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MATHEMATICS

Construction of Smooth Source–Sink Arcs in the Space of Diffeomorphisms of a Two-Dimensional Sphere

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Abstract—It is well known that the mapping class group of the two-dimensional sphere $\mathbb{S}^2$ is isomorphic to the group $\mathbb{Z}_2 = \{-1,+1\}$. At the same time, the class $+1(-1)$ contains all orientation-preserving (orientation-reversing) diffeomorphisms and any two diffeomorphisms of the same class are diffeotopic, that is, they are connected by a smooth arc of diffeomorphisms. On the other hand, each class of maps contains structurally stable diffeomorphisms. It is obvious that in the general case, the arc connecting two diffeotopic structurally stable diffeomorphisms undergoes bifurcations that destroy structural stability. In this direction, it is particular interesting in the question of the existence of a connecting them stable arc – an arc pointwise conjugate to arcs in some of its neighborhood. In general, diffeotopic structurally stable diffeomorphisms of the 2-sphere are not connected by a stable arc. In this paper, the simplest structurally stable diffeomorphisms (source–sink diffeomorphisms) of the 2-sphere are considered. The non-wandering set of such diffeomorphisms consists of two hyperbolic points: the source and the sink. In this paper, the existence of an arc connecting two such orientation-preserving (orientation-reversing) diffeomorphisms and consisting entirely of source-sink diffeomorphisms is constructively proved.

Keywords: source–sink diffeomorphism, smooth arc, stable arc
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1. INTRODUCTION AND FORMULATION OF RESULTS

Consider the two-dimensional sphere

$$\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

and denote by $\mathrm{Diff}(\mathbb{S}^2)$ the space of all diffeomorphisms of the 2-sphere with the C^1-topology.

A *smooth arc* in $\mathrm{Diff}(\mathbb{S}^2)$ is a family $\{\phi_t\}$ of diffeomorphisms of the 2-sphere $\mathbb{S}^2$ that form a diffeotopy $\Phi : \mathbb{S}^2 \times [0,1] \to \mathbb{S}^2$, i.e.,

$$\Phi(x,t) = \phi_t(x), \quad x \in \mathbb{S}^2, \quad t \in [0,1].$$

In this case, the arc $\{\phi_t\}$ is said to *connect the diffeomorphisms* ϕ_0 and ϕ_1.

Diffeomorphisms $f, g \in \mathrm{Diff}(\mathbb{S}^2)$ are called *topologically conjugate* if there exists a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that $hf = gh$.

A diffeomorphism $f \in \mathrm{Diff}(\mathbb{S}^2)$ is called *structurally stable* if there is a neighborhood $U \subset \mathrm{Diff}(\mathbb{S}^2)$ of f such that any diffeomorphism $g \in U$ is topologically conjugate to f.

The relation of being connected by a smooth arc defines an equivalence relation on $\mathrm{Diff}(\mathbb{S}^2)$ and divides it into two equivalence classes consisting of orientation-preserving and orientation-reversing diffeomorphisms, respectively [1]. Each class of maps contains structurally stable diffeomorphisms (e.g., time-1 maps of gradient flows of generic Morse functions). Obviously, in the general case, an arc connecting two diffeotopic structurally stable diffeomorphisms undergoes bifurcations destroying its structural stability. In this context, an issue of particular interest is the existence of an arc whose qualitative properties do not change under small perturbations (of a stable arc), which is mentioned as problem 33 in the Palis–Pugh list of fifty most important problems on dynamical systems [2].

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According to [3], a smooth arc φ_t is called *stable* if it is an interior point of the equivalence class with respect to the following relation: arcs $\{\varphi_t\}$ and $\{\varphi'_t\}$ are called *conjugate* if there exist homeomorphisms $h:[0,1]\to[0,1], H_t:\mathbb{S}^2\to\mathbb{S}^2$ such that $H_t\varphi_t = \varphi'_{h(t)}H_t, t\in[0,1]$, and H_t depends continuously on t.

Generally speaking, diffeotopic structurally stable diffeomorphisms of the 2-sphere are not connected by a stable arc [4]. In this paper, we consider the simplest structurally stable diffeomorphisms of the 2-sphere, namely, source–sink diffeomorphisms. Such diffeomorphisms have exactly two fixed points, a sink and a source, while the orbits of the other points tend asymptotically to the sink in forward time and to the source in reverse time. Orientation-preserving (orientation-reversing) source–sink diffeomorphisms are pairwise topologically, but nonsmoothly, conjugate (see, e.g., [5]).

