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On the Existence of Nash-stable Partition in Leader's Coalition Games

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Abstract

This paper investigates two approaches to determining the leader of a coalition partition: the individual and the collective. In the first approach, each coalition in the partition chooses a representative, and then the leader is chosen from among all the representatives. In the second approach, the leading coalition in the partition is chosen, and then the leader from among members of that coalition is chosen. The leader and the leading coalition are chosen with a certain probability, which is guided by the weight rule or the ranking rule. Both approaches can be encountered in contests, sports competitions, and political elections. The paper delivers results on the existence of Nash-stable partitions depending on the approach and the probability of determining the leader. Cases where the number of coalitions in the partition is fixed and arbitrary are studied. The existence of an equilibrium in weakly dominant strategies is proved for the collective approach and the weight rule, and the necessary and sufficient conditions for a Nash-stable partition to exist were found for the ranking rule. The sufficient conditions for a Nash-stable partition to exist were found for the individual approach and the corresponding probabilistic rules.

Keywords: coalition formation, leader problem, Nash stability

1. Introduction

Agents can form coalitions with each other to achieve their goals in many spheres. Collaboration is a way to get a result an agent cannot accomplish alone. This paper examines the problem of coalition partitioning where each agent is interested in becoming the leader among all agents. Two procedures for determining the leader are examined. Each of the procedures begins with agents breaking up into groups and consists of two steps.

The first procedure for determining the leader is the following. A candidate for the leadership position is selected within each group of agents; the leader is then chosen from among these candidates. In the second procedure, the leading group is chosen and then the leader from among members of that group is chosen. The choice of the leadership candidates, the leading coalition, and the leader among candidates happens with a certain probability. We term the first and the second procedures as the individual and the collective models for determining the leader of the coalition partition, respectively (Fig. 1).

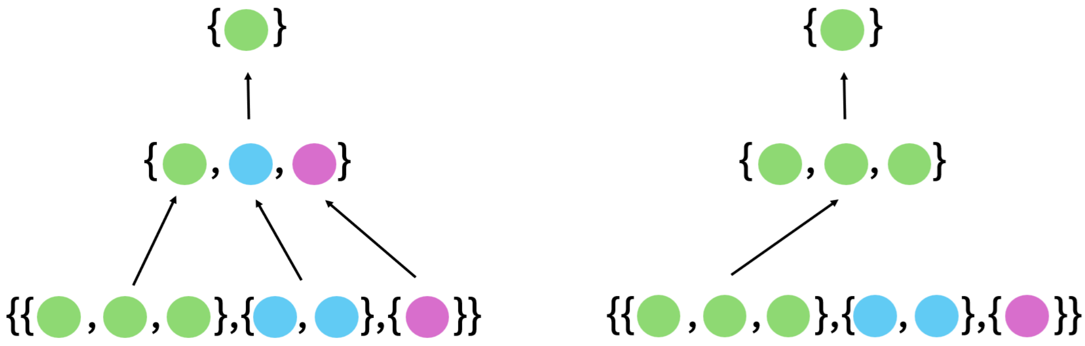

Fig. 1. The individual (left) and the collective (right) models of determining the leader

These models for determining the leader apply to a wide range of human activities. For instance, the individual model often occurs in multiple branch organizations. First, a representative is chosen for each branch, and then the principal representative is appointed from among branch representatives. The collective model of determining the leader is applied in political elections. In the first step, the leading party is determined and in the second step, one of its members is elected to become the leader. Both the collective and the individual models are applied in sports.

Leadership in the economy, politics, or sports largely depends on the agents themselves, but in many cases it is impossible to tell exactly who will become the leader. It is not uncommon that a potential leader is left behind or that luck is on a novice's side. In our study, any agent can become the leader of a coalition partition with some probability.

Suppose we have decided on a model for determining the leader and agents are asked to break up into coalitions. This gives rise to strategic behavior in agents. Each of them is interested in joining a coalition within which they can get into the second round and become the leader at the second step. What kind of a partition will we get depending on the leadership choice model? In the individual model, for instance, each strong agent, i.e. an agent with high weight, hierarchical status, would want to team up with the weakest agents at the first step. In that case, the strong agent has a higher probability of getting into the second round. Realizing this, weak agents may team up with each other in order to raise their individual probabilities of getting into the second round. In the collective model of leadership, however, weak candidates would want to team up with strong agents to ensure that their whole team gets into the second round. Strong agents are aware of that and can team up with other strong agents. In this case, however, the second step will be played among strong agents only and becoming the leader will be a challenge. It is not obvious how candidates should band together depending on the this model. How do agents cooperate with each other depending on the model for determining the leader? How does the probability distribution influence the coalition partition? To answer these questions, we investigate the existence of stable coalition partitions depending on the probability of winning each of the two steps and on the leadership choice model.

If a stable partition does exist, then the impact of the leadership choice model on the cooperation of agents will be reflected in its form. Based on the model-specific form of stable partitions, we can figure out which model can be used in sports events, elections, etc. It turns out there are partitions that are stable in different leadership choice models under some of the probabilities of winning.

1.1 The weight and ranking rules

This article investigates the existence of a stable partition of agents into coalitions for the individual and the collective leadership choice models. This question is approached by using game theory. Collective and individual coalition partition games are introduced. Each of the games models the respective procedure for determining the leader. The player's payoff in the individual and the collective games is their expected chances of becoming the leader. The games are formally defined in Section 2.

We consider cases where the probability of a player becoming the coalition's leader and the probability of a coalition becoming the leading one in the coalition partition are calculated according to the weight and the ranking rules. In the following, we describe these rules.

Let $N = \{1, 2, ..., n\}$ be the set of players. For the weight rule, the probability that the player i becomes the leader in the coalition $K, K \subseteq N, i \in K$ is

$$p_i(K) = \frac{w_i}{\sum\limits_{j \in K} w_j},$$

where $w_j > 0$ is the weight of the player j.

The ranking rule implies there is a rank order of candidates $1 \succ 2 \succ ... \succ n$, and the probability that the player i is the leader in the coalition K is

$$p_i(K) = \frac{\lambda_{R_i(K)}}{\sum\limits_{j=1}^{|K|} \lambda_j},$$

where $\lambda_1 > \lambda_2 > ... > \lambda_n > 0$ and $R_i(K) = 1 + \#\{j | j \succ i, j \in K\}$. The number $R_i(K)$ is the player's ranking in the game K. Note that $i \succ j \Leftrightarrow R_i(K) < R_j(K)\ \forall K \subseteq N$. We say here that player i has a *higher ranking* than player j. The higher the player's ranking in the coalition, the higher the chance of them becoming the leader. Our motivation for taking the ranking rule into consideration is (Dietzenbacher & Kondratev, 2023), in which the ranking rule is applied to sports.

The probability of the coalition K being the leading one in the partition $\pi = \{B_1, B_2, ..., B_m\}$, $K \in \pi, \cup_{j=1}^m B_j = N, B_j \cap B_l = \emptyset, 1 \leq j < l \leq m$ for the weight and the ranking rules, respectively, is determined by the following formulas,

$$p_K(\pi) = \frac{\sum\limits_{j \in K} w_j}{\sum\limits_{j \in N} w_j}, \quad p_K(\pi) = \frac{\sum\limits_{j=1}^{|K|} \lambda_j}{\sum\limits_{B \in \pi} \sum\limits_{j=1}^{|B|} \lambda_j}.$$

We substitute the formulas for $p_i(K)$ and $p_K(\pi)$ into the payoff functions for players of the individual and the collective games. The resultant games are designated as follows: weighted collective game; weighted individual game; ranking collective game; and the ranking individual game. Table 2 gives the corresponding payoff functions. Depending on the leadership choice model and the probabilistic rule, we get different results regarding the existence of stable partitions.

1.2 Literature review

Our study is closely related to (Razin & Piccione, 2009), which examines a model in which social order is built by ranking coalitions of the partition and players within coalitions. The greater the coalition's rank weight, the higher its position in the social order. Similarly, the greater the player's rank weight, the higher their position in the coalition. Like in (Razin & Piccione, 2009), the rank weight is determined for each player and coalition in our ranking collective game. The difference is that a coalition and a player within a coalition can occupy the top position in the social order with a certain probability which is expressed via rank weights. In other words, a player in the model in (Razin & Piccione, 2009) maximizes their societal status, while a player in the ranking collective game maximizes one's chances of leading the coalition partition. In the model from (Razin & Piccione, 2009), not all players can attain the top position in the social order, whereas in the ranking collective game any player can with certain probability become the leader.

The problem of the coalition partition leader is similar in substance to the problem from contest theory (Corchón, 2007; Tullock, 2008). As in contest theory, we set the leadership probability based on the weight rule (Dasgupta & Nti, 1998; Ewerhart, 2017) and record the expected chances of the player becoming the leader of the coalition partition. The difference between these problems is that researchers of contest theory focus on finding the agents' efforts, while the question of interest in coalition formation theory is the existence of stable partitions. Papers on alliance formation in contests (Garfinkel, 2004) combine the research interests of contest theory and coalition formation theory.

Jandoc & Juarez, 2017 consider a resource division model where players first break into coalitions and then compete with each other over a resource. The winning coalition gets the entire resource, but in the next step players are eliminated and the players' powers are recalculated depending on the share of the resource. As in (Jandoc & Juarez, 2017), players who do not make it into the second round of the individual and the collective games drop out. The differences are that our games are two-step games, the players' weights are not re-calculated, and the player's payoff is their expected chance of becoming the leader. The substantial distinction between the models in (Jandoc & Juarez, 2017) and the individual game is that the second round in the latter is not for coalitions but for coalition leaders.

The paper (Watts, 2007) proposes a coalition formation model where the player's payoff depends on their hierarchical status. A finite set of players is distributed among a fixed number of coalitions. The author explores the existence of a segregated equilibrium. Note that in our models it is sometimes beneficial for players with the highest and the lowest hierarchical status to stick together, so the question of segregation is irrelevant in our case. We also consider the case of a fixed number of coalitions. In contrast to our games, the game in (Watts, 2007) is a hedonic one.

In the games constructed here, we investigate the existence of a core partition and a Nash-stable partition. One of the first papers to introduce the concept of the stability of coalition structures in hedonic games is (Dreze & Greenberg, 1980). Some of the early papers on the existence of stable coalition structures are (Aumann & Hart, 1992; Greenberg & Weber, 1985). Greenberg & Weber, 1993 (p. 63) point out the complexity of confirming the existence of stable coalition structures: “there is only a relatively small number of results that guarantee the existence of a ‘stable’ coalition structure" . The similarities and distinctions of the types of stability of coalition structures are explicated in (Bogomolnaia & Jackson, 2002).

1.3 Results

The following results were obtained regarding the existence of a Nash-stable partition with fixed cardinality (NSPC) and a core partition (CP):

- For the strategic weighted collective game, the existence of an equilibrium in weakly dominant strategies is proved. In the case where the weights of all players are different, an equilibrium is shown to exist in dominant strategies, from which it follows obviously that an NSPC and a CP exist (see Theorem 1).
- For the ranking collective game, the necessary and sufficient conditions for the existence of an NSPC were determined and the existence of a CP was demonstrated (see Theorem 2).
- In the weighted individual game, sufficient conditions for the existence of an NSPCwere determined for the case where the number of coalitions is fixed, and a Nash-stable partition (NSP) is shown to exist where the number of coalitions is not fixed (see Theorem 3).
- For the ranking individual game in which the number of coalitions is fixed, the sufficient conditions for the existence of an NSPC were determined (see Theorem 4).

The main results are presented in Table 1.

	Collective	Individual
Weight rule	The existence of a weakly dominant strategy profile is proved (Theorem 1)	The sufficient conditions of an NSPC existence are found (Theorem 3)
Ranking rule	The necessary and sufficient conditions of an NSPC existence are found (Theorem 2)	The sufficient conditions of an NSPC and an NSP existence are found (Theorem 4)

Table 1: Main results

2. Games formulation

2.1 Coalition games

Let $N = \{1, 2, ..., n\}$ be the set of players and $\Pi(N)$ be the set of all partitions N. A non-empty subset of the set N is called a coalition. *A coalition partition game* is a pair (N, H) where $H : \Pi(N) \to \mathbb{R}^n$. The number $H_i(\pi)$, where $\pi \in \Pi(N)$ and $i \in N$, is the payoff of the player i in the coalition partition π. We write $\pi(i)$ to denote the coalition containing the player i in π, i.e. $i \in \pi(i) \in \pi$. *A transversal* of the partition $\pi = \{B_1, B_2, ..., B_m\}$ is the coalition $\{i_1, i_2, ..., i_m\}$, where $i_k \in B_k, \forall k \in \{1, 2, ..., m\}$. Let $M_i(\pi)$ be the set of all transversals of the partition π that contain the player i.

Let us consider two types of coalition partitions. The *harmonious partition* is a partition $\pi = \{B_1, ..., B_m\}$ where $\forall k \in \{1, ..., m\}\ \forall i \in B_k : (i-1) \mod m = k - 1$. In particular, let $|\pi| = 2$, then the harmonious partition is $\pi = \{O, E\} = \{\{1, 3, 5, ...\}, \{2, 4, 6, ...\}\}$. A partition of the form $\pi = \{\{1\}, \{2\}, ..., \{n\}\}$ will be denoted as a *degenerate partition.*

The notation $p_K(\pi), K \in \pi$ denotes the probability that the coalition K is the leading one in the partition π. The number $p_i(K), K \subseteq N, i \in K$ is the probability than the player i is the leader in the coalition K. It follows from the probability properties that $\forall \pi \in \Pi(N)$ we have

$$\forall K \in \pi : p_K(\pi) \geq 0, \sum_{K\in\pi} p_K(\pi) = 1 \text{ and}$$

$$\forall K \subseteq N, i \in K : p_i(K) \geq 0, \sum_{i\in K} p_i(K) = 1.$$

A collective game is a coalition partition game (N, H^C) in which the payoff of the player i is

$$H_i^C(\pi) = p_{\pi(i)}(\pi) \cdot p_i(\pi(i)).$$

In collective games, players break up into non-intersecting coalitions, i.e. form a certain partition π. The coalitions then play against each other and the leading coalition is determined.

The leader is determined among members of the leading coalition and that agent is recognized as the π leader. The payoff of a player in the collective game is their expected chances of leading the coalition partition.

An individual game is a coalition partition game (N, H^I) in which the payoff of the player i is

$$H_i^I(\pi) = \sum_{K \in M_i(\pi)} p_i(K) \cdot \prod_{j \in K} p_j(\pi(j)).$$

In the individual game, players are also distributed among non-intersecting coalitions. Then, a representative is chosen from each coalition and the leader of the coalition partition is determined among all the representatives.

The difference between the individual and the collective games is that they model different two-step procedures for determining the leader of the coalition partition π. Let us consider particular cases of the collective and the individual games.

