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On Isolated Periodic Points of Diffeomorphisms with Expanding Attractors of Codimension 1

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Abstract—In this paper we consider an Ω-stable 3-diffeomorphism whose chain-recurrent set consists of isolated periodic points and hyperbolic 2-dimensional nontrivial attractors. Nontrivial attractors in this case can only be expanding, orientable or not. The most known example from the class under consideration is the DA-diffeomorphism obtained from the algebraic Anosov diffeomorphism, given on a 3-torus, by Smale's surgery. Each such attractor has bunches of degree 1 and 2. We estimate the minimum number of isolated periodic points using information about the structure of attractors. Also, we investigate the topological structure of ambient manifolds for diffeomorphisms with k bunches and k isolated periodic points.

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1. INTRODUCTION AND FORMULATION OF RESULTS

Let M^n be a closed smooth connected n-manifold with a metric d and let $f : M^n \to M^n$ be a diffeomorphism. An invariant compact set $\Lambda \subset M^n$ is called *hyperbolic* if there is a continuous Df-invariant splitting of the tangent bundle $T_\Lambda M^n$ into *stable* and *unstable subbundles* $E^s_\Lambda \oplus E^u_\Lambda$, $\dim E^s_x + \dim E^u_x = n$ $(x \in \Lambda)$ such that for natural k and for some fixed $C_s > 0$, $C_u > 0$, $0 < \lambda < 1$

$$\|Df^k(v)\| \leqslant C_s \lambda^k \|v\|, \qquad v \in E^s_\Lambda,$$
$$\|Df^{-k}(w)\| \leqslant C_u \lambda^k \|w\|, \qquad w \in E^u_\Lambda.$$

Recall that an *ε-chain of length* $m \in \mathbb{N}$, joining points $x, y \in M^n$, for f is called a collection of points $x = x_0, \ldots, x_m = y$ such that $d(f(x_{i-1}), x_i) < \varepsilon$ for $1 \leqslant i \leqslant m$. A point $x \in M^n$ is called *chain recurrent* for f if for any $\varepsilon > 0$ there exists m, depending on $\varepsilon > 0$, and an ε-chain of length m, joining x to itself. The set of all chain recurrent points is called a *chain-recurrent set* and is denoted by $\mathcal{R}_f$.

Summarizing the results in [1–4], we know that the hyperbolicity of $\mathcal{R}_f$ is equivalent to Ω-stability of f, that is, small perturbations of f preserve the chain-recurrent set (equivalently nonwandering set $NW(f)$) structure. Thus, by [5], $\mathcal{R}_f$ consists of a finite number of pairwise disjoint sets, called *basic sets*, each of which is compact, invariant, and topologically transitive (contains a dense orbit). If a basic set is a periodic orbit, then it is named *trivial*. In the opposite case, it is *nontrivial*. If $\dim \Lambda = n - 1$ for some basic set Λ, then it is called a *basic set of codimension* 1.

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A stable and an unstable manifold of a point $x \in \Lambda$, where Λ is a basic set, can be defined in the following way:

$$W^s_x = \{y \in M^n \mid \lim_{k\to+\infty} d\big(f^k(x), f^k(y)\big) = 0\},$$
$$W^u_x = \{y \in M^n \mid \lim_{k\to+\infty} d\big(f^{-k}(x), f^{-k}(y)\big) = 0\}.$$

By [5], W^s_x and W^u_x are injective immersions of $\mathbb{R}^q$ and $\mathbb{R}^{n-q}$, respectively, for some $q \in \{0, 1, \ldots, n\}$. For $r > 0$ we denote by $W^s_{x,r}$ and $W^u_{x,r}$ the immersions of discs $D^q_r \subset \mathbb{R}^q$ and $D^{n-q}_r \subset \mathbb{R}^{n-q}$.

The concept of orientability can be introduced for a basic set Λ with $\dim W^s_x = 1$ or $\dim W^u_x = 1$, $x \in \Lambda$. A nontrivial basic set Λ is called *orientable* if for any point $x \in \Lambda$ and any fixed numbers $\alpha > 0$, $\beta > 0$ the intersection index[1)] $W^u_{x,\alpha} \cap W^s_{x,\beta}$ is the same at all intersection points ($+1$ or -1) [7]. Otherwise, the basic set is called *nonorientable*.