The main result of this work is a constructive proof of the following result.

Theorem 1. *Any two orientation-preserving (orientation-reversing) source–sink diffeomorphisms are connected by a smooth arc consisting of source–sink diffeomorphisms.*

A similar result for orientation-preserving source–sink 3-diffeomorphisms was obtained in [6]. Note that the obtained result does not extend straightforwardly to spheres of dimension higher than three, because several smooth structures may exist on such spheres. For example, it was shown in [6] that for $n = 6$ there are diffeomorphisms of the considered class that cannot be connected by a stable arc.

2. AUXILIARY FACTS

For any subset X of a topological space Y, let $i_X : X\to Y$ denote the *inclusion map*.

For any continuous mapping $\phi : X\to Y$ of arcwise connected topological spaces X and Y, let $\phi_* : \pi_1(X)\to\pi_1(Y)$ denote the *homomorphism induced by* ϕ.

The C^r*-embedding* ($r\geqslant 0$) of a manifold X in a manifold Y is a mapping $\lambda : X\to Y$ such that $\lambda : X\to\lambda(X)$ is a C^r-diffeomorphism. Here, the C^0-embedding is called a *topological embedding*, while the C^r-embedding ($r>0$) is called a *smooth embedding*.

Two continuous mappings $\phi_0 : X\to Y$ and $\phi_1 : X\to Y$ are called *homotopic* if there exists a continuous mapping $\Phi : X\times[0,1]\to Y$ such that $\Phi(x,0)=\phi_0(x)$ and $\Phi(x,1)=\phi_1(x)$. The mapping Φ is said to be a *homotopy* of ϕ_0 and ϕ_1. If X and Y are topological spaces and the mapping $\phi_t(x)=\Phi(x,t)$ is an embedding of X in Y for each $t\in[0,1]$, then the embeddings ϕ_0 and ϕ_1 are called *isotopic*, the mapping Φ is an *isotopy*, and the one-parameter family of embeddings $\{\phi_t\}$ is an *arc* connecting ϕ_0 and ϕ_1. If X and Y are smooth manifolds and the isotopy Φ is a smooth mapping, then Φ is called a *diffeotopy*, and the arc $\{\varphi_t\}$ is called *smooth*.

The *support of the isotopy* Φ (of the arc $\{\phi_t\}$) is the set

$$\operatorname{supp}\{\phi_t\} = \operatorname{cl}\{x\in X : \phi_t(x)\neq\phi_0(x) \text{ for some } t\in(0,1]\}.$$

A smooth arc $\{\varphi_t\}$ is called the *smooth product* of smooth arcs $\{\phi_t\}$ and $\{\psi_t\}$ such that $\phi_1=\psi_0$ if

$$\varphi_t = \begin{cases}\phi_{\tau(2t)}, & 0\leqslant t\leqslant\frac{1}{2},\\ \psi_{\tau(2t-1)}, & \frac{1}{2}\leqslant t\leqslant 1,\end{cases}$$

where $\tau:[0,1]\to[0,1]$ is a smooth monotone mapping such that $\tau(s)=0$ for $0\leqslant s\leqslant\frac{1}{3}$ and $\tau(s)=1$ for $\frac{2}{3}\leqslant s\leqslant 1$. We will write

$$\varphi_t = \phi_t * \psi_t.$$

Let $\operatorname{Diff}(X)$ denote the space of all diffeomorphisms of a smooth manifold X with the C^1-topology. If X is an orientable manifold, then $\operatorname{Diff}_+(X)$ and $\operatorname{Diff}_-(X)$ denote the sets of all orientation-preserving and orientation-reversing diffeomorphisms, respectively, and for any subset $A(X)\subset\operatorname{Diff}(X)$ we set $A_\pm(X)=A(X)\cap\operatorname{Diff}_\pm(X)$.

