Product Technology and Industry Technology: Exploring the Reverse Transformations

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Abstract

One of the main aims of constructing input-output balance models is to assess an impact of exogenous changes in net final demand (certainly at constant prices) on simultaneous behavior of an economy. Nowadays, two approaches to constructing input-output coefficients are widely used in practice, namely, one based on so-called product technology assumption and another based on so-called industry technology assumption. These approaches provide direct transforming supply and use tables (SUT) to symmetric input-output tables (SIOT) in a product-by-product format.

Focus of attention in the article is concentrated on analyzing the reverse transformations that link exogenous changes of final demand in SIOT with corresponding changes of the production and intermediate consumption matrices in initial SUT. Material balance equation, classical Leontief equation and product (or commodity) technology model form the system of equations with production and intermediate consumption matrices as unknowns. It is shown that this system has the solution that guarantees the exogenous changes in final demand to be at constant prices.

In turn, material balance equation, classical Leontief equation and industry technology model constitute another system of equations (with the same unknowns) that can be also resolved with respect to production matrix and intermediate consumption matrix. However, exogenous varying the final demand in obtained solution leads to quantity changes in the intermediate consumption matrix and to price changes in the production matrix. This type of economy's response to exogenous changes in final demand seems to be implausible artifact that is out of economic sense. Thus, there are some certain doubts about plausibility of underlying background for an industry technology assumption and a fixed product sales structure assumption that are widely used for transforming SUT to SIOT.

Keywords: demand-driven input-output model, material balance equation, exogenous changes in final demand, product technology assumption, industry technology assumption, price and quantity changes

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1. Introduction

Consider a common form of demand-driven input-output model written as follows:

$$\mathbf{x} = \mathbf{L}\mathbf{y} \tag{1}$$

where $\mathbf{x}$ is N-dimensional column vector of product outputs, N is the number of products being produced in the economy, $\mathbf{y}$ is N-dimensional column vector of final demand, and $\mathbf{L}$ is nondegenerate square matrix of order N (the Leontief inverse). Clearly, formula (1) expresses the functional dependence of product outputs on final demand components, in which $\mathbf{y}$ plays the role of independent vector variable. Thus, it is implicitly presumed in (1) that vector $\mathbf{y}$ can be varied arbitrarily while all components of the final demand undergo quantity (not price!) changes.

However, vector $\mathbf{y}$ is a part of material balance equation

$$\mathbf{x} = \mathbf{z} + \mathbf{y} \tag{2}$$

where $\mathbf{z}$ is column vector of intermediate product inputs with dimensions $N\times1$. Since $\mathbf{y} = \mathbf{x} - \mathbf{z}$, as it follows from (2), the changes in final demand exert an influence on the differences between product outputs and product inputs. Therefore, exogenous variations of final demand vector $\mathbf{y}$ lead to corresponding changes in $\mathbf{x}$ and $\mathbf{z}$.

Key idea of the Leontief input-output analysis is to eliminate variable $\mathbf{z}$ from equation (2) by substitution of the linear linking relation

$$\mathbf{z} = \mathbf{A}\mathbf{x} \tag{3}$$

where square matrix $\mathbf{A}$ of order N is known in special literature as (Leontief) technical coefficients matrix. The technical coefficients are usually calculated on a base of the given supply and use table for certain time period (say, period 0) that contains supply (production) matrix $\mathbf{X}_0$ and use (intermediate consumption) matrix $\mathbf{Z}_0$ of the same dimension $N\times M$ where M is the number of industries in the economy. Notice that $\mathbf{x} = \mathbf{X}\mathbf{e}_M$ and $\mathbf{z} = \mathbf{Z}\mathbf{e}_M$ where $\mathbf{X}$ and $\mathbf{Z}$ are matrix variables of the same dimension as production matrix $\mathbf{X}_0$ and intermediate consumption matrix $\mathbf{Z}_0$, respectively, and $\mathbf{e}_M$ is $M\times1$ summation column vector with all entries equal to one. Letting $\mathbf{A} = \mathbf{A}(\mathbf{X}_0, \mathbf{Z}_0)$ be a square matrix of order N, and then solving system (2), (3) with respect to the product outputs vector $\mathbf{x}$ leads to the (Leontief) demand-driven input-output model (1) with transformation matrix

$$\mathbf{L} = \left[\mathbf{E}_N - \mathbf{A}(\mathbf{X}_0, \mathbf{Z}_0)\right]^{-1}$$

where $\mathbf{E}_N$ is an identity matrix of order N.

