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Topological Classification of Polar Flows on Four-Dimensional Manifolds

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Abstract—S. Smale has shown that any closed smooth manifold admits a gradient-like flow, which is a structurally stable flow with a finite nonwandering set. Polar flows form a subclass of gradient-like flows characterized by the simplest nonwandering set for the given manifold, consisting of exactly one source, one sink, and a finite number of saddle equilibria. We describe the topology of four-dimensional closed manifolds that admit polar flows without heteroclinic intersections, as well as all classes of topological equivalence of polar flows on each manifold. In particular, we demonstrate that there exists a countable set of nonequivalent flows with a given number $k \geqslant 2$ of saddle equilibria on each manifold, which contrasts with the situation in lower-dimensional analogues.

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1. INTRODUCTION

Let M^n be a closed (compact without boundary) smooth manifold of dimension $n \geqslant 1$ and f^t is a smooth flow on M^n. The flow f^t generates the velocity field $F(x) = \partial f^t(x)/\partial t|_{t=0}$ on M^n. We recall that an equilibrium state $p \in M^n$ of f^t is called *hyperbolic* if all eigenvalues of the Jacobian matrix $(\frac{\partial F}{\partial x})|_p$ have nonzero real parts. A flow f^t on M^n is called *gradient-like* if its nonwandering set Ω_{f^t} consists of a finite number of hyperbolic equilibrium states, and the invariant manifolds of equilibria intersect each other transversely. A nonempty intersection $W^s_p \cap W^u_q$ of invariant manifolds of different saddle equilibria p, q is called *heteroclinic*. It follows from [34] and [35] that gradient-like flows are structurally stable.

Let p be a hyperbolic equilibrium. A number i_p equal to the dimension $\dim W^u_p$ of its unstable manifold W^u_p is called the *Morse index of* p. The point p is called *the source* if $i_p = n$, *the sink* if $i_p = 0$, and *the saddle* in the other cases. A gradient-like flow is called *polar* if its nonwandering set consists of exactly one source, one sink, and an arbitrary finite number of saddles. If f^t is an arbitrary gradient-like flow on M^n, then similar to [31, Theorem 12.1] (using a finite number of saddle-node bifurcations) one may construct a polar flow $\tilde{f}^t$ on the same manifold.

An asymptotic behavior of trajectories of gradient-like flows looks like the behavior of trajectories of gradient flows of Morse functions. Namely, due to [29, 41], for any gradient-like flow $f^t : M^n \to M^n$ there exists *an energy function* $\varphi : M^n \to \mathbb{R}$, that is a smooth function strictly decreasing along all trajectories of f^t different from equilibria and having a nondegenerate critical point at each equilibrium. The sets of gradient-like and gradient flows have a nonempty intersection but

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do not coincide. Specifically, the linearization of gradient flows near any of its equilibria has only real eigenvalues, while a gradient-like flow may have complex eigenvalues. On the other hand, a gradient flow may have nontransversal heteroclinic intersections, while the gradient-like flow does not. Smale in [41, Theorem A] proved that the gradient flow of an arbitrary Morse function can be approximated (in C^1-topology) by a gradient-like flow, which implies, in particular, that any closed manifold admits a gradient-like flow. Due to existence of the energy functions, there is a tight interrelation between the structure of the nonwandering set of a gradient-like flow and the topology of the carrying manifold. Let us explain the nature of this interrelation.

Let $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| \leqslant 1\}$. We recall that *an n-dimensional handle of index $i \in \{0, 1, \ldots, n\}$* is the direct product $H_i^n = \mathbb{B}^i \times \mathbb{B}^{n-i}$. A manifold Y is said to be obtained by *attaching the i-handle* H_i^n to a compact manifold X with nonempty boundary ∂X if Y is obtained from the union $X \cup H_i^n$ by identification of a submanifold $F \in \partial X$ with the set $\partial\mathbb{B}^i \times \mathbb{B}^{n-i}$ by means of a homeomorphism $\psi : \partial\mathbb{B}^i \times \mathbb{B}^{n-i} \to F$. So, any handle is homeomorphic to a ball, and the index i of the handle partially determines the way of attaching the handle, which depends also on the isotopy classes of the set F and the map ψ (for more details, see Section 3.7).

The energy function $\varphi : M^n \to \mathbb{R}$ of a gradient flow f^t, like any Morse function, determines a *handle decomposition* of M^n, that is, a representation of M^n as a finite union of handles, each of which contains exactly one equilibrium state, and the Morse index of the equilibrium coincides with the index of the handle.

In particular, it is known that any closed 3-dimensional manifold M^3 admits a handle decomposition such that the union Q_1 of all 0- and 1-handles is the connected 3-manifold with boundary (*a $(3,1)$-handlebody*), and $M^3 \setminus \operatorname{int} Q_1$ is homeomorphic to Q_1. Such a decomposition is called *the Heegard splitting* of M^3, and the boundary ∂Q_1 of Q_1 is called *the Heegard surface*. The natural question is how to glue two copies of the $(3,1)$-handle body together to obtain the manifold homeomorphic to M^3. The information on the gluing map that allows one to distinguish the resulting manifolds is determined by *a Heegard diagram* that is a special finite set of simple closed curves (meridians of the handlebodies) on the Heegard surface. Fleitas in [10] showed that for any polar flow f^t on M^3 there exists the Heegard surface of M^3 which is simultaneously the cross-section to trajectories of f^t. Moreover, the Heegard diagram of this splitting is formed by the traces of two-dimensional invariant manifolds of saddle equilibria and such a special diagram determines not only the topology of the ambient manifold, but also all classes of topological equivalence for polar flows on M^3. In [10] similar results for the 2-dimensional case are obtained, but since then there has not been any development of the subject for higher dimension. It can be explained, in particular, by the fundamental impossibility of obtaining a system of complete invariants that could distinguish manifolds of dimension 4 and greater.

The end of the 20th century brought several significant works on the topology of four-dimensional manifolds. Kirby, de Sa, Laudenbach and Poenaru have shown in [8, 23, 25] that the handle decomposition of a closed four-dimensional manifold M^4 may be described by some equivalence classes of framed links (that are unions of simple closed curves, or knots, for an exact definition and details, see Section 3.7) smoothly embedded in a three-dimensional sphere $S^3 \subset M^4$. Such a framed link was named the *Kirby diagram*. In contrast to the Heegard diagram, which is a geometrical object, each knot in the Kirby diagram is endowed with some additional information (*the framing*) about the way of gluing 1- and 2-handles. We adapt the Kirby diagram to the solution of the problem of topological classification of polar flows without heteroclinic intersections given on four-dimensional manifolds. The first step in this direction was done in [28], where it was proven that there is exactly one class of topological equivalence among gradient-like flows on M^4 with three equilibria (and, hence, there is exactly one topological type of the manifolds carrying such flows). In [16], a topological classification of polar flows on four-dimensional manifolds was obtained with the restriction that all equilibria of flows have equal Morse indices.

In this paper, we consider a class $\mathcal{P}^4$ of polar flows on closed 4-dimensional manifolds without any restrictions on Morse indices but with the requirement that any $f^t \in \mathcal{P}^4$ has no heteroclinic intersections. The last condition allows us to describe classes of topological equivalence of the flows

using the Kirby diagram of the ambient manifolds, and to describe the interrelation between the structure of the nonwandering set and the topology of the ambient manifold.

2. STATEMENTS OF RESULTS

For a flow $f^t \in \mathcal{P}^4$, let $\Omega^i_{f^t}$ denote the set of all equilibria of f^t whose Morse index is equal to $i \in \{0, \dots, 4\}$ and k_i is the number of point in $\Omega^i_{f^t}$.

Let $S_{f^t} \subset W^s_\omega \setminus \omega$ be a smooth three-dimensional sphere secant to the trajectories of f^t. We set

$$\Lambda^{i-1}_{f^t} = \bigcup_{p \in \Omega^i_{f^t}} (W^u_p \cap S_{f^t}), i \in \{1, 2, 3\}.$$

The set $\Lambda^{i-1}_{f^t}$ is a set of spheres of dimension $(i-1)$ smoothly embedded in an S_{f^t} (n-dimensional sphere can be defined as the boundary of the ball $\mathbb{B}^{n+1}$, so the sphere of dimension 0 is just the pair of points).

In Section 4.2 we provide a definition of spins and framings for elements of the sets $\Lambda^0_{f^t}$ and $\Lambda^1_{f^t}$, respectively (see Definitions 3 and 4).

Definition 1. The set $K_{f^t} = \{\Lambda^0_{f^t}, \Lambda^1_{f^t}, \Lambda^2_{f^t}\}$, where $\Lambda^0_{f^t}, \Lambda^1_{f^t}$ are endowed with spins and framings, is called the Kirby diagram of the flow f^t.

Definition 2. The Kirby diagrams $K_{f^t}, K_{f'^t}$ of the flows f^t, $f'^t \in \mathcal{P}^4$ are equivalent up to homeomorphism if there exists a homeomorphism $h : S_{f^t} \to S_{f'^t}$ such that

1) $h(\Lambda^i_{f^t}) = \Lambda^i_{f'^t}$ for any $i \in \{0, 1, 2\}$;

2) spins of $\lambda^0_p, h(\lambda^0_p)$ coincide for any $\lambda^0_p \subset \Lambda^0_{f^t}$;

3) if $\tilde{\lambda}^1_p$ is the framing of the knot $\lambda^1_p \in \Lambda^1_{f^t}$, then $h(\tilde{\lambda}^1_p)$ is the framing of $h(\lambda^1_p)$.

Theorem 1. *Flows $f^t, f'^t \in \mathcal{P}^4$ are topologically equivalent if and only if the Kirby diagrams $K_{f^t}, K_{f'^t}$ are equivalent up to homeomorphism.*

Before formulating the result describing the topology of the supporting manifold of flows $f^t \in \Psi$, let us recall several definitions.

The framed link $\Lambda^1_{f^t}$ determines a matrix $A_{f^t} \in \mathrm{SL}(k_2, \mathbb{Z})$ of linking numbers, which represents the intersection form of M^4 in the case where M^4 is orientable (see Sections 3.6 and 3.8).

Recall that a connected manifold M^n is called *simply connected* if its fundamental group is trivial. Due to the Poincaré theorem, all three-dimensional simply connected manifolds are diffeomorphic to the sphere. The significant result of Freedman (see [11] and some explanation in Section 3.8) states that four-dimensional simply connected smooth manifolds are completely determined (up to homeomorphism) by their intersection forms.

A *fiber bundle over B with fiber Y* (or *Y-bundle over B*) is a set (E, B, p, Y), where Y, E, B are topological spaces, $p : E \to B$ is a continuous surjective map such that for every point $b \in B$ there is an open neighborhood $U_b \subset B$ and a homeomorphism $h_b : p^{-1}(U_b) \to U_b \times Y$ such that $ph_b^{-1}(x, y) = x$, where $x \in U_b, y \in Y$. The spaces E and B are called *the total space* and *the base* of the fiber bundle. The fiber bundle is called *trivial* if E is homeomorphic to the direct product $B \times Y$. We proved in Proposition 2 that there are exactly two nonhomeomorphic total spaces of $\mathbb{S}^{n-1}$-bundles over $\mathbb{S}^1$: the direct product $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and another one, which we denote by $\mathbb{S}^{n-1} \widetilde{\times} \mathbb{S}^1$.

In Section 5.2, we obtain the following result.

Theorem 2. *Suppose that a closed manifold M^4 admits a flow $f^t \in \mathcal{P}^4$. If M^4 is orientable, then it is homeomorphic to the connected sum of a simply connected manifold N^4 and $k_1 \geqslant 0$ copies of the direct product $\mathbb{S}^3 \times \mathbb{S}^1$. If M^4 is nonorientable, then it is homeomorphic to the connected sum of a simply connected manifold N^4 and $k_1 > 0$ copies of the nontrivial $\mathbb{S}^3$-fiber bundle over $\mathbb{S}^1$.*

By employing Freedman's classification of simply connected four-dimensional manifolds, along with Rochlin's and Donaldson's restrictions on the intersection forms of smooth simply connected manifolds (see Section 3.8 for the review of these results and links), we clarify the topology of the manifold N^4 as follows.

Theorem 3. *The manifold N^4 in the conclusion of Theorem 2 is homeomorphic to the sphere $\mathbb{S}^4$ if and only if $k_2 = 0$. If $k_2 > 0$, then A_{f^t} represents the intersection form of N^4 and one of the following implications holds.*

1) *If the matrix A_{f^t} is definite, then M^4 is homeomorphic to the connected sum of k_2 copies of the complex projective plane $\mathbb{CP}^2$;*

2) *If the matrix A_{f^t} is odd indefinite, then M^4 is homeomorphic to the connected sum of $b_+ > 0$ copies of $\mathbb{CP}^2$ and $b_- > 0$ copies of $\overline{\mathbb{CP}^2}$, where b_+, b_- are the numbers of positive and negative eigenvalues of A_{f^t}, respectively, and $\mathbb{CP}^2, \overline{\mathbb{CP}^2}$ are the complex projective planes with opposite orientations.*

3) *If the matrix A_{f^t} is even indefinite, then M^4 is homeomorphic to the connected sum of $2k \geqslant 0$ copies of Freedman manifold M_{E_8} and $n \geqslant 2k+1$ copies of $\mathbb{S}^2 \times \mathbb{S}^2$, $16k = b_+ - b_-$, $16k + 2n = k_2$.*

In Section 6, we introduce the definition of an admissible Kirby diagram and prove the following result.