Definition 1. A weighted collective game is a collective game (N, H^{WC}) in which the player's payoff is

$$H_i^{WC}(\pi) = \frac{w_i}{\sum\limits_{j \in N} w_j}.$$

Definition 2. A ranking collective game is a collective game (N, H^{RC}) in which the player's payoff is

$$H_i^{RC}(\pi) = \frac{\lambda_{R_i(\pi(i))}}{\sum\limits_{B \in \pi} \sum\limits_{j=1}^{|B|} \lambda_j}.$$

Definition 3. A weighted individual game is an individual game (N, H^{WI}) in which the player's payoff is

$$H_i^{WI}(\pi) = \frac{w_i}{\prod\limits_{B \in \pi} \sum\limits_{j \in B} w_j} \cdot \sum_{K \in M_i(\pi)} \frac{\prod\limits_{j \in K} w_j}{\sum\limits_{l \in K} w_l}.$$

Definition 4. A ranking individual game is an individual game (N, H^{RI}) in which the player's payoff is

$$H_i^{RI}(\pi) = \frac{\lambda_{R_i(\pi(i))}}{\sum\limits_{j=1}^{|\pi|} \lambda_j \cdot \prod\limits_{B \in \pi} \sum\limits_{j \in B} \lambda_j} \cdot \sum_{K \in M_i(\pi)} \lambda_{R_i(K)} \prod_{j \in K \setminus \{i\}} \lambda_{R_j(\pi(j))}.$$

Games from Definitions 1 and 3 are derived from the collective and the individual games, respectively:

$$p_i(K) = \frac{w_i}{\sum\limits_{j \in K} w_j}, \quad p_K(\pi) = \frac{\sum\limits_{j \in K} w_j}{\sum\limits_{j \in N} w_j},$$

and games from Definitions 2 and 4 are derived from the respective games:

$$p_i(K) = \frac{\lambda_{R_i(K)}}{\sum\limits_{j=1}^{|K|} \lambda_j}, \quad p_K(\pi) = \frac{\sum\limits_{j=1}^{|K|} \lambda_j}{\sum\limits_{B\in\pi} \sum\limits_{j=1}^{|B|} \lambda_j}.$$

We distinguish four games, the payoff functions for which are given in Table 2. Note that instead of players' payoffs in the weighted collective game, Table 2 gives the payoffs of players in the strategic weighted collective game described in Section 3.1.

	Collective	Individual
Weight rule	$u_i(s) = \frac{w_{is_i}}{\sum\limits_{j\in N} w_{js_j}}$	$H_i^{WI}(\pi) = \frac{w_i}{\prod\limits_{B\in\pi} \sum\limits_{j\in B} w_j} \cdot \sum\limits_{K\in M_i(\pi)} \frac{\prod\limits_{j\in K} w_j}{\sum\limits_{l\in K} w_l}.$
Rank rule	$H_i^{RC}(\pi) = \frac{\lambda_{R_i(\pi(i))}}{\sum\limits_{B\in\pi} \sum\limits_{j=1}^{\lvert B\rvert} \lambda_j}$	$H_i^{RI}(\pi) = \lambda_{R_i(\pi(i))} \frac{\sum\limits_{K\in M_i(\pi)} \lambda_{R_i(K)} \prod\limits_{j\in K\setminus\{i\}} \lambda_{R_j(\pi(j))}}{\sum\limits_{j=1}^{\lvert\pi\rvert} \lambda_j \cdot \prod\limits_{B\in\pi} \sum\limits_{j\in B} \lambda_j}$

Table 2: Players' payoff functions in the games in question.

Since every player is interested in becoming the leader of the coalition partition, players can migrate between coalitions to augment their payoffs. The question arises as to whether there exists a coalition partition that is stable. The main types of stability investigated in this paper are stated below.

Let $A, B \in \pi, A \neq B$. Denote $\pi_{-A} = \pi \setminus \{A\}, \pi_{-A,B} = \pi \setminus \{A, B\}$. Introduce the sets $D_i(\pi), i \in N$ consisting of coalition structures:

$$D_i(\pi) = \{\pi\} \cup \{\{\pi(i) \setminus \{i\}, A \cup \{i\}, \pi_{-\pi(i),A}\} \mid A = \emptyset \text{ } or \text{ } A \in \pi_{-\pi(i)}\}.$$

The coalition structure π is called a *Nash-stable partition* (NSP) in (N, H) if $\forall i \in N$:

$$H_i(\pi) - H_i(\rho) \geq 0 \; \forall \rho \in D_i(\pi).$$

Introduce the sets $\mathring{D}_i(\pi)$

$$\mathring{D}_i(\pi) = \{\pi\} \cup \{\{\pi(i) \setminus \{i\}, A \cup \{i\}, \pi_{-\pi(i),A}\} \mid \pi(i) \neq \{i\},\ A \in \pi_{-\pi(i)},\ A \neq \emptyset\}.$$

The coalition structure π is called a *Nash-stable partition with fixed cardinality* (NSPC) in

(N, H) if $\forall i \in N$:

$$H_i(\pi) - H_i(\rho) \geq 0 \ \forall \rho \in \mathring{D}_i(\pi).$$

2.2 Simple examples

Let us show how the players' payoffs in the above games can be calculated. Suppose $N = \{1, 2, 3, 4\}$, and the players are to break up into two coalitions. Then, the possible coalition partitions are $\pi_1 = \{\{1, 2\}, \{3, 4\}\}$; $\pi_2 = \{\{1\}, \{2, 3, 4\}\}$; $\pi_3 = \{\{2\}, \{1, 3, 4\}\}$ etc.

In the weighted collective game with the weights $w_1 = 4, w_2 = 3, w_3 = 2, w_4 = 1$, the payoff of the player $i = 1$ in the coalition structure $\pi_1 = \{\{1, 2\}, \{3, 4\}\}$ is

$$H_1^{WC}(\pi_1) = \frac{w_1}{w_1 + w_2 + w_3 + w_4} = \frac{4}{10}.$$

In the weighted individual game with analogous weights, the payoff of the first player in the coalition structure π_1 will be

$$H_1^{WI}(\pi_1) = \frac{w_1}{(w_1 + w_2)(w_3 + w_4)} \cdot \left(\frac{w_3 \cdot w_1}{w_3 + w_1} + \frac{w_4 \cdot w_1}{w_4 + w_1} \right) = \frac{128}{315}.$$

Note that the coalition structure π_1 is an NSPC (see Statement 3, point 2).

In the ranking collective game with the ranking constants $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 2, \lambda_4 = 1$ in the coalition structure π_1, that player's payoff is

$$H_1^{RC}(\pi_1) = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_1 + \lambda_2} = \frac{2}{7}.$$

This partition is not an NSPC, since it is more profitable for the player $i = 1$ to move to the coalition $\{3, 4\}$.

In the ranking individual game with analogous ranking constants, we get

$$H_1^{RI}(\pi_1) = \frac{\lambda_1}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2)} \cdot (\lambda_1 \cdot \lambda_1 + \lambda_2 \cdot \lambda_1) = \frac{16}{49}.$$

The payoffs of all players in the above games for the partition π_1 are given in Table 3.

The player	Weighted Collective	Weighted Individual	Rank Collective	Rank Individual
1	0.4	0.4063	0.2857	0.3265
2	0.3	0.2786	0.2143	0.2449
3	0.2	0.2413	0.2857	0.2449
4	0.1	0.0738	0.2143	0.1837

Table 3: Payoffs of players in the partition $\{\{1, 2\}, \{3, 4\}\}$.

3. Existence of Nash-stable partitions

3.1 Weighted collective game

According to Definition 1, in the weighted collective game, the payoff of the player $i \in N$ in the partition $\pi \in \Pi(N)$ is

$$H_i^{WC}(\pi) = p_{\pi(i)}(\pi) \cdot p_i(\pi(i)) = \frac{w_i}{\sum\limits_{j \in \pi(i)} w_j} \cdot \frac{\sum\limits_{j \in \pi(i)} w_j}{\sum\limits_{j \in N} w_j} = \frac{w_i}{\sum\limits_{j \in N} w_j}.$$

The payoff $H_i^{WC}(\pi)$ does not depend on the coalition partition π. Hence, any partition in the weighted collective game is an NSP. Next, we examine the normal-form strategic game which is an alternative to the weighted collective game.

Let $S = \{1, 2, ..., m\}$ be the set of places where coalitions can be formed. The strategy s_i of the player i consists in choosing the place from the set S. The players who have chosen the same strategy form a coalition. The weight of the player $i \in N$ for the position $j \in S$ is the number $w_{ij} > 0$. We denote the strategy profile as $s = (s_1, s_2, ..., s_n)$.

Definition 5. A strategic weighted collective game is a normal-form game $(N, S, \{u_i\}_{i \in N})$ in which the payoff of the player i is

$$u_i(s) = \frac{w_{is_i}}{\sum\limits_{j \in N} w_{js_j}}.$$

The weight w_{ij_1} of the player i in the place j_1 can be different from the weight w_{ij_2} of the player i in the place j_2. The player is interested in choosing a place where their chances of becoming the leader are the highest. Note also that if the weight w_{ij} does not depend on j, then the strategic weighted collective game is equivalent to the weighted collective game.

Let us introduce some definitions. The strategy s_i is called *dominant* for the player i in the game $(N, S, \{u_i\}_{i \in N})$ if $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})\ \forall s_i' \in S, s_i' \neq s_i,\ \forall s_{-i} \in S^{n-1}$. The strategy s_i is called *weakly dominant* for the player i if $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})\ \forall s_i' \in S,\ \forall s_{-i} \in S^{n-1}$ and at least one of the inequalities is strict.

The normal-form game $(N, S, \{u_i\}_{i \in N})$ is called an *ordinal potential* game (Monderer & Shapley, 1996) if there exists a function $P : S^n \to \mathbb{R}$ such that $\forall i \in N :$

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) > 0 \Leftrightarrow P(s_i, s_{-i}) - P(s_i', s_{-i}) > 0\ \forall s_i, s_i' \in S\ \forall s_{-i} \in S_{-i}.$$

The first main result is Theorem 1.

Theorem 1. *The following is true for the strategic weighted collective game:*

1. *An equilibrium of weakly dominant strategies exists in the game.*

2. *The game is an ordinal potential game with the ordinal potential function*

$$P(s) = \sum_{i \in N} w_{is_i}.$$

The proof of Theorem 1 is given in the Appendix. To illustrate the meaning of this theorem, we suppose that the set S is a set of sports clubs and that each player has a weight specific in each club. Which club should the player choose to become the leader of in the collective contest model?

It follows from the proof of point 1 of Theorem 1 that irrespective of the strategy choices of players in the set $N \setminus \{i\}$ the player $i \in N$ should choose the club where their weight is the greatest. Choosing the club in which the weight of the player i is the greatest is their weakly dominant strategy. If there is only one club for the player i in which their weight is the greatest, then choosing such a club is the dominant strategy of the player i.

Point 2 of Theorem 1 asserts that the strategic weighted collective game is an ordinal potential game. This means it is acyclic, which does not follow from the existence of players' weakly dominant strategies. Acyclicity is important for applications. Suppose the players have erroneously chosen a club in which their weight is not maximal. In that case, due to acyclicity, any sequence of best answers is finite, which means equilibrium will be attained within a finite number of steps.

The papers (Bilò et al., 2023) and (V. Gusev et al., 2024) study a normal-form game in which the player's payoff is determined as:

$$u_i(s) = \frac{w_{is_i}}{\sum_{j \in K_{s_i}(s)} w_{js_j}} \cdot r(s_i),$$

where $K_{s_i}(s)$ is the set of the players in the profile s who have chosen the strategy s_i. If $r(s_i) = 1$ for any strategy s_i, then the player's payoff is their expected chances of becoming the leader in their coalition. What makes the current paper different is that it examines a two-step procedure for determining the leader, while the game from (Bilò et al., 2023) and (V. Gusev et al., 2024) models a one-step procedure. The key distinction is that we demonstrate the existence of an equilibrium in dominant and weakly dominant strategies.

Example 1. Let us consider the strategic weighted collective game where $N = \{1, 2, 3\}$ and the weight matrix has the form

$$\begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix} = \begin{pmatrix} 4 & 1 & 7 \\ 3 & 9 & 2 \\ 1 & 2 & 3 \end{pmatrix},$$

where w_{ij} is the weight of the i-th player in choosing the strategy j. The only equilibrium in this game is the profile $(3, 2, 3)$; i.e. the first and the third players chose strategy 3, and the second player chooses strategy 2. Since all the weights on each line of the matrix are different, this equilibrium is an equilibrium in dominant strategies. The strategy profile (3,2,3) is matched by the ordered coalition structure $(\{\emptyset\}, \{2\}, \{1, 3\})$.

3.2 Ranking collective game

By Definition 2, in the ranking collective game, the payoff of the player $i \in N$ in the partition $\pi \in \Pi(N)$ is

$$H_i^{RC}(\pi) = \frac{\lambda_{R_i(\pi(i))}}{\sum\limits_{B\in\pi}\sum\limits_{j=1}^{|B|}\lambda_j}.$$

In the ranking collective game, the leading coalition is determined by the ranking rule and then the leader of the coalition partition is chosen from among members of that coalition, also following the ranking rule. The payoff of that player is their expected chances of becoming the leader of the coalition partition.

Theorem 2 formulates the necessary and sufficient conditions for the existence of an NSPC. To prove Theorem 2, we formulate Lemma 1.

Lemma 1. *If we suppose that π is an NSPC in the ranking collective game, then the following statements are true:*

1. *The player $i = 1$ is a member of the coalition with the greatest power.*
2. *For the player $i = n$, $i \in B_1 \in \pi$, $|B_2| \geq |B_1| - 1$ is fulfilled for $\forall B_2 \in \pi$.*
3. *For any $B_l, B_r \in \pi$ such that $B_l \neq B_r$ there is $|B_l| \neq |B_r|$.*
4. *$|N|$ is an odd number, $|\pi| = 2$, $\pi = \{B_1, B_2\}$ and $|B_1| = |B_2| + 1$.*
5. *No player can raise their ranking by moving to another coalition, i.e. $\forall i \in N$ $R_i(\pi(i)) < R_i(\rho(i))$, where $\rho \in \mathring{D}_i(\pi)$.*

Lemma 1 characterizes the NSPC structure. Thus, the strongest player in an NSPC, i.e. the player with the lowest index, is always a member of the coalition with the greatest power. As the following will show, the strongest and the weakest players are always members of the same coalition in an NSPC (see point 1 of Theorem 2). Another property characteristic of an NSPC is the condition of a non-increase in ranking. Let us consider examples for Lemma 1.

