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ON SUSPENSIONS OVER GRADIENT-LIKE DIFFEOMORPHISMS OF SURFACES WITH THREE PERIODIC ORBITS

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S. Smale showed that suspensions over conjugate diffeomorphisms are topologically equivalent. Under certain assumptions, the conjugacy of diffeomorphisms is equivalent to the equivalence of suspensions. We show that this criterion holds for gradient-like diffeomorphisms with three periodic orbits on arbitrary orientable surfaces, prove that 3-manifolds admitting suspensions over such diffeomorphisms are small Seifert manifolds, and calculate the homology groups of these manifolds and the number of equivalence classes of flows on each admissible Seifert manifold. Bibliography: 12 *titles. Illustrations*: 1 *figure.*

1 Introduction. Formulation of the Result

The notion of a suspension is an extremely useful construction in the theory of dynamical systems which allows us, for a given diffeomorphism f of a manifold, to construct a flow, called a *suspension* over f. As was shown in [1], suspensions over conjugate diffeomorphisms are topologically equivalent. The converse assertion is false in the general case. However, under certain assumptions, the conjugacy of the corresponding diffeomorphisms is equivalent to the

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equivalence of suspensions. As shown in [2], such a criterion holds if the diffeomorphism is defined on a manifold whose fundamental group does not admit any epimorphism into the group $\mathbb{Z}$. Thus, for orientable surfaces, this criterion holds only on the 2-sphere. In this paper, we show that the criterion is generalized to the class of gradient-like diffeomorphisms with three periodic orbits on arbitrary orientable surfaces.

Let S_p be a closed orientable surface of genus $p>0$, and let G denote the class of orientation–preserving Morse–Smale diffeomorphisms $f\colon S_p\to S_p$ whose non-wandering set consists of exactly three periodic orbits. A complete topological classification of diffeomorphisms in G was obtained in [3] (see also [4], where effective algorithms for distinguishing invariants of such systems are established). In [3], it was also shown that the number of topological conjugacy classes of diffeomorphisms in the class G on a surface of genus p is calculated by the formula $N_p=\varphi(4p)+\varphi(4p+2)$, where $\varphi(n)$ is the Euler function, i.e., the number of integers coprime with n that do not exceed n.

We recall that a suspension over a diffeomorphism $f\colon S_p\to S_p$ is the flow obtained as follows. We define the flow ξ^t on the manifold $S_p\times\mathbb{R}$ by the formula $\xi^t(s,r)=(s,r+t)$ and the diffeomorphism $g\colon S_p\times\mathbb{R}\to S_p\times\mathbb{R}$ by the formula $g(s,r)=(f(s),r-1)$. We set $\Phi=\{g^k,k\in\mathbb{Z}\}$ and $M_f=(S_p\times\mathbb{R})/\Phi$. We denote by $\nu_f\colon S_p\times\mathbb{R}\to M_f$ the natural projection and by f^t the flow on the manifold M_f given by $f^t(x)=\nu_f(\xi^t(\nu_f^{-1}(x)))$. The flow f^t is called the *suspension* over the diffeomorphism f.

We denote by G^t the class of suspensions over diffeomorphisms $f\in G$. According to [5, Theorems 2.1 and 2.2], the non-wandering set of the flow f^t consists of attracting A, repelling R, and saddle S periodic orbits. Let us choose a canonical tubular neighborhood V of the orbit A and two simple closed curves: on $T=\partial V$ a parallel $L\subset T$ (a curve homotopic in V to the orbit A) and a meridian $M\subset T$ (a curve homotopic to zero on V and essential on the torus T). The curves L and M can be thought of as loops at a point of $L\cap M$. Hence their homotopy classes are generators of the group $\pi_1(T)$. If there is no confusion, we use the same notation L and M.

Let ρ denote the metric on a manifold M^n. Every hyperbolic periodic orbit $\mathscr{O}$ of a flow $f^t\colon M^n\to M^n$ possesses *stable* and *unstable manifolds*:

$$W^s_{\mathscr{O}}=\{y\in X\colon \lim_{k\to+\infty} d(\mathscr{O},f^k(y))\to 0\},$$

$$W^u_{\mathscr{O}}=\{y\in X\colon \lim_{k\to-\infty} d(\mathscr{O},f^k(y))\to 0\}.$$

The set $\gamma=W^u_S\cap T$ is a knot on the torus T. We write its homotopy type with respect to the generators L, M of the fundamental group $\pi_1(T)$ as $\langle\gamma\rangle=\langle l,m\rangle$. We note that the choice of the longitude L is unique up to an isotopy. However, the number l is independent of the choice of generators, whereas m depends on the choice of parallels. According to [3, Theorem 2], $l\geqslant 3$ and m can be any integer coprime with l. In this case, a pair of numbers (l,m) uniquely determines the number $d\in\{1,\dots,l-1\}$ by $d\cdot m\equiv 1 \pmod{l}$, independently of the choice of generators. We note, that l and d are coprime.