Theorem 4. *For any admissible Kirby diagram K there exists a flow $f^t \in \mathcal{P}^4$ such that the Kirby diagram K_{f^t} is equivalent to K.*

As an application of Theorems 1, 2, and 4, we claim the following result. Let us denote by $P^+_{k_1,k_2}, P^-_{k_1,k_2} \subset \mathcal{P}^4$ subclasses such that any $f^t \in P^{\pm}_{k_1,k_2}$ have exactly k_1, k_2 saddle equilibria with Morse index equal to 1, 2, and the ambient manifolds of all flows from $P^+_{k_1,k_2}(P^-_{k_1,k_2})$ are orientable (nonorientable).

Theorem 5. 1) *$P^-_{0,k} = \varnothing$ for any $k \geqslant 0$;*

2) *all flows in each class $P^+_{0,0}, P^{\pm}_{1,0}$ are topologically equivalent;*

3) *each class $P^{\pm}_{k,0}$ for $k \geqslant 2$ consists of a finite number of topologically nonequivalent flows;*

4) *the set $P^+_{0,1}$ consists of exactly one equivalence class, while each set $P^{\pm}_{k_1,k_2}$ for $k_1 \geqslant 0, k_2 \geqslant 2$ consists of a countable set of equivalence classes.*

Due to item 4 of Theorem 5, the class of polar flows without heteroclinic intersections with exactly two saddle equilibria contains infinitely many topologically nonequivalent flows given on the same four-dimensional manifold. This contrasts with the analogous cases in two- and three-dimensional manifolds, where there exists only a finite set of equivalence classes for each admissible manifold.

3. AUXILIARY RESULTS

3.1. Preliminary Definitions

A compact connected manifold X^n of dimension n is *orientable* if the homology group $H_n(X^n, \partial X^n, \mathbb{Z})$ is infinite cyclic. A generator μ of $H_n(X^n, \partial X^n, \mathbb{Z})$ is called *the orientation* of X^n. If $H_n(X^n, \partial X^n, \mathbb{Z})$ is not infinite cyclic, then it is trivial, and the manifold X^n is called *nonorientable* (see, for instance, [37, Chapter 2, § G], [18, Section 3.3]).

Proposition 1. *Let X^n be a compact orientable manifold with nonempty connected boundary ∂X^n. Then there exists a canonical isomorphism $\partial_* : H_n(X^n, \partial X^n) \to H_{n-1}(\partial X^n)$. That is, ∂X^n is orientable and $\partial_*(\mu)$ is the orientation of ∂X^n.*

Proof. According to [18, Theorem 2.16], there is the following exact homology sequence of the pair $(X^n, \partial X^n)$

$$\dots \to H_n(X^n) \to H_n(X^n, \partial X^n) \xrightarrow{\partial_*} H_{n-1}(\partial X^n) \xrightarrow{i_*} H_{n-1}(X^n) \to \dots$$

Here i_* is the homomorphism that associates a homology class of a cycle in $H_{n-1}(\partial X^n)$ with the homology class of the same cycle in $H_{n-1}(X^n)$. Since ∂X^n is the boundary of X^n, it follows that $i_*([\partial X^n]) = 0$.

Since $\partial X^n \neq \varnothing$, the group $H_n(X^n)$ is trivial.

Therefore, the exact sequence of pairs determines the exact sequence

$$0 \to H_n(X^n, \partial X^n) \xrightarrow{\partial_*} H_{n-1}(\partial X^n) \to 0.$$

Hence, ∂_* is the isomorphism and $H_{n-1}(\partial X^n)$ is isomorphic to $\mathbb{Z}$. Consequently, ∂X^n is orientable. □

The orientation $\partial_*(\mu)$ of ∂X described in Proposition 1 is called *natural*.

Let μ, ν be the orientations of the compact manifolds X^n, Y^n, respectively. A homeomorphism

$$h : X^n \to Y^n$$

is called *orientation-preserving* if it induces an isomorphism

$$h_* : H_n(X^n, \partial X^n) \to H_n(Y^n, \partial Y^n)$$

such that $h_*(\mu) = \nu$. Otherwise, the homeomorphism h is called *orientation-reversing*.

Let X and Y be compact manifolds with $A \subset \partial X$ and $B \subset \partial Y$ being submanifolds, and let $\xi : A \to B$ be a homeomorphism. Then, the factor space

$$X \cup_\xi Y$$

of $X \sqcup Y$ by a minimal equivalence relation $\sim$ such that $a \sim \xi(a)$ is a manifold that is said to be *obtained by gluing X to Y by means of ξ*.

Recall that for $n \geqslant 1$: $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| \leqslant 1\}$. We call the sets B^n, $\operatorname{int} B^n$, and S^{n-1} that are homeomorphic to $\mathbb{B}^n$, to the interior $\operatorname{int} \mathbb{B}^n$, and to the boundary $\mathbb{S}^{n-1} = \partial \mathbb{B}^n$ of $\mathbb{B}^n$, *the compact ball, the open ball, and the sphere*, respectively.

3.2. $\mathbb{S}^{n-1}$-Bundles over $\mathbb{S}^1$

Let X be a topological space and $\theta : X \to X$ be a homeomorphism. A quotient space

$$Y_\theta = (X \times [0,1])/_\sim$$

by equivalence relation $(x,0) \sim (\theta(x),1)$ is called *the mapping torus* of θ.

Proposition 2. *Let E be the total space of an $\mathbb{S}^{n-1}$-bundle over $\mathbb{S}^1$. Then E is homeomorphic to a mapping torus M_θ of some homeomorphism $\theta : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. There are exactly two nonhomeomorphic total spaces of $\mathbb{S}^{n-1}$-bundles over $\mathbb{S}^1$, one of which is $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ and another one, $\mathbb{S}^1 \widetilde{\times} \mathbb{S}^{n-1}$, is nonorientable.*

Proof. One may represent $\mathbb{S}^1$ as a union of two compact arcs $I \cup J$ such that $I \cap J = \partial I = \partial J$. By virtue of [20, Chapter 4, Corollary 2.5], a locally trivial bundle over contractible space is trivial, so $p^{-1}(I)$, $p^{-1}(J)$ are homeomorphic to $[0,1] \times \mathbb{S}^{n-1}$. The E is homeomorphic to a manifold obtained by gluing together two copies $[0,1] \times \mathbb{S}^{n-1}$, hence it is a mapping torus for some homeomorphism $\theta : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. It is well known that homeomorphisms θ, $\theta' : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ (for $n \geqslant 1$) are isotopic if and only if they are both either orientation-preserving or orientation-reversing. Since the topological type of M_θ depends only on the isotopy class of the homeomorphism θ, there are exactly two (up to homeomorphism) manifolds of type M_θ (see, for instance, [13, Proposition 14.1]). Namely, if $\theta : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is an orientation-preserving homeomorphism, it is isotopic to identity and, consequently, M_θ is homeomorphic to the direct product $\mathbb{S}^{n-1} \times \mathbb{S}^1$. If $\theta, \theta' : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ are two orientation-reversing homeomorphisms, then $M_\theta, M_{\theta'}$ are homeomorphic.

Let us prove that, if $\theta : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is an orientation-reversing homeomorphism, then M_θ is nonorientable (and, hence, nonhomeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$). Denote by $\theta_* : H_{n-1}(\mathbb{S}^{n-1}) \to H_{n-1}(\mathbb{S}^{n-1})$ an automorphism induced by θ. Since θ is orientation-reversing and $H_{n-1}(\mathbb{S}^{n-1})$ is isomorphic to $\mathbb{Z}$, it follows that $\theta_*([S]) = -[S]$ for any class $[S] \in H_{n-1}(\mathbb{S}^{n-1})$.

It follows from [18, Example 2.48] that the sequence below is exact:

$$H_n(\mathbb{S}^{n-1}) \to H_n(M_\theta) \to H_{n-1}(\mathbb{S}^{n-1}) \overset{1-\theta_*}{\to} H_{n-1}(\mathbb{S}^{n-1}) \to H_{n-1}(M_\theta) \to H_{n-2}(\mathbb{S}^{n-1}), \tag{3.1}$$

where $1 - \theta_*$ translates the homology class $[S] \in H_{n-1}(\mathbb{S}^{n-1})$ to the class $[S] - \theta_*([S]) = 2[S] \in H_{n-1}(\mathbb{S}^{n-1})$. Then the sequence (3.1) implies the existence of the following exact sequence:

$$0 \overset{i_1}{\to} H_n(M_\theta) \overset{i_2}{\to} H_{n-1}(\mathbb{S}^{n-1}) \overset{i_3}{\to} \operatorname{Im} i_3 \overset{i_4}{\to} 0, \tag{3.2}$$

where $i_3 = 1 - \theta_*$. Since the sequence (2.2) is exact, $\operatorname{Ker} i_4 = \operatorname{Im} i_3$. By the theorem on homomorphisms, $\operatorname{Im} i_3$ is isomorphic to $H_{n-1}(\mathbb{S}^{n-1})/\operatorname{Ker} i_3$, which is isomorphic to $\mathbb{Z}$. Hence, $\operatorname{Ker} i_3 = 0 = \operatorname{Im} i_2$ and the following sequence is exact:

$$0 \overset{i_1}{\to} H_n(M_\theta) \overset{i_2}{\to} 0,$$

which immediately implies that $H_n(M_\theta) = 0$ and M_θ is nonorientable. □

3.3. Connected Sum

A compact k-dimensional manifold $X \subset M^n$, $k \leqslant n$, is called *locally flat* if for every point $x \in \operatorname{int} X$ ($x \in \partial X$) there exists a neighborhood $U_x \subset M^n$ and a homeomorphism $\psi : U_x \to \mathbb{R}^n$ such that the set $\psi(U_x \cap X) = \mathbb{R}^k \subset \mathbb{R}^n$ ($\mathbb{R}^k_+ \subset \mathbb{R}^n$).

Let M^n_1, M^n_2 be orientable closed manifolds of dimension $n \geqslant 1$ and $B_1 \subset M^n_1$, $B_2 \subset M^n_2$ be compact n-dimensional balls with locally flat boundaries. We fix orientations μ_1 and μ_2 for M^n_1 and M^n_2, respectively, as well as the orientations μ_{S_1} and μ_{S_2} of the spheres ∂B_1 and ∂B_2, which are induced by μ_1 and μ_2. We assume that $\xi : \partial B_1 \to \partial B_2$ is a homeomorphism that induces the isomorphism $\xi_* : H^n(\partial B_1) \to H^n(\partial B_2)$, such that $h_*(\mu_{S_1}) = -\mu_{S_2}$.

The oriented manifold $M^n \# N^n$, obtained by gluing $M^n_1 \setminus \operatorname{int} B_1$ and $M^n_2 \setminus \operatorname{int} B_2$ via ξ, with the orientation induced by μ_1 and μ_2, is called *the connected sum* of M^n_1 and M^n_2. If one of M^n_1 or M^n_2

is nonorientable, the connected sum $M_1^n \# M_2^n$ is defined similarly, but without any restrictions on the gluing map ξ.

The following statement is derived from [22, Lemma 2.1] ([39, Chapter 3]) and [4].

Statement 1. *Let M^n and N^n be arbitrary connected closed manifolds. Then the connected sum $M^n \# N^n$ is well-defined up to homeomorphism, does not depend on the choice of B_1, B_2, or ξ, and the following properties hold:*

1) $M_1^n \# M_2^n \cong M_2^n \# M_1^n$;

2) $M^n \# S^n \cong M^n$;

3) $(M_1^n \# M_2^n) \# M_3^n \cong M_1^n \# (M_2^n \# M_3^n)$.

We will use the notation $\#_g M^n$ for the connected sum of g copies of M^n.

For orientable manifolds of dimensions two and three, the decomposition into connected sum is unique up to the permutation of summands. The following statement shows that this fact is not true for nonorientable manifolds. For $n = 2$, this is a classical result; for $n \geqslant 3$, it can be proved similarly to [19, Lemma 3.17, Theorem 3.21].