A basic set Λ is called an *attractor* if it has a compact *trapping neighborhood* U such that $f(U) \subset \operatorname{int} U$ and $\bigcap\limits_{n=1}^{+\infty} f^n(U) = \Lambda$. Each hyperbolic attractor consists of unstable manifolds of its points by [8]. If $\dim \Lambda = \dim W^u_x$, $x \in \Lambda$, for a hyperbolic attractor Λ, then it is *expanding*.

This article is devoted to the question of the influence of the presence of a nontrivial attractor in the nonwandering set of a dynamical system on the complexity of the system. For example, if the nonwandering set of an Ω-stable diffeomorphism contains the Smale solenoid or 2-dimensional Anosov torus, then it necessarily contains at least another nontrivial basic set [9]. However, some types of hyperbolic attractors can coexist with isolated periodic orbits without additional nontrivial basic sets. A 2-dimensional expanding attractor is one of them. The most famous example of a 3-diffeomorphism with a 2-dimensional expanding attractor is a DA-diffeomorphism, obtained from the Anosov diffeomorphism on a 3-torus by Smale surgery.

Any codimension 1 expanding attractor Λ divides its basin W^s_Λ into a finite number of connected components. Every such component B determines *a bunch* b as the union of unstable manifolds of all periodic points from Λ whose stable separatrix belongs to B. The number k of such so-called *boundary points* is finite and is called *a degree of the bunch* b and b is called *k-bunch* with the *basin* B.

If $n \geqslant 3$ then, by [10, Theorem 2.1], any codimension 1 expanding attractor Λ has 1-bunches or 2-bunches only. Moreover, the following fact takes place.

Statement 1. *If Λ is a hyperbolic expanding attractor of codimension 1 of a diffeomorphism $f : M^n \to M^n$ given on a closed smooth n-manifold M^n, then Λ is nonorientable iff it has a 1-bunch.*

In this paper we consider diffeomorphisms every nontrivial basic set of which is an expanding attractor of codimension 1. We investigate how the number of bunches affects the number of isolated periodic points in the nonwandering set of a 3-diffeomorphism and the structure of its ambient manifold.

An expanding attractor with k bunches of degree 2 can be obtained by applying Smale surgery at k points. The resulting system has k isolated sources. The question is whether there is a

[1)]Let $J^k : \mathbb{R}^k \to M^3$ be immersions, D^k be open balls of finite radii in $\mathbb{R}^k$, $k = 1, 2$. Then the restrictions $J^k : D^k \to M$ are embeddings and their images $W^k = J^k(D^k)$ are smooth embedded submanifolds of the manifold M^3. Let U^k be a tubular neighborhood of W^k, which are images of embeddings in M^3 of spaces of $(3-k)$-dimensional vector bundles on W^k [6, Chapter 4, § 5]. Since the balls D^k are contractible, these bundles are trivial and, hence, $U^2 \setminus W^2$ consists of two connected components U^2_+ and U^2_-. This allows us to define a function $\sigma : U^2_+ \cup U^2_- \to \mathbb{Z}$ such that $\sigma(x) = 1$ if $x \in U^2_+$ and $\sigma(x) = 0$ if $x \in U^2_-$. If submanifolds W^1 and W^2 intersect transversally at a point $x = J^1(t)$, $t \in D^1$, then there exists a number $\delta > 0$ such that $J^1((t - 2\delta, t + 2\delta)) \subset U^2$. The number

$$\operatorname{Ind}_x(W^1, W^2) = \sigma(t + \delta) - \sigma(t - \delta)$$

is called an *intersection index* of submanifolds W^1 and W^2 at the point x. Notice that this definition does not require orientability of the manifold M^3.

diffeomorphism whose nonwandering set consists of such an attractor and isolated periodic points, but the number of points in trivial basic sets is less than k. From this article it follows that the answer is no.

The first example of a 3-diffeomorphism, the only nontrivial basic set of which is a nonorientable 2-dimensional expanding attractor, was constructed in [9] and has 12 isolated periodic points. Theorem 1 and [10, Theorem 2.1] say that there is no such system with less than 12 points.