Example 2. Considering $\pi = \{\{2, 3, 5\}, \{1, 4\}\}$ we see that this partition satisfies points 2, 3, 4, and 5 of Lemma 1, but violates point 1. The partition π is not an NSPC since $H_1(\pi) < H_1(\rho), \rho = \{\{1, 2, 3, 5\}, \{4\}\}$, i.e. the player 1 would benefit from moving to the other coalition. Indeed,

$$H_1^{RC}(\pi) = \frac{\lambda_1}{\sum\limits_{k=1}^{3}\lambda_k + \sum\limits_{k=1}^{2}\lambda_k} < \frac{\lambda_1}{\sum\limits_{k=1}^{4}\lambda_k + \lambda_1} = H_1^{RC}(\rho).$$

Example 3. Consider the partition $\pi = \{\{1,4,7\},\{2,5\},\{3,6\}\}$. This structure satisfies points 1, 2, and 5 of Lemma 1, but violates points 3 and 4. Player 2 would benefit from moving to the coalition $\{3,6\}$, i.e. $H_2^{RC}(\pi) < H_2^{RC}(\rho), \rho = \{\{1,4,7\},\{5\},\{2,3,6\}\}$. Indeed,

$$\frac{\lambda_1}{\sum\limits_{k=1}^{3}\lambda_k + \sum\limits_{k=1}^{2}\lambda_k + \sum\limits_{k=1}^{2}\lambda_k} < \frac{\lambda_1}{\sum\limits_{k=1}^{3}\lambda_k + \lambda_1 + \sum\limits_{k=1}^{3}\lambda_k}.$$

Example 4. Now consider the coalition structure $\pi = \{\{1,2,3\},\{4,5\}\}$. The condition from point 5 of Lemma 1 is violated for π. Players 2 and 3 can attain a higher ranking by moving to the coalition $\{4,5\}$. Let us demonstrate that the partition π is not an NSPC, i.e. $H_3^{RC}(\pi) < H_3^{RC}(\rho), \rho = \{\{1,2\},\{3,4,5\}\}$. Indeed,

$$\frac{\lambda_3}{\sum\limits_{k=1}^{3}\lambda_k + \sum\limits_{k=1}^{2}\lambda_k} < \frac{\lambda_1}{\sum\limits_{k=1}^{2}\lambda_k + \sum\limits_{k=1}^{3}\lambda_k}.$$

We can now characterize the set of NSPCs in the collective ranking game. For an odd n, we define the set of coalition structures Θ as follows:

$$\pi \in \Theta \Leftrightarrow |\pi| = 2, \forall B \in \pi : \{2i-1, 2i\} \not\subset B,\ |\pi(1)| \geq |B|\ \forall i = 1,2,..,\frac{n-1}{2}.$$

For example, the coalition structures $\pi_1 = \{\{1,3,5\},\{2,4\}\}, \pi_2 = \{\{1,4,5\},\{2,3\}\}$ belong to Θ, while the structures $\pi_3 = \{\{1,2,5\},\{3,4\}\}$ and $\pi_4 = \{\{1,3,5\},\{2,4,6\}\}$ do not.

Lemma 2. *If π is an NSPC in the ranking collective game, then $\pi \in \Theta$.*

It follows from Lemma 2 that any NSPC can be derived from some NSPC by swapping players from the pairs $\{2i-1, 2i\}$ $\forall i = 1,2,..,\frac{n-1}{2}$. Thus, due to Lemma 2, the partition $\pi_1 = \{\{1,3,5,7\},\{2,4,6\}\}$ is an NSPC. Swapping the third and the fourth players, we again get a Nash-stable partition, i.e. $\pi_2 = \{\{1,4,5,7\},\{2,3,6\}\}$ is an NSPC. Lemma 1 and Lemma 2 help prove the uniqueness of an NSPC and the necessary and sufficient conditions for its existence. The following theorem applies.

Theorem 2. *The following is true in the ranking collective game:*

1. *Suppose an NSPC exists, then $|N|$ is odd and an NSPC is a harmonious partition $\{O, E\}$.*
2. *An NSPC exists iff the ranking constants $\lambda_i,\ i = 1,...,n$ meet the condition*

$$\frac{2\sum\limits_{k=1}^{(n-3)/2}\lambda_k + \lambda_{\frac{n-1}{2}} + \lambda_{\frac{(n+1)}{2}} + \lambda_{\frac{(n+3)}{2}}}{2\sum\limits_{k=1}^{(n-3)/2}\lambda_k + \lambda_{\frac{(n-1)}{2}} + \lambda_{\frac{(n+1)}{2}} + \lambda_{\frac{n-1}{2}}} \geq \max_{j=1,...,(n-1)/2}\left(\frac{\lambda_{j+1}}{\lambda_j}\right).$$

3. *For any λ_i, $i = 1, .., n$ an NSP does not exist.*

It follows from Theorem 2 that a harmonious partition in a ranking collective game with an odd number of players and two coalitions is an NSPC. On the other hand, if the number of coalitions is not fixed, then there is always a player who would gain from moving out to an empty coalition (see proof of point 3 of Theorem 2 in the Appendix). In practice, this means that agents are not interested in collaborating.

3.3 Weighted individual game

According to Definition 3, the payoff of the player $i \in N$ in the partition $\pi \in \Pi(N)$ in the weighted individual game is

$$H_i^{WI}(\pi) = \frac{w_i}{\prod\limits_{B\in\pi} \sum\limits_{j\in B} w_j} \cdot \sum_{K\in M_i(\pi)} \frac{\prod\limits_{j\in K} w_j}{\sum\limits_{l\in K} w_l}.$$

In the weighted individual game, a representative is chosen from each coalition according to the weight rule, and then the leader of the coalition partition is chosen from among these representatives, also following the weight rule.

The weighted individual game has several features in common with weighted congestion games (Harks & Klimm, 2012, 2015; Milchtaich, 1996), but the player's payoff in the weighted individual game depends not only on the sum of weights of players from one coalition but also on the entire coalition partition. Let us formulate the definition of the potential coalition partition game.

The potential game is a game (N, H) for which there exists a potential function $P : \Pi(N) \to \mathbb{R}$ such that $\forall i \in N\ \forall \pi \in \Pi(N)$:

$$H_i(\pi) - H_i(\rho) = P(\pi) - P(\rho)\ \forall \rho \in D_i(\pi).$$

The definition of the potential game for normal-form games was given in (Monderer & Shapley, 1996). The potential-game nature of coalition partition games was investigated in (V. V. Gusev, 2021).

Statement 1. *The following statements are true:*

1. *Let $|N| = 3$ and $|\pi| = 2$. Then, an NSPC exists in the individual game.*

2. *Let $|N| = 4$. Then, the partition $\pi = \{A, B\}$ in the weighted individual game is an NSPC iff $|A| = |B|$.*

3. *If we have $\forall i, j \in N\ w_i = w_j$ in the weighted individual game, then this game is a potential game with the potential function of the form:*

$$P(\pi) = \frac{1}{|\pi|} \sum_{K\in\pi} \sum_{i=1}^{|K|} \frac{1}{|i|}.$$

Let us explore the existence of an NSPC in the weighted individual game for $|\pi| = 2$. It is possible to prove the following lemma.

Lemma 3. *We denote*

$$f_i(x) = \frac{x}{x + w_i},$$

where $x > 0$ and $w_i > 0$ is the weight of the player i, which does not depend on x. In that case, an NSPC in the weighted individual game with two coalitions exists iff there exists an NSPC in the coalition partition game $(N, \{u_i\}_{i\in N})$ with two coalitions, where

$$u_i(\pi) = \frac{f_i\left(\sum\limits_{i\in\pi(i)\setminus\{i\}} w_i\right)}{\sum\limits_{i\in\pi(i)\setminus\{i\}} f_i(w_i)}.$$

Example 5. Suppose $\{\{1, 3, 5\}, \{2, 4, 6\}\}$ is an NSPC in the weighted individual game. Hence, for player 1 the condition $u_1(\{\{1, 3, 5\}, \{2, 4, 6\}\}) \geq u_1(\{\{3, 5\}, \{1, 2, 4, 6\}\})$ is satisfied iff

$$\frac{f_1(w_3 + w_5)}{f_1(w_3) + f_1(w_5)} \geq \frac{f_1(w_2 + w_4 + w_6)}{f_1(w_2) + f_1(w_4) + f_1(w_6)},$$

and for player 2 the condition $u_2(\{\{1, 3, 5\}, \{2, 4, 6\}\}) \geq u_2(\{\{1, 2, 3, 5\}, \{4, 6\}\})$ is satisfied iff

$$\frac{f_2(w_4 + w_6)}{f_2(w_4) + f_2(w_6)} \geq \frac{f_2(w_1 + w_3 + w_5)}{f_2(w_1) + f_2(w_3) + f_2(w_5)}.$$

The same inequalities must be fulfilled for all players in the coalition partition $\{\{1, 3, 5\}, \{2, 4, 6\}\}$.

Be reminded that the harmonious partition for $|\pi| = 2$ is $\pi = \{O, E\}$, where $O = \{1, 3, 5, ...\}$ and $E = \{2, 4, 6, ...\}$. The following theorem applies.

Theorem 3. *The following is true for the weighted individual game:*

1. *Let $|\pi| = 2$ and $\forall i, j, k \in N \quad w_i + w_j \geq w_k$. If n is even, then the harmonious partition is an NSPC. If n is odd, then π is an NSPC.*

2. *A degenerate partition is an NSP.*

Note that according to point 1 of Theorem 2, an NSPC in the ranking collective game is a harmonious partition for an odd number of players. According to point 1 of Theorem 3, a harmonious partition in the weighted individual game is an NSPC for an even number of players. The condition $w_i + w_j \geq w_k$, $\forall i, j, k \in N$ means that the total weight of a two-player coalition is not smaller than the weight of each individual player.

3.4 Ranking individual game

By Definition 4, in the ranking individual game, the payoff of the player $i \in N$ in the partition $\pi \in \Pi(N)$ is

$$H_i^{RI}(\pi) = \lambda_{R_i(\pi(i))} \cdot \frac{\sum\limits_{K \in M_i(\pi)} \lambda_{R_i(K)} \prod\limits_{j \in K \setminus \{i\}} \lambda_{R_j(\pi(j))}}{\sum\limits_{j=1}^{|\pi|} \lambda_j \cdot \prod\limits_{B \in \pi} \sum\limits_{j \in B} \lambda_j}.$$

In the ranking individual game, a representative is chosen from each coalition according to the ranking rule, and then the leader of the coalition partition is chosen from among these representatives, also following the ranking rule.

Statement 2. *In the ranking individual game with two coalitions an NSPC exists iff an NSPC exists in the game (N, H) with two coalitions where*

$$H_i(\pi) = \frac{\lambda_{R_i(\pi(i))}}{\sum\limits_{k=1}^{|\pi(i)|} \lambda_k} \cdot \left(\frac{\sum\limits_{k=1}^{|\pi(i)|-1} \lambda_k}{\sum\limits_{l \in \pi(i) \setminus i} \sum\limits_{k=1}^{|\pi(i)|-1} \lambda_k \lambda_{R_i(i,l)}} \right).$$

Statement 2 demonstrates that the study of an NSPC in the ranking individual game comes down to exploring an NSPC in a coalition partition game where the player's payoff depends only on the coalition of which that player is a member. The corresponding fact was demonstrated above for the weighed individual game in Lemma 3.

Theorem 4. *The following is true for the ranking individual game:*

1. *A harmonious partition $\{O, E\}$ is an NSPC if*

$$\frac{\sum\limits_{j=1}^{|O|} \lambda_j \cdot \lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|O|} \lambda_j} \geq \frac{\sum\limits_{j=1}^{|E \setminus \{i\}|} \lambda_j \cdot \lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|E \setminus \{i\}|} \lambda_j}$$

   is fulfilled for $\forall i \in E$.

2. *A degenerate partition is the only possible NSP. For an NSP to exist it is necessary and sufficient that $\forall \lambda_i$ we have*

$$\frac{\sum\limits_{k=1}^{|N|-1} \lambda_k}{\sum\limits_{k=1}^{|N|} \lambda_k} \geq \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot \max_i \left(\frac{\lambda_{i-1}}{\lambda_i} \right) \forall i \in N.$$

Theorem 4 asserts that only a degenerate partition can be an NSP, which also demonstrates the unwillingness of players to cooperate. The condition in point 1 of Theorem 4 is sufficient but not necessary.

4. Core partition

This section investigates the existence of a core partition in the games in question.

Let $\pi = \{B_1, B_2, ..., B_m\}$ and $K \subseteq N, K \neq \emptyset$. We denote $\pi_K = \{K, B_1 \backslash K, B_2 \backslash K, ..., B_m \backslash K\}$. The coalition K is said to be *blocking* for π in the game (N, H) if $H_i(\pi_K) > H_i(\pi)\ \forall i \in K$. The coalition structure for which no blocking coalition exists will be called *a core partition* (CP).

Let $(N, S, \{u_i\}_{i \in N})$ be a normal-form game, $s \in S^n, K \subseteq N, K \neq \emptyset, x \in S$. The profile $s' = s'(K, x)$ is derived from the profile s by changing the strategies of players in the coalition K to the strategy x. We say that the profile s is *blocked* by the coalition K and the strategy x if $u_i(s) > u_i(s')\ \forall i \in K \cup K_x(s)$, where $K_x(s)$ is the set of players who have chosen the strategy x in the profile s. *The core strategy profile* is a strategy profile without blocking coalitions and strategies.

One of the sufficient conditions for the existence of a CP is the top-coalition property introduced in (Banerjee et al., 2001). However, the method of finding the top coalition was applied in an earlier paper (Farrell & Scotchmer, 1988). The top-coalition property is used in the analysis of hedonic games, and it was shown in (Gallo & Inarra, 2018) that noncircular hedonic games fulfill this property. The games examined in our paper are not hedonic, so the existence of a CP was demonstrated without the top-coalition property. The following theorem is true for the games here.

Theorem 5. *The following statements are true:*

1. *In the strategic weighted collective game, there exists a core-strategy profile for which each player chooses a strategy that maximizes their weight.*
2. *A degenerate partition is a CP in the ranking collective, weighted individual, and ranking individual games.*

Point 1 of Theorem 5 shows that in the strategic weighted collective game, the weakly dominant strategy profile is a core strategy profile. Point 2 of Theorem 5 implies that a degenerate partition is a CP. This means cooperation is not beneficial for players. Furthermore, a degenerate partition is also an NSP in problems with a non-fixed number of coalitions.

5. A comparison between the games in relation to harmonious partition

There are some essential differences between the collective and the individual games. The question arises of what advantages one of the games may have over the other. Let us introduce the game comparison criterion. Suppose the strongest player is the player 1, $1 \in N$. We say that the coalition partition game $\Gamma = (N, \{H_i\}_{i \in N})$ *dominates* the game $\Gamma' = (N, \{H'_i\}_{i \in N})$ in the partition π if $H_1(\pi) > H'_1(\pi)$. We denote this fact as $\Gamma \succ_\pi \Gamma'$. In other words, the partition-dominant game is the one that maximizes the strongest player's chances of becoming the leader.