Statement 2. *If M^n is a nonorientable closed manifold of dimension $n \geqslant 2$, then the manifolds $M^n \# (\mathbb{S}^{n-1} \times \mathbb{S}^1), M^n \# (\mathbb{S}^{n-1} \widetilde{\times} \mathbb{S}^1)$ are homeomorphic.*

Corollary 1. *$\#_m (\mathbb{S}^{n-1} \times \mathbb{S}^1) \# (\mathbb{S}^{n-1} \widetilde{\times} \mathbb{S}^1)$ is homeomorphic to $\#_{m+1} (\mathbb{S}^{n-1} \widetilde{\times} \mathbb{S}^1)$.*

3.4. Surgery Along a Sphere S^{n-1}

The main tool for the proof of Theorem 2 is the surgery along a sphere, which is determined as follows.

Let M^n be a closed topological manifold and $S^{n-1} \subset M^n$ be a locally flat sphere. It follows from the generalized Schoenflies theorem [3, Theorem 3] and [17, Proposition 2] that S^{n-1} is bicollared, meaning that there exists a locally flat embedding $e : \mathbb{S}^{n-1} \times [-1, 1] \to M^n$ such that $e(\mathbb{S}^{n-1} \times \{0\}) = S^{n-1}$. Set $A^n = e(\mathbb{S}^{n-1} \times [-1, 1])$ and denote by $\widehat{M}^n$ the closed manifold obtained by gluing a copy of the ball $\mathbb{B}^n$ to each connected component of the boundary of $M^n \setminus \mathrm{int} A^n$. We will say that the manifold $\widehat{M}^n$ is obtained from M^n by *surgery along* S^{n-1}.

An *inverse surgery* is the operation of obtaining a closed manifold M^n from a closed manifold $\widehat{M}^n$ by removing two disjoint locally flat balls $B_+^n, B_-^n \subset \widehat{M}^n$ and gluing the annulus $\mathbb{A}^n = \mathbb{S}^{n-1} \times [-1, 1]$ to $\widehat{M}^n \setminus \mathrm{int}(B_+^n \cup B_-^n)$ via a homeomorphism $\theta : \partial \mathbb{A}^n \to \partial(B_-^n \cup B_+^n)$. The result of the inverse surgery depends on the isotopy class of θ. In particular, the following proposition holds.

Proposition 3. *Let M^n be a closed manifold obtained from the sphere $\mathbb{S}^n$ by inverse surgery. Then M^n is homeomorphic to either $\mathbb{S}^{n-1} \times \mathbb{S}^1$ or $\mathbb{S}^{n-1} \widetilde{\times} \mathbb{S}^1$.*

Proof. Let B_-^n and $B_+^n \subset \mathbb{S}^n$ be locally flat balls. By the annulus theorem, the set $\mathbb{S}^n \setminus \mathrm{int}(B_-^n \cup B_+^n)$ is homeomorphic to $\mathbb{S}^{n-1} \times [0, 1]$. Therefore, the proposition immediately follows from Proposition 2. □

Due to Proposition 3, we may denote the manifold M^n described there by M_θ. Let $p : \big(\mathbb{S}^n \setminus \mathrm{int}(B_-^n \cup B_+^n)\big) \sqcup \mathbb{A}^n \to M_\theta$ be a natural projection, $\widetilde{X} \subset \mathbb{S}^n \setminus (B_-^n \cup B_+^n)$ be a compact locally flat ball, $\widetilde{B}^n = \mathbb{S}^n \setminus \mathrm{int} \widetilde{X}$, and let

$$X = p(\widetilde{X}), B^n = p(\widetilde{B}^n),\ Y = \big(B^n \setminus \mathrm{int}(B_-^n \cup B_+^n)\big) \cup_\theta \mathbb{A}^n = M_\theta \setminus \mathrm{int} X.$$

Since p translates $\widetilde{X}$ into M_θ homeomorphically, X is a locally flat ball (see Fig. 1). The following statement is a consequence of the homogeneity theorem (see [39, Lemma 3.33]) and the annulus theorem.

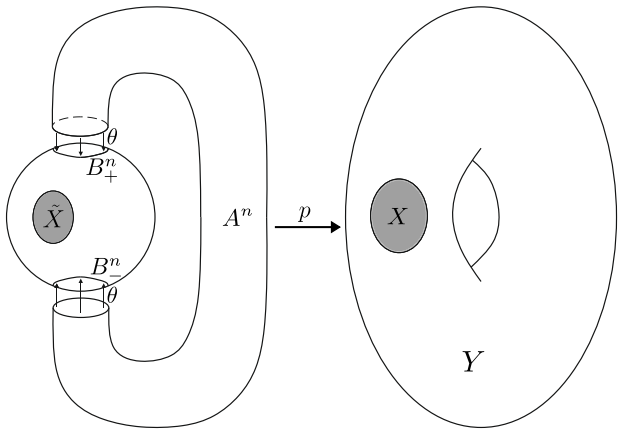

Fig. 1. A manifold M_θ without the ball X.

Statement 3. *Let B_1, $B_2 \subset M^n$ be compact n-dimensional balls with locally flat boundaries, and let $X \subset M^n \setminus (B_1 \cup B_2)$ be a compact n-dimensional manifold. Then, there exists an isotopy $h_t : M^n \to M^n$ relative to X such that $h_0 = \mathrm{id}_{M^n}$ and $h_1(B_1) = B_2$.*

We immediately derive the following statement from Statement 3.

Corollary 2. *Let $D^n \subset M_\theta$ be a locally flat ball. Then $M_\theta \setminus \mathrm{int} D^n$ is homeomorphic to Y.*

Lemma 1. *Let $\widehat{M}^n$ be obtained from the connected manifold M^n by surgery along $S^{n-1} \subset M^n$. Then the following alternatives hold:*

1) *If $\widehat{M}^n$ is disconnected, then $M^n = M_1 \# M_2$, where M_1, M_2 are the connected components of $\widehat{M}^n$;*

2) *If $\widehat{M}^n$ is connected and M^n is orientable, then M^n is homeomorphic to $\widehat{M}^n \#(\mathbb{S}^{n-1} \times \mathbb{S}^1)$;*

3) *If $\widehat{M}^n$ is connected and M^n is nonorientable, then M^n is homeomorphic either to $\widehat{M}^n \#(\mathbb{S}^{n-1} \times \mathbb{S}^1)$ or to $\widehat{M}^n \#(\mathbb{S}^{n-1} \tilde{\times} \mathbb{S}^1)$.*

Proof. If $\widehat{M}^n$ is disconnected, then the statement follows immediately from the definition of the connected sum. If $\widehat{M}^n$ is connected and M^n is orientable, then the statement is proven in [27, Lemma 7]. Therefore, we need to prove item 3. Suppose that M^n is nonorientable and $\widehat{M}^n$ is connected.

By definition, there are disjoint locally flat balls $B^n_-, B^n_+ \subset \widehat{M}^n$ such that M^n is the result of gluing the annulus $\mathbb{A}^n = \mathbb{S}^{n-1} \times [-1, 1]$ to $\widehat{M}^n \setminus \mathrm{int}(B^n_- \cup B^n_+)$ via a homeomorphism $\theta : \partial \mathbb{A}^n \to \partial(B^n_- \cup B^n_+)$. Let $C \subset \widehat{M}^n \setminus \mathrm{int} B^n_-$ be a collar of ∂B^n_-. It follows from Statement 3 that there exists a homeomorphism $h : \widehat{M}^n \to \widehat{M}^n$ such that $h|_{B^n_-} = \mathrm{id}_{B^n_-}$ and $h(B^n_+) \subset \mathrm{int}\, C$. So, without loss of generality, we can assume that $B^n_+ \subset \mathrm{int}\, C$ (otherwise, consider $h(B^n_+)$ and $h\theta$ instead of B^n_+ and θ). Thus, $B^n_- \cup B^n_+$ belongs to the interior of a locally flat ball $B^n = B^n_- \cup C$. The manifold $(B^n \setminus \mathrm{int}(B^n_- \cup B^n_+)) \cup_\theta \mathbb{A}^n$ is homeomorphic to the manifold $Y = M^n_\theta \setminus \mathrm{int}\, D^n$ described in Corollary 2. Let $\xi : \big(B^n \setminus \mathrm{int}(B^n_- \cup B^n_+)\big) \cup_\theta \mathbb{A}^n \to Y$ be a homeomorphism, and $i : \partial B^n \to \widehat{M}^n$ be an inclusion map. Then we have

$$M^n = \big(\widehat{M}^n \setminus \mathrm{int}(B^n_- \cup B^n_+)\big) \cup_\theta \mathbb{A}^n = (\widehat{M}^n \setminus \mathrm{int} B^n) \cup_i \Big(\big(B^n \setminus \mathrm{int}(B^n_- \cup B^n_+)\big) \cup_\theta \mathbb{A}^n\Big) =$$

$$= (\widehat{M}^n \setminus \mathrm{int} B^n) \cup_\xi (M_\theta \setminus D^n) = M^n \# M_\theta.$$

□

3.5. Extension of Homeomorphisms

The statement below is proven in [13, Proposition 14.3].

Statement 4. *Let $e_i, e_i' : \mathbb{B}^n \to \operatorname{int}\mathbb{B}^n$, $i \in \{1, \dots, k\}$ be orientation-preserving topological embeddings such that:*

1) *the spheres $e_i(\partial\mathbb{B}^n), e_i'(\partial\mathbb{B}^n)$ are locally flat in $\mathbb{B}^n$ for all $i \in \{1, \dots, k\}$;*

2) *$e_i(\mathbb{B}^n) \cap e_j(\mathbb{B}^n) = \varnothing$, $e_i'(\mathbb{B}^n) \cap e_j'(\mathbb{B}^n) = \varnothing$ for any $i, j \in \{1, \dots, k\}$, $i \neq j$.*

Then there exists a homeomorphism $\theta : \mathbb{B}^n \to \mathbb{B}^n$ such that:

1) *$h|_{\partial\mathbb{B}^n} = \mathrm{id}$;*

2) *$he_i = e_i'$, $i \in \{1, \dots, k\}$.*

Statement 5. *Let $\psi_0, \psi_1 : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be orientation-preserving (or orientation-reversing) homeomorphisms. Then there exists a homeomorphism $\Psi : \mathbb{S}^{n-1} \times [0,1] \to \mathbb{S}^{n-1} \times [0,1]$ such that:*

1) *$\Psi|_{\mathbb{S}^{n-1}\times\{0\}} = \psi_0$;*

2) *$\Psi|_{\mathbb{S}^{n-1}\times\{1\}} = \psi_1$.*

Proof. Two homeomorphisms of the sphere $\mathbb{S}^{n-1}$ are isotopic if and only if they are either both orientation-preserving or both orientation-reversing (see [9]). Hence, under the condition of the lemma, there exists an isotopy $\psi_t : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ connecting the homeomorphisms ψ_0 and ψ_1. Then the desired homeomorphism is defined by the formula

$$\Psi(x, s) = \big(\psi_s(x), s\big), \quad x \in \mathbb{S}^{n-1},\ s \in [0,1].$$

□

Statement 6. *Let $e, e' : \mathbb{S}^{n-1} \times [-1,1] \to \operatorname{int}(\mathbb{S}^{n-1} \times [-1,1])$ be orientation-preserving locally flat embeddings such that $e(\mathbb{S}^{n-1} \times \{0\}) = e'(\mathbb{S}^{n-1} \times \{0\}) = \mathbb{S}^{n-1} \times \{0\}$. Then there exists a homeomorphism $\theta : \mathbb{S}^{n-1} \times [-1,1] \to \mathbb{S}^{n-1} \times [-1,1]$ such that:*

1) *$\theta e = e'$;*

2) *$\theta|_{\mathbb{S}^{n-1}\times\{-1,1\}} = \mathrm{id}\,|_{\mathbb{S}^{n-1}\times\{-1,1\}}$.*

3.6. Knots and Links on S^3

Let us recall that an *m-component link* on a sphere S^3 is defined as a disjoint union of m locally flat circles. A link consisting of a single component is called a *knot*.

Following [6, 37], we will say that links $L, L' \subset S^3$ are *equivalent* if there exists a homeomorphism $h : S^3 \to S^3$ such that $h(L) = L'$, and *isotopic* if there exists an isotopy $h_t : S^3 \to S^3$ such that h_0 is the identity homeomorphism id_{S^3} and $h_1(L) = L'$. It is evident that isotopic links are equivalent. In [9, Theorems 15, 16], it is shown that, up to isotopy, there are exactly two classes of homeomorphisms of the sphere S^3: those that preserve orientation and those that reverse orientation. Hence, it follows that if the links $L, L' \subset S^3$ are equivalent by means of an orientation-preserving homeomorphism of the sphere, then they are also isotopic.