The main result is the following theorem.

Theorem 1. *Let $f : M^3 \to M^3$ be an Ω-stable diffeomorphism given on a closed 3-manifold, and let Λ be a nonempty set of nontrivial basic sets of f. If Λ consists of expanding attractors of codimension 1 having a total of k_1 bunches of degree 1 and k_2 bunches of degree 2, then the number of points in the set $NW(f) \setminus \Lambda$ is no less than $\frac{3}{2}k_1 + k_2$ and this estimate is exact.*

Corollary 1. *If the nonwandering set $NW(f)$ of an Ω-stable diffeomorphism $f : M^3 \to M^3$ consists of 2-dimensional expanding attractors with k bunches in total and k isolated periodic points, then*

- *each nontrivial attractor and M^3 are orientable;*
- $\dim W^u_p = 1$ *for every isolated saddle point p;*
- *each connected component of the set $M^3 \setminus \Lambda$ is homeomorphic to a punctured 3-sphere.*

It is clear from Corollary 1 that in a subclass of diffeomorphisms with an orientable attractor Λ and a nonorientable manifold M^3 the estimates from Theorem 1 cannot be reached. Indeed, if Λ is orientable, then $k_1 = 0$ by Statement 1 and the number of points in the set $NW(f) \setminus \Lambda$ is no less than k_2. But this number cannot be equal to k_2, because if it is, then M^3 is orientable by Corollary 1. For this case the following theorem takes place.

Theorem 2. *Let an Ω-stable diffeomorphism $f : M^3 \to M^3$ be given on a closed nonorientable manifold M^3 and a set of nontrivial basic sets consists of expanding orientable 2-dimensional attractors having a total of k bunches, then the number of isolated periodic points is no less than $k + 2$.*

A simple structure of the orbit space of the restriction of f to the set $W^s_\Lambda \setminus \Lambda$ gives us a way to obtain an Ω-stable system without nontrivial basic sets from the considered one. We will describe a procedure of transition from a cascade with codimension 1 expanding attractors to a corresponding regular system in Section 2. Section 3 gives a proof of estimates from Theorem 1 and Theorem 2. A proof of Corollary 1 directly follows from the proof of Theorem 1. Finally, in Section 4 we show that the estimates are exact.

2. TRANSITION TO A REGULAR SYSTEM

In this section we will show how to obtain a system $\widetilde{f} : \widetilde{M}^3 \to \widetilde{M}^3$ with regular dynamics from a system $f : M^3 \to M^3$ with codimension 1 expanding attractors and isolated periodic points. The classical technique for working with orientable expanding attractors of codimension 1 is to replace them by hyperbolic sinks. This is facilitated by the structure of the orbit spaces of the action of the diffeomorphism on the bunch basins. We adapt this technique to the case where the attractor is nonorientable, that is, it has bunches of degree 1. This allows us to use the Lefschetz formula for the desired estimates.

Let Λ be a set of nontrivial attractors of f and U_Λ be its trapping neighborhood. The boundary of U_Λ consists of k_1 copies of[2)] $\mathbb{R}P^2$ and k_2 copies of $\mathbb{S}^2$ ([11, Lemma 2.2]) as in Fig. 1. Let $M^3 \setminus \operatorname{int} U_\Lambda = M^+ \sqcup M^-$, where M^+ and M^- are compact subsets of M^3 (one of them can be empty) such that ∂M^+ consists of k^+ 2-spheres, ∂M^- consists of $k_1^- > 0$ copies of $\mathbb{R}P^2$ and k_2^-

[2)] $\mathbb{R}P^2$ is the real projective plane.

