ON THE RK-PREORDER ON C-CONES OF RK-MINIMAL ULTRAFILTERS

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Many works in the theory of ultrafilters consider different (pre)orders on the set $\boldsymbol{\beta}X$ (of ultrafilters on the set X). Apparently, the Rudin-Keisler and Comfort preorders on $\beta\omega$ are most well studied, see, e.g., [1, 2, 3], but there are still many open problems in this area. In this paper we describe the Rudin-Keisler preorder on the lower cones of RK-minimal ultrafilters with respect to the Comfort preorder.

1 Basic definitions

For any set X the set of all subsets of X is denoted by $\mathscr{P}(X)$. An *ultrafilter* on X is a set $\mathfrak{u} \subseteq \mathscr{P}(X)$ such that

1. $\emptyset \notin \mathfrak{u}$;
2. if $A \in \mathfrak{u}$ and $B \in \mathfrak{u}$, then $A \cap B \in \mathfrak{u}$;
3. if $A \in \mathfrak{u}$ and $A \subseteq B$, then $B \in \mathfrak{u}$;
4. $A \in \mathfrak{u}$ or $X \setminus A \in \mathfrak{u}$

for all $A, B \subseteq X$. The set of ultrafilters on X is usually denoted by $\boldsymbol{\beta}X$ and provided with a natural topology with the base

$$\{\{\mathfrak{u} \in \boldsymbol{\beta}X : A \in \mathfrak{u}\} : A \subseteq X\}.$$

This topological space is compact, Hausdorff, zero-dimensional and extremely disconnected. An ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}X$ is *principal* if $\mathfrak{u} = \{A \subseteq X : a \in A\}$ for some $a \in X$. Principal ultrafilters on X are usually identified with elements of X, so $\boldsymbol{\beta} X$ is considered as an extension of X (called a *Stone–Čech compactification* of X). For any function $f : X \to \boldsymbol{\beta} Y$, the *ultrafiter extension* $\widetilde{f} : \boldsymbol{\beta} X \to \boldsymbol{\beta} Y$ is defined by the formula

$$\widetilde{f}(\mathfrak{u}) = \{S \subseteq Y : (\forall A \in \mathfrak{u})\,(\exists a \in A)\, S \in f(a)\}$$

for all $\mathfrak{u} \in \boldsymbol{\beta} X$. We obtain an equivalent definition if we put

$$\widetilde{f}(\mathfrak{u}) = \{S \subseteq Y : (\exists A \in \mathfrak{u})\,(\forall a \in A)\, S \in f(a)\}.$$

The function $\widetilde{f}$ is the unique continuous (with respect to the natural topology) function from $\boldsymbol{\beta} X$ to $\boldsymbol{\beta} Y$ which extends the function f. Considering functions $f : X \to Y$ as functions from X to $\boldsymbol{\beta} Y$ with a range consisting of principal ultrafilters, we also have the definition of the ultrafilter extension $\widetilde{f} : \boldsymbol{\beta} X \to \boldsymbol{\beta} Y$ for each function $f : X \to Y$.

The *Rudin-Keisler preorder* (or RK-*preorder*) on $\boldsymbol{\beta} X$ is the binary relation $\leqslant_{\mathrm{RK}} \subseteq \boldsymbol{\beta} X \times \boldsymbol{\beta} X$ defined by

$$\mathfrak{u} \leqslant_{\mathrm{RK}} \mathfrak{v} \Leftrightarrow \widetilde{f}(\mathfrak{v}) = \mathfrak{u} \text{ for some } f : X \to X.$$

An ultrafilter $\mathfrak{u} \in \boldsymbol{\beta} X$ is called RK-*minimal* if it is non-principal and

$$\mathfrak{v} \leqslant_{\mathrm{RK}} \mathfrak{u} \;\Rightarrow\; \mathfrak{v} \text{ is principal or } \mathfrak{u} \leqslant_{\mathrm{RK}} \mathfrak{v}$$

for any $\mathfrak{v} \in \boldsymbol{\beta} X$. There are many different characterizations of RK-minimal ultrafilters, see [4], Theorem 9.6, and also [5]. In particular, a non-principal ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}\omega$ is RK-minimal if and only if it is a Ramsey ultrafilter and if and only if it is a quasi-normal ultrafilter.