Proposition 4. *Any locally flat link $L \subset S^3$ is equivalent to a smooth one.*

Proof. Due to the results of Bing and Moise, see [30, Theorem 6.1], [1, Theorems 7, 8], a locally flat link L in a three-dimensional manifold (as well as any compact locally flat embedded subset) is a subpolyhedron with respect to some triangulation of this manifold. Therefore, a locally flat link can be considered as piecewise linearly embedded. From [24, Theorem 7.3], it follows that there exists a homeomorphism $h : S^3 \to S^3$ such that $h(L)$ is a smooth submanifold of S^3. □

Corollary 3. *For any locally flat knot $\lambda \subset S^3$ there exists a neighborhood $\Pi_\lambda \subset M^3$ which is homeomorphic to the solid torus $\mathbb{S}^1 \times \mathbb{B}^2$.*

We will call the solid torus Π_λ *the tubular neighborhood of the knot* λ. A circle $\mathbb{S}^1 \times \{O\}$ is called *the core* of the solid torus $\mathbb{S}^1 \times \mathbb{B}^2$. A simple closed curve $m \in \partial(\mathbb{S}^1 \times \mathbb{B}^2)$ that is homotopic to the curve $\{x\} \times \partial\mathbb{B}^2$, where $x \in \mathbb{S}^1$, is called *the meridian* of the solid torus $\mathbb{S}^1 \times \mathbb{B}^2$.

Throughout this text, unless otherwise noted, we consider locally flat knots and links.

Let us recall the definition of the linking number of a pair of knots $\lambda, \widetilde{\lambda} \subset S^3$. The meridian μ_λ of the solid torus $\Pi_\lambda = \psi(\Pi_1)$ can be chosen as a generator of the homology group $H_1(S^3 \setminus \lambda, \mathbb{Z})$, which is isomorphic to the group $\mathbb{Z}$ (see, for example, [38, Problem 3.9]). Therefore, given a fixed orientation of the knots μ_λ, $\widetilde{\lambda}$, and the sphere S^3, there exists a unique number $n \in \mathbb{Z}$ such that $[\widetilde{\lambda}] = n[\mu_\lambda]$, where $[\widetilde{\lambda}]$ and $[\mu_\lambda]$ are the homology classes of the knots $\widetilde{\lambda}$ and μ_λ, respectively. This number is called the *linking number* of λ and $\widetilde{\lambda}$ (see, for example, [15, §4.5]) and is denoted by $\mathrm{lk}(\lambda, \widetilde{\lambda})$. The linking number of the knots can be interpreted as the number of twists of the knot $\widetilde{\lambda}$ around the meridian μ_λ; it is an isotopy invariant of the pair λ, $\widetilde{\lambda}$ and changes sign if the orientation of λ or S^3 is reversed.

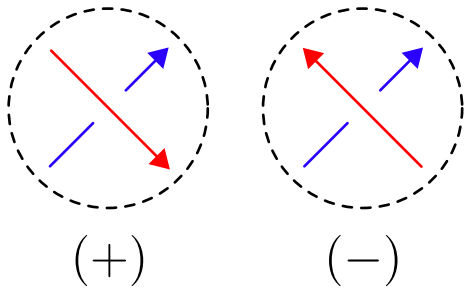

Fig. 2. Signs of the crossings in the link diagram.

A link diagram of a link $L \subset S^3$ is defined as a set of curves that forms its projection onto the plane, having a finite number of intersections and self-intersections, such that at each intersection point exactly four segments converge. Figure 2 shows all possible types, up to isotopy, of such crossing for oriented knots. We will say that the left crossing is positive, while the right crossing is negative.

Using a diagram of the link $\{\lambda, \widetilde{\lambda}\}$, one can find $\mathrm{lk}(\lambda, \widetilde{\lambda})$ as follows.

Statement 7 ([15, Exercise 4.5.4]). $\mathrm{lk}(\lambda, \widetilde{\lambda})$ *equals the algebraic sum of positive and negative crossings of the diagram of the link $\{\lambda, \widetilde{\lambda}\}$ at which $\widetilde{\lambda}$ passes below λ.*

A link is called *trivial* if its diagram consists of a finite number of pairwise nonintersecting circles (without self-intersections). A trivial link consisting of a single component is called a *trivial knot*. A knot that is not trivial is called *nontrivial*.

Let $L \subset \mathbb{S}^3$ be an m-component link, N_L be a collection of pairwise disjoint tubular neighborhoods of its components, and $\psi : \mathbb{S}^1 \times \partial\mathbb{B}^2 \times \mathbb{Z}_m \to \partial N_L$ be an orientation-preserving homeomorphism. The manifold

$$M^3 = (\mathbb{S}^3 \setminus \operatorname{int} N_L) \cup \psi(\mathbb{S}^1 \times \mathbb{B}^2 \times \mathbb{Z}_m)$$

is said to be obtained by means of the *Dehn surgery along the link* L. Dehn surgery along a knot λ, in which the meridian of the solid torus $\mathbb{S}^1 \times \mathbb{B}^2$ is glued to the meridian of the tubular neighborhood of the knot λ, is called *trivial*. The manifold obtained by means of a trivial surgery along any knot is homeomorphic to a sphere.

3.7. A Handle Decomposition and Kirby Diagram of Manifolds

Recall that the set $H_i^n = \mathbb{B}^i \times \mathbb{B}^{n-i}$ is called *the handle of dimension* n *and index* i (*or an* i*-handle*). The set $F_i = \partial\mathbb{B}^i \times \mathbb{B}^{n-i}$ is called the *foot of the handle* H_i^n. The spheres $\partial\mathbb{B}^i \times \{O\}$, $\{O\} \times \partial\mathbb{B}^{n-i}$ are called *the attaching sphere* and *the belt sphere*, respectively. Let us remark that the attaching sphere of the 1-handle is a pair of points.

Let X be a compact manifold with boundary and $\psi : F_i \to \partial X$ be a locally flat embedding. We say that the manifold $Y = X \cup_\psi H_i^n$ is obtained by *gluing the handle* H_i^n. The embedding ψ is called *the attaching map*. The representation of the manifold Y as a union of handles is called its *handle decomposition*.

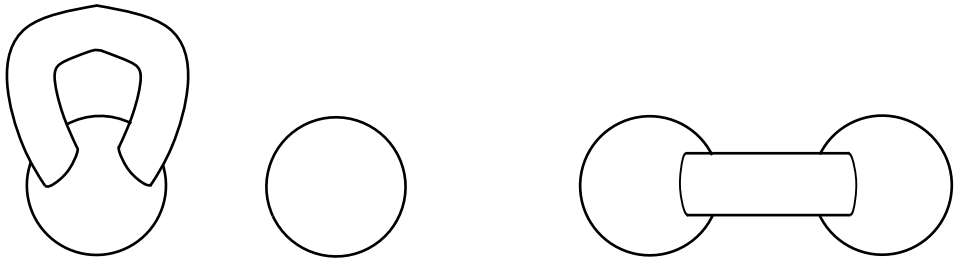

Fig. 3. Gluing the 1-handle to a pair of balls.

Proposition 5. *Let* X_k^n *be a connected* n*-dimensional* ($n \geqslant 1$) *manifold with handle decomposition consisting of* $k+1$ *handles of index* 0 *and* k *handles of index* 1*. Then* X_k^n *is homeomorphic to* $\mathbb{B}^n$*.*

Proof. X_0^n is the ball H_0^n. Suppose that the proposition is true for X_{k-1}^n. To obtain X_k^n, we should glue the 1-handle to the union of two balls: X_{k-1}^n and one more 0-handle H_0^n. There are two possibilities to do it. They are illustrated in Fig. 3. The manifold depicted on the left is disconnected, which contradicts the condition. The manifold depicted on the right is connected and homeomorphic to the ball $\mathbb{B}^n$. □

Proposition 6. *Let* X^n *be a manifold obtained by gluing* k_1 *copies of* 1*-handle and* k_{n-1} *copies of* $(n-1)$*-handle to the ball* $\mathbb{B}^n$ *in such a way that the images of foots are pairwise disjoint and* ∂X^n *is connected. Let* ψ *be the attaching map. Then*

1) $k_1 \geqslant k_{n-1}$ *and* ∂X^n *is homeomorphic to a connected sum of the sphere* $\mathbb{S}^{n-1}$ *and* $(k_1 - k_{n-1})$ *copies of* $\mathbb{S}^{n-2}$*-bundle over* $\mathbb{S}^1$*;*

2) *if* $k_1 = k_{n-1}$*, then for any* 1*-handle* H_1^n *there exists an* $(n-1)$*-handle* H_{n-1}^n *such the connected components of* $\psi(F_1)$ *belongs to different connected components of* $\mathbb{S}^{n-1} \setminus \psi(F_{n-1})$*;*

Proof. Since the foot images are disjoint, we can assume that the $(n-1)$-handles were glued first. Let us denote by Y_k^n the manifolds obtained by gluing k copies of $(n-1)$-handle to $\mathbb{B}^n$ and prove by induction on k that ∂Y_k^n is the disjoint union of $(k+1)$ spheres. Let $k = 1$. Then $\psi(F_{n-1}) \subset \mathbb{S}^{n-1}$ is the annulus with locally flat boundary. By the generalized Schoenflies theorem, $\mathbb{S}^{n-1} \setminus \operatorname{int} \psi(F_{n-1})$ is a pair of balls. By definition, ∂Y_1^n is the result of attaching the set $\partial H_{n-1}^n \setminus \operatorname{int} F_{n-1}$ to $\mathbb{S}^{n-1} \setminus \operatorname{int} \psi(F_{n-1})$. The set $\partial H_{n-1}^n \setminus \operatorname{int} F_{n-1}$ is also the union of two balls. It follows from the Alexander theorem that the manifold obtained by gluing two n-balls along the

boundary is homeomorphic to an n-sphere. Hence, ∂Y^n_1 is the union of two $(n-1)$-dimensional spheres. Assume that the statement holds for $Y_{k_{n-1}-1}$. Then $\partial Y^n_{k_{n-1}-1}$ is the disjoint union of k_{n-1} spheres. To obtain $Y^n_{k_{n-1}}$, we should glue one more handle H^n_{n-1} to $Y^n_{k_{n-1}-1}$. Then the image $\psi(F_{n-1})$ lies in one of the connected components of $\partial Y^n_{k_{n-1}-1}$. The same arguments as for the base of induction prove that the boundary of $\partial Y^n_{k_{n-1}}$ consists of $k_{n-1}+1$ spheres.

To obtain X^n, we should attach the remaining 1-handles to $Y^n_{k_{n-1}}$. The foot F_1 is a disjoint union of two balls. Attaching both balls to the same connected component of $\partial Y^n_{k_{n-1}}$ does not change the number of connected components, while attaching two balls to different connected components decreases the number of connected components by one. Hence, since ∂X^n is connected, $k_1 \geqslant k_{n-1}$.

The boundary of the manifold, obtained from a manifold Z^n by gluing a 1-handle is the result of inverse surgery applied to ∂Z^n. Then the statement of item 1 immediately follows from Lemma 1 and Proposition 3. If $k_1 = k_{n-1}$, each attachment of the 1-handle decreases the number of connected components of the boundary by one, which immediately implies that the arrangement of the foot images is as stated in item 2. □

The boundary of the 2-handle H^4_2 is the three-dimensional sphere, naturally represented as the union of two solid tori $\Pi_1 = \partial\mathbb{B}^2 \times \mathbb{B}^2$ and $\Pi_2 = \mathbb{B}^2 \times \partial\mathbb{B}^2$ with a common boundary. In this case, the attaching and the belt spheres of the handle H^4_2 are cores of solid tori Π_1 and Π_2, respectively. Denote by m_2 the meridian of Π_2. Let $\psi : \Pi_1 \to \mathbb{S}^3$ be the attaching map. Then the image $\lambda = \psi(\partial\mathbb{B}^2 \times \{O\})$ of the attaching sphere is a knot. We call a knot $\widetilde{\lambda} = \psi(m_2)$ *the framing of the knot* λ and say that the handle H^4_2 is attached along the link $\{\lambda, \widetilde{\lambda}\}$. If we attach several copies of 2-handles to H^4_0, then the collection of all corresponding framed knots forms a *a framed link* $L \in \mathbb{S}^3$. We denote by $Y_{_L}$ the manifold obtained in this way and say that the handles were attached *along the framed link* L. Let us remark that the boundary of $Y_{_L}$ is obtained by Dehn surgery along L.

To each framed m-component link $L = \{\lambda_i, \tilde{\lambda}_i\}$, we associate a *linking matrix* $A_{_L}$ which is a symmetric integer matrix with elements $a_{ij} = \mathrm{lk}(\lambda_i, \lambda_j)$ for $i \neq j$, and $a_{ii} = \mathrm{lk}(\lambda_i, \widetilde{\lambda}_i)$ for $i, j \in \{1, \ldots, m\}$.

The proof of the following statement for smooth embeddings is given in [16, Proposition 3]. Its validity for topological embeddings follows from the smooth analogue and the fact of the existence of an isotopy between any homeomorphism of a three-dimensional manifold to a diffeomorphism which follows from [5, 32].