In this paper, we establish the following classification result.

Theorem 1.1. *Flows f^t, $f'^t\in G^t$ are topologically equivalent if and only if $(l,d)=(l',d')$.*

Thus, the pair of numbers (l,d) is a complete topological invariant of the flow $f^t\in G^t$. According to [3, Theorem 1], the pair of numbers (l,d) is also a complete topological invariant of the corresponding diffeomorphism $f\in G$, which directly implies the following fact.

Corollary 1.1. *Flows f^t, $f'^t \in G^t$ are topologically equivalent if and only if the corresponding diffeomorphisms f, $f' \in G$ are topologically conjugate.*

The connection between gradient-like diffeomorphisms $f \in G$ and periodic homeomorphisms, established in [3], allows us to describe the topology of manifolds M_f admitting flows $f^t \in G^t$ as follows (see also [6] where a larger class of flows was considered).

Theorem 1.2. *Let a flow $f^t \in G^t$ have parameters (l, d). Then M_f is a small Seifert manifold of one of the following types*:

(1) $M_f \cong M(\mathbb{S}^2, (2,1), (l,d), (l, l/2 - d))$ *if l is even and $l/2$ is even,*

(2) $M_f \cong M(\mathbb{S}^2, (2,1), (l,d), (l/2, (l/2 - d)/2))$ *if l is even and $l/2$ is odd,*

(3) $M_f \cong M(\mathbb{S}^2, (2,1), (l,d), (2l, l - 2d))$ *if l is odd.*

In this paper, we calculate the number of equivalence classes of the flows under consideration on all admissible manifolds.

Theorem 1.3. *The supporting manifold of any flow f^t from the class G^t, which is the suspension over the diffeomorphism $f\colon S_p \to S_p \in G$, is homeomorphic to exactly one of the following manifolds:*

(1) $A_{p,d} = M(\mathbb{S}^2, (2,1), (4p, d), (4p, 2p - d))$, $p \in \mathbb{N}$, $d \in \{1, \dots, p-1\}$, $(d, 4p) = 1$,

(2) $B_{p,d} = M(\mathbb{S}^2, (2,1), (4p+2, d), (2p+1, p-(d-1)/2), p \in \mathbb{N}, d \in \{1, \dots, 2p-1\}, (d, 4p+2) = 1$.

Moreover,

(1) *each manifold $A_{p,d}$ admits exactly two classes of topological equivalence of flows in set G^t represented by flows with parameters $(4p, d)$ and $(4p, 2p - d)$,*

(2) *each manifold $B_{p,d}$ admits exactly two classes of topological equivalence of flows in the set G^t, represented by flows with parameters $(4p+2, d)$ and $(2p+1, p-(d-1)/2)$.*

We also calculate all homology groups with integer coefficients of the manifolds $A_{p,d}$ and $B_{p,d}$.

Theorem 1.4. *Homology groups with integer coefficients of the manifolds $A_{p,d}$ and $B_{p,d}$ are isomorphic to the following groups*:

(1) $H_3(A_{p,d}) \cong H_2(A_{p,d}) \cong \mathbb{Z}$ *and* $H_1(A_{p,d}) \cong \mathbb{Z} \times \mathbb{Z}_2$,

(2) $H_3(B_{p,d}) \cong H_2(B_{p,d}) \cong H_1(B_{p,d}) \cong \mathbb{Z}$.

2 Preliminaries

2.1. Gradient-like diffeomorphisms. Let S_p be a closed orientable surface of genus $p \geqslant 0$ with metric ρ. Homeomorphisms $f, f'\colon S_p \to S_p$ are *topologically conjugate* if there is a homeomorphism $h : S_p \to S_p$ such that $f'h = hf$. A point $x \in S_p$ is called *wandering* for a homeomorphism f if there is an open neighborhood U_x of the point x such that $f^n(U_x) \cap U_x = \varnothing$ for all $n \in \mathbb{N}$. Otherwise, the point is called *non-wandering*. The set of non-wandering points of

f is called the *non-wandering set* and is denoted by Ω_f. If the set Ω_f is finite, then each point $r \in \Omega_f$ is periodic with some period $m_r \in \mathbb{N}$.