Proposition 1 (Thom's isotopy extension theorem [7], Theorem 5.8). *Let Y be a smooth manifold without boundary, X be a smooth compact submanifold of Y, and $\{\phi_t : X\to Y, t\in[0,1]\}$ be a smooth arc such that ϕ_0 is the inclusion map of X into Y. Then, for any compact set $Z\subset Y$ containing the set $\bigcup_{t\in[0,1]}\phi_t(X)$, there exists a smooth arc $\{\varphi_t\}\subset\operatorname{Diff}(Y)$ such that $\varphi_0=id$, $\varphi_t|_X=\phi_t|_X$ for every $t\in[0,1]$, and $\varphi_t|_{Y\setminus Z}=id$.*

Proposition 2 ([8], Lemma de fragmentation). *Let $U=\{U_j\}$ be an open covering of a closed manifold X and $\varphi : X\to X$ be a diffeomorphism diffeotopic to the identity map. Then φ can be decomposed into a composition of finitely many diffeomorphisms diffeotopic to the identity map,*

$$\varphi = \phi_q\ldots\phi_2\phi_1,$$

such that $\operatorname{supp}\{\phi_{i,t}\}\subset U_{j(i)}, i\in\{1,\ldots,q\}$, where $U_{j(i)}\in U$ and $\{\phi_{i,t}\}$ is a smooth arc connecting the identity map to the diffeomorphism ϕ_i.

3. MAPPING CLASS GROUPS

The *mapping class group of a topological space X* is the group of equivalence classes of homeomorphisms of X up to isotopy. If X is a smooth manifold, then the equivalence class group of diffeomorphisms of X up to diffeotopy is denoted by $\pi_0(\mathrm{Diff}(X))$.

Proposition 3 (see [1]). *The mapping class group of the sphere satisfies* $\pi_0(\mathrm{Diff}(\mathbb{S}^2)) \cong \mathbb{Z}_2$. *Here, the classes coincide with the sets* $\mathrm{Diff}_+(\mathbb{S}^2)$ *and* $\mathrm{Diff}_-(\mathbb{S}^2)$*, respectively.*

To prove the main result, we will also need the mapping class groups of the two-dimensional torus $\mathbb{T}^2$ and the Klein bottle $\mathbb{K}^2$.

Proposition 4 (see [9]). *The mapping class group of the two-dimensional torus satisfies* $\pi_0(\mathrm{Diff}(\mathbb{T}^2)) \cong GL(2,\mathbb{Z})$. *Here, the classes coincide with the sets* $\{h \in \mathrm{Diff}(\mathbb{T}^2) : h_* = A \in GL(2,\mathbb{Z})\}$.

To describe a representative of each class in $\pi_0(\mathrm{Diff}(\mathbb{K}^2))$, we represent $\mathbb{K}^2$ as a quotient space $C/_\sim$, where $C = \{(e^{i2\pi\theta}, t) : \theta \in [0,1], 0 \leqslant t \leqslant 1\}$ and $\sim$ is a minimum equivalence relation satisfying the condition

$$(e^{i2\pi\theta}, 0) \sim (e^{i2\pi(1-\theta)}, 1).$$

Let $p : C \to \mathbb{K}^2$ be the natural projection. The diffeomorphisms $\bar{\alpha}, \bar{\beta} : C \to C$ are defined by the formulas

$$\bar{\alpha}(e^{i2\pi\theta}, t) = (e^{i2\pi\theta}, 1 - t),$$

$$\bar{\beta}(e^{i2\pi\theta}, t) = \begin{cases} (e^{i2\pi\theta}, t), & 0 \leqslant t \leqslant \dfrac{1}{2}, \\ (e^{i(2\pi\theta + 4\pi t)}), t) & \dfrac{1}{2} < t \leqslant 1. \end{cases} \quad (1)$$

Let $\alpha = p\bar{\alpha}p^{-1}, \beta = p\bar{\beta}p^{-1} : \mathbb{K}^2 \to \mathbb{K}^2$.

Proposition 5 (see [10]). *The mapping class group of the Klein bottle satisfies* $\pi_0(\mathrm{Diff}(\mathbb{K}^2)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. *Here, representatives of each of the four classes are the diffeomorphisms* $id, \alpha, \beta, \alpha\beta$*, respectively.*

4. LOCALLY MODEL DIFFEOMORPHISMS

Recall that $\mathrm{Diff}(\mathbb{S}^2)$ denotes the set of all diffeomorphisms of the two-dimensional sphere

$$\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

By Proposition 3, the group $\pi_0(\mathrm{Diff}(\mathbb{S}^2))$ consists of two equivalence classes, $\mathrm{Diff}_+(\mathbb{S}^2)$ and $\mathrm{Diff}_-(\mathbb{S}^2)$, of orientation-preserving and orientation-reversing diffeomorphisms of the 2-sphere, respectively.

Let $J(\mathbb{S}^2) \subset \mathrm{Diff}(\mathbb{S}^2)$ denote the set of all source–sink diffeomorphisms and $NS(\mathbb{S}^2) \subset J(\mathbb{S}^2)$ denote those of them having a source and a sink at the north pole $N(0,0,1)$ and the south pole $S(0,0,-1)$, respectively.

