MSRI Mathematical Circles Library

Mathematics via Problems

PART 3: Combinatorics

Mikhail B. Skopenkov & Alexey A. Zaslavsky

Translated from Russian by Sergei G. Shubin and Paul Zeitz

SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE
Berkeley, California

AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island

7.4.5. Prove that there are infinitely many integers m and n such that $\left|\sqrt{2}-\frac{m}{n}\right|$ is less than

(a) $\frac{1}{2n}$; (b) $\frac{1}{999n}$; (c) $\frac{1}{n^2}$; (d)* $\frac{1}{\sqrt{5}n^2}$.

7.4.6.* (a) For any $\varepsilon > 0$ there exists a countable set of intervals, the sum of whose lengths is less than ε, such that, if α is not in any of the intervals and c is any positive number, there exist $m, n \in \mathbb{Z}$, $n > 0$, such that $\left|\alpha - \frac{m}{n}\right| < \frac{1}{cn^{2+\varepsilon}}$.

(Rigorous reformulation: for any $\varepsilon > 0$, the set of real numbers that are not $(2+\varepsilon)$-approximated has measure 0.)

(b) **Hurwitz-Borel theorem.** For any irrational number α, there are infinitely many $m/n \in \mathbb{Q}$, such that $\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2\sqrt{5}}$.

(c) The number $\sqrt{5}$ in the Hurwitz-Borel theorem cannot be increased: for any $c > \sqrt{5}$ there is an irrational number α such that the inequality $\left|\alpha - \frac{m}{n}\right| < \frac{1}{cn^2}$ holds for only a finite number of $m/n \in \mathbb{Q}$,

Suggestions, solutions, and answers

7.4.3. This problem is analyzed in [**Bol78**] and [**Ar98**].

7.4.5. (b) For any positive integers N, k and any irrational number α there are at least k different fractions $m/n \in \mathbb{Q}$ for which $n \le Nk$ and $\left|\alpha - \frac{m}{n}\right| < \frac{1}{Nn}$. For details, see the suggestion to problem 7.5.14 below.

5. The pigeonhole principle and its application to geometry[1] (3) *By I. V. Arzhantsev*

The area of a figure

We will call a planar figure A *simple* if it can be cut into a finite number of triangles. Its area $S(A)$ is defined as the sum of the areas of the corresponding triangles.

Recall that a point $(x_0, y_0) \in A$ is called an *interior* point of A if there is a circle with center (x_0, y_0) entirely lying in A.

It is easy to verify that the function "area" on the set of simple figures has the following properties:

- if A has interior points, then $S(A) > 0$;
- if A is the union of simple figures A_1 and A_2 without common interior points, then $S(A) = S(A_1) + S(A_2)$;
- congruent figures have the same area;
- the area of a unit square is 1.

[1]Based on [**Yad**].

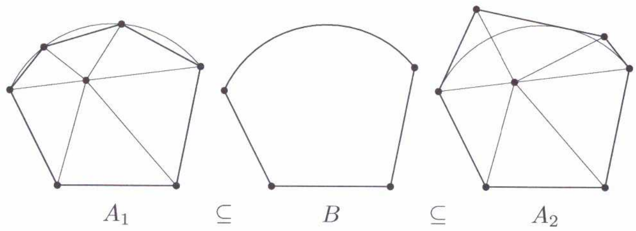

FIGURE 9

More generally, a planar set B is called *measurable* if for any $\varepsilon > 0$ there exist simple figures A_1 and A_2 such that $A_1 \subseteq B \subseteq A_2$ and $S(A_2) - S(A_1) < \varepsilon$ (see Fig. 9). For measurable sets, one can also define the concept of area and prove that the area is the only function on the set of measurable sets that has the four properties listed above.

Note that not every plane set is measurable (see, for example, problem 7.5.2). To those who want to learn more about the concept of area and its generalizations we can recommend the book [**Leb**].

7.5.1. Prove that a bounded figure whose boundary consists of a finite number of segments and arcs of circles is measurable.

Recall that a planar set is called *bounded* if it is contained in some circle.

7.5.2. Prove that any measurable set is bounded.

The pigeonhole principle for areas

The following geometric statement resembles the well-known "pigeonhole principle" and is therefore usually called the *geometric pigeonhole principle* or the *pigeonhole principle for areas.*

7.5.3. Pigeonhole principle for areas. Let A be a measurable set, and let $A_1, \dots, A_m$ be measurable subsets of A. Suppose that

$$S(A) < S(A_1) + S(A_2) + \cdots + S(A_m).$$

Then at least two of the sets $A_1, \dots, A_m$ have a common interior point.

Suggestion. Assume, to the contrary, that the sets have no common interior points. Then

$$S(A_1 \cup A_2 \cup \cdots \cup A_m) = S(A_1) + S(A_2) + \cdots + S(A_m).$$

Since $A_1, \dots, A_m \subseteq A$ and the complement $A - (A_1 \cup A_2 \cup \cdots \cup A_m)$ are measurable, we have

$$S(A_1 \cup A_2 \cup \cdots \cup A_m) \leq S(A),$$

a contradiction.