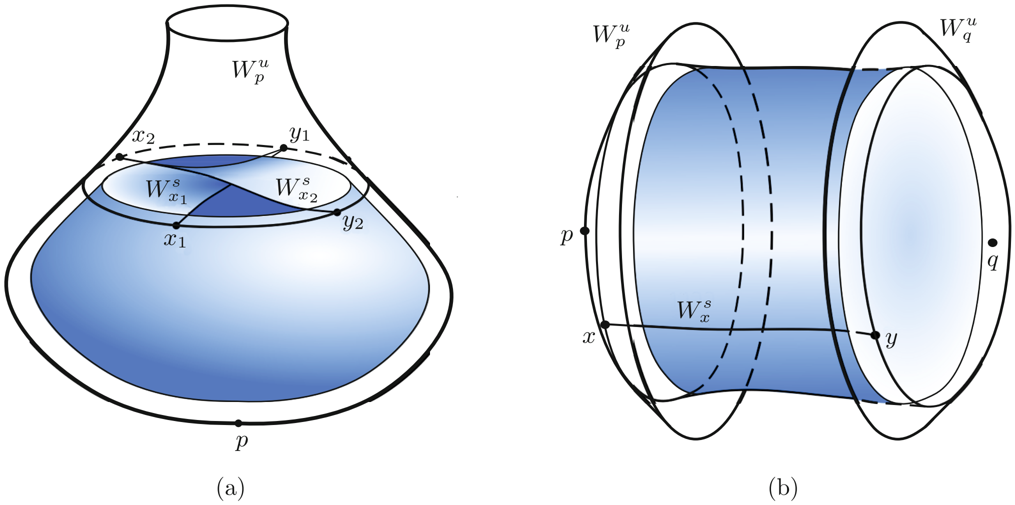

Fig. 1. Components of the boundary of a trapping neighborhood near bunches of different degrees.

copies of $\mathbb{S}^2$. Notice that each connected component of M^- is nonorientable [12] and hence there exists a double cover $\pi : \widehat{M}^- \to M^-$ [9] such that $\partial\widehat{M}^-$ consists of $\hat{k}^- = k_1^- + 2k_2^-$ 2-spheres. There is the following division of $\widetilde{M}^3$ on disjoint closed submanifolds $\widetilde{M}^+$ and $\widetilde{M}^-$:

- $\widetilde{M}^+ = M^+ \cup_{h^+} (D \times \mathbb{Z}_{k^+})$, where $D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leqslant 1\}$, $h^+ : \partial M^+ \to \partial(D \times \mathbb{Z}_{k^+})$ is a diffeomorphism;
- $\widetilde{M}^- = \widehat{M}^- \cup_{h^-} (D \times \mathbb{Z}_{\hat{k}^-})$, $h^- : \partial\widehat{M}^- \to \partial(D \times \mathbb{Z}_{\hat{k}^-})$ is a diffeomorphism.

Let us introduce the following notation:

- $\mathcal{M}^+ = \bigcup\limits_{m=1}^{+\infty} f^m(M^+)$, $\mathcal{M}^- = \bigcup\limits_{m=1}^{+\infty} f^m(M^-)$;
- $\pi : \widehat{\mathcal{M}}^- \to \mathcal{M}^-$ is a double cover of $\mathcal{M}^-$;
- $\widehat{\mathcal{M}} = \mathcal{M}^+ \cup \widehat{\mathcal{M}}^-$, $k = k^+ + \hat{k}^-$;
- $\widehat{f} : \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$ is a diffeomorphism such that $\widehat{f}|_{\mathcal{M}^+} = f|_{\mathcal{M}^+}$ and $\widehat{f}|_{\mathcal{M}^-}$ is a lift of $f|_{\mathcal{M}^-}$.

Also, let O be the center of the disk D.

Theorem 3. *There exists a diffeomorphism $\widetilde{f} : \widetilde{M}^3 \to \widetilde{M}^3$ which has k sinks at the points $O \times \mathbb{Z}_k \subset \widetilde{M}^3$ and $\widetilde{f}|_{\widetilde{M}^3 \setminus (O \times \mathbb{Z}_k)}$ is topologically conjugated with $\widehat{f}$.*