The *model* diffeomorphism $g_\pm \in NS_\pm(\mathbb{S}^2)$ is defined by the formula

$$g_\pm(x_1, x_2, x_3) = \left(\frac{4x_1}{5 - 3x_3}, \frac{4x_2}{\pm(5 - 3x_3)}, \frac{5x_3 - 3}{5 - 3x_3}\right).$$

Note that on $\mathbb{S}^2 \setminus \{N\}$ the diffeomorphism $g_\pm$ is smoothly conjugate to a linear diffeomorphism of the plane, $\bar{g}_\pm \in \mathrm{Diff}_\pm(\mathbb{R}^2)$, defined by the formula

$$\bar{g}_\pm(x_1, x_2) = \left(\frac{x_1}{2}, \pm\frac{x_2}{2}\right).$$

Specifically, $\bar{g}_\pm = \vartheta_N g_\pm \vartheta_N^{-1}$, where $\vartheta_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$ is the stereographic projection defined by the formula

$$\vartheta_N(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}\right). \quad (2)$$

A diffeomorphism $h \in NS(\mathbb{S}^2)$ is called a *locally model* diffeomorphism of the sphere $\mathbb{S}^2$ if there are neighborhoods U_h^N, U_h^S of the points N, S for which $h|_{U_h^N \cup U_h^S} = g|_{U_h^N \cup U_h^S}$, where $g \in \{g_+, g_-\}$. Let $E_g \subset NS(\mathbb{S}^2)$ denote the set of locally model diffeomorphisms of the 2-sphere.

Lemma 1. *For any diffeomorphism* $h \in E_g$ *there exists a unique homeomorphism* $\gamma_h : \mathbb{S}^2 \to \mathbb{S}^2$ *with the following properties:*

- $\gamma_h h = g\gamma_h$;
- $\gamma_h|_{U_h^N} = id$;
- $\gamma_h|_{\mathbb{S}^2 \setminus S}$ *is a diffeomorphism.*

Proof. Since any diffeomorphism $h \in E_g$ coincides with g in a neighborhood U_h^N of the point N, we set $\gamma_h|_{U_h^N} = id$. Since γ_h makes h conjugate to g on the entire sphere $\mathbb{S}^2$, we have

$$\gamma_h h^k(x) = g^k \gamma_h(x), \quad x \in \mathbb{S}^2, \quad k \in \mathbb{Z}. \quad (3)$$

For any point $x \in \mathbb{S}^2 \setminus \{S\}$, there exists $k \in \mathbb{Z}$ such that $h^{-k}(x) \in U_h^N$; therefore,

$$\gamma_h(x) = g^k h^{-k}(x), \quad x \in \mathbb{S}^2 \setminus \{S\}. \quad (4)$$

Constructed by continuity, the diffeomorphism extends to the point S by the condition $\gamma_h(S) = S$.

Let $V = \mathbb{S}^2 \backslash \{N, S\}$. Denote by $\hat{V}_g$ the space of orbits of the action of g on V, and let $p_g : V \to \hat{V}_g$ denote the natural projection. By construction, the surface $\hat{V}_g$ is homeomorphic to the Klein bottle $\mathbb{K}^2$ if $g = g_-$ and homeomorphic to the torus $\mathbb{T}^2$ if $g = g_+$. Let $a = \vartheta_N^{-1}(Ox_1)$ and $b = \vartheta_N^{-1}(\mathbb{S}^1)$ be curve on V. By construction, the closed curve b is a generator of the fundamental group $\pi_1(V)$. The generators of the fundamental group $\pi_1(\hat{V}_g)$ are defined as

$$\hat{a}_g = p_g(a), \quad \hat{b}_g = p_g(b). \tag{5}$$

The natural projection $p_g : V \to \hat{V}_g$ induces an epimorphism $\eta_g : \pi_1(\hat{V}_g) \to \mathbb{Z}$ as follows. Let $\hat{c}$ be a loop in $\hat{V}_g$ such that $\hat{c}(0) = \hat{c}(1) = \hat{x}_0$. By the monodromy theorem (see, e.g., [11]), there is a unique path c in V starting at the point $x_0 = c(0) \in p_g^{-1}(\hat{x}_0)$ that is a lift of the path $\hat{c}$. Therefore, there exists a unique $k \in \mathbb{Z}$ such that $x_1 = c(1) = g^k(x_0)$ and the mapping η_g given by the formula $\eta_g([\hat{c}]) = k$ is well defined (i.e., does not depend on the choice of a loop in the class $[\hat{c}]$). By construction,