The equivalence relation $\leqslant_{\mathrm{RK}} \cap \leqslant_{\mathrm{RK}}^{-1}$ is denoted by $\approx_{\mathrm{RK}}$. The equivalence class of an ultrafilter $\mathfrak{u} \in \boldsymbol{\beta} X$ with respect to the relation $\approx_{\mathrm{RK}}$ is called *a type* of ultrafilter $\mathfrak{u}$ and is denoted by $\tau(\mathfrak{u})$, see [4]. The Rudin-Keisler preorder naturally extends to the quotient set $\boldsymbol{\beta} X/_{\approx_{\mathrm{RK}}}$: $\tau(\mathfrak{u}) \leqslant_{\mathrm{RK}} \tau(\mathfrak{v}) \Leftrightarrow \mathfrak{u} \leqslant_{\mathrm{RK}} \mathfrak{v}$ for all types $\tau(\mathfrak{u})$ and $\tau(\mathfrak{v})$ of ultrafilters $\mathfrak{u}$ and $\mathfrak{v}$, respectively. Obviously, $\leqslant_{\mathrm{RK}}$ is a partial order on $\boldsymbol{\beta} X/_{\approx_{\mathrm{RK}}}$. Therefore, we call the relation $\leqslant_{\mathrm{RK}}$ on the set $\boldsymbol{\beta} X/_{\approx_{\mathrm{RK}}}$ the *Rudin-Keisler order* (or RK-*order*).

To define the Comfort preorder on $\boldsymbol{\beta} X$ we need some topological concepts. Let $\mathfrak{u} \in \boldsymbol{\beta} X$. A point $y \in Y$ of a topological space (Y, T) is called the $\mathfrak{u}$-limit of a function $f : X \to Y$ if for any neighborhood U of y the set $\{x \in X : f(x) \in U\}$ belongs to $\mathfrak{u}$. The $\mathfrak{u}$-limit of a function f is denoted by the symbol $\mathfrak{u}\text{-}\lim f$. A topological space (Y, T) is called $\mathfrak{u}$-*compact* if for any $f : X \to Y$ there exists $\mathfrak{u}\text{-}\lim f \in Y$. The *Comfort preorder* $\leqslant_{\mathrm{C}}$ on $\boldsymbol{\beta} X$ is defined as follows: for all ultrafilters $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta} X$, $\mathfrak{u} \leqslant_C \mathfrak{v}$ iff any $\mathfrak{v}$-compact topological space (Y, T) is $\mathfrak{u}$-compact.

It is well known that $\leqslant_{\mathrm{RK}} \subseteq \leqslant_{\mathrm{C}}$, and hence $\approx_{\mathrm{RK}}$ is a congruence of the structure $(\boldsymbol{\beta}X; \leqslant_{\mathrm{C}})$. Thus, we can assume that the Comfort preorder is defined on $\boldsymbol{\beta}X/_{\approx_{\mathrm{RK}}}$. More information can be found in the [2, 3].

The C-*cone* of an ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}X$ is the set

$$\mathrm{Con}_{\mathrm{C}}(\mathfrak{u}) = \{\tau(\mathfrak{v}) : \mathfrak{v} \in \boldsymbol{\beta}X \wedge \mathfrak{v} \leqslant_{\mathrm{C}} \mathfrak{u}\}.$$

An ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}X$ is called C-*minimal* if it is non-principal and

$$\mathfrak{v} \leqslant_{\mathrm{C}} \mathfrak{u} \Rightarrow \mathfrak{v} \text{ is principal or } \mathfrak{u} \leqslant_{\mathrm{C}} \mathfrak{v}$$

for any $\mathfrak{v} \in \boldsymbol{\beta}X$. It is well known (see [2]) that if the type of ultrafilter $\mathfrak{v} \in \boldsymbol{\beta}\omega \setminus \omega$ belongs to the C-cone of some RK-minimal ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}\omega$, then $\mathfrak{v}$ is a C-minimal ultrafilter. The inverse implication remains an open problem.

2 Main result

For all posets $\mathfrak{A} = (A, \leqslant_0)$ and $\mathfrak{B} = (B, \leqslant_1)$, their *sum* is the poset $\mathfrak{A} + \mathfrak{B} = (C, \leqslant_2)$, where $C = A \cup B'$, $A \cap B' = \emptyset$, $(A, \leqslant_2) = \mathfrak{A}$, $(B', \leqslant_2) \cong \mathfrak{B}$, and $a \leqslant_2 b$ for all $a \in A$ and $b \in B'$.

For any model $\mathfrak{M}$ and ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}X$, the ultrapower of $\mathfrak{M}$ modulo $\mathfrak{u}$ is denoted by $\prod_{\mathfrak{u}} \mathfrak{M}$.

