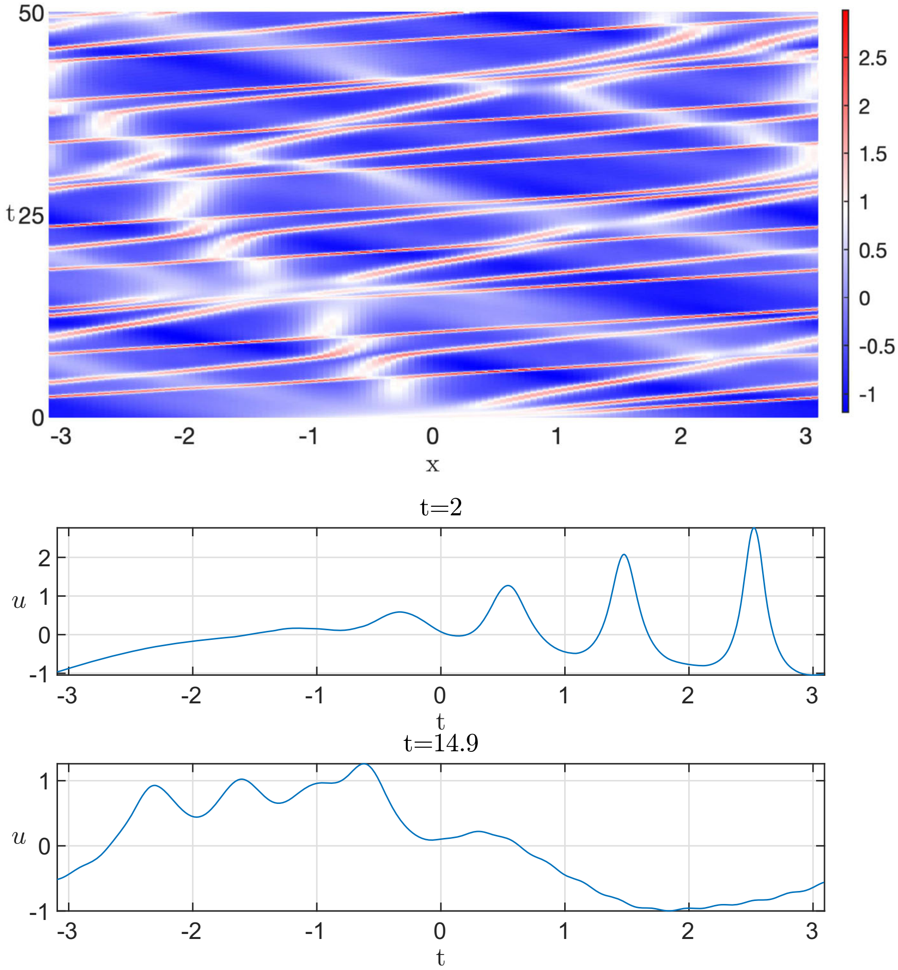


Fig. 6 The evolution of the cosine wave with $\mu = 0.01$ and $\beta = +1$

within the KdV equation. To measure the degree of recurrence we introduce the energy function

$$\epsilon(t) = \sum_{k=0}^{N} \left| |\widehat{u}(k,t)|^2 - |\widehat{u}(k,0)|^2 \right|, \quad (13)$$

where N is the highest mode in the discrete solution. This quantity measure the degree of recurrence, if $\epsilon(T) = 0$ for any $T > 0$, this means that the solution recurs exactly to the initial state in the sense of energy sharing. It does not mean that the wave recovers its initial shape though. Moreover, the nonlinearity is responsible for the energy transfer among harmonics. Consequently, in the linearized problem, there is no energy transfer, and thus the energy function defined in Eq. (13) will be zero. Another important quantity is the modal energy associated with the wavenumber k, defined as

$$E_k(t) = |\widehat{u}(k,t)|^2. \quad (14)$$

This quantity is computed using the Fast Fourier Transform (FFT) [34], and allows us to evaluate the energy distribution across wavenumbers at any given time.

In the next sections we focus on the recurrence phenomenon as we vary the the dispersion parameter (μ) within the Gardner equation with positive and negative cubic nonlinearity.

4.1 Positive cubic nonlinearity

We begin by considering the Gardner equation with a positive cubic nonlinearity for various values of μ. The wave pattern is highly sensitive to the choice of μ, with nonlinear effects becoming more pronounced as μ decreases.