As demonstrated in Theorems 2, 3, and 4, a harmonious partition is either always or under certain conditions an NSPC in the weighted collective, ranking collective, weighted individual, and ranking individual games. Let us study the necessary and sufficient conditions for the weighted individual game to dominate the weighted collective game and for the ranking individual game to dominate the ranking collective game in a harmonious partition.

Let us consider the functions

$$\phi(K) = \frac{\prod\limits_{j\in K} w_j}{\sum\limits_{l\in K} w_l}, \quad \phi_N(\pi) = \frac{\prod\limits_{B\in\pi}\sum\limits_{j\in B} w_j}{\sum\limits_{j\in N} w_j}.$$

Statement 3. *The following is true:*

1. *The weighted individual game dominates the weighted collective game in the partition π iff*

$$\sum_{K\in M_1(\pi)} \phi(K) \geq \phi_N(\pi).$$

2. *Let $|N|$ be odd. In this case, the ranking individual game dominates the ranking collective game in a harmonious partition iff*

$$\frac{\sum\limits_{k=1}^{(n-1)/2} \lambda_k}{\sum\limits_{k=1}^{(n+1)/2} \lambda_k} \geq \frac{\lambda_2}{\lambda_1}.$$

Point 2 of Statement 3 suggests the number of players is odd since it follows from point 1 of Theorem 2 that a harmonious partition can be an NSPC only with an odd number of players. It follows from point 2 of Statement 3 that where the number of players is sufficiently large, the ranking individual game dominates the ranking collective game. Since a CP in the ranking individual and collective games is a degenerate partition, the question arises of game domination in this partition. However, the payoff probabilities of all players in this coalition structure coincide, so the games do not dominate each other in a CP.

6. Discussion

Theorems on the existence of an NSPC in the weighted collective, ranking collective, weighted individual, and ranking individual games were formulated in this paper. These games model the political and economic processes of group formation.

It was shown for the strategic weighed collective game that the profile in which players choose the strategy that maximizes their weight is an equilibrium in weakly dominant strategies. This means that agents should choose the team in which they will be valued the most, i.e. where their weight is the greatest. Suppose, e.g., economic agents are choosing a company, then the agent's strategy is to choose one company out of their finite set. The objective of each agent is to maximize their expected chances of becoming the most successful employee among all employees of all companies. Each company appraises agents differently depending on their professional qualities. The more agents with high professional qualities there are in the company, the higher are its chances of becoming the market leader. In real life, many people want to be employed by the largest company, i.e. the one with the greatest total weight. Theorem 1 asserts that agents should not pay attention to the size of the company and its employees, but only take into account the agent's value in each specific company.

In the general case of the ranking collective game in the general case, an NSPC does not necessarily exist. We described the necessary and sufficient conditions for the existence of an NSPC—only a harmonious partition with an odd number of players and two coalitions can be an NSPC. In practice, this fact strongly limits the use of the ranking rule. Thus, organizers of a group contest guided by the collective ranking rule must always invite an odd number of players and allow only two teams to participate. Otherwise, any such partition will not be an NSPC. Furthermore, an NSP does not exist in the ranking collective game. This happens because in any partition π, $|\pi| < n$ there will be a coalition B, $|B| \geq 2$ such that cooperation will not be beneficial for a player $i \in B$. A degenerate partition in that case is not an NSP either.

For the weighted individual game, if $|\pi| = 2$ and the sum of the weights of any two players is greater than the weight of a third player, then an NSPC exists. Otherwise, if all the weights are equal, the game is a potential game. A degenerate partition is an NSP and a CP.

The sufficient condition for a harmonious partition to be an NSPC was found for the ranking individual game. Only a degenerate partition can be an NSP.

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Appendix

Proof of Theorem 1.

1. We represent the payoff of the player i as follows,

$$u_i(s) = \frac{w_{is_i}}{\sum\limits_{j\in N} w_{js_j}} = \frac{w_{is_i}}{w_{is_i} + \sum\limits_{j\in N\setminus\{i\}} w_{js_j}} = \frac{w_{is_i}}{w_{is_i} + c(s_{-i})},$$

where $c(s_{-i}) = \sum\limits_{j\in N\setminus\{i\}} w_{js_j}$ does not depend on the strategy of the player i. We denote $s_i^* \in \arg\max\limits_{j\in S} w_{ij}$. Let us show that $\forall i \in N$ s_i^* is a weakly dominant strategy of the player i. Due to monotonicity of the function f, where $f(x) = \dfrac{x}{x+\alpha}$, where α does not depend on x, and since $s_i^* \in \arg\max\limits_{j\in S} w_{ij}$ $\forall i \in N$ we have

$$u_i(s_i^*, s_{-i}) = \frac{w_{is_i^*}}{w_{is_i^*} + c(s_{-i})} \geq \frac{w_{is_i}}{w_{is_i} + c(s_{-i})} = u(s_i, s_{-i}) \ \forall s_{-i} \in S^{n-1} \ \forall s_i \in S.$$

Hence, $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ $\forall s_{-i} \in S^{n-1}$ $\forall s_i \in S$. Then, by definition, s_i^* is a weakly dominant strategy of the player i.

2. We transform the difference of the payoffs of the player i,

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = \frac{w_{is_i}}{w_{is_i} + c(s_{-i})} - \frac{w_{is_i'}}{w_{is_i'} + c(s_{-i})}$$

$$= \frac{w_{is_i} + c(s_{-i}) - c(s_{-i})}{w_{is_i} + c(s_{-i})} - \frac{w_{is_i'} + c(s_{-i}) - c(s_{-i})}{w_{is_i'} + c(s_{-i})}$$

$$= c(s_{-i}) \cdot \left(\frac{1}{w_{is_i'} + c(s_{-i})} - \frac{1}{w_{is_i} + c(s_{-i})} \right)$$

$$= \frac{c(s_{-i})}{(w_{is_i'} + c(s_{-i})) \cdot (w_{is_i} + c(s_{-i}))} \cdot \left(w_{is_i} + c(s_{-i}) - w_{is_i'} - c(s_{-i}) \right)$$

$$= \frac{c(s_{-i})}{P(s_i, s_{-i}) \cdot P(s_i', s_{-i})} \left(P(s_i, s_{-i}) - P(s_i', s_{-i}) \right).$$

In other words, $\forall i \in N$ $\forall s_i, s_i' \in S$ $\forall s_{-i} \in S^{n-1}$

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = \frac{c(s_{-i})}{P(s_i, s_{-i}) \cdot P(s_i', s_{-i})} \left(P(s_i, s_{-i}) - P(s_i', s_{-i}) \right).$$

Since $\forall i \in N$ $\forall s_i, s_i' \in S$ $\forall s_{-i} \in S^{n-1}$

$$\frac{c(s_{-i})}{P(s_i, s_{-i}) \cdot P(s'_i, s_{-i})} > 0,$$

then

$$u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) > 0 \Leftrightarrow P(s_i, s_{-i}) - P(s'_i, s_{-i}) > 0.$$

This means, by definition, that the strategic weighted collective game is an ordinal potential game.

□

Proof of Lemma 1. Firstly, we remark that the partition π in the ranking collective game is an NSPC iff $\forall B_1 \in \pi \ \forall i \in B_1 \ \forall B_2 \in \pi : B_1 \neq B_2$ the inequality

$$\left(\sum_{k=1}^{|B_1|-1} \lambda_k + \sum_{k=1}^{|B_2|} \lambda_k + \sum_{B \in \pi \setminus \{B_1, B_2\}} \sum_{k=1}^{|B|} \lambda_k \right) \left(\lambda_{R_i(B_1)} - \lambda_{R_i(B_2 \cup \{i\})} \right) \geq \lambda_{R_i(B_2 \cup \{i\})} \cdot \lambda_{|B_1|} - \lambda_{R_i(B_1)} \cdot \lambda_{|B_2|+1}$$

is fulfilled. Indeed, let π be an NSPC and $\pi = \{B_1, B_2, ..., B_m\}$. Without loss of generality, we assume $|B_1| \geq 2$, $i \in B_1$, $\rho = \{B_1 \setminus \{i\}, B_2 \cup \{i\}, B_3 ..., B_m\}$. Since π is an NSPC, then

$$H_i^{RC}(\pi) \geq H_i^{RC}(\rho) \Leftrightarrow \frac{\lambda_{R_i(B_1)}}{\sum\limits_{k=1}^{|B_1|} \lambda_k + \sum\limits_{k=1}^{|B_2|} \lambda_k + ... + \sum\limits_{k=1}^{|B_m|} \lambda_k} \geq \frac{\lambda_{R_i(B_2 \cup \{i\})}}{\sum\limits_{k=1}^{|B_1|-1} \lambda_k + \sum\limits_{k=1}^{|B_2|+1} \lambda_k + ... + \sum\limits_{k=1}^{|B_m|} \lambda_k},$$

where $\lambda_{R_i(B_1)}$ is the ranking constant before the migration of the player i to another coalition, and $\lambda_{R_i(B_2 \cup \{i\})}$ is the ranking constant after the migration. This leads us to

$$\lambda_{R_i(B_1)} \left(\lambda_{|B_2|+1} + \sum_{k=1}^{|B_1|-1} \lambda_k + \sum_{k=1}^{|B_2|} \lambda_k + \sum_{B \in \pi \setminus \{B_1, B_2\}} \sum_{k=1}^{|B|} \lambda_k \right)$$

$$\geq \lambda_{R_i(B_2 \cup \{i\})} \left(\lambda_{|B_1|} + \sum_{k=1}^{|B_1|-1} \lambda_k + \sum_{k=1}^{|B_2|} \lambda_k + \sum_{B \in \pi \setminus \{B_1, B_2\}} \sum_{k=1}^{|B|} \lambda_k \right),$$

$$\left(\sum_{k=1}^{|B_1|-1} \lambda_k + \sum_{k=1}^{|B_2|} \lambda_k + \sum_{B \in \pi \setminus \{B_1, B_2\}} \sum_{k=1}^{|B|} \lambda_k \right) \left(\lambda_{R_i(B_1)} - \lambda_{R_i(B_2 \cup \{i\})} \right) \geq \lambda_{R_i(B_2 \cup \{i\})} \cdot \lambda_{|B_1|} - \lambda_{R_i(B_1)} \cdot \lambda_{|B_2|+1}.$$

Proceeding from this inequality, we prove the validity of points 1 — 5 of this lemma.

1. Let $i = 1$, $i \in B_1 \in \pi$ and the player i moves to the coalition $B_2 \in \pi$. Observe that the player i preserves his/her ranking in any coalition, wherefore we have

$$\left(\sum_{k=1}^{|B_1|-1} \lambda_k + \sum_{k=1}^{|B_2|} \lambda_k + \sum_{B \in \pi \setminus \{B_1, B_2\}} \sum_{k=1}^{|B|} \lambda_k \right) (\lambda_1 - \lambda_1) \geq \lambda_1 \cdot \lambda_{|B_1|} - \lambda_1 \cdot \lambda_{|B_2|+1}.$$

Hence, $\lambda_{|B_2|+1} \geq \lambda_{|B_1|}$, but this is possible if and only if $|B_1| > |B_2| \; \forall B_2 \in \pi, B_2 \neq B_1$ is fulfilled for the player i.

2. Let $i = n,\ i \in B_1 \in \pi$, then

$$\left(\sum_{k=1}^{|B_1|-1} \lambda_k + \sum_{k=1}^{|B_2|} \lambda_k + \sum_{B \in \pi \setminus \{B_1, B_2\}} \sum_{k=1}^{|B|} \lambda_k \right) \left(\lambda_{|B_1|} - \lambda_{|B_2|+1} \right) \geq \lambda_{|B_2|+1} \cdot \lambda_{|B_1|} - \lambda_{|B_1|} \cdot \lambda_{|B_2|+1}.$$

Hence, $\lambda_{|B_1|} \geq \lambda_{|B_2|+1}$, but this is possible if and only if $|B_2| \geq |B_1| - 1$ is fulfilled for the player n, i.e. the power of the coalition B_2 cannot be greater than B_1 by more than 1.

3. Let $\pi = \{B_1, B_2, ..., B_m\}$. Without loss of generality, we show that $|B_1| > |B_2| > |B_3| > ... > |B_m|$ is fulfilled. Let us suppose the opposite. Say there are two coalitions of equal powers B_r and B_l, so that $|B_r| = |B_l|$. Let j be the player with the lowest index (the strongest) among all players of the coalitions B_r and B_l. If $j \in B_r$, then we find that j would benefit from moving to the coalition B_l and if $j \in B_l$, then the player would gain from moving to B_r. Indeed, without loss of generality, we suppose that $j \in B_r$, hence $R_j(B_r) = 1$ and $R_j(B_l \cup \{j\}) = 1$. Then

$$\left(\sum_{k=1}^{|B_r|-1} \lambda_k + \sum_{k=1}^{|B_l|} \lambda_k + \sum_{B \in \pi \setminus \{B_l, B_r\}} \sum_{k=1}^{|B|} \lambda_k \right) (\lambda_1 - \lambda_1) \geq \lambda_1 \cdot \lambda_{|B_r|} - \lambda_1 \cdot \lambda_{|B_l|+1}.$$

This means that $\lambda_{|B_l|+1} \geq \lambda_{|B_r|}$, and $|B_r| > |B_l|$, which contradicts the supposition $|B_r| = |B_l|$.

4. Let $\pi = \{B_1, B_2, ..., B_m\}$ be an NSPC. We will show that $|\pi| = 2$. Let us suppose the opposite, i.e. $|\pi| > 2$. As a consequence of point 1 of Lemma 1, $1 \in B_1$, where B_1 is the coalition of the greatest power, and from point 3 of Lemma 1 it follows that all coalitions in the partition π have different powers. Let us examine the payoff function for the weakest player $j \in B_1$, whose ranking is $|B_1|$. We will show that the player j gains from moving to the coalition with the lowest power B_m. Since π is an NSPC, then

$$\left(\sum_{k=1}^{|B_1|-1} \lambda_k + \sum_{k=1}^{|B_2|} \lambda_k + \sum_{B \in \pi \setminus \{B_1, B_2\}} \sum_{k=1}^{|B|} \lambda_k \right) \left(\lambda_{|B_1|} - \lambda_{|B_m|+1} \right) \geq \lambda_{|B_m|+1} \cdot \lambda_{|B_1|} - \lambda_{|B_1|} \cdot \lambda_{|B_m|+1}.$$

Hence, $|B_m| \geq |B_1| - 1$. It follows from point 1 of Lemma 1 that $|B_1| \geq |B| + 1 \; \forall B \in \pi$. Then, $|B_1| \geq |B_m| + 1$. Thus, $|B_m| = |B_1| - 1$. If $|\pi| > 2$ and $|B_m| = |B_1| - 1$, then coalitions of equal power exist in the partition π. However, according to point 3 of Lemma 2, all coalitions have different powers. We have a contradiction. Hence, $|\pi| = 2$.