For any limit ordinal α and non-decreasing sequence $\{\mathfrak{M}_\beta\}_{\beta<\alpha}$ of models in the same signature, the direct limit of $\{\mathfrak{M}_\beta\}_{\beta<\alpha}$ is denoted by $\lim\limits_{\beta\to\alpha} \mathfrak{M}_\beta$.

For any poset $\mathfrak{A}$, ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}X$, and ordinal α, define the *overbuilding ultralimit* $\operatorname*{olim}\limits_{\mathfrak{u},\alpha}$ of $\mathfrak{A}$ of rank α modulo $\mathfrak{u}$ by recursion on α:

i. $\operatorname*{olim}\limits_{\mathfrak{u},0} \mathfrak{A} = \mathfrak{A}$;

ii. if $\alpha = \beta + 1$, $\operatorname*{olim}\limits_{\mathfrak{u},\beta} \mathfrak{A} = (A, \leqslant_0)$, and $\prod\limits_{\mathfrak{u}} \operatorname*{olim}\limits_{\mathfrak{u},\beta} \mathfrak{A} = (B, \leqslant_1)$ then

$$\operatorname*{olim}_{\mathfrak{u},\alpha} \mathfrak{A} = \operatorname*{olim}_{\mathfrak{u},\beta} \mathfrak{A} + \mathfrak{B}$$

where $\mathfrak{B}$ is the submodel of $\prod\limits_{\mathfrak{u}} \operatorname*{olim}\limits_{\mathfrak{u},\beta} \mathfrak{A}$ with the universe $\{b \in B : b \cap A = \emptyset\}$;

iii. if α is a limit ordinal, then $\operatorname*{olim}\limits_{\mathfrak{u},\alpha} \mathfrak{A} = \lim\limits_{\beta\to\alpha} \operatorname*{olim}\limits_{\mathfrak{u},\beta} \mathfrak{A}$.

This construction resembles the construction of a *limiting ultrapower* of a model (also called an *ultralit* of a model), but does not coincide with it. In particular, an overbuilding ultralimit of positive rank of a finite poset $\mathfrak{A}$ is not isomorphic to $\mathfrak{A}$.

Denote the one-element poset $(1, \leqslant)$ by $\mathfrak{O}$.

Theorem 1. *For any* RK-*minimal ultrafilter* $\mathfrak{u} \in \boldsymbol{\beta}\omega$

$$(\mathrm{Con}_{\mathrm{C}}(\mathfrak{u}), \leqslant_{\mathrm{RK}}) \cong \operatorname*{olim}_{\mathfrak{u},\omega_1} \mathfrak{O}.$$

Sketch of proof. First, we establish the "ordinal stratification" of the Comfort preorder on $\boldsymbol{\beta}\omega/_{\approx_{\mathrm{RK}}}$ (essentially introduced in [8, 9]). For any ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}\omega$ and ordinal α we define the sets $U_\alpha(\mathfrak{u}), U_{<\alpha}(\mathfrak{u}) \subseteq \boldsymbol{\beta}\omega/_{\approx_{\mathrm{RK}}}$:

i. $U_0(\mathfrak{u}) = \{\tau(0)\}$,

ii. for $\alpha > 0$, we put $U_{<\alpha}(\mathfrak{u}) = \bigcup_{\beta<\alpha} U_\beta(\mathfrak{u})$ and

$$U_\alpha(\mathfrak{u}) = \{\tau(\widetilde{f}(\mathfrak{u})) : f \in (\boldsymbol{\beta}\omega)^\omega \text{ and } (\forall i < \omega)\, \tau(f(i)) \in U_{<\alpha}(\mathfrak{u})\}.$$

We prove that for each ultrafilters $\mathfrak{u} \in \boldsymbol{\beta}\omega$

$$\mathrm{Con}_{\mathrm{C}}(\mathfrak{u}) = U_{<\omega_1}(\mathfrak{u}). \tag{1}$$

Next, we show that if an ultrafilter $\mathfrak{u}$ is RK-minimal, then we can restrict ourselves to injective functions $f : \omega \to \boldsymbol{\beta}\omega$ with a discrete range when constructing the sets $U_\alpha(\mathfrak{u})$. A set $W \subseteq \boldsymbol{\beta}X$ is *discrete* if there is a partition $\{A_{\mathfrak{w}}\}_{\mathfrak{w}\in W}$ of X such that $A_{\mathfrak{w}} \in \mathfrak{w}$ for all $\mathfrak{w} \in W$. Let DF be a set of all injective functions $f : \omega \to \boldsymbol{\beta}\omega$ with a discrete range. For any ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}\omega$ and ordinal $\alpha > 0$ we define the sets $V_\alpha(\mathfrak{u}), V_{<\alpha}(\mathfrak{u}) \subseteq \boldsymbol{\beta}\omega/_{\approx_{\mathrm{RK}}}$:

i. $V_1(\mathfrak{u}) = \{\tau(\mathfrak{u})\}$,

ii. for $\alpha > 1$, we put $V_{<\alpha}(\mathfrak{u}) = \bigcup_{\beta<\alpha} V_\beta(\mathfrak{u})$ and

$$V_\alpha(\mathfrak{u}) = \{\tau(\widetilde{f}(\mathfrak{u})) : f \in \mathrm{DF} \text{ and } (\forall i < \omega)\, \tau(f(i)) \in V_{<\alpha}(\mathfrak{u})\}.$$

We prove that for any positive ordinal α and RK-minimal ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}\omega$

$$U_\alpha(\mathfrak{u}) = V_\alpha(\mathfrak{u}) \cup \{\tau(0)\}. \tag{2}$$

Finally, we will need the fact that for all functions $f, g \in \mathrm{DF}$ and ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}\omega$

$$\widetilde{f}(\mathfrak{u}) \leqslant_{\mathrm{RK}} \widetilde{g}(\mathfrak{u}) \Leftrightarrow \{i < \omega : f(i) \leqslant_{\mathrm{RK}} g(i)\} \in \mathfrak{u} \tag{3}$$

(see, e.g., [10]).

Using the facts (1) – (3), the theorem can be easily proved by induction on α. $\square$

The equivalence relation $\leqslant_{\mathrm{C}} \cap \leqslant_{\mathrm{C}}^{-1}$ on $\boldsymbol{\beta}X/_{\approx_{\mathrm{RK}}}$ is denoted by $\approx_{\mathrm{C}}$. For any $\mathfrak{u} \in \boldsymbol{\beta}X$, let $[\mathfrak{u}]_{\mathrm{C}} = \{\tau(\mathfrak{v}) : \mathfrak{v} \in \boldsymbol{\beta}X \text{ and } \tau(\mathfrak{v}) \approx_{\mathrm{C}} \tau(\mathfrak{u})\}$. It is easy to see that for any RK-minimal ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}\omega$ and non-principal ultrafilter $\mathfrak{v} \leqslant_{\mathrm{C}} \mathfrak{u}$ we have: $\mathfrak{u} \leqslant_{\mathrm{RK}} \mathfrak{v}$ and, so,

$$[\mathfrak{v}]_{\mathrm{C}} \cup \{\tau(0)\} = \mathrm{Con}_{\mathrm{C}}(\mathfrak{u}).$$

Therefore, theorem 1 immediately entails the following corollary.

Corollary 1. *Let* $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$. *If* $\mathfrak{u}$ *is* RK-*minimal and* $\tau(\mathfrak{u}) \in [\mathfrak{v}]_{\mathrm{C}}$ *then*

$$([\mathfrak{v}]_{\mathrm{C}}, \leqslant_{\mathrm{RK}}) \cong \operatorname*{olim}_{\mathfrak{u},\omega_1} \mathfrak{O}.$$

Discussion. Can the poset $\operatorname*{olim}_{\mathfrak{u},\omega_1} \mathfrak{O}$ be described more explicitly? Note, e.g., that $\operatorname*{olim}_{\mathfrak{u},\omega+1} \mathfrak{O}$ is isomorphic to the ultrapower of $(\omega, \leqslant)$ modulo $\mathfrak{u}$ where $\leqslant$ is the natural ordering of ω. Are the posets $\operatorname*{olim}_{\mathfrak{u},\omega_1} \mathfrak{O}$ and $\operatorname*{olim}_{\mathfrak{v},\omega_1} \mathfrak{O}$ isomorphic for all RK-minimal ultrafilters $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$? Let us call a C-minimal ultrafilter $\mathfrak{v} \in \boldsymbol{\beta}\omega$ a *normal C-minimal ultrafilter* if $\tau(\mathfrak{u}) \in [\mathfrak{v}]_{\mathrm{C}}$ for some RK-minimal ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}\omega$. Is the statement inverse to Corollary 1 true? In other words, is it true that the condition "there exists an RK-minimal ultrafilter $\mathfrak{u} \in \boldsymbol{\beta}\omega$ for which $([\mathfrak{v}]_{\mathrm{C}}, \leqslant_{\mathrm{RK}}) \cong \operatorname*{olim}_{\mathfrak{u},\omega_1} \mathfrak{O}$" exactly characterises normal C-minimal ultrafilters $\mathfrak{v} \in \boldsymbol{\beta}\omega$?

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