Statement 8. *Let manifolds $Y_{_L}$ and $Y'_{_L}$ be obtained by gluing 2-handles to H^4_0 using locally flat embeddings $\psi, \psi' : \Pi_1 \times \mathbb{Z}_m \to \bigcup\limits_{\lambda \subset L} \Pi_\lambda$ such that $\mathrm{lk}(\lambda, \widetilde{\lambda}) = \mathrm{lk}(\lambda, \widetilde{\lambda}')$ for every knot $\lambda \in L$. Then $Y_{_L}$ and $Y'_{_L}$ are homeomorphic.*

The following technical statement shows that Statement 8 does not depend on the choice of a neighborhood of the knot $\lambda \subset L$.

Statement 9. *Let $\lambda \subset \mathbb{S}^3$ be a locally flat knot, and let $\xi_0, \xi, \xi' : \mathbb{S}^1 \times \mathbb{B}^2 \to \operatorname{int}\Pi_\lambda$ be locally flat embeddings such that:*

1) $\xi_0(\mathbb{S}^1 \times \{O\}) = \xi(\mathbb{S}^1 \times \{O\}) = \xi'(\mathbb{S}^1 \times \{O\}) = \lambda$, *where O is the center of the disk $\mathbb{B}^2$;*
2) $\xi(\mathbb{S}^1 \times \mathbb{B}^2), \xi'(\mathbb{S}^1 \times \mathbb{B}^2) \subset \operatorname{int}\xi_0(\mathbb{S}^1 \times \mathbb{B}^2)$;
3) $\mathrm{lk}\big(\xi(\mathbb{S}^1 \times \{x\}), \lambda\big) = \mathrm{lk}\big(\xi'(\mathbb{S}^1 \times \{x\}), \lambda\big)$, *for a fixed point $x \in \partial\mathbb{B}^2$.*

Then there exists a homeomorphism $\theta : \mathbb{S}^3 \to \mathbb{S}^3$ such that:

1) $\theta\xi = \xi'$;
2) $\theta|_{\mathbb{S}^3 \setminus \operatorname{int}\Pi_\lambda} = \mathrm{id}$.

Suppose that a closed oriented manifold M^4 has a handle decomposition with exactly 1 handle H_0^4 of index 0, exactly 1 handle of index 4 and some number of 2-and 3-handles. It follows from [25] that the topology of M^4 does not depend on the way of the gluing 3-handles. Gluing the 1-handle is determined by the image of its foot on ∂H_0^4 that can be seen as a couple of dots. As was shown above, gluing the 2-handle is determined either by the framed knot $\lambda \in \partial H_0^4$ disjoint from the dots or as a framed arc $l \in \partial H_0^4$ connecting the dots. In the last case the attaching sphere of H_2^4 intersects ∂H_2^4 along l and the remaining part of it lies in the handle H_1^4. The union of all dots, framed arcs and knots is called *the Kirby diagram of* M^4.

Serre (see [15, Theorem 4.2.12]) showed that any two handle decompositions of an arbitrary compact manifold can be obtained from each other by a sequence of layered isotopies and only two additional operations: the insertion (or cancellation) of a removable pair of handles and the sliding of handles. Each operation can be described as a perturbation of the Kirby diagram of the manifold M^4, which is called a Kirby move. Let us describe them.

A pair of handles H_1^4 and H_2^4 of indices 1 and 2, respectively, is called a *removable pair of handles* if the image of the attached sphere of H_2^4 intersects the belt sphere of H_1^4 transversely (on the boundary of ∂H_1^4) at a single point. Then this pair of handles is represented in the Kirby diagram as the pair of dots connected by a single arc. Transformation of the Kirby diagram by adding or removing such pairs of dots and arcs is called *the first Kirby move*. A similar definition applies to the removable pair of handles of indices 2 and 3.

Sliding of 2-handles leads to a transformation of the Kirby diagram, which is called the *second Kirby move*. We describe it below.

Let $L \subset \mathbb{S}^3$ be a framed link in which the knots $\lambda_i, \lambda_j \subset L$ have framings $\widetilde{\lambda}_i, \widetilde{\lambda}_j$ with linking numbers n_i, n_j, respectively. Let $e : [0,1] \times [0,1] \to \mathbb{S}^3$ be a locally flat embedding such that the intersection of the set $R = e([0,1] \times [0,1])$ with L consists of two segments $I_0 = e(\{0\} \times [0,1]) \subset \lambda_i$ and $I_1 = e(\{1\} \times [0,1]) \subset \widetilde{\lambda}_j$. The knot $\lambda_i' = \lambda_i \cup \widetilde{\lambda}_j \cup \partial R \setminus \operatorname{int}(I_0 \cup I_1)$ is called the *ribbon sum* of the knots $\lambda_i, \widetilde{\lambda}_j$. Let $n_i' = n_i + n_j \pm 2\operatorname{lk}(\lambda_i, \lambda_j)$, where the "+" sign is taken if the orientation of the segments I_0, I_1, induced by the orientation of the edge of the ribbon $[0,1] \times [0,1]$, is opposite to the orientation of both knots $\lambda_i, \widetilde{\lambda}_j$, and the "−" sign is taken in the opposite case. The replacement in the endowed link L of the pair $(\lambda_i, \widetilde{\lambda}_i)$ with the pair $(\lambda_i', \widetilde{\lambda}_i')$ with linking number n_i' is called the *second type Kirby move*.

Statement 10 ([23]). *If the framed link L' is obtained from the link L via the second Kirby move, then the three-dimensional manifolds obtained by Dehn surgery along the links L, L' are diffeomorphic.*

We say that the Kirby diagrams are *equivalent up to Kirby moves* if one can be transformed into the other by a finite sequence of first and second type Kirby moves and isotopies.

Statement 11 ([8, 23]). *Simply connected manifolds M^4, N^4 are homeomorphic if and only if their Kirby diagrams are equivalent up to Kirby moves.*

Note that there are examples of framed links that are not equivalent by means of the second type Kirby move alone (see the comment at the end of [26, §3.4]). Therefore, the conditions of Statement 11 cannot generally be weakened by excluding first type Kirby moves from consideration.

3.8. The Intersection form of a Simply Connected Four-Dimensional Manifold

Let M^4 be a simply connected closed four-dimensional manifold. Since M^4 is simply connected, it is orientable and the homology groups $H_1(M^4, \mathbb{Z}), H_3(M^4, \mathbb{Z})$ are trivial. By virtue of [21, Corollary 8.4], for any manifold N the torsion group of $H^2(N, \mathbb{Z})$ is isomorphic to the torsion group of $H_1(N, \mathbb{Z})$. Then triviality of $H_1(M^4, \mathbb{Z})$ entails the fact that $H^2(M^4, \mathbb{Z})$ is free. By Poincaré duality, $H_2(M^4, \mathbb{Z})$ is isomorphic to $H^2(M^4, \mathbb{Z})$ and $H_2(M^4, \mathbb{Z})$ is free as well. Hence, it is isomorphic to $\mathbb{Z}^m$ for some $m \geqslant 0$.

Recall that the symmetric bilinear form $Q_{M^4} : H_2(M^4, \mathbb{Z}) \times H_2(M^4, \mathbb{Z}) \to \mathbb{Z}$, which assigns to each pair of classes $(a, b) \in H_2(M^4, \mathbb{Z}) \times H_2(M^4, \mathbb{Z})$ their homological intersection index (see the definition in [38, § 5.3]), is called the *intersection form* of the manifold M^4.

It is known that the intersection form of a closed four-dimensional manifold is unimodular (see [15, Exercise 1.2.10]). From the results of J. Whitehead and L. S. Pontryagin (see [15, Theorem 1.2.25] and the comments on that theorem), it follows that two such manifolds M^4 and N^4 are homotopy equivalent if and only if their intersection forms are *equivalent*, that is, there exist orientations of M^4 and N^4 such that the matrices defining their intersection forms in some bases are congruent via an integer matrix. M. Freedman proved the following statement.

Statement 12. *For every unimodular symmetric bilinear form Q, there exists a simply connected closed topological four-manifold M^4 whose intersection form is equivalent to Q. If the form Q is even, then this manifold is unique up to homeomorphism; if the form Q is odd, there exist exactly two nonhomeomorphic topological manifolds with the given intersection form, and at most one of them admits a smooth structure. Therefore, smooth simply connected closed manifolds are determined by their intersection form up to homeomorphism.*

Below, we provide a brief overview of the necessary conditions that the intersection form of a closed smooth simply connected manifold must satisfy. More detailed information can be found, for instance, in [15, § 1.2] and [40, § 4.4].

Let us recall the classification of integer forms. Let Q be a symmetric bilinear form defined on a finitely generated free Abelian group A. By diagonalizing the matrix of the form Q over the field of real numbers, we denote by $b_+(b_-)$ the number of positive (negative) entries of the resulting diagonal matrix. The numbers $\sigma(Q) = b_+ - b_-$ and $\mathrm{rk}(Q) = b_+ + b_-$ are referred to as the *signature* and the *rank* of the form Q, respectively. The form Q is called *even* if $Q(a, a)$ is even for every $a \in A$ and *odd* otherwise. The form Q is said to be *positive* (*negative*) *definite* if $\mathrm{rk}(Q) = \sigma(Q)$ (respectively, $\mathrm{rk}(Q) = -\sigma(Q)$). In other cases, the form Q is termed *indefinite*.

The *direct sum* $Q = Q_1 \oplus Q_2$ of forms Q_1, Q_2 defined on groups A_1, A_2, respectively, is defined on $A = A_1 \oplus A_2$ in the following manner. If elements $a, b \in A$ are expressed as $a = a_1 + a_2, b = b_1 + b_2$, where $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$, then $Q(a, b) = Q_1(a_1, b_1) + Q_2(a_2, b_2)$.

If $k \geqslant 0$, then kQ denotes the k-fold sum $\oplus_k Q$; for a negative k, we take kQ to be equal to $|k|(-Q)$. We denote by $\langle n \rangle$ the form on $\mathbb{Z}$ represented by the matrix (n) of size 1×1. We denote by H the form on the module $A = \mathbb{Z} \oplus \mathbb{Z}$ represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The form E_8 on $\mathbb{Z}^8$ is defined by the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

By Statement 12, the forms H, E_8 can be realized as intersection forms of some closed simply connected topological four-dimensional manifolds. The first one is $\mathbb{S}^2 \times \mathbb{S}^2$ and the second one is called a *Freedman E_8-manifold*, denoted by M_{E_8} (see [40, § 3.2]). It follows from the Rokhlin theorem below that the manifold M_{E_8} does not admit any smooth structure.

The following restriction on the intersection form of a smooth simply connected manifold was discovered by S. Donaldson.

Statement 13 ([7]). *If the manifold M^4 is smooth and the form Q_{M^4} is positive (negative) definite, then Q_{M^4} is equivalent to the form* $\mathrm{rk}(Q)\langle 1\rangle$ *(respectively,* $\mathrm{rk}(Q)\langle -1\rangle$*).*

Statement 14 ([15, Theorem 1.2.21]). *Let Q be a symmetric, indefinite, and unimodular form of rank* $\mathrm{rk}(Q)$*. If Q is odd, then it is isomorphic to the form $b_+\langle 1\rangle \oplus b_-\langle -1\rangle$. If the form Q is even, then its signature $\sigma(Q)$ is a multiple of 8 and Q is isomorphic to the form $\frac{\sigma(Q)}{8}E_8 \oplus \frac{\mathrm{rk}(Q)-|\sigma(Q)|}{2}H$.*

The forms $k\langle 1\rangle$, $k\langle -1\rangle$, and $b_+\langle 1\rangle \oplus b_-\langle -1\rangle$ are realized by smooth connected sums of the required number of copies of the complex projective plane $\mathbb{CP}^2$, $\overline{\mathbb{CP}^2}$ with opposite orientations.

By Freedman theorem (see Statement 12), simply connected manifolds that have equivalent intersection forms are homeomorphic. Thus, any smooth manifold with definite and odd indefinite intersection form is homeomorphic to one of such connected sums, and the following statement holds.

Proposition 7. *Any smooth manifold M^4 with a definite or odd indefinite intersection form over $\mathbb{Z}^k$ is homeomorphic to a connected sum of k copies of $\mathbb{CP}^2$ or a connected sum of $\mu_1 \geqslant 0$ copies of $\mathbb{CP}^2$ and $\mu_2 \geqslant 0$ copies of $\overline{\mathbb{CP}^2}$, where $\mu_1 + \mu_2 = k$.*

At present, it is unknown whether all even indefinite forms that are necessarily isomorphic to the forms described in Statement 14 are realized by smooth manifolds.

The *signature and rank of a simply connected manifold M^4* are defined as the signature and rank of its intersection form Q_{M^4}. Rokhlin proved the following statement (Rokhlin theorem).