$$\eta_g([\hat{a}_g]) = 1, \quad \eta_g([\hat{b}_g]) = 0. \tag{6}$$

For any $r \in \mathbb{R}$, let $\bar{B}_r = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le r^2\}$, $\bar{K}_r = \mathrm{cl}(\bar{B}_r \backslash \bar{B}_{\frac{r}{2}})$, and $B_r = \vartheta_N^{-1}(\bar{B}_r), K_r = \vartheta_N^{-1}(\bar{K}_r)$. Then, for any diffeomorphism $h \in E_g$, there is a real number $r_h > 0$ such that $h|_{B_{r_h}} = g|_{B_{r_h}}$. It follows that

$$\gamma_h g|_{B_{r_h}} = g\gamma_h|_{B_{r_h}}. \tag{7}$$

Relation (7) uniquely defines a diffeomorphism $\nu_h : V \to V$ commuting with the diffeomorphism g

$$\nu_h g = g\nu_h \tag{8}$$

and coinciding with γ_h on B_{r_h}, i.e.,

$$\nu_h|_{B_{r_h}} = \gamma_h|_{B_{r_h}}. \tag{9}$$

Then (see, e.g., [11], Theorem 5.5), there exists a unique orientation-preserving diffeomorphism $\hat{\nu}_h : \hat{V}_g \to \hat{V}_g$ for which ν_h is covering, i.e.,

$$\hat{\nu}_h p_g = p_g \nu_h. \tag{10}$$

The following lemma describes the action of the resulting diffeomorphism on the generators.

Lemma 2. *The diffeomorphism $\hat{\nu}_h$ induces an isomorphism $\hat{\nu}_{h*} : \pi_1(\hat{V}_g) \to \pi_1(\hat{V}_g)$ with the following properties*:

1. $\eta_g \hat{\nu}_{h*}([\hat{a}_g]) = 1$;
2. $\hat{\nu}_{h*}([\hat{b}_g]) = [\hat{b}_g]$.

Proof. From the definition of the epimorphism η_g and formula (10), it follows straightforwardly that

$$\eta_g \hat{\nu}_{h*}([\hat{c}]) = \eta_g([\hat{c}]). \tag{11}$$

Then $\eta_g \hat{\nu}_{h*}([\hat{a}_g]) = \eta_g([\hat{a}_g])$ and, hence, in view of (6), $\eta_g \hat{\nu}_{h*}([\hat{a}_g]) = 1$. Since $\pi_1(V) = \langle b \rangle = \{b^n : n \in \mathbb{Z}\}$, we have $\nu_{h*}([b]) = [b]$, whence, in view of (5) and (11), $\hat{\nu}_{h*}[\hat{b}_g] = [\hat{b}_g]$.

Let $h \in E_g$, and let $w : \mathbb{S}^2 \to \mathbb{S}^2$ be a diffeomorphism of the sphere that is the identity outside some ring $K_r, r < \frac{r_h}{2}$. We set

$$\hat{w} = p_g w (p_g|_{K_r \backslash \partial B_r})^{-1}(\hat{x}). \tag{12}$$

By construction, $wh \in E_g$, and the following lemma provides the relation between the diffeomorphisms $\hat{\nu}_{wh}$ and $\hat{\nu}_h$.

Lemma 3. $\hat{\nu}_{wh} = \hat{\nu}_h \hat{w}^{-1}$.

Proof. It follows from formula (10) that

$$\hat{\nu}_{wh}(\hat{x}) = p_g \nu_{wh} (p_g|_{K_r \backslash \partial B_r})^{-1}(\hat{x}), \quad \hat{x} \in \hat{V}_g. \tag{13}$$

Then $x = (p_g|_{K_r \backslash \partial B_r})^{-1}(\hat{x}) \in K_r$. It is directly verified that the diffeomorphisms wh and h coincide on the entire sphere $\mathbb{S}^2$, except for the interior of the ring K_{2r}; therefore, $r_{wh} = r_h$ and $U_{wh}^N = U_h^N$. Then formulas (9) and (13) imply that

$$\hat{\nu}_{wh}(\hat{x}) = p_g \gamma_{wh}(x). \tag{14}$$

Let $k \in \mathbb{Z}$ be a number such that $(wh)^{-k}(x) \in U_h^N$. Then formula (4) yields

$$\gamma_{wh}(x) = g^k (wh)^{-k}(x). \tag{15}$$

Since the diffeomorphisms $(wh)^{-1}$ and h^{-1} coincide on the entire sphere $\mathbb{S}^2$, except for the interior of the ring K_r, we have

$$(wh)^{-k} = h^{-k} w^{-1}. \tag{16}$$

Substituting (16) into (15) and taking into account formula (9), we obtain

$$\gamma_{wh}(x) = g^k h^{-k} w^{-1}(x) = \gamma_h w^{-1}(x) = \nu_h w^{-1}(x). \tag{17}$$