Proof. Let $\mathcal{B}^+$ and $\mathcal{B}^-$ be sets of the bunch basins in the sets $\mathcal{M}^+$ and $\mathcal{M}^-$, respectively. Let also $\widehat{\mathcal{B}}^- = \pi^{-1}(\mathcal{B}^-)$ and $\widehat{\mathcal{B}} = \mathcal{B}^+ \cup \widehat{\mathcal{B}}^-$. Since the bunch basins are periodic, there exists a division of the set $\widehat{\mathcal{B}}$ on subsets $\widehat{\mathcal{B}}_i$, $i = 1, \ldots, l$, each of which has a minimum natural number m_i such that the set $\widehat{\mathcal{B}}_i = \bigcup\limits_{j=1}^{m_i} f^j(\widehat{B}_i)$, where $\widehat{B}_i$ is some connected component of $\widehat{\mathcal{B}}$. Then $m_1 + \cdots + m_l = k$. It follows from [11, Lemma 2.2] that each $\widehat{B}_i$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$ and hence the orbit space of $f|_{\widehat{\mathcal{B}}_i}$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ if $\widehat{f}^{m_i}|_{\widehat{B}_i}$ preserves orientation or to $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ if $\widehat{f}^{m_i}|_{\widehat{B}_i}$ reverses orientation. Notice that periodic hyperbolic sinks have the same orbit spaces in their basins.

Let $g_i : \mathbb{R}^3 \times \mathbb{Z}_{m_i} \to \mathbb{R}^3 \times \mathbb{Z}_{m_i}$ be a diffeomorphism with m_i sinks at the origins $O \times \mathbb{Z}_{m_i}$, $g_i = (\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, t+1 \mod m_i)$ if $\widehat{f}^{m_i}|_{\widehat{\mathcal{B}}_i}$ preserves orientation, and $g_i = (-\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, t+1 \mod m_i)$ otherwise. Since manifolds $\widehat{\mathcal{B}}_i$ and $(\mathbb{R}^3 \setminus O) \times \mathbb{Z}_{m_i}$ have the same number of connected components and orbit spaces of $\widehat{f}|_{\widehat{\mathcal{B}}_i}$ and $g_i|_{(\mathbb{R}^3 \setminus O)\times\mathbb{Z}_{m_i}}$ are diffeomorphic, it follows that for each i there exists a diffeomorphism $h_i : \widehat{\mathcal{B}}_i \to (\mathbb{R}^3 \setminus O) \times \mathbb{Z}_{m_i}$, conjugated $\widehat{f}|_{\widehat{\mathcal{B}}_i}$ with $g_i|_{(\mathbb{R}^3 \setminus O)\times\mathbb{Z}_{m_i}}$ by [13, Statement 10.35 2)].

Then diffeomorphisms $g : \mathbb{R}^3 \times \mathbb{Z}_k \to \mathbb{R}^3 \times \mathbb{Z}_k$ and $h : \widehat{B} \to (\mathbb{R}^3 \setminus O) \times \mathbb{Z}_k$ can be composed of g_i and h_i. Moreover, h can be chosen in such a way that $h(U_\Lambda) = \mathbb{S}^2 \times \mathbb{Z}_k$, where $\mathbb{S}^2 \subset \mathbb{R}^3$ is a standard 2-sphere. Then $\widetilde{M}^3 = \widehat{\mathcal{M}} \sqcup_h (\mathbb{R}^3 \times \mathbb{Z}_k)$ with a natural projection $q : \widehat{\mathcal{M}} \sqcup (\mathbb{R}^3 \times \mathbb{Z}_k) \to \widetilde{M}^3$. The desired diffeomorphism $\widetilde{f}$ coincides with $q\widehat{f}(q|_{\widehat{\mathcal{M}}})^{-1}$ on the set $q(\widehat{\mathcal{M}})$ and with $qg(q|_{\mathbb{R}^3\times\mathbb{Z}_k})^{-1}$ on the sets $q(\mathbb{R}^3 \times \mathbb{Z}_k)$. Notice that by the construction $\widetilde{f}$ has k sinks more than $\widehat{f}$. □

3. LOW ESTIMATE OF TRIVIAL BASIC SETS NUMBER

In this section we will prove the estimate from Theorem 1. Let $f : M^3 \to M^3$ be an Ω-stable diffeomorphism, given on a closed connected 3-manifold. Throughout this section we will assume that all isolated periodic points and also boundary periodic points are fixed, because this does not affect the lower estimates: an appropriate degree of the initial system satisfies these properties and has the same number of isolated periodic points. Let $R_f = \Lambda \cup p_1 \cup p_2 \cup \ldots \cup p_m$, where Λ is a union of expanding attractors of codimension 1 with k_1 bunches of degree 1 and k_2 bunches of degree 2 in total and p_i is a fixed point, $i \in \{1, 2, \ldots, m\}$. Below we will prove that $m \geqslant \frac{3}{2}k_1 + k_2$.