Thus, constructing demand-driven input-output model for analytic purposes seemingly comes down to a choice of appropriate pattern for technical coefficients matrix. There are many ways to define the coefficients pattern known in special literature. "It is standard to derive input-output constructs from alternative assumptions" (Kop Jansen and ten Raa, 1990, p. 214). Nevertheless, two approaches to constructing input-output coefficients for model with exogenous final demand are most widely used in practice, namely, one based on so-called product technology assumption and another one based on so-called industry technology assumption (see Eurostat, 2008; United Nations, 2018).

Main scope of the article is to assess and interpret the consequences of exogenous final demand varying in input-output model (1) for production matrix $\mathbf{X}$ and intermediate consumption matrix $\mathbf{Z}$ at choosing input-output coefficients $\mathbf{A} = \mathbf{A}(\mathbf{X}_0, \mathbf{Z}_0)$ under product technology assumption and, in turn, under industry technology assumption. The Leontief input-output model with exogenous final demand

$$\mathbf{X}\mathbf{e}_M = \left[\mathbf{E}_N - \mathbf{A}(\mathbf{X}_0, \mathbf{Z}_0)\right]^{-1}\mathbf{y} \tag{4}$$

and material balance equation

$$\mathbf{X}\mathbf{e}_M = \mathbf{Z}\mathbf{e}_M + \mathbf{y} \tag{5}$$

serve as a toolbox for analyzing concomitant changes in product outputs and intermediate inputs following the changes in final demand.

2. Product technology assumption

The product technology pattern (or the commodity technology model – see, e.g., Kop Jansen and ten Raa, 1990) can be presenting in our denotations as follows:

$$\mathbf{A}(\mathbf{X}_0, \mathbf{Z}_0) = \mathbf{Z}_0\mathbf{X}_0^{-1}. \tag{6}$$

Obviously, the pattern is valid if number of products N coincides with number of industries M, i.e., $N = M = K$, and square production matrix $\mathbf{X}_0$ of order K is invertible. The latter does not seem to be too restrictive because the actual production matrices use to be strictly diagonally dominant in practice. Note that in accordance with product technology assumption each product is produced in its own specific way in all industries where it is produced (see, e.g., Eurostat, 2008).

Substituting the pattern (6) into the Leontief demand-driven input-output model (4) yields

$$\mathbf{X}\mathbf{e}_K = \left(\mathbf{E}_K - \mathbf{Z}_0\mathbf{X}_0^{-1}\right)^{-1}\mathbf{y} = \mathbf{X}_0\left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1}\mathbf{y}\,.$$

Since this equation should be fulfilled at any vector of final demand, it is possible to express the production matrix **X** in left-hand side as

$$\mathbf{X} = \mathbf{X}_0\left\langle\left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1}\mathbf{y}\right\rangle = \mathbf{X}_0\hat{\mathbf{q}} \tag{7}$$

where angled bracketing around vector's designation (or putting a "hat" over vector's symbol) denotes a diagonal matrix with the vector on its main diagonal and zeros elsewhere (see Miller and Blair, 2009, p. 697).

Substitution the production matrix (7) into material balance equation (5) and rearrangement the terms gives

$$\begin{aligned}\mathbf{Z}\mathbf{e}_K &= \mathbf{X}\mathbf{e}_K - \mathbf{y} = \mathbf{X}_0\left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1}\mathbf{y} - \mathbf{y} = \left[\mathbf{X}_0\left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1} - \mathbf{E}_K\right]\mathbf{y} = \\ &= \left[\mathbf{X}_0 - \left(\mathbf{X}_0 - \mathbf{Z}_0\right)\right]\left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1}\mathbf{y} = \mathbf{Z}_0\left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1}\mathbf{y}.\end{aligned}$$

This equation should be fulfilled at any final demand vector **y**, hence, intermediate consumption matrix **Z** in left-hand side can be determined as

$$\mathbf{Z} = \mathbf{Z}_0\left\langle\left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1}\mathbf{y}\right\rangle = \mathbf{Z}_0\hat{\mathbf{q}}\,. \tag{8}$$