Substituting (17) into (14) and taking into account formulas (10) and (12), we derive

$$\hat{\nu}_{wh}(\hat{x}) = p_g \nu_h w^{-1}(x)$$
$$= p_g \nu_h (p_g|_{K_r \backslash \partial B_r})^{-1} p_g w^{-1} (p_g|_{K_r \backslash \partial B_r})^{-1}(\hat{x}) = \hat{\nu}_h \hat{w}^{-1}(\hat{x}).$$

□

Lemma 4. *Any diffeomorphism $h \in E_g$ is connected by a smooth arc $\{\phi_t\} \subset NS(\mathbb{S}^2)$ with the diffeomorphism g.*

Proof. Lemma 1 implies that a homeomorphism γ_h coinciding on B_{r_h} with a diffeomorphism $\nu_h : V \to V$ is a diffeomorphism everywhere, except possibly the point S. If ν_h extends smoothly to S so that $\nu_h(S) = S$, then, by Proposition 3, there exists a smooth arc $\{\rho_t\} \subset \mathrm{Diff}_+(\mathbb{S}^2)$ such that $\rho_0 = \nu_h$, $\rho_1 = \mathrm{id}$. Then the sought arc ϕ_t is given by the formula

$$\phi_t = \rho_t^{-1} g \rho_t.$$

The case where the mapping ν_h is not smooth at S is divided into two subcases, depending on the diffeotopy class of $\hat{\nu}_h : \hat{V}_g \to \hat{V}_g$: (I) $\hat{\nu}_h$ is diffeotopic to the identity map, and (II) $\hat{\nu}_h$ is not diffeotopic to the identity map.

In case (I), following the line of reasoning used above, it suffices to construct an arc h_t connecting h with a diffeomorphism h_1 such that ν_{h_1} is a diffeomorphism.

Consider an open covering $U = \{U_1, \cdots, U_q\}$ of the orbit space $\hat{V}_g$ such that $p_g^{-1}(U_i) \subset K_{r_i}$ for some $r_i \in \mathbb{R}$. By Lemma 2, there exists a decomposition of $\hat{\nu}_h$ into a composition of finitely many diffeomorphisms

$$\hat{\nu}_h = \hat{w}_q \ldots \hat{w}_2 \hat{w}_1$$

that are diffeotopic to the identity map and such that $\mathrm{supp}\{\hat{w}_{i,t}\} \subset U_{j(i)}, i \in \{1, \ldots, q\}$, where $U_{j(i)} \in U$ and $\{\hat{w}_{i,t}\}$ is a smooth arc connecting the identity map to the diffeomorphism $\hat{w}_i$.

Let $w_{i,t} : \mathbb{S}^2 \to \mathbb{S}^2$ be a diffeomorphism of the sphere that is the identity map outside the ring $K_{r_{j(i)}}$ and is defined on $K_{r_{j(i)}}$ by the formula $w_{i,t}(x) = (p_g|_{K_{r_{j(i)}} \setminus \partial B_{r_{j(i)}}})^{-1} \hat{w}_{i,t} p_g(x)$. Without loss of generality, we assume that the values $r_{j(i)}$ are such that $r_{j(i+1)} < \frac{r_{j(i)}}{2}$, $i = 1, \ldots, q$ and $r_{j(1)} < \frac{r_h}{2}$. Let us show that $h_t = w_{q,t} \ldots w_{1,t} h : \mathbb{S}^2 \to \mathbb{S}^2$ is the sought arc.

Indeed, by construction, $h_t \in E_g$ for any $t \in [0,1]$. By Lemma 3, $\hat{\nu}_{h_1} = \hat{\nu}_h \hat{w}_1^{-1} \hat{w}_2^{-1} \ldots \hat{w}_q^{-1} = \mathrm{id}$. This implies that $\nu_{h_1} = g^n$ for some $n \in \mathbb{N}$; therefore, ν_{h_1} is a diffeomorphism.