For $\mu = 0.1$, the initial cosine is slightly deformed due to nonlinear effects. However, no fission of the

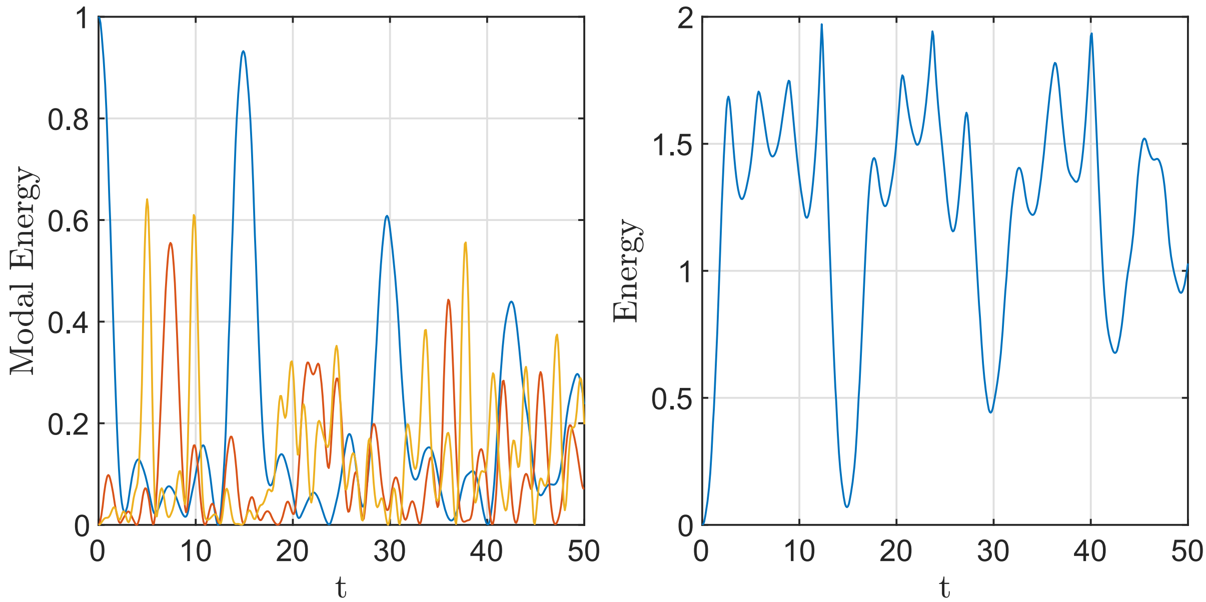
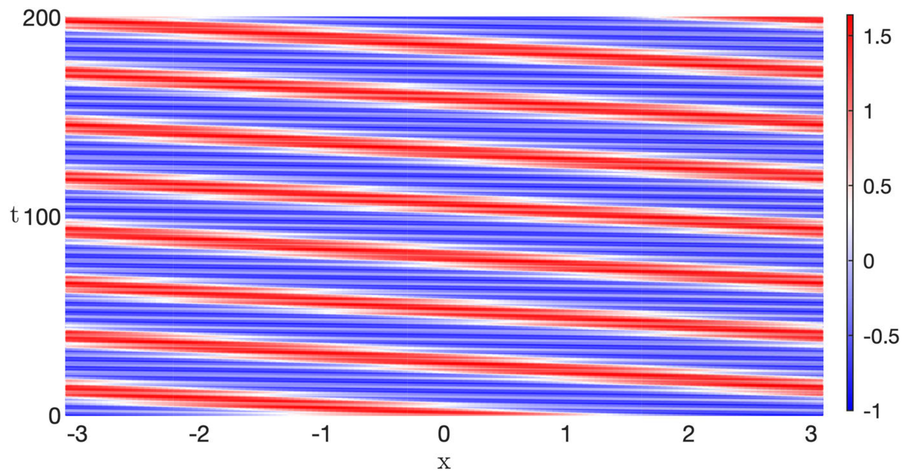


Fig. 7 The modal energy of the first three harmonic modes $k = 1$, $k = 2$, $k = 3$ in blue, red and yellow respectively and the variation of the energy defined in Eq. (13) as a function of time for $\mu = 0.01$ and $\beta = +1$

Fig. 8 The evolution of the cosine wave with $\mu = 0.1$ and $\beta = -1$



sinusoidal wave is observed. This scenario is exhibited in Fig. 2. The energy at the first harmonic is initially transferred to other the second and third harmonics, then at $t = 5$ and $t = 10$ the first mode recovers about 97% of the energy. This is not when the first recurrence takes place though. At time $t = 15.1$ the first harmonic regains up to 99% of its initial energy state and the same at time $t = 30.2$ and $t = 40.3$, which indicates a "super recurrence", see Fig. 3.