It is easy to show that $\mathbf{q} = \mathbf{e}_K$ at $\mathbf{y} = \mathbf{y}_0 = \mathbf{X}_0\mathbf{e}_K - \mathbf{Z}_0\mathbf{e}_K$. Indeed, it follows directly from $\mathbf{y}_0 = \left(\mathbf{X}_0 - \mathbf{Z}_0\right)\mathbf{e}_K$ so that $\mathbf{q} = \left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1}\mathbf{y}_0 = \mathbf{e}_K$, or $\hat{\mathbf{q}} = \mathbf{E}_K$.

Thus, at choosing input-output coefficients according to *product technology* pattern, an arbitrary variation of final demand vector leads to the changes in production and intermediate consumption matrices described by simple *post*multiplication formulas $\mathbf{X} = \mathbf{X}_0\hat{\mathbf{q}}_{\mathrm{PT}}$, $\mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}_{\mathrm{PT}}$ where

$$\mathbf{q}_{\mathrm{PT}} = \left(\mathbf{X}_0 - \mathbf{Z}_0\right)^{-1}\mathbf{y}\,, \tag{9}$$

and subindex "PT" means an association of the vector with product technology pattern. Note that all K components of vector $\mathbf{q}_{\mathrm{PT}}$ are dimensionless.

3. Industry technology assumption

Kop Jansen and ten Raa (1990) studied the industry technology model that in our denotations becomes

$$\mathbf{A}(\mathbf{X}_0, \mathbf{Z}_0) = \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1} \tag{10}$$

where putting a prime after matrix's (vector's) symbol denotes a transpose of this matrix (vector). It is easy to see that this pattern for technical coefficients matrix is valid at any combinations of number of products N and number of industries M. Note that according to industry technology assumption each industry has its own specific way of production, irrespective of its product mix (see, e.g., Eurostat, 2008).

Substituting industry technology pattern (10) into the Leontief demand-driven input-output model (4) and rearranging the terms yields

$$\mathbf{X}\mathbf{e}_M = \left(\mathbf{E}_N - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \langle \mathbf{X}_0 \mathbf{e}_M \rangle^{-1}\right)^{-1} \mathbf{y} = \langle \mathbf{X}_0 \mathbf{e}_M \rangle \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0\right)^{-1} \mathbf{y} =$$
$$= \left\langle \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0\right)^{-1} \mathbf{y} \right\rangle \mathbf{X}_0 \mathbf{e}_M$$

where the obvious quasi-commutativity property $\hat{\mathbf{a}}\mathbf{b} = \hat{\mathbf{b}}\mathbf{a}$ for a pair of N-dimensional column vectors $\mathbf{a}$ and $\mathbf{b}$ is used. The latter equation should be fulfilled at any vector of final demand, as earlier. Therefore,

$$\mathbf{X} = \left\langle \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0\right)^{-1} \mathbf{y} \right\rangle \mathbf{X}_0 = \hat{\mathbf{p}} \mathbf{X}_0 \,. \tag{11}$$

Substitution the production matrix (11) into material balance equation (5) and rearrangement the terms gives

$$\mathbf{Z}\mathbf{e}_M = \mathbf{X}\mathbf{e}_M - \mathbf{y} = \left[\langle \mathbf{X}_0 \mathbf{e}_M \rangle \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0\right)^{-1} - \mathbf{E}_N\right] \mathbf{y} =$$
$$= \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \langle \mathbf{X}_0 \mathbf{e}_M \rangle + \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0\right) \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0\right)^{-1} \mathbf{y} =$$
$$= \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0\right)^{-1} \mathbf{y}.$$

Again, as earlier, this equation should be fulfilled at any final demand vector $\mathbf{y}$, hence, intermediate consumption matrix $\mathbf{Z}$ in left-hand side can be derived as

$$\mathbf{Z} = \mathbf{Z}_0 \left\langle \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \left(\langle \mathbf{X}_0 \mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0\right)^{-1} \mathbf{y} \right\rangle = \mathbf{Z}_0 \hat{\mathbf{q}} = \mathbf{Z}_0 \left\langle \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \, \mathbf{p} \right\rangle . \tag{12}$$