Statement 15 ([36]). *If the manifold M^4 is smooth and the form Q_{M^4} is even, then the signature of M^4 is a multiple of* 16*.*

From the Rokhlin theorem, it follows in particular that the manifold M_{E_8} is nonsmoothable. Moreover, if a smooth manifold has an intersection form congruent to $m(\pm E_8) \oplus nH$, then m is an even number. The following statement provides a lower bound for the number n.

Statement 16 ([12]). *If a smooth manifold M^4 has an intersection form Q_{M^4} that is equivalent to the form $2kE_8 \oplus nH$, then $n \geqslant 2|k| + 1$.*

Proposition 8. *Any smooth manifold M^4 with an even indefinite intersection form over $\mathbb{Z}^k$ is homeomorphic to a connected sum of $2m \geqslant 0$ copies of Freedman manifold M_{E_8} and $n \geqslant 2|m| + 1$ copies of $\mathbb{S}^2 \times \mathbb{S}^2$, where $16m + 2n = k$.*

4. NECESSARY AND SUFFICIENT CONDITIONS OF TOPOLOGICAL EQUIVALENCE

4.1. Local Conjugacy and Canonical Neighborhoods of Equilibria

The proof of the following statement can be found in [33, Lemmas 7.2, 7.3 § 7, Chapter 2].

Statement 17. *Let f^t be a smooth flow on a closed manifold M^n, and let p be a saddle hyperbolic equilibrium f^t with Morse index i_p. Then there exists a compact neighborhood $V_p \subset M^n$ of the point p and continuous maps $\pi_u : V_p \to B^u_p$, $\pi_s : V_p \to V_p \cap B^s_p$ such that:*

1) *For any points $x \in B^u_p$ and $y \in B^s_p$, the complete preimages $\pi_u^{-1}(x)$ and $\pi_s^{-1}(y)$ are compact balls of dimensions $4 - i_p$ and i_p, respectively, and these balls intersect transversely.*
2) *$\pi_u^{-1}(p) \subset B^s_p$ and $\pi_s^{-1}(p) \subset B^u_p$.*
3) *$f^{-t}\big(\pi_u^{-1}(x)\big) \subset \pi_u^{-1}\big(f^{-t}(x)\big)$ and $f^t\big(\pi_s^{-1}(x)\big) \supset \pi_s^{-1}\big(f^t(x)\big)$.*

The set V_p is called the *compact canonical neighborhood of the equilibrium state* p.

The structure of the direct product $B^u_p \times B^s_p$ determines the natural representation of the boundary ∂V_p of the canonical neighborhood V_p as the union of two compact submanifolds $\Pi^u_p = \partial B^u_p \times B^s_p$ and $\Pi^s_p = B^u_p \times \partial B^s_p$, which share a common boundary. In the case when $n = 4$ and $i_p = 2$, the manifolds Π^u_p and Π^s_p are solid tori.

The following proposition is a variation of the classical Grobman – Hartman theorem.

Proposition 9. *Let p and p' be saddle equilibrium states of smooth flows f^t, f'^t, respectively, having the same Morse index, and let V_p and $V_{p'}$ be their compact canonical neighborhoods. Then there exists an orientation-preserving homeomorphism $h_{p,p'} : V_p \to V_{p'}$ such that:*

1) *$h_{p,p'} f^t = f'^t h_{p,p'}$ for all t for which both sides of the equality are defined;*

2) *restrictions $h_{p,p'}|_{B^u_p}, h_{p,p'}|_{B^s_p}$ may be chosen to be either orientation-preserving or orientation-reversing simultaneously;*

3) *If $n = 4$ and $i_p = 2$, then $h_{p,p'}$ maps the meridians of the solid tori Π^u_p and Π^s_p to the meridians of the solid tori $\Pi^u_{p'}$ and $\Pi^s_{p'}$, respectively.*

Proof. Let $\pi_u : V_p \to B^u_p$ and $\pi_s : V_p \to B^s_p$ be continuous mappings defined in Statement 17. Denote by $\pi'_u : V_{p'} \to B^u_{p'}$ and $\pi'_s : V_{p'} \to B^s_{p'}$ the analogous mappings for the flow f'^t.

Let $g_u : \partial B^u_p \to \partial B^u_{p'}$ be a homeomorphism. To each point $x \in B^u_p$, we assign a time $t_x \in \mathbb{R}$ such that $f^{t_x}(x) \in \partial B^u_p$. Define the homeomorphism $h_u : B^u_p \to B^u_{p'}$ by

$$h_u(x) = \begin{cases} f'^{-t_x}\Big(g_u\big(f^{t_x}(x)\big)\Big), & x \in B^u_p \setminus p \\ p', & x = p. \end{cases}$$

Similarly, define the homeomorphism $h_s : B^s_p \to B^s_{p'}$ such that $h_s f^t|_{B^s_p} = f'^t h_s|_{B^s_p}$. Then the desired homeomorphism $h_{p,p'}$ is defined by the relation

$$h_{p,p'}(z) = {\pi'_u}^{-1}\Big(h_u\big(\pi_u(z)\big)\Big) \cap {\pi'_s}^{-1}\Big(h_s\big(\pi_s(z)\big)\Big), \quad z \in V_p.$$

We may choose homeomorphisms h_s, h_u to be orientation-preserving or orientation-reversing simultaneously, then the total homeomorphism $h_{p,p'}$ is orientation-preserving.

The knots $\partial B^u_p \times \{y\}$ and $\{x\} \times \partial B^s_p$, where $(x, y) \in \partial B^u_p \times \partial B^s_p$, are the meridians of the solid tori $\Pi^s_p, \Pi^u_p \subset \partial V_p$, respectively. By construction, the homeomorphism $h_{p,p'}$ maps these nodes to the nodes $\partial B^u_{p'} \times \{y'\}$ and $\{x'\} \times \partial B^s_{p'}$, which are the meridians of the solid tori $\Pi^s_{p'}, \Pi^u_{p'} \subset \partial V_{p'}$. □

4.2. Kirby Diagram of the Flow $f^t \in \mathcal{P}^4$

We say that a compact three-dimensional manifold Σ, locally flat in a manifold X^n, is a *cross-section* for a flow f^t defined on X^n if there exists a topological embedding $e : \Sigma \times [-1, 1] \to X^n$ such that $e(\Sigma \times \{0\}) = \Sigma$ and for any point $(x, 0) \in \Sigma \times [-1, 1]$ the segment $e(\{x\} \times [-1, 1])$ belongs to the trajectory of the point $e(x, 0)$.

Let ω be a sink equilibrium of $f^t \in \mathcal{P}^4$ and Σ_{f^t} denotes the set of all saddle equilibria of the flow f^t.

Proposition 10. *Any closed connected three-dimensional cross-section $S \subset W^s_\omega \setminus \omega$ is homeomorphic to the sphere $\mathbb{S}^3$.*

Proof. From the definition of the cross-section S, it follows that for any point $x \in W^s_\omega \setminus \omega$, there exists a unique time t_x such that $f^{t_x}(x) \in S$. Then, the map $r : W^s_\omega \setminus \omega \to S$, defined by the formula $r(x) = f^{t_x}(x)$, is a retraction. The map $F : W^s_\omega \setminus \omega \times [0,1] \to W^s_\omega \setminus \omega$ determined by the formula $F(x,s) = f^{st_x}(x)$ is a homotopy between the identity map of $W^s_\omega \setminus \omega$ and the map r. Hence, by definition, the manifold S is a deformation retract of $W^s_\omega \setminus \omega$. Therefore, the cross-section S is homotopically equivalent to the space $W^s_\omega \setminus \omega$. From hyperbolicity, it follows that the space $W^s_\omega \setminus \omega$ is homeomorphic to $\mathbb{R}^4 \setminus \{O\}$; consequently, it is homotopically equivalent to the sphere $\mathbb{S}^3$. Thus, S is a closed connected three-dimensional manifold, homotopically equivalent to a sphere. By the Poincaré – Perelman theorem, S is homeomorphic to a sphere. □

Lemma 2. *For any flow $f^t \in \mathcal{P}^4$ there exist a locally flat cross-section $S_{f^t} \subset W^s_\omega \setminus \omega$ and a set of pairwise disjoint compact canonical neighborhoods $\{V_p, p \in \Sigma_{f^t}\}$ such that*

$$V_p \cap S_{f^t} = \Pi^u_p$$

for any saddle equilibrium $p \in \Sigma_{f^t}$.

Proof. From Proposition 10, it follows that there exists a secant sphere $S_\omega \subset W^s_\omega \setminus \omega$.

Let $p \in \Omega^1_{f^t}$ and V_p be its compact canonical neighborhood. By considering the iteration $f^T(V_p)$ for some $T \in \mathbb{R}$, we can ensure that $f^t(V_p) \cap S_\omega = \varnothing$. In this case, $f^T(V_p)$ serves as the canonical neighborhood of the point p. Thus, without loss of generality, we can assume that $V_p \cap S_\omega = \varnothing$. We modify the cross-section S_ω so that the resulting cross-section satisfies property 2.

We connect the set Π^u_p with S_ω via segments of the trajectories of the flow f^t, denoting the geometric locus of the endpoints of these trajectories as $\widetilde{\Pi}$, such that $\widetilde{\Pi} \subset S_\omega$. By definition, $\widetilde{\Pi}$ is a submanifold of the manifold S_ω, hence there exists an embedding $e : \partial\widetilde{\Pi} \times [0,1] \to S_\omega$ such that $e(\partial\widetilde{\Pi} \times [0,1]) \cap \widetilde{\Pi} = e(\partial\widetilde{\Pi} \times \{0\})$ and $e(\partial\widetilde{\Pi} \times [0,1]) \cap V_q = \varnothing$ for any saddle equilibrium state q different from p.

We define a continuous function $T : \widetilde{\Pi} \to \mathbb{R}$ that assigns to each point $x \in \widetilde{\Pi}$ the time t_x such that $f^{t_x}(x) \subset \Pi^u_p$. We then extend this function continuously to the set $K = e(\partial\widetilde{\Pi} \times [0,1])$ so that $T(x) = 0$ for all points $x \in e(\partial\widetilde{\Pi} \times \{1\})$.

The new cross-section will be taken as the set (see Fig. 4):

$$\widetilde{S}_\omega = \big(S_\omega \setminus (\widetilde{\Pi} \cup K)\big) \cup \left(\bigcup_{x \in \widetilde{\Pi} \cup K} f^{T(x)}(x) \right).$$

By performing a similar procedure for each saddle p, after finitely many steps, we obtain the desired cross-section S_{f^t}. □

Corollary 4. *For any $p \in S_{f^t}$ the set $\lambda_p = W^u_p \cap S_{f^t}$ is a locally flat sphere of dimension $(i_p - 1)$.*

Set

$$\Lambda^{i-1}_{f^t} = \bigcup_{p \in \Omega^i_{f^t}} (W^u_p \cap S_{f^t}), i \in \{1,2,3\}$$

It follows from the generalized Schoenflies theorem that S_{f^t} bounds a ball $V_\omega \subset W^s_\omega$. Throughout the remainder of this paper we orient S_{f^t} as the boundary of the ball V_ω.

The set $Q = V_\omega \cup \bigcup\limits_{p \in \Sigma_{f^t}} V_p$ is obtained from V_ω by attaching the handles $\{V_p\}$ to V_ω along the sets $\Lambda^0_{f^t}, \Lambda^1_{f^t}, \Lambda^2_{f^t} \subset S_{f^t} = \partial V_\omega$. Let us endow each of these sets by information about the attaching map. If the Morse index of the point p equals 1, then the set $\lambda^0_p \subset \Lambda^0_{f^t}$ is the pair of points (v_1, v_2),

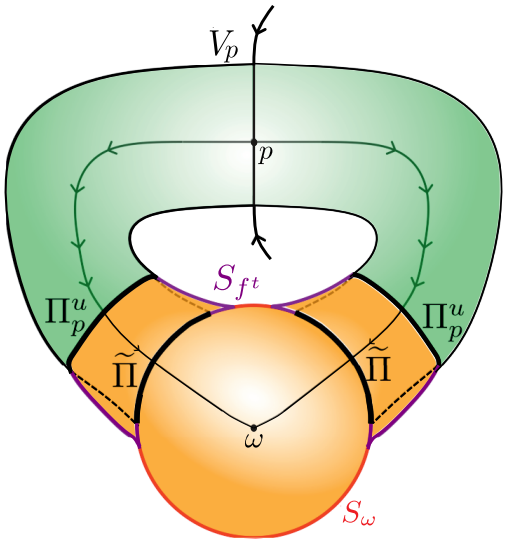

Fig. 4. Construction of the cross-section S_{f^t}.

and the set Π^u_p consists of two disjoint compact three-dimensional balls $B_{p,1}$ and $B_{p,2}$ centered at (v_1, v_2), respectively. We denote by μ_1, μ_2 the orientation of $B_{p,1}$ and $B_{p,2}$ induced by the orientation of the handle V_p. On the other hand, $B_{p,i} \subset S_{f^t}$ and the orientation of the sphere S_{f^t} induces the orientation $\tilde{\mu}_i$ of $B_{p,i}$.