7.5.4. Let A be a measurable set, and let $A_1, \ldots, A_m$ be measurable subsets of A. Suppose that

$$nS(A) < S(A_1) + S(A_2) + \cdots + S(A_m)$$

for some positive integer $n < m$. Then at least $n+1$ of $A_1, \ldots, A_m$ have a common interior point.

Suggestion. If no $n+1$ sets share an interior point, then each interior point of the set $A_1 \cup \cdots \cup A_m$ is "counted" no more than n times in the sum

$$S(A_1) + S(A_2) + \cdots + S(A_m),$$

and therefore

$$S(A_1) + S(A_2) + \cdots + S(A_m) \leq nS(A).$$

7.5.5. A unit square contains a set whose area is more than $\frac{1}{2}$. Prove that this set contains two points, symmetric about the center of the square.

7.5.6. The area of a set on the sphere is greater than half of the area of the sphere. Prove that this set covers a pair of diametrically opposite points on the sphere.

The theorems of Blichfeldt and Minkowski

Fix a rectangular Cartesian coordinate system on the plane and through each point with integer coordinates draw two lines, parallel to the coordinate axes. The resulting system of lines is called an *integer lattice*, and points with integer coordinates are called *lattice points*. The integer lattice cuts the plane into unit squares.

Consider an integer lattice and a measurable plane set. The number of lattice points covered by the set depends not only on the shape of the set, but also its location. For example, there are sets with arbitrarily large area that do not cover a single lattice point (give an example!).

7.5.7. Blichfeldt's theorem. Let A be a measurable set on the coordinate plane with area greater than n. Then A can be translated so that it covers at least $n+1$ lattice points.

Suggestion. The integer lattice cuts A into a finite number of pieces (the figure A is bounded!). The condition $S(A) > n$ shows that the number of pieces is not less than $n+1$. Place all the squares that our figure intersects "in a single deck" (see Fig. 10). We will get at least $n+1$ shapes inside a unit square with the total area greater than n.

Applying problem 7.5.4 to the unit square we see that there is a point P that belongs to at least $n+1$ pieces of our set. It suffices to translate A by the vector that connects P with the lattice point.

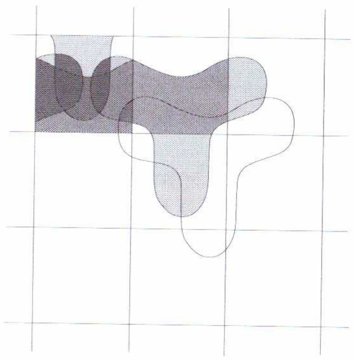

FIGURE 10

For $n = 1$, Blichfeldt's theorem can be reformulated as follows:

7.5.8. Let A be a measurable figure on the coordinate plane whose area is more than 1. Then A contains two distinct points (x_1, y_1) and (x_2, y_2) such that $x_2 - x_1$ and $y_2 - y_1$ are integers.

Recall that a plane figure A is called *convex* if the segment joining any two of its points lies entirely in A.

The following theorem, due to the German mathematician Hermann Minkowski, appears in geometric number theory.

7.5.9. Minkowski's theorem. Let A be a convex measurable set with area greater than 4 that is symmetric with respect to the origin. Then A contains a point with integer coordinates different from the origin.

Suggestion. Apply a homothety with the center at the origin and coefficient $\frac{1}{2}$ to A, obtaining the set B, whose area is greater than 1. By Blichfeldt's theorem, B contains distinct points (x_1, y_1) and (x_2, y_2), for which $x_2 - x_1$ and $y_2 - y_1$ are integers. By symmetry, $(-x_1, -y_1)$ also lies in B, and because B is convex, the midpoint O of the segment connecting $(-x_1, -y_1)$ and (x_2, y_2) also lies in B. The point O has the coordinates $\left(\frac{x_2-x_1}{2}, \frac{y_2-y_1}{2}\right)$. Therefore, the point with coordinates $(x_2 - x_1, y_2 - y_1)$ lies in A.

7.5.10. Show by an example that the condition $S(A) > 4$ in Minkowski's theorem cannot be replaced by $S(A) \geq 4$.

7.5.11. Let A be a measurable set on the coordinate plane whose area is less than n. Prove that A can be translated so that it covers at most $n - 1$ lattice points.

7.5.12. Let A be a convex measurable set that is symmetric with respect to the origin and has area greater than $4n$. Prove that A contains at least $2n+1$ lattice points.

Dirichlet's theorem on approximation of irrational numbers

7.5.13. Dirichlet's theorem. For an arbitrary irrational number α and an arbitrary natural number s there exist integers x and y such that $0 < x \leq s$ and

$$|\alpha x - y| < \frac{1}{s}.$$

Suggestion. We give a sketch of a proof using Minkowski's theorem. A direct proof can be obtained by following the suggestion to problem 7.5.14.