On checking the initial condition, setting $\mathbf{y} = \mathbf{y}_0 = \mathbf{X}_0 \mathbf{e}_M - \mathbf{Z}_0 \mathbf{e}_M$ in (11) gives the following equation with respect to unknown vector **p**:

$$\langle \mathbf{X}_0\mathbf{e}_M \rangle \mathbf{p} - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \mathbf{p} = \mathbf{X}_0\mathbf{e}_M - \mathbf{Z}_0\mathbf{e}_M .$$

Since $\langle \mathbf{X}_0\mathbf{e}_M \rangle \mathbf{e}_N = \mathbf{X}_0\mathbf{e}_M$ and $\mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \mathbf{e}_N = \mathbf{Z}_0 \langle \mathbf{X}'_0 \mathbf{e}_N \rangle^{-1} \mathbf{X}'_0 \mathbf{e}_N = \mathbf{Z}_0\mathbf{e}_M$, the solution to this equation is $\mathbf{p} = \mathbf{e}_N$, from which and (12) it directly follows that

$$\mathbf{q} = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0\, \mathbf{p} = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \mathbf{e}_N = \mathbf{e}_M .$$

Thus, at choosing input-output coefficients according to *industry technology* pattern (10) an arbitrary variation of final demand vector induces the changes in production and intermediate consumption matrices respectively described by *pre*multiplication formula $\mathbf{X} = \hat{\mathbf{p}}_{\mathrm{IT}} \mathbf{X}_0$ (in contrast to product technology case) and *post*multiplication formula $\mathbf{Z} = \mathbf{Z}_0 \hat{\mathbf{q}}_{\mathrm{IT}}$ (as in product technology case) where

$$\mathbf{p}_{\mathrm{IT}} = \left(\langle \mathbf{X}_0\mathbf{e}_M \rangle - \mathbf{Z}_0 \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0 \right)^{-1} \mathbf{y} , \qquad \mathbf{q}_{\mathrm{IT}} = \langle \mathbf{e}'_N \mathbf{X}_0 \rangle^{-1} \mathbf{X}'_0\, \mathbf{p}_{\mathrm{IT}} , \tag{13}$$

and subindex "IT" means an association of the vector with industry technology pattern. Note that all components of vectors $\mathbf{p}_{\mathrm{IT}}$ and $\mathbf{q}_{\mathrm{IT}}$ are dimensionless in accordance with (11) and (12), respectively.

4. Economic interpretation of the obtained results

There are some distinctions between the formal results obtained above in Section 2 (product technology case) and Section 3 (industry technology case). The substantial structural distinction lies in the expressions derived for production matrix, namely, $\mathbf{X} = \mathbf{X}_0 \hat{\mathbf{q}}_{\mathrm{PT}}$ in product technology case whereas $\mathbf{X} = \hat{\mathbf{p}}_{\mathrm{IT}} \mathbf{X}_0$ in industry technology case. Recall that $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$ are diagonal matrices of appropriate orders.

To clarify a role of vector **p** in arbitrary varying of final demand together with production and intermediate consumption matrices, consider the Leontief price model

$$\mathbf{p}' = \mathbf{p}'\mathbf{A} + \mathbf{v}' \langle \mathbf{e}'_K \mathbf{X}_0 \rangle^{-1}$$

where **p** is price index vector with dimensions $K{\times}1$, $\mathbf{A} = \mathbf{Z}_0 \langle \mathbf{e}'_K \mathbf{X}_0 \rangle^{-1}$ is technical coefficients matrix, and **v** is value added vector with dimensions $K{\times}1$ (see, e.g., Miller and Blair, 2009). Substituting technical coefficients matrix **A** into the price model yields the financial balance equation $\mathbf{p}' \langle \mathbf{e}'_K \mathbf{X}_0 \rangle = \mathbf{p}'\mathbf{Z}_0 + \mathbf{v}'$ from which $\mathbf{X} = \hat{\mathbf{p}} \langle \mathbf{e}'_K \mathbf{X}_0 \rangle$ and $\mathbf{Z} = \hat{\mathbf{p}} \mathbf{Z}_0$ (recall that in Leontief price model the production matrix is assumed to be diagonal). Hence, *price changes* in production matrix and intermediate consumption matrix are described by the initial matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ *pre*multiplying by diagonal matrix of price indices.