Proof. Via the transition to a regular system, described in Section 2, we will obtain an Ω-stable diffeomorphism $\widetilde{f} : \widetilde{M}^3 \to \widetilde{M}^3$ with a finite chain-recurrent set on a closed manifold $\widetilde{M}^3$. Notice that all chain-recurrent points of $\widetilde{f}$ are fixed. Let M be a connected component of $\mathcal{M} = M^3 \setminus \Lambda$. Notice that M is f-invariant. There exists a connected component $\widetilde{M}$ of $\widetilde{M}^3$ corresponding to M.

Let us denote a number of 1- and 2-bunch basins, contained in M, by l_1 and l_2, respectively. M, and hence $\widetilde{M}$, can be one of 2 types (see Section 2): (1) $M \subset \mathcal{M}^+$ and (2) $M \subset \mathcal{M}^-$. In the first case $l_1 = 0$ and $\widetilde{f}|_{\widetilde{M}}$ has l_2 sinks more than $\widehat{f}|_M$. In the second case $l_1 > 0$ and even and $\widetilde{f}|_{\widetilde{M}}$ has $l_1 + 2l_2$ sinks more than $\widehat{f}|_{\pi^{-1}(M)}$.

Let C_j, $j = 0, 1, 2, 3$, be the number of fixed points p of $\widetilde{f}|_{\widetilde{M}}$ with $\dim W^u_p = j$, for example, let C_0 be the number of sinks. Also, $\widetilde{f}|_{\widetilde{M}}$ has at least 1 source, since it is Ω-stable. Then, by the Lefschetz formula, the alternating sum of C_j is equal to 0:

$$C_3 - C_2 + C_1 - C_0 = 0.$$

At the same time, since $\widetilde{M}$ is connected, it follows that $C_1 - C_0 + 1 \geqslant 0$ [14]. If $\widetilde{M}$ of type (1), then $C_0 \geqslant l_2 > 0$ and there are no additional restrictions. The finding of the minimum of the sum $C_0 + C_1 + C_2 + C_3$ is a linear programming problem, it can be solved by a simplex method. Then the minimum of fixed points of $\widetilde{f}|_{\widetilde{M}}$ can be reached if $C_3 = 1$, $C_2 = 0$, $C_1 = l_2 - 1$, $C_0 = l_2$. Therefore, $f|_M$ has at least l_2 isolated fixed points if $\widetilde{M}$ is of type (1). It follows from [15] that $\widetilde{M}$ is homeomorphic to $\mathbb{S}^3$ in this case.

If $\widetilde{M}$ is of type (2), then C_1, C_2, and C_3 are even, because the isolated periodic points of $f|_M$ are doubled in this case. Also, $C_0 \geqslant l_1 + 2l_2 > 0$. Without loss of generality we suppose that 1-dimensional separatrices of saddles do not intersect[3]. Then we can arrange points in the

[3] Each Ω-stable diffeomorphism with a finite chain-recurrent set has an ε-close Morse – Smale diffeomorphism with the same number of chain-recurrent points. Therefore, we can consider this Morse – Smale diffeomorphism instead of the initial one to calculate the desired estimates.

nonwandering set of $\widetilde{f}|_{\widetilde{M}}$ to agree with the Smale relation[4)]. Moreover, the order can be chosen in such a way that each saddle of index 1 comes before all saddles of index 2. Thus, we have $\omega_1 \prec \ldots \prec \omega_{C_0} \prec \sigma_1 \prec \ldots \prec \sigma_{C_1} \prec \beta_1 \prec \ldots \prec \beta_{C_2} \prec \alpha_1 \prec \ldots \prec \alpha_{C_3}$, where each ω_i is a sink, each σ_i is a saddle of index 1, each β_i is a saddle of index 2, and each α_i is a source.