As the nonlinearity becomes stronger (e.g., for small values such as $\mu = 0.05$), the initial state fission into a series of solitons (see Fig. 4) and most energy from the first harmonic is transferred to the second harmonic due to wave steepness as show in Fig. 5 (left). The first recurrence occurs at time $t = 19.5$ where the first mode recovers about 98 % of its initial energy, however the second recurrence $t = 29$ is not as good as the first since the recovered energy is about 95%.

Further decreasing the dispersion parameter, the recurrence no longer happens. At an early stage, the initial cosine splits into several solitons that interact with each other elastically causing a phase shift in their

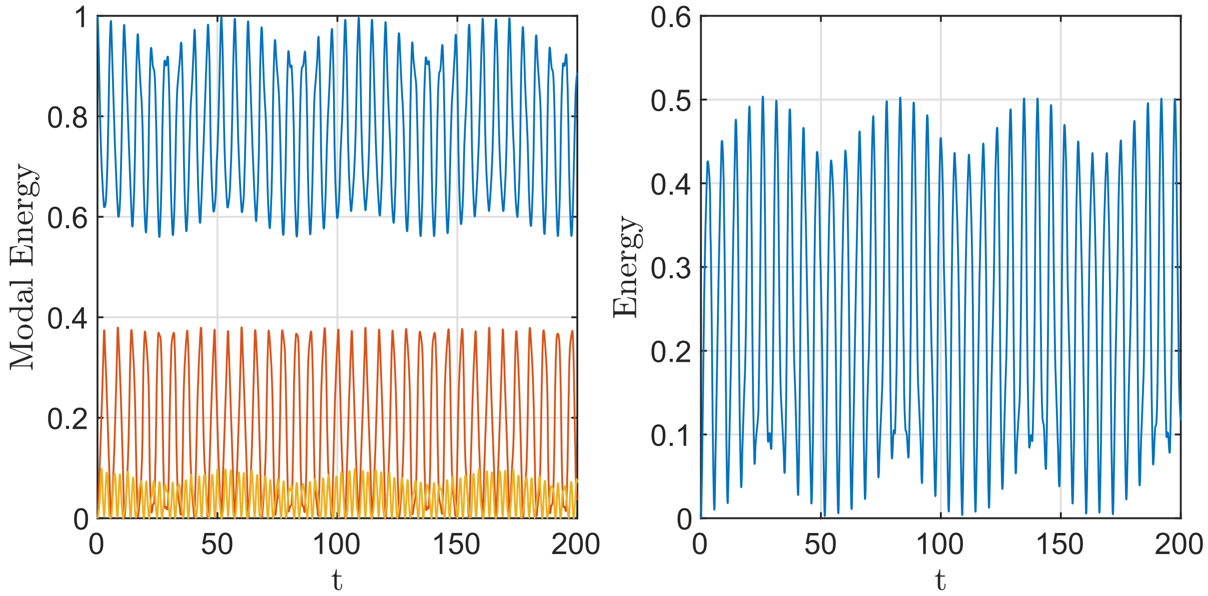
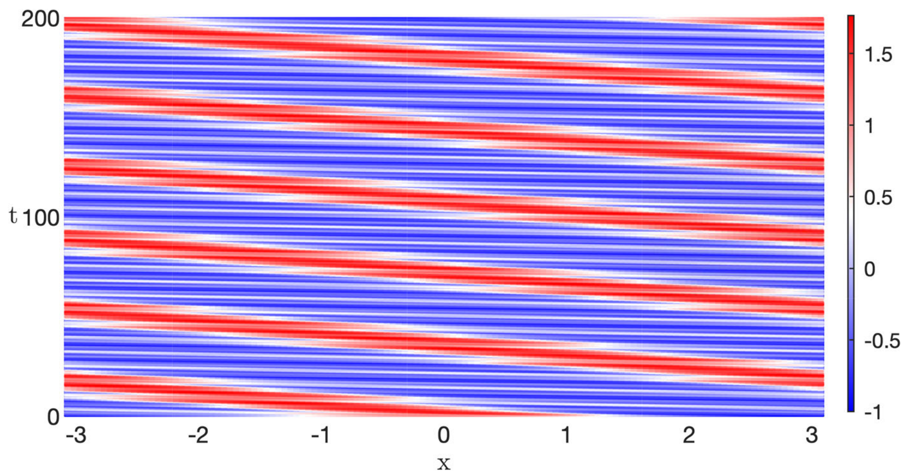


Fig. 9 The modal energy of the first three harmonic modes $k = 1, k = 2, k = 3$ in blue, red and yellow respectively and the variation of the energy defined in Eq. (13) as a function of time for $\mu = 0.1$ and $\beta = -1$

Fig. 10 The evolution of the cosine wave with $\mu = 0.05$ and $\beta = -1$



crests as shown in Fig. 6. About 93 % of the energy is recovered at $t = 14.9$, however our simulations show that first mode only recovers about 60 % of its energy at $t = 29.8$, which does not feature a recurrence. The energy is spread out more evenly in other harmonics, see Fig. 7.