In case (II), following the line of reasoning used above, it suffices to construct an arc h_t connecting h with a diffeomorphism h_1 such that $\hat{\nu}_{h_1}$ is diffeotopic to the identity map. The following two cases are possible: (i) $g = g_+$ and (ii) $g = g_-$.

In case (i), $\hat{V}_g \cong \mathbb{T}^2$. Proposition 4 and Lemma 2 imply that the isomorphism induced by the diffeomorphism $\hat{\nu}_h$ is defined by the matrix $\begin{pmatrix} 1 & n_0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Z})$, where $n_0 \neq 0$. For a fixed $r < \frac{r_h}{2} \in \mathbb{R}$, the diffeomorphism $\mu : \mathbb{R} \to [0,1]$ is defined by the formula

$$\mu(x) = \begin{cases} 0, & x \geqslant r, \\ 1 - \left(1 + e^{\frac{r^3\left(\frac{3r}{4} - x\right)}{8\left(x - \frac{r}{2}\right)^2 (x - r)^2}}\right)^{-1}, & \frac{r}{2} < x < r. \\ 1, & x \leqslant \frac{r}{2} \end{cases}$$

In the plane $\mathbb{R}^2$ we introduce polar coordinates (ρ, φ) and define a diffeomorphism $\bar{\theta}_{n_0,t} : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$\bar{\theta}_{n_0,t}(\rho e^{i\varphi}) = \rho e^{i(\varphi + 2n_0 \pi t \mu(\rho))}.$$

Let $\theta_{n_0,t} = \vartheta_N^{-1} \bar{\theta}_{n_0,t} \vartheta_N : \mathbb{S}^2 \setminus \{S\} \to \mathbb{S}^2 \setminus \{S\}$. Then $\theta_{n_0,t}$ smoothly extends to $\mathbb{S}^2$ by the condition $\theta_{n_0,t}(S) = S$ and $h_t = \theta_{n_0,t} h$ is the sought arc.

In case (ii), $\hat{V}_g \cong \mathbb{K}^2$. Then Proposition 5 and Lemma 2 imply that the isomorphism induced by the diffeomorphism $\hat{\nu}_h$ belongs to the class of the mapping β. Then $h_t = \theta_{1,t} h$ is the sought arc (the figure shows the curves $a_t = \bigcup_{n \in \mathbb{N}} h_t^n (a \cap U_h^N)$ for $t = 0, \frac{1}{2}, 1$).

5. PROOF OF THEOREM 1

Consider orientation-preserving (orientation-reversing) source–sink diffeomorphisms $f, f' \in J(\mathbb{S}^2)$. Let us show that there exists an arc connecting f, f'

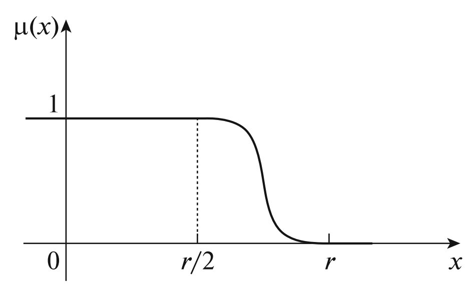

Fig. 1. Graph of the function $\mu(x)$.

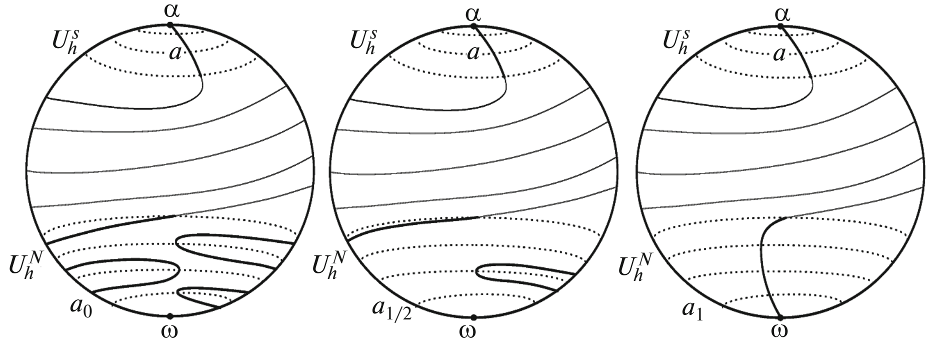

Fig. 2. Curves $a_t = \bigcup_{n\in\mathbb{N}} h_t^n(a \cap U_h^N)$ for $t = 0, \frac{1}{2}, 1$.

that consists entirely of $J(\mathbb{S}^2)$ diffeomorphisms. For this purpose, in the lemma below, we construct arcs connecting f, f' with locally model diffeomorphisms $h_f, h_{f'} \in E_{g_+}$ ($h_f, h_{f'} \in E_{g_-}$). Then the sought arc is the product of the constructed smooth arcs and the arcs connecting $h_f, h_{f'}$ with the model diffeomorphism g_+ (g_-); the existence of $h_f, h_{f'}$ follows from Lemma 4.