It follows from [16] that a set $\mathcal{A} = \bigcup\limits_{i=1}^{c_1} cl(W^u_{\sigma_i})$ is 1-dimensional and connected. Since π is a double cover, there is an involution φ on the set $\mathcal{A} \setminus (\omega_1 \cup \ldots \cup \omega_{C_0})$, which is a lift of an identity map on $\pi(\mathcal{A} \setminus (\omega_1 \cup \ldots \cup \omega_{C_0}))$ and swaps preimages of points. The involution φ can be extended by continuity on the whole $\mathcal{A}$. Moreover, a set of fixed points of the extended involution φ coincides with the set of sinks corresponding to 1-bunches.

Let $\mathcal{A}^* = \mathcal{A}/_{\varphi}$. Since a natural projection is a continuous map, the connectedness of $\mathcal{A}$ implies the connectedness of $\mathcal{A}^*$. $\mathcal{A}^*$ contains $(C_0 + l_1)/2$ sinks and hence at least $(C_0 + l_1)/2 - 1$ saddles of index 1 are needed. Therefore, $\mathcal{A}$ contains at least $(C_0 + l_1 - 2)$ saddles of index 1, i. e., $C_1 \geqslant C_0 + l_1 - 2$.

Let us solve a linear programming task for this case:

$$C_3 - C_2 + C_1 - C_0 = 0,$$

$$C_0 \geqslant l_1 + 2l_2,$$

$$C_1 - C_0 \geqslant l_1 - 2,$$

$$C_3 \geqslant 2.$$

The optimal values are: $C_0 = l_1 + 2l_2$, $C_1 = 2l_1 + 2l_2 - 2$, $C_2 = l_1$, and $C_3 = 2$. Then there are at least $l_1 + l_2 - 1$ saddles of index 1, $l_1/2$ saddles of index 2, and 1 source at the component M.

Summing over all connected components of $\mathcal{M}$, we find that f has at least $\frac{3}{2}k_1 + k_2$ isolated periodic points: at least s sources, $(k_1 + k_2 - s)$ saddles of index 1, and $k_1/2$ saddles of index 2, where s is the number of connected components of $\mathcal{M}$. □

Below we will prove Theorem 2.

Proof. If M^3 is nonorientable, but Λ contains only orientable attractors, then by [10] W^s_Λ is homeomorphic to a punctured 3-torus, $\mathcal{M}^- = \varnothing$, and there exists a nonorientable connected component M of the set $\mathcal{M}^+$. Then the corresponding manifold $\widetilde{M}$ is also nonorientable, and $\widetilde{f}|_{\widetilde{M}}$ has saddles of different indices [15], that is, $C_2 > 0$ and $C_1 > 0$. There are two optimal possibilities: 1 source, 1 saddle of index 2, l_2 saddles of index 1, and l_2 sinks or 2 sources, 1 saddle of index 2, $l_2 - 1$ saddles of index 1, and l_2 sinks — for the both possibilities the total number of points in the nonwandering set of $\widetilde{f}|_{\widetilde{M}}$ is $2l_2 + 2$, so $f|_M$ has at least $l_2 + 2$ isolated periodic points. □

4. ACHIEVABILITY OF THE ESTIMATES

In this section we present realizations of diffeomorphisms with a minimum number of trivial basic sets, i. e., we will prove the second part of Theorems 1 and 2. First of all, we will answer the question: how to obtain an Ω-stable cascade $f : M^3 \to M^3$ with a set of expanding attractors of codimension 1 Λ with $k_1 \geqslant 0$ bunches of degree 1 and $k_2 \geqslant 0$ bunches of degree 2 in total $(k_1 + k_2 > 0)$ and $\frac{3}{2}k_1 + k_2$ periodic points outside of Λ.

Let f be a diffeomorphism of the class under consideration with the following properties:

- all bunches and isolated periodic points are fixed;

[4)]Let Λ_1 and Λ_2 be basic sets of an Ω-stable diffeomorphism $f : M \to M$. $\Lambda_1 \prec \Lambda_2$ if $W^s_{\Lambda_1} \cap W^u_{\Lambda_2} \neq \varnothing$.