4.2 Negative cubic nonlinearity

The effects of weak dispersion are initially considered. Figure 8 illustrates the evolution of the cosine wave with dispersion $\mu = 0.1$. The energy from the first mode is mainly transferred to the second mode. At time $t = 51.7$, approximately 99 % of the energy of the first mode is recovered, as shown in Fig. 9. The next recurrence occurs at $t = 108.8$. This indicates that while the energy of the first mode is nearly fully recovered, the process is not periodic, unlike the Gardner equation with positive cubic nonlinearity.

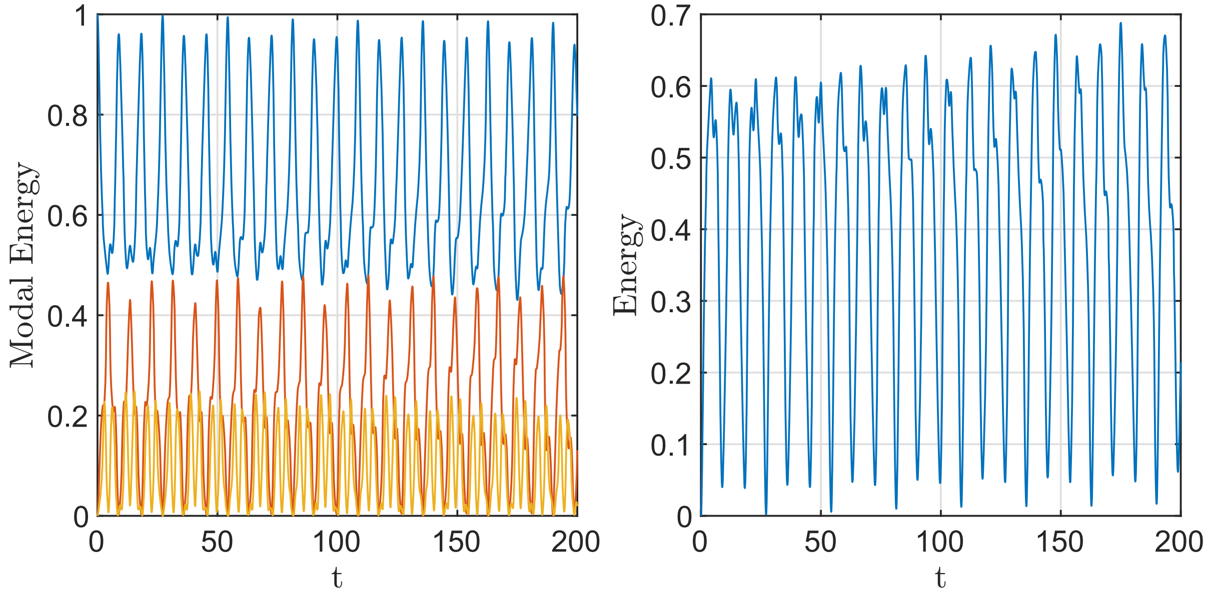
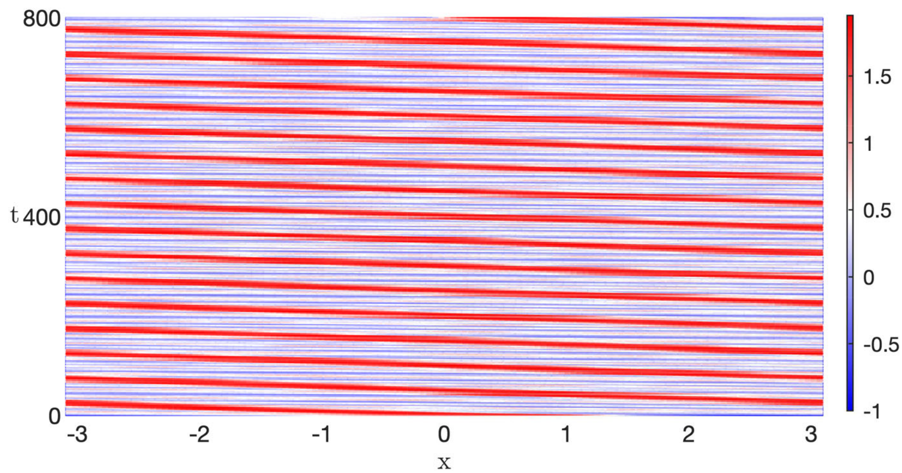