Lemma 5. *Any diffeomorphism $f \in J(\mathbb{S}^2)$ is connected by a smooth arc $\{\phi_t\} \subset NS(\mathbb{S}^2)$ with a diffeomorphism $h \in E_g$.*

Proof. Assume that $f \in J(\mathbb{S}^2)$ and the non-wandering set of f consists of a source α and a sink ω. According to [12], there exists a smooth arc $\{H_t \in \mathrm{Diff}_+(\mathbb{S}^2)\}$ with the following properties: $H_0 = \mathrm{id}$, $H_1(N) = \alpha$, and $H_1(S) = \omega$. Then $H_t^{-1} f H_t$ is a smooth arc connecting f with the diffeomorphism $H_1^{-1} f H_1 \in NS(\mathbb{S}^2)$.

In view of what was said above, without loss of generality, we assume that $f \in NS(\mathbb{S}^2)$. Then, to prove the lemma, it suffices to construct an arc $\{\phi_t\} \subset NS(\mathbb{S}^2)$ connecting $f \in NS(\mathbb{S}^2)$ with a diffeomorphism $h \in E_g$. We show how to construct an arc $\{\phi_t^S\} \subset NS(\mathbb{S}^2)$ connecting $f \in NS(\mathbb{S}^2)$ with a diffeomorphism $h_S \in NS(\mathbb{S}^2)$ that coincides with f in a neighborhood of the pole N and with g in a neighborhood of the pole S. An arc $\{\phi_t^N\} \subset NS(\mathbb{S}^2)$ connecting h_S with $h \in E_g$ is constructed similarly. Then the sought arc is $\{\phi_t = \phi_t^S * \phi_t^N\}$.

To construct the arc $\{\phi_t^S\}$, we set $\bar{f} = \vartheta_N f \vartheta_N^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$. Then the diffeomorphism $\bar{f}$ is a contraction to a hyperbolic point O. By the Franks lemma [13, 14], we may assume that the diffeomorphism $\bar{f}$ coincides in a neighborhood of O with a linear mapping $\bar{Q} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by a matrix Q that is either diagonal or has the form $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, where $0 < \alpha^2 + \beta^2 < 1$. If $\bar{Q} = \bar{g}$, then the lemma is proved. Otherwise, by Proposition 5.4 from [15], there exists an arc $\bar{Q}_t : \mathbb{R}^2 \to \mathbb{R}^2$ consisting of linear contractions to O defined by matrices Q_t such that $\bar{Q}_t(\bar{B}_r) \subset \mathrm{int}\,\bar{B}_r$ for any $r > 0$ and $\bar{Q}_0 = \bar{Q}, \bar{Q}_1 = \bar{g}$. Consider an arc $\{\xi_t = \bar{Q}_t \bar{Q}^{-1}\} \subset \mathrm{Diff}_+(\mathbb{R}^2)$ connecting the identity map $\xi_0 = id$ with the diffeomorphism $\xi_1 = \bar{g}\bar{Q}^{-1}$. Let $r_1 > r_2$ be positive numbers such that

$$\bar{f}|_{\bar{B}_{r_1}} = \bar{Q}\,|_{\bar{B}_{r_1}}, \quad \bar{Q}(\bar{B}_{r_1}) \subset \bar{B}_{r_2}.$$

Then, by Proposition 1, there exists an arc $\{\Xi_t\} \subset \mathrm{Diff}_+(\mathbb{R}^2)$ such that $\Xi_0 = id$, $\Xi_t|_{\bar{Q}(\bar{B}_{r_1})} = \xi_t|_{\bar{Q}(\bar{B}_{r_1})}$, and $\Xi_t|_{\mathbb{R}^2 \setminus \bar{B}_{r_1}} = \mathrm{id}|_{\mathbb{R}^2 \setminus \bar{B}_{r_1}}$. Then $\{\bar{\phi}_t^S = \Xi_t \bar{f}\}$ is the sought arc.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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