Definition 3. We say that the pair $(v_1, v_2) \in \Lambda^0_{f^t}$ has the spin + if either $\mu_1 = \tilde{\mu}_1$ and $\mu_2 = \tilde{\mu}_2$ or $\mu_1 = -\tilde{\mu}_1$ and $\mu_2 = -\tilde{\mu}_2$. The pair $(v_1, v_2) \in \Lambda^0_{f^t}$ has the spin − otherwise.

If the Morse index of the point p is 2, then the set $\lambda^1_p \subset \Lambda^1_{f^t}$ is a locally flat knot, while the sets Π^u_p and Π^s_p are solid tori.

Definition 4. The meridian $\widetilde{\lambda}_p$ of the solid torus Π^s_p is called the framing of the knot λ^1_p.

Recall that the set $K_{f^t} = \{\Lambda^0_{f^t}, \Lambda^1_{f^t}, \Lambda^2_{f^t}\}$, where $\Lambda^0_{f^t}, \Lambda^1_{f^t}$ are endowed with spins and the framings, is called the Kirby diagram of the flow f^t. The equivalence relation of the Kirby diagrams $K_{f^t}, K_{f'^t}$ of the flows f^t, $f'^t \in \mathcal{P}^4$ is given in Definition 2.

4.3. Proof of Theorem 1

In this section, we prove that two flows $f^t, f'^t \in \mathcal{P}^4_0$ are topological equivalent if and only if their Kirby diagrams are equivalent up to homeomorphism. The necessity of the condition directly follows from the definition of topological equivalence. Let us prove the sufficiency.

Let $h : S_{f^t} \to S_{f'^t}$ be a homeomorphism which satisfies Definition 2. We aim to construct a homeomorphism $H : M^4 \to M^4$ such that $f^t = H^{-1} f'^t H$. So, we show that the flows f^t, f'^t are topologically conjugated and, as a consequence, are topologically equivalent.

Let us remark that the definition of the spin of the set $\lambda^0_p = (v_{p,1}, v_{p,2})$ does not depend on the choice of the orientation of S_{f^t}. Then without loss of generality we assume that h is orientation-preserving. Due to Statements 4, 6, 9 we may also assume that $h(\Pi^u_p) = \Pi^u_{p'}$ for any pair $p \in \Sigma_{f^t}$, $p' \in \Sigma_{f'^t}$ such that $h(\lambda^i_p) = \lambda^i_{p'}$.

Due to Proposition 9, there exists a homeomorphism $h_{p,p'} : V_p \to V_{p'}$ that conjugates the restrictions of the flows f^t and f'^t in the neighborhoods V_p and $V_{p'}$. Define a homeomorphism

$\eta_p : \Pi^u_p \to \Pi^u_{p'}$ by $\eta = h^{-1}h_{p,p'}$. Let us show that there exists a homeomorphism $\theta_p : S_{f'^t} \to S_{f'^t}$ such that $\theta_p|_{\Pi^u_p} = \eta_p|_{\Pi^u_p}$ and θ_p is the identity outside a small neighborhood of Π^u_p.

For $p \in \Omega^1_p$, the homeomorphism h preserves spins of pairs of Λ^0. Then, with an appropriate choice of $h_{p,p'}$ described in Proposition 9, the restriction of η_p on each ball from the pair Π^u_p is orientation-preserving. Then, due to Statement 4, $h_{p,p'}$ extends up to the desired homeomorphism θ_p. For $p \in \Omega^2_p$, the existence of the desired homeomorphism $\theta_p : S_{f'^t} \to S_{f'^t}$ follows from the properties of $h, h_{p,p'}$ and Statement 9. The arguments for point $p \in \Omega^3_{f^t}$ are similar.

Let $h' : S_{f'^t} \to S_{f'^t}$ denote the homeomorphism that is the superposition of the homeomorphisms θ_p constructed for each pair of saddle points $p \in \Sigma_{f^t}, p' \in \Sigma_{f'^t}$. We set $\tilde{h} = h'h$. By construction,

$$\tilde{h}|_{V_p \cap S_{f^t}} = h_{p,p'}|_{V_p \cap S_{f^t}}$$

for each saddle equilibrium $p \in \Sigma_{f^t}$.

For each point $z \in M^4 \setminus \Omega_{f^t}$, we associate the time $t_z \in \mathbb{R}$ such that $f^{t_z}(z) \subset V_\omega \cup \bigcup\limits_{p \in \Sigma_{f^t}} V_p$ and set

$$H(z) = \begin{cases} f'^{-t_z}\Big(\tilde{h}\big(f^{t_z}(z)\big)\Big), & f^{t_z}(z) \in S_{f^t}; \\ f'^{-t_z}\Big(h_{p,p'}\big(f^{t_z}(z)\big)\Big), & f^{t_z}(z) \in V_p. \end{cases}$$

We note that, although the time t_z is generally defined ambiguously for those points z for which $f^{t_z}(z) \in V_p$, the mapping H is a continuous bijection. Indeed, let τ_z, ν_z be such that $f^{\tau_z}(z) \in V_p$ and $f^{\nu_z}(z) \in V_p$ with $t = \nu_z - \tau_z > 0$. Then we can write:

$$H(z) = f'^{-\tau_z}\Big(h_{p,p'}\big(f^{\tau_z}(z)\big)\Big) = f'^{-\tau_z}\Big(f'^{-t}\big(h_{p,p'}(f^t(f^{\tau_z}(z)))\big)\Big) = f'^{-\nu_z}\Big(h_{p,p'}\big(f^{\nu_z}(z)\big)\Big).$$

The constructed map extends by continuity to the set Ω_{f^t} up to the desired homeomorphism.

5. TOPOLOGY OF MANIFOLDS CARRYING FLOWS FROM $\mathcal{P}^4$

5.1. Handle Decomposition of M^4

Let M^4 be the carrying manifold for $f^t \in \mathcal{P}^4$. As is shown in Section 4.2, the union V_ω, V_α and $\{V_p\}$ determines a handle decomposition of M^4. Here we study the properties of this decomposition necessary to prove Theorems 2 and 4.

We set

$$N_i = \bigcup_{p \in \Omega^i_{f^t}} V_p, \ Q_1 = V_\omega \cup N_1, \ Q_2 = Q_1 \cup N_2$$

and recall that

$$Q = V_\omega \cup \Big(\cup_{p \in \Sigma_{f^t}} V_p\Big) = Q_2 \cup N_3.$$

Lemma 3. 1) *∂Q is homeomorphic to the sphere $\mathbb{S}^3$;*

2) *$k_1 = k_3$;*

3) *for any pair of points $\lambda^0_p = (v_1, v_2)$ lying in the set $\Lambda^0_{f^t}$, there exists a sphere $\lambda^2_q \subset \Lambda^2_{f^t}$ such that v_1, v_2 belongs to different connected components of $S_{f^t} \setminus \lambda^2_q$.*

4) *Dehn surgery of S_{f^t} along the framed link $\Lambda^1_{f^t}$ yields a sphere;*

5) *M^4 is orientable if and only if ∂Q_1 is orientable;*

6) *∂Q_1 is homeomorphic to $\partial(V_\alpha \cup N_3)$ and both are homeomorphic either to $\#_{k_1}(\mathbb{S}^2 \times \mathbb{S}^1)$ or to $\#_{k_1}(\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1)$;*

Proof. By construction, ∂Q is the cross-section for the flow f^t and belongs to $W^u_\alpha \setminus \alpha$. Applying Proposition 10 to f^{-t}, we find that ∂Q is homeomorphic to $\mathbb{S}^3$ and prove Statement 1 of the lemma.

Set $P = V_\omega \cup N_1 \cup N_3$. Then ∂P is also the boundary of $V_\alpha \cup N_2$. Therefore, ∂P is obtained from the sphere ∂V_α by the Dehn surgery, which implies that it is connected. Hence, by Statement 1 of Proposition 6, $k_1 \geqslant k_3$. By applying similar arguments to the flow f^{-t}, we find that $k_3 \geqslant k_1$. Then $k_1 = k_3$, and Statement 3 of the lemma directly follows from Statement 2 of Proposition 6.

Since $k_1 = k_3$, due to Statement 1 of Proposition 6 ∂P is a sphere. Since $Q = P \cup N_2$, ∂Q is obtained from ∂P by Dehn surgery along $\Lambda^1_{f^t}$. We proved that ∂Q is homeomorphic to the sphere $\mathbb{S}^3$. This proves Statement 4 of the lemma.

Manifold M^4 is orientable if and only if $M^4 \setminus \operatorname{int} V_\alpha$ is orientable. $M^4 \setminus \operatorname{int} V_\alpha$ can be obtained from $V_\omega \cup N_2 \cup N_3$ by attaching the union of 1-handles N_1. $V_\omega \cup N_2 \cup N_3$ is homotopically equivalent to a bouquet of 2-dimensional and 3-dimensional spheres. Hence, it is simply connected and, consequently, orientable. Then the orientability of $M^4 \setminus \operatorname{int} V_\alpha$ depends only on the gluing of N_1. So, $\partial Q_1, \partial(V_\alpha \cup N^3)$, M^4 are orientable or nonorientable simultaneously. Consequently, taking into account Statement 2 and Corollary 1, $\partial Q_1, \partial(V_\alpha \cup N^3)$ are homeomorphic either to $\#_{k_1}(\mathbb{S}^2 \times \mathbb{S}^1)$ or to $\#_{k_1}(\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1)$ simultaneously. Statements 5 and 6 are proved. □

Corollary 5. *M^4 is orientable if and only if either $k_1 = 0$ or spin of any pair $\lambda^0_p \in \Lambda^0_p$ equals $+$.*

The following corollary proves the first statement of Theorem 5.

Corollary 6. *$P^-_{0,k} = \varnothing$ for any $k \geqslant 0$.*

5.2. Proof of Theorem 2

Let $f^t \in \mathcal{P}^4$ be a flow on M^4. Let us prove that M^4 is homeomorphic to a connected sum of a simply connected manifold N^4 and k_1 copies of $\mathbb{S}^3 \times \mathbb{S}^1$ or $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$.

Suppose p is a saddle equilibrium with Morse index 1 of the flow $f^t \in \mathcal{P}^4$. Then the closure $\operatorname{cl} W^s_p$ of W^s_p is the locally flat sphere $S_p = W^s_p \cup \alpha$ with the neighborhood $N_p = V_\alpha \cup V_p$. The set N_p is a ball with the 3-handle attached. Hence, N_p is homeomorphic to $\mathbb{S}^3 \times [-1, 1]$. Moreover, due to the definitions of V_α and V_p, the boundary spheres of N_p are cross-sections for f^t, and the vector field induced by $f^t|_{\partial N_p}$ is directed "outward" N_p. Let M_p be a manifold obtained from M^4 by performing surgery along S_p. Then $M_p = (M^4 \setminus \operatorname{int} N_p) \cup_\psi (\mathbb{B}^4 \times \mathbb{S}^0)$. We define a flow f^t_p on M_p such that $f^t_p|_{M^4 \setminus \operatorname{int} N_p}$ coincides with $f^t|_{M^4 \setminus \operatorname{int} N_p}$ and have exactly one source equilibria at each ball in $\mathbb{B}^4 \times \mathbb{S}^0$.

The flow f^t_p is gradient-like with the same number of equilibrium states as f^t, but has one more source and one less index 1 saddle than f^t. We will repeat the surgical operation until we exhaust all the index 1 saddles. After k_1 steps, we obtain a manifold N^4 and a gradient-like flow g^t on N^4 such that the nonwandering set of g^t consists of exactly one sink ω, k_2 saddles of Morse index 2, k_3 saddles of Morse index 3, and $k_1 + 1 = k_3 + 1$ sources. Similar to Lemma 2, we may construct the family $\{V_q : q \in \Omega_{g^t}\}$ of the compact canonical neighborhoods of the equilibria of the flow g^t that forms the handle decomposition of N^4. Let P_i denotes the union of handles with indices from 0 to i. The boundary of P_2 is obtained from the sphere ∂P_0 by Dehn surgery, hence ∂P_2 is connected. Then $\partial(N^4 \setminus \operatorname{int} P_2)$ is connected. Considering g^{-t}, one can see that $N^4 \setminus \operatorname{int} P_2$ is the connected union of $(k_3 + 1)$ handles of index 0 and k_3 handles of index 1. Due to Proposition 5, $N^4 \setminus \operatorname{int} P_2$ is homeomorphic to the ball. Consequently, N^4 is connected. Moreover, N^4 is simply connected. Indeed, any loop $\gamma \in N^4$ is homotopic to a loop in P_2. P_2 is homotopically equivalent

to a bouquet of two-dimensional spheres, therefore it is simply connected. Consequently, N^4 is also simply connected.