Consider

$$A = \Big\{(x, y) \colon |\alpha x - y| < \frac{1}{s}, \quad |x| \leq s + \frac{1}{2}\Big\}.$$

This set is a parallelogram whose area is

$$\frac{2}{s} \cdot 2\Big(s + \frac{1}{2}\Big) = 4\Big(1 + \frac{1}{2s}\Big) > 4.$$

This figure is convex and symmetrical with respect to the origin (see Fig. 11). Minkowski's theorem states that in A there is a point with integer coordinates other than $(0, 0)$. We can assume that the first coordinate of this point is positive (explain this!). Thus, the theorem is proved.

7.5.14. Prove that for arbitrary irrational number α and natural number s there is a rational number $\frac{m}{n}$ such that $0 < n \leq s$ and $\left|\alpha - \frac{m}{n}\right| < \frac{1}{ns}$.

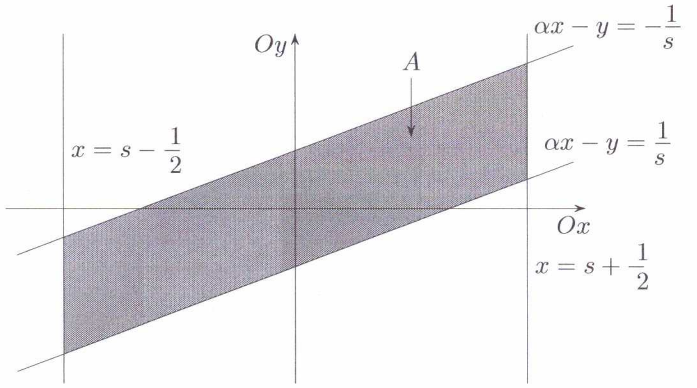

FIGURE 11

7.5.15. Prove that for an arbitrary irrational number α there are infinitely many rational numbers $\frac{m}{n}$ such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2}.$$

Suggestions, solutions, and answers

7.5.1. Given a disk, one can circumscribe a regular n-gon around it and also inscribe a regular n-gon in it. The difference in the areas of these two polygons can be made arbitrarily small by choosing a sufficiently large n. Consequently, disks and segments of disks are measurable.

7.5.2. Any simple set is obviously bounded. Since a measurable set is contained in a simple set, it is also bounded.

7.5.5. Let F be the given figure, and let F' be the figure symmetric to it with respect to the center of the square. Then $S(F) + S(F') > 1$ and by the pigeonhole principle for areas (problem 7.5.3) there exists a point $X \in F \cap F'$. Obviously, X and the point X' symmetric to X form the required pair.

7.5.6. Consider a figure that is symmetrical to a given one relative to the center of the sphere, and repeat the arguments of the previous problem.

7.5.10. Consider the open square $\{(x, y)\colon |x| < 1, |y| < 1\}$.

7.5.11. Note that the half-open square $-k \leq x, y < k$ covers exactly $4k^2$ lattice points when translated by any vector. Choose k large enough so that A is contained in some such half-open square K. By Blichfeldt's theorem, one can translate the set $K - A$ to cover at least $4k^2 - n + 1$ lattice points. Since all these lattice points lie in the translated image of the square K, the image of A will cover at most $n - 1$ nodes.

7.5.12. Apply a homothety with center at the origin and coefficient $\frac{1}{2}$ to A, getting the figure B whose area is greater than n. It follows from Blichfeldt's theorem that B contains distinct points $(x_0, y_0), \ldots, (x_n, y_n)$, for which all the differences $x_i - x_j$ and $y_i - y_j$ are integers. We can assume that $x_0 \geq x_1 \geq \cdots \geq x_n$ and that among the points (x_i, y_i) for which $x_i = x_0$, the maximum value of the second coordinate is y_0. As in the proof of Minkowski's theorem, it can be shown that A contains distinct points $(0, 0)$, $(x_0 - x_i, y_0 - y_i)$, $(x_i - x_0, y_i - y_0)$.

7.5.14. It suffices to divide both sides of the inequality in Dirichlet's theorem by x.

Alternatively, you can solve this problem without using geometric considerations: consider the fractional parts of the numbers α, 2α, $\ldots$, $s\alpha$ and divide the segment $[0, 1]$ into s equal parts. There are two cases possible.

1. Each of the s segments contains exactly one of the numbers α, $2\alpha, \ldots, s\alpha$. Then for some $n \leq s$ the inequality $\{n\alpha\} < 1/s$ holds, and the desired number has the form m/n, where $m = [n\alpha]$.

2. The fractional parts of the numbers $n_1\alpha$ and $n_2\alpha$ lie in one segment. Then the desired number is m/n, where $m = |[n_1\alpha] - [n_2\alpha]|$, $n = |n_1 - n_2|$.