We will now show that N is odd. Let us suppose the opposite, i.e. $|N|$ is odd. We have proved previously that $|\pi| = 2$. Let $1 \in B_1$. It follows from point 1, point 3 of Lemma 1 and the parity of $|N|$ that $|B_1| > |B_2| + 1$. Note that the weakest player $j \in B_1$ would benefit from moving to B_2. Indeed, applying the necessary conditions for an NSPC, we get that

$$\left(\sum_{k=1}^{|B_1|-1} \lambda_k + \sum_{k=1}^{|B_2|} \lambda_k\right) \left(\lambda_{|B_1|} - \lambda_{R_j(B_2\cup\{j\})}\right) \geq \lambda_{R_j(B_2\cup\{j\})} \cdot \lambda_{|B_1|} - \lambda_{|B_1|} \cdot \lambda_{|B_2|+1}.$$

Since $\lambda_{|B_1|} < \lambda_{R_j(B_2\cup\{j\})}$, then the left-hand side of the inequality is negative, which means the right-hand side of this inequality is also negative, i.e. $\lambda_{R_j(B_2\cup\{j\})} \cdot \lambda_{|B_1|} < \lambda_{|B_1|} \cdot \lambda_{|B_2|+1}$. If follows that $\lambda_{R_j(B_2\cup\{j\})} < \lambda_{|B_2|+1}$. We have a contradiction.

5. It follows from point 4 of this lemma that $|N|$ is an odd number and $|\pi| = 2$. Let $\pi = \{B_1, B_2\}$ and $|B_1| = |B_2| + 1$. Suppose there happens to be a player $i \in B_1$ for whom $R_i(B_1) < R_i(B_2 \cup \{i\})$. Then, we have

$$\left(\sum_{k=1}^{|B_1|-1} \lambda_k + \sum_{k=1}^{|B_2|} \lambda_k\right) \left(\lambda_{R_i(B_1)} - \lambda_{R_i(B_2\cup\{i\})}\right) \geq \lambda_{R_i(B_2\cup\{i\})} \cdot \lambda_{|B_1|} - \lambda_{R_i(B_1)} \cdot \lambda_{|B_2|+1}.$$

   Since $\lambda_{R_i(B_1)} - \lambda_{R_i(B_2\cup\{i\})} < 0$, the left-hand side of the inequality is a negative number. The right-hand side of the inequality however is a positive number. Indeed, point 4 of Lemma 1 gives us that $\lambda_{|B_1|} = \lambda_{|B_2|+1}$. Hence, considering that $\lambda_{R_i(B_2\cup\{i\})} > \lambda_{R_i(B_1)}$, we get that

$$\lambda_{R_i(B_2\cup\{i\})} \cdot \lambda_{|B_1|} > \lambda_{R_i(B_1)} \cdot \lambda_{|B_2|+1}.$$

   Thus, the right-hand side of the inequality is a positive number and the left-hand side is a negative number. We have a contradiction.

   Next, we suppose there is a player $j \in B_2$ for whom $R_i(B_2) < R_j(B_1 \cup \{j\})$. We apply the inequality

$$\left(\sum_{k=1}^{|B_2|-1} \lambda_k + \sum_{k=1}^{|B_1|} \lambda_k\right) \left(\lambda_{R_j(B_2)} - \lambda_{R_j(B_1\cup\{j\})}\right) \geq \lambda_{R_j(B_1\cup\{j\})} \cdot \lambda_{|B_2|} - \lambda_{R_j(B_2)} \cdot \lambda_{|B_1|+1}.$$

   Since $\lambda_{R_j(B_2)} - \lambda_{R_j(B_1\cup\{j\})} < 0$, the left-hand side of the inequality is a negative number. The right-hand side of the inequality however is a positive number. Indeed, point 4 of Lemma 1 gives us that $\lambda_{|B_1|+1} < \lambda_{|B_2|}$. Hence, considering that $\lambda_{R_j(B_1\cup\{j\})} > \lambda_{R_j(B_2)}$, we get that

$$\lambda_{R_j(B_1\cup\{j\})} \cdot \lambda_{|B_2|} > \lambda_{R_j(B_2)} \cdot \lambda_{|B_1|+1}.$$

   Thus, the right-hand side of the inequality is a positive number and the left-hand side is a negative number. We have a contradiction.

□

Proof of Lemma 2. Let π be an NSPC. Suppose that $\pi \notin \Theta$. This means that within one coalition of the partition π there will be a pair of players $\{2i-1, 2i\}$ for some $i \in \{1, 2, .., \frac{n-1}{2}\}$. It then follows from point 1 and point 4 of Lemma 1 that in the second coalition of the partition π there will be a pair $\{2j-1, 2j\}$ where $i \neq j$, $j = 1, 2, .., \frac{n-1}{2}$. Without loss of generality, we suppose that $i > j$. Note that by joining the opposing coalition, the player $2i$ can raise one's ranking, since $2i < 2j - 1 < 2j$. However, this contradicts point 5 of Lemma 1. Consequently, $\pi \in \Theta$. $\square$

Proof of Theorem 2.

1. It follows from Lemma 2 that any an NSPC π belongs to the set Θ. Let π^* be a harmonious partition. Note that $\pi^* \in \Theta$. Let us suppose that the set Θ has an NSPC $\pi = \{A, B\}$, $|A| = |B| + 1$, $\pi \neq \pi^*$. Then, $\forall j \in B$ $H_j^{RC}(\pi) \geq H_j^{RC}(\rho)$ is true for $\rho \in \mathring{D}_j(\pi)$, i.e.

$$\frac{\lambda_{R_j(B)}}{\sum\limits_{k=1}^{|A|} \lambda_k + \sum\limits_{k=1}^{|B|} \lambda_k} \geq \frac{\lambda_{R_j(A \cup \{j\})}}{\sum\limits_{k=1}^{|B|-1} \lambda_k + \sum\limits_{k=1}^{|A|+1} \lambda_k},$$

$$\frac{\lambda_{R_j(B)}}{2\sum\limits_{k=1}^{|A|-2} \lambda_k + \lambda_{|A|} + \lambda_{|A|-1}} \geq \frac{\lambda_{R_j(A \cup \{j\})}}{2\sum\limits_{k=1}^{|A|-2} \lambda_k + \lambda_{|A|} + \lambda_{|A|+1}}.$$

Since $\pi \neq \pi^*$, then the coalition B will contain a player i whose ranking will be preserved when moving to the coalition A, i.e. $\lambda_{R_i(B)} = \lambda_{R_i(A \cup \{i\})}$, and the numerator in the inequality above can be cancelled out. Hence,

$$\frac{1}{2\sum\limits_{k=1}^{|A|-2} \lambda_k + \lambda_{|A|} + \lambda_{|A|-1}} \geq \frac{1}{2\sum\limits_{k=1}^{|A|-2} \lambda_k + \lambda_{|A|} + \lambda_{|A|+1}}.$$

However, $\lambda_{|A|-1} > \lambda_{|A|+1}$, so the left-hand fraction is strictly smaller than the right-hand fraction. We have a contradiction. Consequently, the only possible an NSPC in the set Θ is a harmonious partition.

2. If we suppose that an NSPC exists, then it follows from point 1 of Theorem 2 that $|N|$ is odd and the an NSPC is a harmonious partition of the form $\pi = \{O, E\}$. Note that not a single player i from the coalition O would want to move to the coalition E, since this would not change their payoff. Indeed, any player $i \in O$ keeps one's ranking upon moving and, since $|O| = |E| + 1$, the equality

$$H_i^{RC}(\pi) = \frac{\lambda_{R_i(O)}}{\sum\limits_{k=1}^{|O|} \lambda_k + \sum\limits_{k=1}^{|E|} \lambda_k} = \frac{\lambda_{R_i(E \cup \{i\})}}{\sum\limits_{k=1}^{|O|-1} \lambda_k + \sum\limits_{k=1}^{|E|+1} \lambda_k} = H_i(\rho)$$

applies. Thus, a harmonious partition is an NSPC iff no player j from the coalition E would want to move to the coalition O, i.e. it is necessary and sufficient that

$$\frac{\lambda_{R_j(E)}}{2\sum\limits_{k=1}^{|O|-2}\lambda_k+\lambda_{|O|}+\lambda_{|O|-1}} \geq \frac{\lambda_{R_j(O\cup\{j\})}}{2\sum\limits_{k=1}^{|O|-2}\lambda_k+\lambda_{|O|}+\lambda_{|O|+1}}$$

or

$$\frac{2\sum\limits_{k=1}^{|O|-2}\lambda_k+\lambda_{|O|-1}+\lambda_{|O|}+\lambda_{|O|+1}}{2\sum\limits_{k=1}^{|O|-2}\lambda_k+\lambda_{|O|-1}+\lambda_{|O|}+\lambda_{|O|-1}} \geq \frac{\lambda_{R_j(O\cup\{j\})}}{\lambda_{R_j(E)}}.$$

In a harmonious partition, however, $\forall j\in E\ \lambda_{R_j(E)}=\lambda_{R_j(O\cup\{j\})-1}$. Since the inequality above is fulfilled for $\forall j\in B$, it is necessary and sufficient that

$$\frac{2\sum\limits_{k=1}^{|O|-2}\lambda_k+\lambda_{|O|-1}+\lambda_{|O|}+\lambda_{|O|+1}}{2\sum\limits_{k=1}^{|O|-2}\lambda_k+\lambda_{|O|-1}+\lambda_{|O|}+\lambda_{|O|-1}} \geq \max_{j=1,\dots,|E|}\left(\frac{\lambda_{j+1}}{\lambda_j}\right).$$

This gives us that

$$\frac{2\sum\limits_{k=1}^{(n-3)/2}\lambda_k+\lambda_{\frac{n-1}{2}}+\lambda_{\frac{(n+1)}{2}}+\lambda_{\frac{(n+3)}{2}}}{2\sum\limits_{k=1}^{(n-3)/2}\lambda_k+\lambda_{\frac{(n-1)}{2}}+\lambda_{\frac{(n+1)}{2}}+\lambda_{\frac{n-1}{2}}} \geq \max_{j=1,\dots,(n-1)/2}\left(\frac{\lambda_{j+1}}{\lambda_j}\right).$$

3. Consider an arbitrary non-degenerate partition $\pi=\{B_1,...,B_m\}$ where a certain coalition B_1 fulfills $|B_1|>1$. Let us show that the player $i\in B_1$ with the ranking f will benefit from moving out to an empty coalition, i.e. $H_i^{RC}(\pi)<H_i^{RC}(\rho)$, where $\rho=\{\{i\},B_1\setminus\{i\},B_2,...,B_m\}$. But $H_i^{RC}(\pi)<H_i^{RC}(\rho)\ \Leftrightarrow$

$$\frac{\lambda_{|B_1|}}{\lambda_{|B_1|}+\sum\limits_{k=1}^{|B_1|-1}\lambda_k+\sum\limits_{k=1}^{|B_2|}\lambda_k+...+\sum\limits_{k=1}^{|B_m|}\lambda_k} < \frac{\lambda_1}{\lambda_1+\sum\limits_{k=1}^{|B_1|-1}\lambda_k+\sum\limits_{k=1}^{|B_2|}\lambda_k+...+\sum\limits_{k=1}^{|B_m|}\lambda_k}.$$

The left-hand and right-hand sides of the inequality represent a monotonically increasing function $f(x)=\dfrac{x}{x+c}$, where c does not depend on x. Since $\lambda_{|B_1|}<\lambda_1$, the inequality is fulfilled. Hence, any coalition B such that $|B|>1$ will contain a player willing to move out to an empty coalition. Thus, the only possible an NSP is the degenerate partition $\{\{1\},...,\{n\}\}$. However, it is not an NSP, since it is profitable for player 1 to join player 2. Indeed,

$$H_1^{RC}(\{\{1\},\{2\},...,\{n\}\})<H_1^{RC}(\{\{1,2\},...,\{n\}\})\Leftrightarrow\frac{\lambda_1}{n\lambda_1}<\frac{\lambda_1}{\lambda_1+\lambda_2+(n-2)\lambda_1}.$$

The inequality is true since $\lambda_1 > \lambda_2$. Thus, $\forall \lambda_i, i = 1, ..., n$ an NSP does not exist.

$\square$

Proof of Statement 1.

1. Let $N = \{i, j, k\}$. We will show that $\pi = \{\{i, j\}, \{k\}\}$ is an NSPC. Since only coalition structures with two coalitions are permissible, only the player i or j can join the player k. We have

$$H_i^I(\{\{i,j\},\{k\}\}) = p_i(\{i,j\}) \cdot p_i(\{i,k\}) = H_i^I(\{\{j\},\{i,k\}\}),$$

$$H_j^I(\{\{i,j\},\{k\}\}) = p_j(\{i,j\}) \cdot p_j(\{j,k\}) = H_j^I(\{\{i\},\{j,k\}\}).$$

Hence, the payoffs of the players i and j will not change if they move to another coalition. This means that π is an NSPC.

2. Let $N = \{i, j, k, l\}$. We will show that $\pi = \{\{i, j\}, \{k, l\}\}$ is an NSPC. Since only coalition structures made up of two non-empty coalitions are permissible, then $H_i^{WI}(\{\{i,j\},\{k,l\}\}) \geq H_i^{WI}(\{\{j\},\{i,k,l\}\}) \Leftrightarrow$

$$\frac{w_i}{w_i + w_j}\left(\frac{w_k}{w_k + w_l} \cdot \frac{w_i}{w_i + w_k} + \frac{w_l}{w_k + w_l} \cdot \frac{w_i}{w_i + w_l}\right) \geq \frac{w_i}{w_i + w_k + w_l} \cdot \frac{w_i}{w_i + w_j}.$$

Hence,

$$\frac{w_k}{w_k + w_i} + \frac{w_l}{w_l + w_i} \geq \frac{w_k + w_l}{w_k + w_l + w_i}.$$

The latter inequality if fulfilled according to point 1 of Lemma 3. Similar reasoning is applied to the players j, k, l.

3. Let $\forall i, j \in N \; w_i = w_j = w$. Then, $\forall \pi \in \prod(N) \; \forall i \in N \; \forall \rho \in \mathring{D}_i(\pi)$

$$H_i^{WI}(\pi) - H_i^{WI}(\rho) = \frac{w_i}{\prod\limits_{B\in\pi}\sum\limits_{j\in B} w_j} \cdot \sum_{K\in M_i(\pi)} \frac{\prod\limits_{j\in K} w_j}{\sum\limits_{l\in K} w_l} - \frac{w_i}{\prod\limits_{B\in\rho}\sum\limits_{j\in B} w_j} \cdot \sum_{K\in M_i(\rho)} \frac{\prod\limits_{j\in K} w_j}{\sum\limits_{l\in K} w_l}$$

$$= \frac{w}{w^{|\pi|}\prod\limits_{B\in\pi}|B|}\left(\sum_{K\in M_i(\pi)} \frac{w^{|K|}}{w|K|}\right) - \frac{w}{w^{|\rho|}\prod\limits_{B\in\rho}|B|}\left(\sum_{K\in M_i(\rho)} \frac{w^{|K|}}{w|K|}\right)$$

$$= \frac{1}{|\pi(i)||\pi|} - \frac{1}{|\rho(i)||\rho|} = \frac{1}{|\pi|}\sum_{K\in\pi}\sum_{i=1}^{|K|}\frac{1}{|i|} - \frac{1}{|\rho|}\sum_{K\in\rho}\sum_{i=1}^{|K|}\frac{1}{|i|} = P(\pi) - P(\rho).$$

Hence, by definition, this game is a potential game.