Furthermore, for establishing a role of vector $\mathbf{q}$ in arbitrary varying of final demand together with production and intermediate consumption matrices, consider the Ghosh quantity model

$$\mathbf{q} = \mathbf{B}\mathbf{q} + \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1}\mathbf{y}$$

where $\mathbf{q}$ is quantity index vector with dimensions $K\times 1$, and $\mathbf{B} = \langle \mathbf{X}_0\mathbf{e}_K \rangle^{-1}\mathbf{Z}_0$ is allocation coefficients matrix (see Miller and Blair, 2009; Motorin, 2017). Substituting allocation coefficients matrix $\mathbf{B}$ into the quantity model yields the material balance equation $\langle \mathbf{X}_0\mathbf{e}_K \rangle\mathbf{q} = \mathbf{Z}_0\mathbf{q} + \mathbf{y}$ from which $\mathbf{X} = \langle \mathbf{X}_0\mathbf{e}_K \rangle\hat{\mathbf{q}}$ and $\mathbf{Z} = \mathbf{Z}_0\hat{\mathbf{q}}$ (in the Ghosh quantity model the production matrix is also assumed to be diagonal). Therefore, *quantity changes* in production matrix and intermediate consumption matrix are described by the initial matrices $\mathbf{X}_0$ and $\mathbf{Z}_0$ *post*multiplying by diagonal matrix of quantity indices.

Thus, arbitrary variations of final demand vector in input-output model (1) within a *product technology* pattern (PT-model) are translated into the *quantity changes* in production and intermediate consumption matrices. Each component of vector (9) represents an index of output growth in corresponding industry induced by a change of final demand as well as an index for intermediate inputs growth in the same industry caused by the industry output growth. In other words, the PT-model with exogenous final demand operates *at constant prices* because $\hat{\mathbf{p}}_{\mathrm{PT}} = \mathbf{E}_K$. It is worth to mention here that this conclusion exactly corresponds to price invariance axiom fulfillment established by Kop Jansen and ten Raa (1990) for product technology pattern (6).

To illustrate the PT-model operating, consider an obvious parametric scheme of forming the final demand at constant prices

$$\mathbf{y} = \mathbf{X}_0\mathbf{q} - \mathbf{Z}_0\mathbf{q}$$

where $\mathbf{q}$ now is arbitrarily varying vector of quantity indices. Substitution the scheme into the material balance equation $\mathbf{X}\mathbf{e}_K - \mathbf{Z}\mathbf{e}_K = \mathbf{y}$ and rearrangement the terms gives $\mathbf{X}\mathbf{e}_K - \mathbf{X}_0\mathbf{q} = \mathbf{Z}\mathbf{e}_K - \mathbf{Z}_0\mathbf{q}$. The latter equation should be fulfilled at any vector $\mathbf{q}$ and therefore one can set down $\mathbf{X}\mathbf{e}_K = \mathbf{X}_0\mathbf{q}$ and $\mathbf{Z}\mathbf{e}_K = \mathbf{Z}_0\mathbf{q}$. Solving the equation $\mathbf{X}_0\mathbf{q} - \mathbf{Z}_0\mathbf{q} = \mathbf{y}$ with respect to $\mathbf{q}$ for any given final demand vector $\mathbf{y}$ yields the familiar formula

$\mathbf{q} = (\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y} = \mathbf{q}_{\text{PT}}$ from which we get the Leontief demand-driven input-output model (4) under product technology assumption (6), namely

$$\mathbf{X}\mathbf{e}_K = \mathbf{X}\mathbf{q} = \mathbf{X}_0(\mathbf{X}_0 - \mathbf{Z}_0)^{-1}\mathbf{y} = \left(\mathbf{E}_K - \mathbf{Z}_0\mathbf{X}_0^{-1}\right)^{-1}\mathbf{y}\,.$$

Thus, if a researcher intends to deal with demand-driven input-output model *at constant prices* (this is quite naturally!) then the product technology pattern should be chosen as a sole opportunity to make the model with exogenous final demand operational. Therefore, the mentioned above problem of choosing appropriate pattern for technical coefficients matrix in model with exogenous final demand actually does not exist.