Fig. 11 The modal energy of the first three harmonic modes $k = 1$, $k = 2$, $k = 3$ in blue, red and yellow respectively and the variation of the energy defined in Eq. (13) as a function of time for $\mu = 0.05$ and $\beta = -1$

Fig. 12 The evolution of the cosine wave with $\mu = 0.01$ and $\beta = -1$



5 Conclusion

In this study, we have investigated the FPUT recurrence phenomenon within the context of the Gardner equation. We began by analyzing a simplified case involving two harmonics with time-dependent amplitudes, from which we derived a dynamical system to describe the amplitude evolution of these harmonics. For this scenario, we found that exact FPUT recurrence occurs at the equilibrium center of the dynamical system. However, for the single harmonic case, the same analytical approach proved inadequate, necessitating a numerical analysis. Our results demonstrate that the Gardner equation with positive cubic nonlinearity exhibits the recurrence phenomenon under both strong and intermediate dispersion. In contrast, with negative cubic nonlinearity, while the initial mode can occasionally recover its energy under strong and intermediate dispersion, it does not exhibit sustained periodic motion over time.

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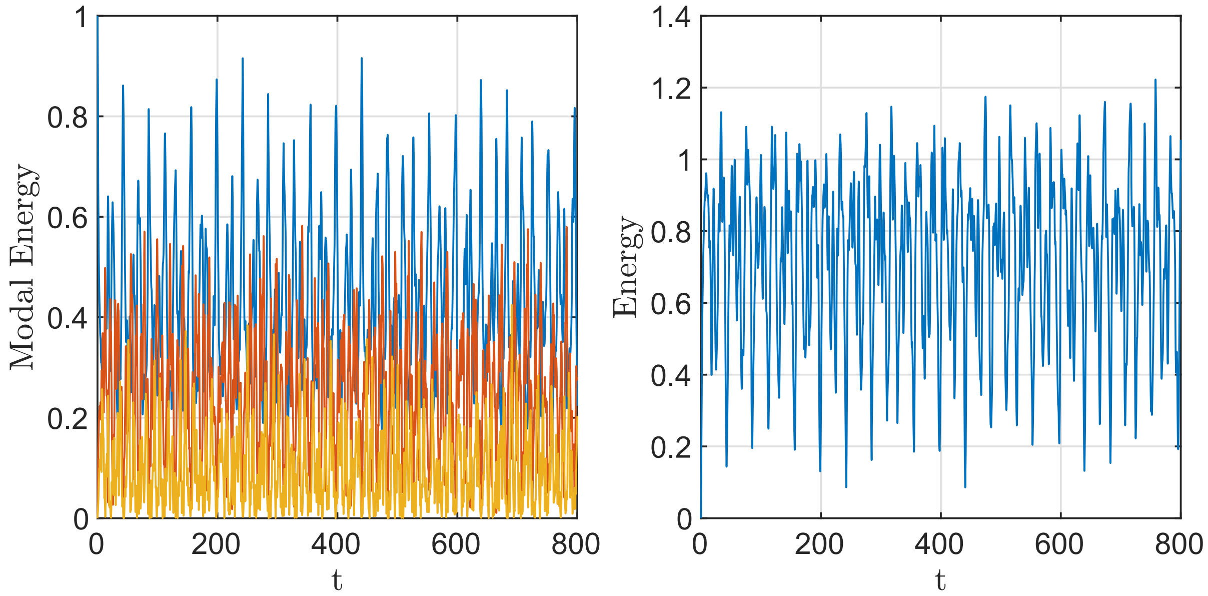

Fig. 13 The modal energy of the first three harmonic modes $k = 1$, $k = 2$, $k = 3$ in blue, red and yellow respectively and the variation of the energy defined in Eq. (13) as a function of time for $\mu = 0.01$ and $\beta = -1$

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Data availability Data sharing is not applicable to this article as all parameters used in the numerical experiments are informed in this paper.

Declarations

Conflict of interest The authors state that there is no Conflict of interest.

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