According to Lemma 1 and Proposition 2, the ambient manifold M^4 of f^t is homeomorphic to $N^4\#\underbrace{(\mathbb{S}^3\times\mathbb{S}^1)\#\cdots\#(\mathbb{S}^3\times\mathbb{S}^1)}_{k_1}$ if it is orientable, and to $N^4\#\underbrace{(\mathbb{S}^3\widetilde{\times}\mathbb{S}^1)\#\cdots\#(\mathbb{S}^3\widetilde{\times}\mathbb{S}^1)}_{k_1}$ if it is nonorientable.

5.3. Proof of Theorem 3

Let N^4 and g^t be the simply connected manifold and the gradient-like flow on N^4 constructed in Section 5.2 as the result of the sequence of surgeries along locally flat spheres. Let us remark that all surgeries keep the set $V_\omega\cup N_2$ and, consequently, the sets $\Lambda^1_{f^t},\Lambda^2_{f^t}\subset\partial V_\omega$. Then the set $K_{g^t}=\{\Lambda^2_{f^t},\Lambda^3_{f^t}\}$ is the Kirby diagram of the flow g^t. Let $A_{\Lambda^2_{f^t}}$ be the linking matrix defined by the framed link $\Lambda^2_{f^t}$ (see the definition of this matrix in Section 3.7).

In the previous section we proved that N^4 admits a handle decomposition without handles of indices 1 and 3. Due to [15, Proposition 4.5.11], for such N^4, the matrix $A_{\Lambda^2_{f^t}}$ represents the intersection form Q of the manifold N^4 in some basis. Now Theorem 3 immediately follows from Propositions 7 and 8.

6. REALIZATION OF TOPOLOGICAL EQUIVALENCE CLASSES

This section presents an algorithm for constructing a representative of each class of topological equivalence of flows from the class $\mathcal{P}^4$, proving Theorem 4.

Let $\Lambda^0,\Lambda^1,\Lambda^2\subset\mathbb{S}^3$ be disjoint unions of pairs of points, smooth framed knots and smooth two-dimensional spheres, respectively.

Definition 5. A set $K=\{\Lambda^0,\Lambda^1,\Lambda^2\}$ is called the admissible Kirby diagram if it satisfies the following conditions:

1) each pair of points $\lambda^0\in\Lambda^0$ is marked by a spin $\pm$;

2) for any pair of points $\lambda^0=(v_1,v_2)$ there exists a sphere $\lambda^2\subset\Lambda^2$ such that v_1 and v_2 belong to different connected components of $\mathbb{S}^3\setminus\lambda^2$;

3) the number $k\geqslant 0$ of pairs of points in Λ^0 equals the number of spheres constituting the set Λ^2;

4) Dehn surgery along the framed link Λ^1 yields sphere S^3.

Proposition 4 and Lemma 3 immediately give the following fact.

Proposition 11. *The Kirby diagram of any $f^t\in\mathcal{P}^4$ is admissible.*

6.1. Prove of Theorem 4

We prove here that for any admissible Kirby diagram K there exists a flow $f^t\in\mathcal{P}^4$ whose Kirby diagram K_{f^t} is isomorphic to K.

Let F_i denote a vector field on the handle $H^4_i=\mathbb{B}^i\times\mathbb{B}^{4-i}$ given by the equations $\dot{x}=x,\dot{y}=-y$, where $x\in\mathbb{B}^i\times\{O\},y\in\{O\}\times\mathbb{B}^{4-i}$.

Let $K=\{\Lambda^0,\Lambda^1,\Lambda^2\}$ be an admissible Kirby diagram. Gluing the handles of indices $1,2,3$ along sets $\Lambda^0,\Lambda^1,\Lambda^2\subset\mathbb{S}^3$ to $H^4_0=\mathbb{B}^4$ by means of diffeomorphisms which provide a smooth gluing of the vector fields F_0 and $\{F_i;\,i=1,2,3\}$, we obtain the manifold Q with a boundary and the vector field F on Q with a singular point of Morse index $i\in\{0,1,2,3\}$ in each handle H^4_i, transversal to ∂Q. By Definition 5, ∂Q is homeomorphic to sphere $\mathbb{S}^3$. Then smoothing the corners of Q and gluing the handle H^4_4 which carries the vector field F_4, we obtain a closed smooth manifold M^4 and a flow $f^t\in\mathcal{P}^4_0$ whose Kirby diagram K_{f^t} is equivalent to K up to homeomorphism.

6.2. Admissible Diagrams of Flows in $P^{\pm}_{k,0}$

Figure 5 shows all admissible nonequivalent Kirby diagrams of the flows from $P^{\delta}_{k,0}$ for $\delta = \pm$, $k \in \{1, 2\}$, which, by virtue of Proposition 11 and Theorem 1, immediately proves Statement 2 of Theorem 5. For $k \geqslant 2$ and $\delta = -$, it is possible to choose spins of pairs of points in the set Λ^0 in a different way, as shown in Fig. 5b. According to Theorem 1, the diagrams in Fig. 5 determine topologically nonequivalent flows.

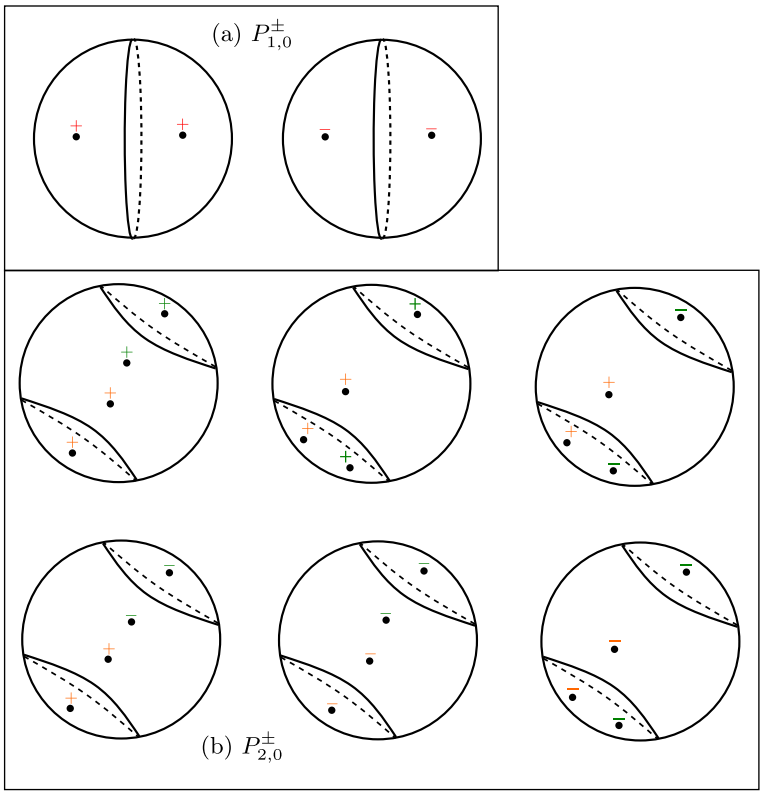

Fig. 5. (a)–(c): Admissible Kirby diagrams of the flows from the class (*a*) $P^{\pm}_{1,0}$, (*b*) $P^{\pm}_{2,0}$.

Due to the Jordan – Brouwer theorem [2], for any two locally flat spheres $S^2, \widetilde{S}^2 \subset \mathbb{S}^3$ there is a homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$ such that $h(S^2) = \widetilde{S}^2$. That is why for any k there are finitely many nonequivalent admissible Kirby diagrams with empty set Λ^2. This fact proves Statement 3 of Theorem 5.

6.3. Nonequivalent Flows from $P^{\pm}_{k_1,k_2}$, $k_2 \geqslant 2$

It follows from [28] that the class $P^{+}_{0,1}$ is nonempty, all flows in $P^{+}_{0,1}$ are topologically equivalent, and the ambient manifold of such flows is the complex projective plane $\mathbb{CP}^2$ (see also [16]).

Figure 6 shows three framed links that can be realized, using Theorem 4, as the Kirby diagrams of flows $a^t, b^t, c^t \subset P^{+}_{0,2}$. Each link consists of two framed knots. We present only the knots themselves, defining their framings by integers. An integer n near the diagram of the knot λ means that the knot has a framing $\widetilde{\lambda}$ such that $\mathrm{lk}(\lambda, \widetilde{\lambda}) = n$.

In [16] it is proved that the ambient manifolds of the flows a^t, b^t, c^t are $\mathbb{CP}^2 \# \mathbb{CP}^2$, $\overline{\mathbb{CP}^2} \# \mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$, respectively. Here $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ denotes the complex projective plane with the opposite

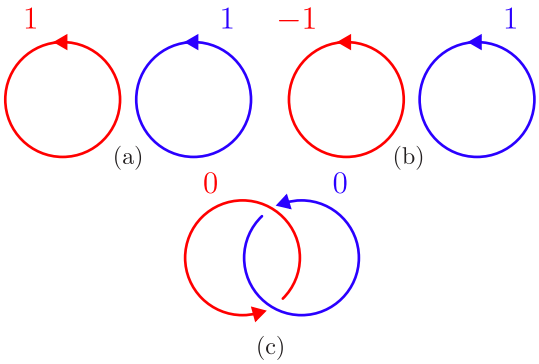

Fig. 6. (a)–(c): Kirby diagrams of the flow $a^t, b^t, c^t \in P_{0,2}$ with the ambient manifolds $\mathbb{CP}^2\#\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2\#\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, resp.

orientations. All three manifolds are pairwise nonhomeomorphic and exhaust, due to Theorem 3, all manifolds admitting flows from $P^+_{0,2}$.

Figures 7 and 8 show the sequence of second Kirby moves applied to framed links $\Lambda^1_{b^t}, \Lambda^1_{c^t}$. Each Kirby move in Fig. 7 increases the number of positive crossings, where one of the components of the link passes below another one. According to Statement 7, the linking number of the link components also increases. Hence, each move produces a link that is not equivalent to the previous one up to homeomorphism. According to Theorems 1 and 4, every Kirby diagram of this series determines a flow on $\overline{\mathbb{CP}}^2\#\mathbb{CP}^2$ that is nonequivalent to the previous one. For link $\Lambda^1_{a^t}$ the construction is similar.

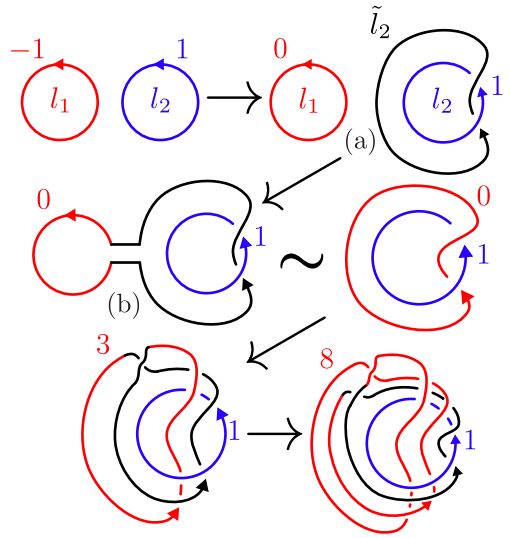

Fig. 7. A sequence of second Kirby moves on $\overline{\mathbb{CP}^2}\#\mathbb{CP}^2$ generated a countable set of topologically nonequivalent flows.

Each Kirby move in Fig. 8 preserves the components of the link $\Lambda^1_{c^t}$, but changes the framing of the first knot. Hence, each move produces a link that is nonequivalent to the previous one up to homeomorphism. According to Theorems 1 and 4, every Kirby diagram of this series determines a flow on $\mathbb{S}^2 \times \mathbb{S}^2$ that is nonequivalent to the previous one.

Hence, we have shown that there is a countable set of pairwise nonequivalent flows with two saddles on each manifold. Taking a connecting sum with every such flow, we obtain a countable set of nonequivalent flows for any $k_1, k_2 \geqslant 2$ given on the same manifold. This observation completes the proof of Theorem 5.

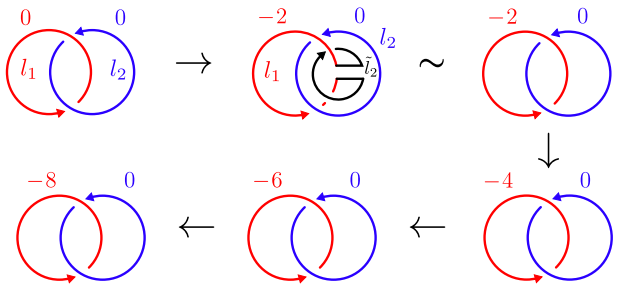

Fig. 8. A sequence of second Kirby moves on $\mathbb{S}^2 \times \mathbb{S}^2$ generated a countable set of topologically nonequivalent flows.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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