- if $k_2 > 0$, then M^+ is connected and has k_2 boundary components, otherwise M^+ is empty;
- if $k_1 > 0$, then each nontrivial attractor has 1-bunches and M^- has $k_1/2$ connected components, each of which is homeomorphic to $\mathbb{R}P^2 \times [-1, 1]$.

The corresponding regular system $\widetilde{f}|_{\widetilde{M}^+}$ for the set M^+ realizing the minimum can be as in Fig. 2. It has k_2 sinks, $k_2 - 1$ saddles, and 1 source.

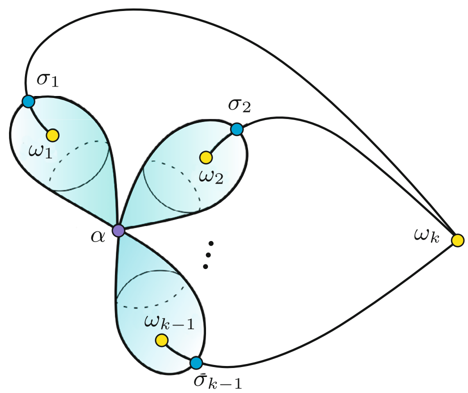

Fig. 2. Morse – Smale system for $\mathcal{M}^+$, realizing low estimates.

If $k_1 > 0$, all bunches of degree 1 are divided into pairs in such a way that after gluing the cylinders $\mathbb{R}P^2 \times [-1, 1]$ to a trapping neighborhood of Λ we will obtain a connected manifold M^3. Let the restriction $f|_M$ of the desired diffeomorphism f on each connected component M of $\mathcal{M}^-$ be topologically conjugated to a diffeomorphism $(g_1 \times g_2)$, where $g_1 : \mathbb{R}P^2 \to \mathbb{R}P^2$ is as in Fig. 3 and $g_2 : \mathbb{R} \to \mathbb{R}$ such that $g_2(x) = 2x$.

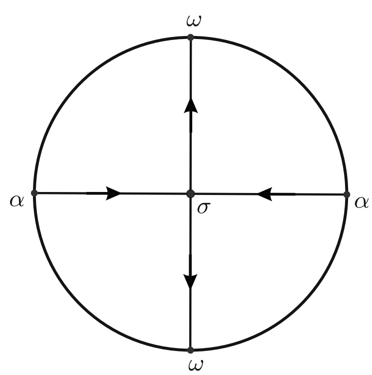

Fig. 3. Morse – Smale system on $\mathbb{R}P^2$.

Achievability of the estimate from Theorem 2 will be shown with a diffeomorphism $f : M^3 \to M^3$ with the following properties:

- f has only 1 nontrivial attractor Λ, which is connected and has k_2 bunches of degree 2;
- all bunches and isolated periodic points of f are fixed;
- a set M^+ consists of k_2 connected components.

Let the corresponding regular system $\widetilde{f} : \widetilde{M}^3 \to \widetilde{M}^3$ be given on $\mathbb{S}^3 \times \mathbb{Z}_{k_2-1} \sqcup \mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$, the dynamics on each 3-sphere be “sink-source” and on the $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ be as in Fig. 4. Therefore, $f : M^3 \to M^3$ has exactly $k_2 + 2$: k_2 sources and 2 saddles of different indices — isolated chain-recurrent points, and M^3 is nonorientable.

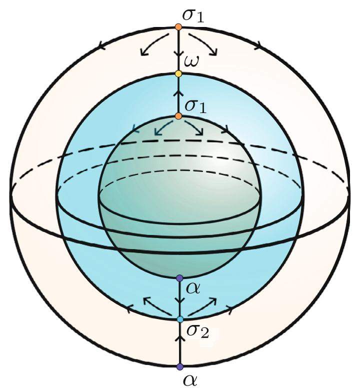

Fig. 4. Morse – Smale system on $\mathbb{S}^2\widetilde{\times}\mathbb{S}^1$.

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CONFLICT OF INTEREST

The author declares that she has no conflicts of interest.

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