$\square$

Proof of Lemma 3. Let $N = \{1, 2, ..., n\}, \pi = \{B_1, B_2\}$. For a fixed player i the partition π can be given in the form $\pi = \{\pi(i), \rho(i) \setminus \{i\}\}$. When the player i moves in the partition π from one coalition to another, we get the partition $\rho = \{\rho(i), \pi(i) \setminus \{i\}\}$. Simplifying the player's payoff, we get

$$H_i^{WI}(\pi) = \sum_{j \in \rho(i) \setminus \{i\}} \frac{w_i}{w_j + w_i} \cdot \frac{w_i}{\sum_{l \in \pi(i)} w_l} \cdot \frac{w_j}{\sum_{l \in \rho(i) \setminus \{i\}} w_l}$$
$$= \frac{w_i^2}{\sum_{l \in \pi(i)} w_l \cdot \sum_{l \in \rho(i) \setminus \{i\}} w_l} \cdot \sum_{j \in \rho(i) \setminus \{i\}} \frac{w_j}{w_j + w_i}.$$

The payoff of the player i in the partition ρ is

$$H_i^{WI}(\rho) = \frac{w_i^2}{\sum_{l \in \rho(i)} w_l \cdot \sum_{l \in \pi(i) \setminus \{i\}} w_l} \cdot \sum_{j \in \pi(i) \setminus \{i\}} \frac{w_j}{w_j + w_i}.$$

Then,

$$H_i^{WI}(\pi) \geq H_i^{WI}(\rho) \Leftrightarrow$$
$$\frac{w_i^2}{\sum_{l \in \pi(i)} w_l \cdot \sum_{l \in \rho(i) \setminus \{i\}} w_l} \cdot \sum_{j \in \rho(i) \setminus \{i\}} \frac{w_j}{w_j + w_i} \geq \frac{w_i^2}{\sum_{l \in \rho(i)} w_l \cdot \sum_{l \in \pi(i) \setminus \{i\}} w_l} \cdot \sum_{j \in \pi(i) \setminus \{i\}} \frac{w_j}{w_j + w_i} \Leftrightarrow$$
$$\frac{\sum_{l \in \pi(i) \setminus \{i\}} w_l}{\sum_{l \in \pi(i)} w_l} \cdot \frac{1}{\sum_{j \in \pi(i) \setminus \{i\}} \frac{w_j}{w_j + w_i}} \geq \frac{\sum_{l \in \rho(i) \setminus \{i\}} w_l}{\sum_{l \in \rho(i)} w_l} \cdot \frac{1}{\sum_{j \in \rho(i) \setminus \{i\}} \frac{w_j}{w_j + w_i}}.$$

We denote $f_i(x) = \frac{x}{x + w_i}$. In that case, the above inequality is equivalent to the inequality

$$\frac{f_i\left(\sum_{l \in \pi(i) \setminus \{i\}} w_l\right)}{\sum_{l \in \pi(i) \setminus \{i\}} f_i(w_l)} \geq \frac{f_i\left(\sum_{l \in \rho(i) \setminus \{i\}} w_l\right)}{\sum_{l \in \rho(i) \setminus \{i\}} f_i(w_l)}.$$

Hence, an NSPC in the weighted individual game with two coalitions is an NSPC in the auxiliary coalition partition game $(N, \{u\}_{i \in N})$, where

$$u_i(\pi) = \frac{f_i\left(\sum_{l \in \pi(i) \setminus \{i\}} w_l\right)}{\sum_{l \in \pi(i) \setminus \{i\}} f_i(w_l)}.$$

□

Proof of Lemma 4.

1. Consider the function $F_i(x, y) = f_i(x) + f_i(y) - f_i(x + y)$ where $i \in N, x > 0, y > 0$. Then,

$$\frac{\partial F_i(x, y)}{\partial x} = \frac{w_i}{(x + w_i)^2} - \frac{w_i}{(x + y + w_i)^2} > 0$$

and

$$\frac{\partial F_i(x, y)}{\partial y} = \frac{w_i}{(y + w_i)^2} - \frac{w_i}{(x + y + w_i)^2} > 0.$$

Hence, $\forall x, y > 0$ monotonicity suggesting that $F_i(x, y) > F_i(0, y)$. Consequently, $\forall x, y > 0$ we have

$$f_i(x) + f_i(y) - f_i(x + y) > f_i(0) + f_i(y) - f_i(0 + y).$$

Therefore,

$$f_i(x) + f_i(y) > f_i(x + y).$$

It follows from this inequality that

$$f_i\left(\sum_{k=1}^{m} x_k\right) = f_i\left(x_1 + \sum_{k=2}^{m} x_k\right) < f_i(x_1) + f_i\left(\sum_{k=2}^{m} x_k\right) = f_i(x_1) + f_i\left(x_2 + \sum_{k=3}^{m} x_k\right)$$

$$< f_i(x_1) + f_i(x_2) + f_i\left(\sum_{k=3}^{m} x_k\right) < ... < \sum_{k=1}^{m} f_i\left(x_k\right).$$

Thus,

$$f_i\left(\sum_{k=1}^{m} x_k\right) < \sum_{k=1}^{m} f_i\left(x_k\right).$$

Let us consider the functions $g_i(x_1, x_2, ..., x_m) = f_i\left(\sum\limits_{k=1}^{m} x_k\right)$ and $h_i(x_1, x_2, ..., x_m) = \sum\limits_{k=1}^{m} f_i\left(x_k\right)$. Then, it follows from point 1 of this lemma that $g_i(x_1, x_2, ..., x_m) < h_i(x_1, x_2, ..., x_m)$ $\forall x_k > 0$,
$k \in \{1, 2, ..., m\}$. Note that

$$\frac{\partial g_i}{\partial x_j} > 0, \frac{\partial h_i}{\partial x_j} > 0,$$

where $j \in \{1, ..., m\}$, and

$$\frac{\partial g_i}{\partial x_j} = \frac{w_i}{\left(\sum\limits_{k=1}^{m} x_k + w_i\right)^2} < \frac{w_i}{(x_j + w_i)^2} = \frac{\partial h_i}{\partial x_j}.$$

Let us demonstrate that the function $g_i(x_j, x_{-j})/h_i(x_j, x_{i-j})$ decreases monotonically as x_j increases. Indeed, we can show that

$$\frac{\partial}{\partial x_j}\left(\frac{g_i(x_j, x_{-j})}{h_i(x_j, x_{i-j})}\right) = \frac{g_i \cdot \frac{\partial h_i}{\partial x_j} - h_i \cdot \frac{\partial g_i}{\partial x_j}}{h_i^2} < 0.$$

To do so, it is enough to note that

$$\frac{\partial g_i}{\partial x_j} \cdot h_i - \frac{\partial h_i}{\partial x_j} \cdot g_i < 0.$$

Hence, if $x_j^{'} < x_j$, then

$$\frac{f_i\left(x_j + \sum\limits_{k\in K} x_k\right)}{f_i(x_j) + \sum\limits_{k\in K} f_i(x_k)} < \frac{f_i\left(x_j^{'} + \sum\limits_{k\in K} x_k\right)}{f_i(x_j^{'}) + \sum\limits_{k\in K} f_i(x_k)}.$$

Thence we have

$$\frac{f_i\left(\sum\limits_{k=1}^{m} x_k\right)}{\sum\limits_{k=1}^{m} f_i\left(x_k^{'}\right)} = \frac{f_i(x_1 + x_2 + x_3 + ... + x_m)}{f_i(x_1) + f_i(x_2) + f_i(x_3) + ... + f_i(x_m)}$$

$$< \frac{f_i(x_1^{'} + x_2 + x_3 + ... + x_m)}{f_i(x_1^{'}) + f_i(x_2) + f_i(x_3) + ... + f_i(x_m)} < \frac{f_i(x_1^{'} + x_2^{'} + x_3 + ... + x_m)}{f_i(x_1^{'}) + f_i(x_2^{'}) + f_i(x_3) + ... + f_i(x_m)} < ...$$

$$< \frac{f_i(x_1^{'} + x_2^{'} + x_3^{'} + ... + x_m^{'})}{f_i(x_1^{'}) + f_i(x_2^{'}) + f_i(x_3^{'}) + ... + f_i(x_m^{'})} = \frac{f_i(\sum\limits_{k=1}^{m} x_k^{'})}{\sum\limits_{k=1}^{m} f_i(x_k^{'})}$$

So, if $0 < x_k^{'} < x_k, k = 1, ..., m$, then

$$\frac{f_i\left(\sum\limits_{k=1}^{m} x_k\right)}{\sum\limits_{k=1}^{m} f_i\left(x_k\right)} < \frac{f_i\left(\sum\limits_{k=1}^{m} x_k^{'}\right)}{\sum\limits_{k=1}^{m} f_i(x_k^{'})}.$$

2. If follows from point 1 of this lemma that $\forall x_k > 0$, $k = 1, ..., m$ we have

$$\frac{f_i(x_1 + ... + x_{m-1} + x_m)}{f_i(x_1) + ... + f_i(x_{m-1}) + f_i(x_m)} < \frac{f_i(x_1 + ... + x_{m-1} + x_m)}{f_i(x_1) + ... + f_i(x_{m-1} + x_m)}.$$

Since $x_{m-1} < x_{m-1} + x_m$, then, applying p. 2 of this lemma, we get that

$$\frac{f_i(x_1 + ... + x_{m-1} + x_m)}{f_i(x_1) + ... + f_i(x_{m-1} + x_m)} < \frac{f_i(x_1 + ... + x_{m-1})}{f_i(x_1) + ... + f_i(x_{m-1})}.$$

Thus,

$$\frac{f_i(x_1 + ... + x_{m-1} + x_m)}{f_i(x_1) + ... + f_i(x_{m-1}) + f_i(x_m)} < \frac{f_i(x_1 + ... + x_{m-1})}{f_i(x_1) + ... + f_i(x_{m-1})}.$$

Since $x_{m-2} < x_{m-2} + x_{m-1}$, then, applying in the same manner p. 1 and p. 2 of this lemma, we get that

$$\frac{f_i(x_1 + ... + x_{m-2} + x_{m-1})}{f_i(x_1) + ... + f_i(x_{m-2}) + f_i(x_{m-1})} < \frac{f_i(x_1 + ... + x_{m-2})}{f_i(x_1) + ... + f_i(x_{m-2})}.$$

Continuing this way, we get that

$$\frac{f_i\left(\sum\limits_{k=1}^{m} x_k\right)}{\sum\limits_{k=1}^{m} f_i(x_k)} < \frac{f_i\left(\sum\limits_{k=1}^{m-1} x_k\right)}{\sum\limits_{k=1}^{m-1} f_i(x_k)} < ... < \frac{f_i(x_1+x_2)}{f_i(x_1)+f_i(x_2)} < \frac{f_i(x_1)}{f_i(x_1)}.$$

□

Proof of Theorem 3.

1. Let us consider two cases, where the number of players is even (1.1) and odd (1.2).

   1.1 We will show that $\pi = \{O, E\}$ is an NSPC in the game $(N, \{u_i\}_{i\in N})$, where $O = \{1, 3, 5, ..., n-1\}$, $E = \{2, 4, 6, ..., n\}$. We will also show that $\forall i \in O$ the inequality

$$u_i(O) = \frac{f_i(w_1 + w_3 + ... + w_{i-2} + w_{i+2} + ... + w_{n-1})}{f_i(w_1) + f_i(w_3) + ... + f_i(w_{i-2}) + f_i(w_{i+2}) + ... + f_i(w_{n-1})}$$

$$\geq \frac{f_i(w_2 + w_4 + ... + w_{n-2} + w_n)}{f_i(w_2) + f_i(w_4) + ... + f_i(w_{n-2} + w_n)}$$

   is true.

   Indeed, suppose that $x_1 = w_1,\ x_1' = w_{n-2} + w_n,\ x_2 = w_3,\ x_2' = w_3$, etc. Since it is stipulated that $\forall i, j, k\ w_i + w_j \geq w_k$, then according to point 2 of Lemma 4 the inequality above is fulfilled. It follows from point 1 of Lemma 4 that $f_i(w_{n-2}) + f_i(w_n) \geq f_i(w_{n-2} + w_n)$, so the inequality

$$\frac{f_i(w_2 + w_4 + ... + w_{n-2} + w_n)}{f_i(w_2) + f_i(w_4) + ... + f_i(w_{n-2} + w_n)} \geq \frac{f_i(w_2 + w_4 + ... + w_{n-2} + w_n)}{f_i(w_2) + f_i(w_4) + ... + f_i(w_{n-2}) + f_i(w_n)}$$

   is true.

   This leads us to the conclusion that $\forall i \in O$

$$u_i(O) = \frac{f_i(w_1 + w_3 + ... + w_{i-2} + w_{i+2} + ... + w_{n-1})}{f_i(w_1) + f_i(w_3) + ... + f_i(w_{i-2}) + f_i(w_{i+2}) + ... + f_i(w_{n-1})}$$

$$\geq \frac{f_i(w_2 + w_4 + ... + w_n)}{f_i(w_2) + f_i(w_4) + ... + f_i(w_{n-2}) + f_i(w_n)} = u_i(E \cup \{i\}).$$

   On the other hand, since $w_2 < w_1,\ w_4 < w_3$, etc., then it follows from point 2 of Lemma 4 that $\forall i \in E$

$$u_i(E) = \frac{f_i(w_2 + w_4 + ... + w_{i-2} + w_{i+2} + ... + w_n)}{f_i(w_2) + f_i(w_4) + ... f_i(w_{i-2}) + f_i(w_{i+2}) + ... + f_i(w_{n-2}) + f_i(w_n)}$$

$$\geq \frac{f_i(w_1 + w_3 ... + w_{n-1})}{f_i(w_1) + ... + f_i(w_{n-1})} = u_i(O \cup \{i\}).$$

Thus, the partition $\{O, E\}$ in the game $(N, \{u_i\}_{i\in N})$ is an NSPC. Therefore, according to Lemma 3, the partition $\{O, E\}$ is an NSPC in the weighted individual game with two coalitions.