In turn, arbitrary variations of final demand vector in input-output model (1) within an *industry technology* pattern (IT-model) are transferred to the *price changes* in production matrix and at the same time to the *quantity changes* in intermediate consumption matrix. Note that each entry of vector $\mathbf{p}_{\text{IT}}$ represents a price index for output of corresponding product that does not vary along the row of all producing-and-consuming industries. As it is follows from the first formula (13), this price-induced output change is a part of response to exogenous change of final demand in the IT-model. The another part of the response is related to the quantity changes in intermediate consumption matrix, whence follows that each element of vector in the second formula (13), namely $\mathbf{q}_{\text{IT}} = \langle \mathbf{e}'_N\mathbf{X}_0\rangle^{-1}\mathbf{X}'_0\,\mathbf{p}_{\text{IT}}$, should be considered as an index for *intermediate inputs growth* in corresponding industry caused by the industry component of *price-induced output change* mentioned above (?!). This mixed situation of combining price and quantity changes in a presence of the direct linkage (13) between vectors $\mathbf{p}_{\text{IT}}$ and $\mathbf{q}_{\text{IT}}$ seems to be an implausible artifact that is out of economic sense.

5. Evaluating the quantitative distinctions between two models

In general, input-output model (1) within a product technology pattern

$$\mathbf{X}_{\text{PT}}\mathbf{e}_K = \left(\mathbf{E}_K - \mathbf{Z}_0\mathbf{X}_0^{-1}\right)^{-1}\mathbf{y} = \mathbf{L}_{\text{PT}}\mathbf{y} \tag{14}$$

and input-output model (1) within a industry technology pattern

$$\mathbf{X}_{\text{IT}}\mathbf{e}_K = \left(\mathbf{E}_K - \mathbf{Z}_0\langle \mathbf{e}'_K\mathbf{X}_0\rangle^{-1}\mathbf{X}'_0\langle \mathbf{X}_0\mathbf{e}_K\rangle^{-1}\right)^{-1}\mathbf{y} = \mathbf{L}_{\text{IT}}\mathbf{y} \tag{15}$$

determine the different product output vectors at any given vector of final demand. It is easy to admit that difference

$$\Delta\mathbf{x}_{\text{PT}-\text{IT}} = \mathbf{X}_{\text{PT}}\mathbf{e}_K - \mathbf{X}_{\text{IT}}\mathbf{e}_K = (\mathbf{L}_{\text{PT}} - \mathbf{L}_{\text{IT}})\,\mathbf{y} = \mathbf{L}_{\text{PT}}\left(\mathbf{L}_{\text{IT}}^{-1} - \mathbf{L}_{\text{PT}}^{-1}\right)\mathbf{L}_{\text{IT}}\mathbf{y} \tag{16}$$

between equations (14) and (15) can serve as a proper measure of quantitative distinctions between PT-model and IT-model. The vector measure $\Delta\mathbf{x}_{\text{PT}-\text{IT}}$ could be considered as defect of IT-model in comparison with PT-model that operates at constant prices.

"Leontief input-output economics derive their significance largely from the fact that output multipliers measuring the combined effects of the direct and indirect repercussions of a change in final demand were readily calculated" (Steenge A.E., 1990, p. 377). So another reasonable measure of quantitative distinctions between the models seems to be the difference

$$\Delta\mathbf{m}'_{\text{PT}-\text{IT}} = \mathbf{m}'_{\text{PT}} - \mathbf{m}'_{\text{IT}} = \mathbf{e}'_K(\mathbf{L}_{\text{PT}} - \mathbf{L}_{\text{IT}}) = \mathbf{e}'_K\mathbf{L}_{\text{PT}}\left(\mathbf{L}_{\text{IT}}^{-1} - \mathbf{L}_{\text{PT}}^{-1}\right)\mathbf{L}_{\text{IT}} \tag{17}$$

where $\mathbf{m}'_{\text{PT}}$ and $\mathbf{m}'_{\text{IT}}$ are the row vectors of output multipliers calculated from PT-model and IT-model, respectively (see Miller and Blair, 2009).