1.2 Let us demonstrate that $\pi = \{O\backslash\{n\}, E\cup\{n\}\}$ is an NSPC in the game $(N, \{u_i\}_{i\in N})$. It follows from the condition $\forall i, j, k\ w_i + w_j \geq w_k$ and points 2 and 3 of Lemma 4 that $\forall i \in O$

$$u_i(O) = \frac{f_i(w_1 + w_3 + ... + w_{i-2} + w_{i+2} + ... + w_{n-1})}{f_i(w_1) + f_i(w_3) + ... + f_i(w_{i-2}) + f_i(w_{i+2}) + ... + f_i(w_{n-1})}$$

$$\geq \frac{f_i(w_2 + w_4 + ... + w_{n-1} + w_n)}{f_i(w_2) + f_i(w_4) + ... + f_i(w_{n-1} + w_n)}$$

$$\geq \frac{f_i(w_2 + w_4 + ... + w_{n-1} + w_n)}{f_i(w_2) + f_i(w_4) + ... + f_i(w_{n-1}) + f_i(w_n)} = u_i(E \cup \{i\}).$$

On the other hand, it follows from point 2 of Lemma 4 that $\forall i \in E$

$$u_i(E) = \frac{f_i(w_2 + w_4 + ... + w_{n-1} + w_n)}{f_i(w_2) + f_i(w_4) + ... + f_i(w_{n-1}) + f_i(w_n)} \geq \frac{f_i(w_1 + w_3 ... + w_{n-2})}{f_i(w_1) + ... + f_i(w_{n-2})} = u_i(O\cup\{i\}).$$

Thus, the partition $\{O \setminus \{n\}, E \cup \{n\}\}$ in the game $(N, \{u_i\}_{i\in N})$ is an NSPC. Then, according to Lemma 3, the partition $\{O \setminus \{n\}, E \cup \{n\}\}$ is an NSPC in the weighted individual game with two coalitions.

2. Let us show that the degenerate partition $\pi = \{\{1\}, \{2\}, ..., \{n\}\}$ is an NSP. Indeed, $H_i^{WI}(\pi) \geq H_i^{WI}(\rho)\ \forall i \in N,\ \rho \in D_i(\pi) \Leftrightarrow \forall i \in N$

$$\frac{w_i}{\sum\limits_{k\in N} w_k} \geq \frac{w_i}{w_i + w_j} \cdot \frac{w_i}{\sum\limits_{k\in N\backslash\{j\}} w_k}\ \forall i \in N.$$

Hence,

$$\frac{\sum\limits_{k\in N\backslash\{j\}} w_k}{\sum\limits_{j\in N} w_k} \geq \frac{w_i}{w_i + w_j} \Leftrightarrow f_j(\sum_{k\in N\backslash\{j\}} w_k) \geq f_j(w_i).$$

Since $\sum\limits_{k\in N\backslash\{j\}} w_k > w_i$, we can conclude due the monotonicity of the function $f_j(x)$ that the inequality is true. Thus, $\forall i \in N\ H_i^{WI}(\pi) > H_i^{WI}(\rho)\ \forall \rho \in D_i(\pi)$.

□

Proof of Statement 2. By definition, an NSPC exists iff $\forall i \in N\ H_i(\pi) - H_i(\rho) \geq 0\ \forall \rho \in \mathring{D}_i(\pi) \Leftrightarrow$

$$\frac{\lambda_{R_i(\pi(i))}}{\sum\limits_{k=1}^{|\pi(i)|}\lambda_k}\cdot\left(\frac{\sum\limits_{l\in\rho(i)\setminus i}\sum\limits_{k=1}^{|\rho(i)\setminus i|}\lambda_k\lambda_{R_i(i,l)}}{\sum\limits_{k=1}^{|\rho(i)\setminus i|}\lambda_k(\lambda_1+\lambda_2)}\right)\geq\frac{\lambda_{R_i(\rho(i))}}{\sum\limits_{k=1}^{|\rho(i)|}\lambda_k}\cdot\left(\frac{\sum\limits_{l\in\pi(i)\setminus i}\sum\limits_{k=1}^{|\pi(i)-1|}\lambda_k\lambda_{R_i(i,l)}}{\sum\limits_{k=1}^{|\pi(i)-1|}\lambda_k(\lambda_1+\lambda_2)}\right).$$

Therefore,

$$H_i(\pi(i))=\frac{\lambda_{R_i(\pi(i))}}{\sum\limits_{k=1}^{|\pi(i)|}\lambda_k}\cdot\left(\frac{\sum\limits_{k=1}^{|\pi(i)-1|}\lambda_k}{\sum\limits_{l\in\pi(i)\setminus i}\sum\limits_{k=1}^{|\pi(i)-1|}\lambda_k\lambda_{R_i(i,l)}}\right)\geq\frac{\lambda_{R_i(\rho(i))}}{\sum\limits_{k=1}^{|\rho(i)|}\lambda_k}\cdot\left(\frac{\sum\limits_{k=1}^{|\rho(i)\setminus i|}\lambda_k}{\sum\limits_{l\in\rho(i)\setminus i}\sum\limits_{k=1}^{|\rho(i)\setminus i|}\lambda_k\lambda_{R_i(i,l)}}\right)=H_i(\rho(i)).$$

□

Proof of Theorem 4.

1. By definition, the fact that $\pi=\{O,E\}$ is an NSPC means that $H_i^{RI}(\pi)\geq H_i^{RI}(\rho)$, $\forall i\in N$, $\rho\in\mathring{D}_i(\pi)\Leftrightarrow$

$$\frac{\lambda_{R_i(\pi(i))}}{\sum\limits_{j=1}^{|\pi(i)|}\lambda_j}\cdot\left(\frac{\sum\limits_{j=1}^{|\pi\setminus\pi(i)|}\lambda_j\cdot\lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|\pi\setminus\pi(i)|}\lambda_j\cdot(\lambda_1+\lambda_2)}\right)\geq\frac{\lambda_{R_i(\rho(i))}}{\sum\limits_{j=1}^{|\rho(i)|}\lambda_j}\cdot\left(\frac{\sum\limits_{j=1}^{|\pi(i)\setminus\{i\}|}\lambda_j\cdot\lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|\pi(i)\setminus\{i\}|}\lambda_j\cdot(\lambda_1+\lambda_2)}\right).$$

Since $|\pi(i)|\leq|\rho(i)|$ and $\forall i\in O$ $\lambda_{R_i(O)}=\lambda_{R_i(E)\cup\{i\}}$, as well as $\forall i\in E$ $\lambda_{R_i(E)}>\lambda_{R_i(O)\cup\{i\}}$, we have

$$\frac{\lambda_{R_i(\pi(i))}}{\sum\limits_{j=1}^{|\pi(i)|}\lambda_j}\geq\frac{\lambda_{R_i(\rho(i))}}{\sum\limits_{j=1}^{|\rho(i)|}\lambda_j}.$$

Let us show that the value in parentheses on the left-hand side is greater than the value in parentheses on the right-hand side, i.e.

$$\frac{\sum\limits_{j=1}^{|\pi\setminus\pi(i)|}\lambda_j\cdot\lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|\pi\setminus\pi(i)|}\lambda_j\cdot(\lambda_1+\lambda_2)}\geq\frac{\sum\limits_{j=1}^{|\pi(i)\setminus\{i\}|}\lambda_j\cdot\lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|\pi(i)\setminus\{i\}|}\lambda_j\cdot(\lambda_1+\lambda_2)}.$$

To this end, we will demonstrate that the inequality

$$\frac{\sum\limits_{j=1}^{|K|+1}\lambda_j\cdot\lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|K|+1}\lambda_j}\geq\frac{\sum\limits_{j=1}^{|K|}\lambda_j\cdot\lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|K|}\lambda_j}$$

occurs when $K \subset N$, $i \in N \setminus K$.

Indeed, the inequality $\lambda_{R_i(i,|K|+1)} \geq \lambda_{R_i(i,j)}$ $\forall j \in K$ leads to

$$\lambda_{R_i(i,|K|+1)} \cdot \sum_{j=1}^{|K|} \lambda_j \geq \sum_{j=1}^{|K|} \lambda_j \cdot \lambda_{R_i(i,j)}.$$

Therefore,

$$1 + \frac{\lambda_{|K|+1} \cdot \lambda_{R_i(i,|K|+1)}}{\sum\limits_{j=1}^{|K|} \lambda_j \cdot \lambda_{R_i(i,j)}} \geq 1 + \frac{\lambda_{|K|+1}}{\sum\limits_{j=1}^{|K|} \lambda_j},$$

$$\frac{\sum\limits_{j=1}^{|K|+1} \lambda_j \cdot \lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|K|} \lambda_j \cdot \lambda_{R_i(i,j)}} \geq \frac{\sum\limits_{j=1}^{|K|+1} \lambda_j}{\sum\limits_{j=1}^{|K|} \lambda_j}.$$

Finally, we get the necessary inequality

$$\frac{\sum\limits_{j=1}^{|K|+1} \lambda_j \cdot \lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|K|+1} \lambda_j} \geq \frac{\sum\limits_{j=1}^{|K|} \lambda_j \cdot \lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|K|} \lambda_j}.$$

Then, while $|\pi \setminus \pi(i)| \geq |\pi(i) \setminus \{i\}|$, we have

$$\left(\frac{\sum\limits_{j=1}^{|\pi \setminus \pi(i)|} \lambda_j \cdot \lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|\pi \setminus \pi(i)|} \lambda_j \cdot (\lambda_1 + \lambda_2)} \right) \geq \left(\frac{\sum\limits_{j=1}^{|\pi(i) \setminus \{i\}|} \lambda_j \cdot \lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|\pi(i) \setminus \{i\}|} \lambda_j \cdot (\lambda_1 + \lambda_2)} \right).$$

Thus, $\forall \lambda_i$ $i = 1, ..., n$ no player from the O coalition will want to join the E coalition. Let's show that no one from the E coalition wants to switch to O, i.e. $H_i^{RI}(\pi) \geq H_i^{RI}(\rho)$, $\forall i \in E$, $\rho \in \mathring{D}_i(\pi) \Leftrightarrow$

$$\frac{\lambda_{R_i(E)}}{\sum\limits_{j=1}^{|E|} \lambda_j} \cdot \left(\frac{\sum\limits_{j=1}^{|O|} \lambda_j \cdot \lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|O|} \lambda_j \cdot (\lambda_1 + \lambda_2)} \right) \geq \frac{\lambda_{R_i(O \cup \{i\})}}{\sum\limits_{j=1}^{|O \cup \{i\}|} \lambda_j} \cdot \left(\frac{\sum\limits_{j=1}^{|E \setminus \{i\}|} \lambda_j \cdot \lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|E \setminus \{i\}|} \lambda_j \cdot (\lambda_1 + \lambda_2)} \right).$$

Since $\lambda_{R_i(E)} > \lambda_{R_i(E)}$ and $|E| < |O \cup \{i\}|$, then

$$\frac{\lambda_{R_i(E)}}{\sum\limits_{j=1}^{|E|}\lambda_j} > \frac{\lambda_{R_i(O\cup\{i\})}}{\sum\limits_{j=1}^{|O\cup\{i\}|}\lambda_j}.$$

According to the condition of this theorem $\forall i \in E$ we have

$$\frac{\sum\limits_{j=1}^{|O|}\lambda_j\cdot\lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|O|}\lambda_j\cdot(\lambda_1+\lambda_2)} \geq \frac{\sum\limits_{j=1}^{|E\setminus\{i\}|}\lambda_j\cdot\lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|E\setminus\{i\}|}\lambda_j\cdot(\lambda_1+\lambda_2)}.$$

This leads us to

$$\frac{\lambda_{R_i(E)}}{\sum\limits_{j=1}^{|E|}\lambda_j}\cdot\left(\frac{\sum\limits_{j=1}^{|O|}\lambda_j\cdot\lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|O|}\lambda_j\cdot(\lambda_1+\lambda_2)}\right) \geq \frac{\lambda_{R_i(O\cup\{i\})}}{\sum\limits_{j=1}^{|O\cup\{i\}|}\lambda_j}\cdot\left(\frac{\sum\limits_{j=1}^{|E\setminus\{i\}|}\lambda_j\cdot\lambda_{R_i(i,j)}}{\sum\limits_{j=1}^{|E\setminus\{i\}|}\lambda_j\cdot(\lambda_1+\lambda_2)}\right),$$

which is equivalent to $H_i^{RI}(\pi) \geq H_i^{RI}(\rho)$, $\forall i \in E$, $\rho \in \mathring{D}_i(\pi)$.

2. Suppose $\pi = \{B_1, B_2, ..., B_m\}$ is a certain coalition partition. We start with showing that if player 1 is part of a coalition B_1 where $|B_1| > 1$, then he/she would benefit from moving out to an empty coalition, i.e. $H_1(B_1, B_2, ..., B_m) < H_1(\{1\}, B_1\setminus\{1\}, B_2, ..., B_m) \Leftrightarrow$

$$\frac{\lambda_1}{\sum\limits_{k=1}^{|B_1|}\lambda_k}\cdot\frac{\lambda_1}{\sum\limits_{k=1}^{|\pi|}\lambda_k} < \frac{\lambda_1}{\sum\limits_{k=1}^{|\pi|+1}\lambda_k},$$

which is fulfilled since

$$\frac{\lambda_1}{\sum\limits_{k=1}^{|B_1|}\lambda_k} < \frac{\sum\limits_{k=1}^{|\pi|}\lambda_k}{\sum\limits_{k=1}^{|\pi|+1}\lambda_k}$$

is true.

Thus, it is always beneficial for player 1 to move out to an empty coalition. If after such a move, there is at least one coalition of power greater than one, then it will definitely comprise a player j who would benefit from moving to an empty coalition. Here is the reasoning. Let $N' = \bigcup\limits_{\substack{B\in\pi\\|B|\neq 1}} B$ and $j \in B_2 \subseteq N'$: $R_j(N') = 1$. It is possible here that $B_2 = B_1 \setminus 1$.

We have $H_j(\{1\}, B_1\setminus\{1\}, B_2, ..., B_m) < H_j(\{1\}, \{j\}, B_1\setminus\{1\}, B_2\setminus\{j\}, ..., B_m) \Leftrightarrow$

$$\frac{\lambda_1}{\sum\limits_{k=1}^{|B_2|} \lambda_k} \cdot \frac{\lambda_{R_j(N')}}{\sum\limits_{k=1}^{|\pi|+1} \lambda_k} < \frac{\lambda_{R_j(N')}}{\sum\limits_{k=1}^{|\pi|+2} \lambda_k}.$$

The inequality is valid since the inequality

$$\frac{\lambda_1}{\sum\limits_{k=1}^{|B_2|} \lambda_k} = \frac{\lambda_1}{\lambda_1 + \sum\limits_{k=2}^{|B_2|} \lambda_k} < \frac{\sum\limits_{k=1}^{|\pi|+1} \lambda_k}{\sum\limits_{k=1}^{|\pi|+1} \lambda_k + \lambda_{|\pi|+2}} = \frac{\sum\limits_{k=1}^{|\pi|+1} \lambda_k}{\sum\limits_{k=1}^{|\pi|+2} \lambda_k}$$

is true. Continuing in the same manner, we find that the only possible an NSP is a degenerate partition.

Next, we prove the necessary and sufficient conditions for an NSP. Since an NSP can only have a degenerate form, it is necessary and sufficient to show that not a single player in the degenerate partition π would want to join a player with a greater ranking. Thus, π is an NSP $\Leftrightarrow$ $H_i(\pi) \geq H_i(\rho)\ \forall i \in N\ R_j(N) \succ R_i(N)\ \forall \rho \in D_i(\pi)$, i.e.

$$\frac{\lambda_{R_i(N)}}{\sum\limits_{k=1}^{|N|} \lambda_k} \geq \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_{R_i(N)-1}}{\sum\limits_{k=1}^{|N|-1} \lambda_k}\ \forall i \in N$$

or

$$\frac{\sum\limits_{k=1}^{|N|-1} \lambda_k}{\sum\limits_{k=1}^{|N|} \lambda_k} \geq \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_{R_i(N)-1}}{\lambda_{R_i(N)}}\ \forall i \in N.$$

Thence we get the necessary and sufficient conditions

$$\frac{\sum\limits_{k=1}^{|N|-1} \lambda_k}{\sum\limits_{k=1}^{|N|} \lambda_k} \geq \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot \max_i \left(\frac{\lambda_{i-1}}{\lambda_i}\right)\ \forall i \in N.$$

□

Proof of Theorem 5.

1. The existence of a core strategy profile follows from point 1 of Theorem 1. Furthermore, if $w_{is_i} \neq w_{is_j}\ \forall s_j \in S$, then equilibrium in dominant strategies exists in the game and, hence, there exists a unique core partition.

2. Consider the ranking collective game. Let us show that a degenerate partition is a CP. It follows from the proof of point 3 of Theorem 2 that only a degenerate partition can

be a CP. Suppose that for $\{\{1\},\{2\},...,\{n\}\}$ there is a blocking coalition B. It that case however, there will be a player $j \in B$ with a ranking $|B|$ who would benefit from moving to an empty coalition (see proof of Theorem 2 p. 3). We have a contradiction. Hence, a CP exists and has the form $\{\{1\},\{2\},...,\{n\}\}$.

Let us consider the weighted individual game. Suppose that the partition $\pi = \{\{1\},...,\{n\}\}$ is not a CP. Then, there is a blocking coalition B such that $H_i^{WI}(\pi) < H_i^{WI}(\rho)\ \forall i \in B$, $\rho = \{B, \pi_{-B}\} \Leftrightarrow$

$$\frac{w_i}{\sum\limits_{k \in N} w_k} < \frac{w_i}{w_i + \sum\limits_{k \in B \setminus \{i\}} w_k} \cdot \frac{w_i}{w_i + \sum\limits_{k \in N \setminus B} w_k}\ \forall i \in B.$$

Therefore,

$$\frac{w_i + \sum\limits_{k \in N \setminus B} w_k}{\sum\limits_{k \in N} w_k} < \frac{w_i}{w_i + \sum\limits_{k \in B \setminus \{i\}} w_k},$$

and

$$\frac{w_i + \sum\limits_{k \in N \setminus B} w_k}{w_i + \sum\limits_{k \in N \setminus B} w_k + \sum\limits_{k \in B \setminus \{i\}} w_k} < \frac{w_i}{w_i + \sum\limits_{k \in B \setminus \{i\}} w_k},$$

Let us define the function

$$f_A(x) = \frac{x}{x + \sum\limits_{j \in A} w_j},$$

where $A \subseteq N, A \neq \emptyset$. Then, the left-hand side of the inequality represents the function $f_{B \setminus \{i\}}(w_i + \sum\limits_{k \in N \setminus B} w_k)$, and $f_{B \setminus \{i\}}(w_i)$ is written on the right-hand side. Since $w_i + \sum\limits_{k \in N \setminus B} w_k > w_i$, then the conclusion following from the monotonicity of the function $f_A(x)$ is that the inequality does not hold. We have a contradiction. A blocking coalition does not exist.

Now consider the ranking individual game. Let us show that a degenerate partition is a CP. Let us assume this is not true. Let B block $\{\{1\},\{2\},...,\{n\}\}$, then $H_i^{RC}(\pi) < H_i^{RC}(\rho)\ \forall i \in B$, $\rho = \{B, \pi_{-B}\} \Leftrightarrow$

$$\frac{\frac{\lambda_{R_i(N)}}{|N|}}{\sum\limits_{k=1}^{|N|} \lambda_k} < \frac{\frac{\lambda_{R_i(B)}}{|B|}}{\sum\limits_{k=1}^{|B|} \lambda_k} \cdot \frac{\frac{\lambda_{R_i((N \setminus B) \cup \{i\})}}{|(N \setminus B) \cup \{i\}|}}{\sum\limits_{k=1}^{|(N \setminus B) \cup \{i\}|} \lambda_k}\ \forall i \in B.$$

Take a player $j \in B$ such that $R_j(B) = 1$. Note that $R_j(N) = R_j((N \setminus B) \cup \{j\})$ is true for the player j. For instance, the ranking of the player $j = 3$ in the coalition $N = \{1, 2, 3, 4, 5, 6\}$ is 3, $R_j(N) = 3$. Suppose the blocking coalition has the form $B = \{3, 4, 5\}$. Then, players team up into a coalition, forming a partition ρ. In that case, the transversal in which player 3 wins within one's coalition has the form $(N \setminus B) \cup \{3\} = \{1, 2, 3, 6\}$ and player 3 preserves one's ranking within that coalition, i.e. $R_3((N \setminus B) \cup \{3\})$. Thus, for the player $j \in B$ the inequality above can be divided by $\lambda_{R_j}(N)$. We get the inequality

$$\frac{\sum\limits_{k=1}^{|(N\setminus B)\cup\{i\}|} \lambda_k}{\sum\limits_{k=1}^{|N|} \lambda_k} < \frac{\lambda_1}{\sum\limits_{k=1}^{|B|} \lambda_k}.$$

We transform this inequality into

$$\frac{\sum\limits_{k=1}^{|N|-|B|+1} \lambda_k}{\sum\limits_{k=1}^{|N|-|B|+1} \lambda_k + \sum\limits_{k=|N|-|B|+2}^{|N|} \lambda_k} < \frac{\lambda_1}{\lambda_1 + \sum\limits_{k=1}^{|B|-1} \lambda_k}$$

Note that the left-hand and the right-hand sides of the inequality represent a function of the form $f(x) = \dfrac{x}{x+c}$. Indeed, on the left-hand side of the inequality, we set $x = \sum\limits_{k=1}^{|N|-|B|+1} \lambda_k$, $c = \sum\limits_{k=|N|-|B|+2}^{|N|} \lambda_k$. On the right-hand side, we set $x = \lambda_1$, $c = \sum\limits_{k=1}^{|B|-1} \lambda_k$. Also, $\sum\limits_{k=1}^{|N|-|B|+1} \lambda_k > \lambda_1$ and $\sum\limits_{k=1}^{|B|-1} \lambda_k > \sum\limits_{k=|N|-|B|+2}^{|N|} \lambda_k$. However, the function $f(x)$ increases monotonically when x increases and decreases monotonically when c increases. Hence, the left-hand side of the inequality is strictly greater than the right-hand side of the inequality. We have a contradiction. Therefore, a blocking coalition does not exist.

□

Proof of Statement 3.

1. Let $\pi = \{B_1, B_2, ..., B_m\}$ and, without loss of generality, we assume that $i \in B_1$. We denote by $H_1^{WI}(\pi)$ the payoff of player 1 in the weighted individual game in the partition π, and by $H_1^{WC}(\pi)$ the payoff of player 1 in the weighted collective game in the partition π. In that case, the condition $H_1^{WI}(\pi) \geq H_1^{WC}(\pi)$ is equivalent to

$$\frac{w_i}{\prod\limits_{B\in\pi} \sum\limits_{j\in B} w_j} \cdot \sum_{K\in M_i(\pi)} \frac{\prod\limits_{j\in K} w_j}{\sum\limits_{l\in K} w_l} \geq \frac{w_i}{\sum\limits_{j\in N} w_j}.$$

This leads to

$$\sum_{K\in M_i(\pi)} \frac{\prod\limits_{j\in K} w_j}{\sum\limits_{l\in K} w_l} \geq \frac{\prod\limits_{B\in\pi}\sum\limits_{j\in B} w_j}{\sum\limits_{j\in N} w_j}.$$

The function $\sum\limits_{K\in M_i(\pi)} \phi(K)$ is written on the left-hand side of the inequality, and the function $\phi_N(\pi)$ is written on the right-hand side.

2. Let $\pi = \{B_1, B_2, ..., B_m\}$. By definition, dominance means that $H_1^{RI}(\pi) \geq H_1^{RC}(\pi) \Leftrightarrow$

$$\frac{\lambda_1}{\prod\limits_{B_j\in\pi}\sum\limits_{k=1}^{|B_j|}\lambda_k}\left(\sum_{K\in M_i(\pi)} \frac{\prod\limits_{j\in K}\lambda_j}{\sum\limits_{l\in K}\lambda_l}\right) \geq \frac{\lambda_1}{\sum\limits_{B_j\in\pi}\sum\limits_{k=1}^{|B_j|}\lambda_k}.$$

Note that where $i = 1$, we have

$$\frac{\lambda_1}{\prod\limits_{B_j\in\pi}\sum\limits_{k=1}^{|B_j|}\lambda_k}\left(\sum_{K\in M_i(\pi)} \frac{\prod\limits_{j\in K}\lambda_j}{\sum\limits_{l\in K}\lambda_l}\right) = \frac{\lambda_1}{\sum\limits_{k=1}^{|\pi(1)|}\lambda_k}\cdot\frac{\lambda_1}{\sum\limits_{k=1}^{|\pi|}\lambda_k}.$$

This leads to

$$\frac{\lambda_1}{\sum\limits_{k=1}^{|\pi(1)|}\lambda_k\cdot\sum\limits_{k=1}^{|\pi|}\lambda_k} \geq \frac{1}{\sum\limits_{B_j\in\pi}\sum\limits_{k=1}^{|B_j|}\lambda_k}.$$

The condition is that π is an NSPC in the ranking collective and the ranking individual games. It follows from point 1 of Theorem 2 that only a harmonious partition $\pi = \{O, E\} = \{\{1, 3, 5, ..., n\}, \{2, 4, 6, ..., n-1\}\}, 1 \in O$ can be an NSPC.

It is therefore enough to show that

$$\frac{\lambda_1}{\sum\limits_{k=1}^{|O|}\lambda_k\cdot(\lambda_1+\lambda_2)} \geq \frac{1}{\sum\limits_{k=1}^{|O|}\lambda_k + \sum\limits_{k=1}^{|O|-1}\lambda_k}$$

or

$$\lambda_1\cdot\sum_{k=1}^{|O|-1}\lambda_k \geq \lambda_2\sum_{k=1}^{|O|}\lambda_k.$$

This gives us the inequality

$$\frac{\sum\limits_{k=1}^{|O|-1}\lambda_k}{\sum\limits_{k=1}^{|O|}\lambda_k} \geq \frac{\lambda_2}{\lambda_1},$$

which is equivalent to

$$\frac{\sum_{k=1}^{(n-1)/2} \lambda_k}{\sum_{k=1}^{(n+1)/2} \lambda_k} \geq \frac{\lambda_2}{\lambda_1}.$$

□

References

Aumann, R. J., & Hart, S. (1992). *Handbook of game theory with economic applications* (Vol. 2). Elsevier.

Banerjee, S., Konishi, H., & Sönmez, T. (2001). Core in a simple coalition formation game. *Social Choice and Welfare*, 18(1), 135–153.

Bilò, V., Gourvès, L., & Monnot, J. (2023). Project games. *Theoretical Computer Science*, 940, 97–111.

Bogomolnaia, A., & Jackson, M. O. (2002). The stability of hedonic coalition structures. *Games and Economic Behavior*, 38(2), 201–230.

Corchón, L. C. (2007). The theory of contests: A survey. *Review of economic design*, 11, 69–100.

Dasgupta, A., & Nti, K. O. (1998). Designing an optimal contest. *European Journal of Political Economy*, 14(4), 587–603.

Dietzenbacher, B. J., & Kondratev, A. Y. (2023). Fair and consistent prize allocation in competitions. *Management Science*, 69(6), 3319–3339.

Dreze, J. H., & Greenberg, J. (1980). Hedonic coalitions: Optimality and stability. *Econometrica: Journal of the Econometric Society*, 987–1003.

Ewerhart, C. (2017). The lottery contest is a best-response potential game. *Economics Letters*, 155, 168–171.

Farrell, J., & Scotchmer, S. (1988). Partnerships. *The Quarterly Journal of Economics*, 103(2), 279–297.

Gallo, O., & Inarra, E. (2018). Rationing rules and stable coalition structures. *Theoretical Economics*, 13(3), 933–950.

Garfinkel, M. R. (2004). Stable alliance formation in distributional conflict. *European Journal of Political Economy*, 20(4), 829–852.

Greenberg, J., & Weber, S. (1985). Multiparty equilibria under proportional representation. *American Political Science Review*, 79(3), 693–703.

Greenberg, J., & Weber, S. (1993). Stable coalition structures with a unidimensional set of alternatives. *Journal of Economic Theory*, 60(1), 62–82.

Gusev, V., Nesterov, A., Reshetov, M., & Suzdaltsev, A. (2024). The existence of a pure-strategy nash equilibrium in a discrete ponds dilemma. *Games and Economic Behavior*, 147, 38–51.

Gusev, V. V. (2021). Nash-stable coalition partition and potential functions in games with coalition structure. *European Journal of Operational Research*, 295(3), 1180–1188.

Harks, T., & Klimm, M. (2012). On the existence of pure nash equilibria in weighted congestion games. *Mathematics of Operations Research*, 37(3), 419–436.

Harks, T., & Klimm, M. (2015). Equilibria in a class of aggregative location games. *Journal of Mathematical Economics*, 61, 211–220.

Jandoc, K., & Juarez, R. (2017). Self-enforcing coalitions with power accumulation. *International Journal of Game Theory*, 46, 327–355.

Milchtaich, I. (1996). Congestion games with player-specific payoff functions. *Games and economic behavior*, 13(1), 111–124.

Monderer, D., & Shapley, L. S. (1996). Potential games. *Games and economic behavior*, 14(1), 124–143.

Razin, R., & Piccione, M. (2009). Coalition formation under power relations. *Theoretical Economics*, 4(1), 1–15.

Tullock, G. (2008). Efficient rent seeking. In *40 years of research on rent seeking 1* (pp. 105–120). Springer.

Watts, A. (2007). Formation of segregated and integrated groups. *International Journal of Game Theory*, 35, 505–519.

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