The matrix in parentheses from equations (16) and (17) after appropriate transformation can be represented as follows:

$$\mathbf{L}_{\text{IT}}^{-1} - \mathbf{L}_{\text{PT}}^{-1} = \mathbf{Z}_0\left(\mathbf{E}_K - \langle\mathbf{e}'_K\mathbf{X}_0\rangle^{-1}\mathbf{X}'_0\langle\mathbf{X}_0\mathbf{e}_K\rangle^{-1}\mathbf{X}_0\right)\mathbf{X}_0^{-1} = \mathbf{Z}_0(\mathbf{E}_K - \mathbf{C}'_0\mathbf{D}_0)\,\mathbf{X}_0^{-1} \tag{18}$$

where matrix $\mathbf{C}_0 = \mathbf{X}_0\langle\mathbf{e}'_K\mathbf{X}_0\rangle^{-1}$ and matrix $\mathbf{D}_0 = \langle\mathbf{X}_0\mathbf{e}_K\rangle^{-1}\mathbf{X}_0$ are known in input-output literature as product-mix matrix and the transpose of market shares matrix, respectively (see Eurostat, 2008; Miller and Blair, 2009). Clearly, all elements of matrices $\mathbf{C}_0$ and $\mathbf{D}_0$ are non-negative as well as the elements of production matrix $\mathbf{X}_0$.

It is easy to show that matrix (18) becomes null one if and only if the production matrix is diagonal together with the matrices $\mathbf{C}_0$ and $\mathbf{D}_0$. Hence, in the diagonal case the input-output models (14) and (15) determine the same product output vector at any given vector of final demand as well as the same vector of output multipliers, i.e. PT-model and IT-model become equivalent.

In general, after some substitutions and subsequent transformations of (16) and (17) using (18), we get two linkage equations

$$\mathbf{X}_{\text{PT}}\mathbf{e}_K = \left(\mathbf{E}_K + \mathbf{L}_{\text{PT}}\mathbf{Z}_0\mathbf{X}_0^{-1} - \mathbf{L}_{\text{PT}}\mathbf{Z}_0\mathbf{C}'_0\mathbf{D}_0\mathbf{X}_0^{-1}\right)\mathbf{X}_{\text{IT}}\mathbf{e}_K\,,$$

$$\mathbf{m}'_{\text{IT}} = \mathbf{m}'_{\text{PT}}\left(\mathbf{E}_K - \mathbf{Z}_0\mathbf{X}_0^{-1}\mathbf{L}_{\text{IT}} + \mathbf{Z}_0\mathbf{C}'_0\mathbf{D}_0\mathbf{X}_0^{-1}\mathbf{L}_{\text{IT}}\right)$$

together with obvious concomitant difficulties of their further formal analysis. Nevertheless, without pretending to be mathematically rigorous, it can be argued that the more the production matrix deviates from its diagonal form, the greater the defects of IT-model in comparison with PT-model (16) and (17) will be.

6. Concluding remarks

Practical applications of demand-driven input-output model (1) are actually based on a principal opportunity to arbitrarily vary the final demand vector in right-hand side of (1) provided that all its components undergo quantity changes. This seems to be a necessary requirement for constructing the proper transformation matrix **L** applicable to solving various problems in main branches of modern input-output theory such as multiplier analysis, impact analysis, structural decomposition analysis, value-added chain analysis, etc.

Formally, the demand-driven input-output model (1) within a product technology pattern (6) is fully (mathematically and economically) consistent with the above requirement. As it is shown in Section 2 and 4, the model do operates at constant prices. Nevertheless, well-known (but not indisputable!) problem of negative cell entries in the product technology (see, e.g., United Nations, 2018, Annex B to Chapter 12) yet does not allow to consider a product technology approach as universal way of constructing demand-driven input-output models and transforming the supply and use tables into symmetric input-output tables.

To continue, the demand-driven input-output model (1) within an industry technology pattern (10) do certainly violate the above requirement in general with the exception of a case when the production matrix is diagonal (because a price index and a growth index are obviously indiscernible for diagonal output matrix). Unfortunately, model operating generates an informational and logical "price'n'quantity" gap between initial supply and use table and the resulting input-output table/model. It casts some doubt on plausibility of an industry technology assumption and a fixed product sales structure assumption (see Eurostat, 2008) often used in converting supply and use tables to symmetric input-output tables in product-by-product and industry-by-industry format, respectively.

Declarations

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Consent for publication

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Not applicable

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Endnotes

Not applicable (there are no endnotes in the text)

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