Let f be a diffeomorphism. A point $r \in \Omega_f$ is called *hyperbolic* if the absolute values of all eigenvalues of the Jacobian matrix $\left(\frac{\partial f^{m_r}}{\partial x}\right)\Big|_r$ are not equal to 1. If the absolute values of all eigenvalues are less (greater) than 1, then the point r is called a *sink* (a *source*). Sinks and sources are referred to as *nodes*. If a hyperbolic periodic point is not a node, then it is a *saddle point*.

For a hyperbolic periodic point r of a diffeomorphism f we denote by q_r the number of eigenvalues of the Jacobian matrix $\left(\frac{\partial f^{m_r}}{\partial x}\right)\Big|_r$ modulo larger than 1. The hyperbolic structure of a periodic point r implies the existence of *stable* $W^s_r = \{x \in S_p : \lim\limits_{k\to+\infty} \rho(f^{k\cdot m_r}(x), r) = 0\}$ and *unstable* $W^u_r = \{x \in S_p : \lim\limits_{k\to+\infty} \rho(f^{-k\cdot m_r}(x), r) = 0\}$ manifolds that are smooth embeddings of $\mathbb{R}^{2-q_r}$ and $\mathbb{R}^{q_r}$ respectively.

Stable and unstable manifolds are called *invariant manifolds*. The connected component of the set $W^u_r \setminus r$ ($W^s_r \setminus r$) is called the *unstable* (*stable*) *separatrix*. A diffeomorphism $f\colon S_p \to S_p$ is called a *Morse–Smale diffeomorphism* if Ω_f is finite and hyperbolic and the invariant manifolds of periodic points intersect transversally.

Periodic data of a periodic orbit $\mathcal{O}_r$ of a periodic point r is a set of numbers (m_r, q_r, ν_r) such that m_r is the period of r, $q_r = \dim W^u_r$, and ν_r is the orientation type of r, i.e., $\nu_r = +1$ ($\nu_r = -1$) if $f^{m_r}|_{W^u_r}$ preserves (changes) orientation. For orientation–preserving diffeomorphisms the orientation type of all nodes is equal to $+1$, and the orientation type of saddle points can be $+1$ or -1.

2.2. Seifert manifolds. We recall some facts from the theory of Seifert manifolds (see [7] for details).

A solid torus $\mathbb{V} = \mathbb{D}^2 \times \mathbb{S}^1$ divided into fibers $\{x\} \times \mathbb{S}^1$ is called a *trivially fibered solid torus*. We consider the solid torus $\mathbb{V}$ as a cylinder $\mathbb{D}^2 \times [0,1]$ with the bases glued together by rotation by an angle $2\pi m/l$ for coprime integers m, l, $l > 1$. The partition of the cylinder into segments $\{x\} \times [0,1]$ determines the fibration of the solid torus into circles, called *fibers*. The segment $\{0\} \times [0,1]$ generates a *singular* fiber; all other (*nonsingular*) fibers wraps l times around the singular fiber and m times around the meridian of the torus. The number l is called the *multiplicity* of the singular fiber. A solid torus with such a partition into fibers is called a *nontrivially fibered solid torus* with orbital invariants (l, m).

By a *Seifert manifold* we mean a compact, orientable 3-manifold M divided into disjoint simple closed curves (fibers) in such a way that each fiber has a neighborhood foliated by fibers, which is fiberwise homeomorphic to the fibered solid torus. Such a partition is called *Seifert fibration*. Fibers mapped under such a homeomorphism to the core of a nontrivially fibered solid torus are called *singular*.

The *base* of a Seifert manifold M is a compact surface $\Sigma = M/_\sim$, where $\sim$ is an equivalence relation such that $x \sim y$ if and only if x and y belong to the same fiber.

The base of any Seifert manifold is a compact surface that is closed if and only if the manifold M is closed. We say that a Seifert manifold is *small* if its base is 2-sphere and it has at most three singular fibers.

The Seifert fibration M with base Σ and orbital invariants $(l_1, m_1), \ldots, (l_s, m_s)$, $s \in \mathbb{N}$ is usually written as $M(\Sigma,(l_1,d_1),\ldots,(l_s,d_s))$, where $m_i\cdot d_i\equiv 1 \pmod{l_i}$, $i\in\{1,\ldots,s\}$. The orientation on fibers of a Seifert manifold is uniquely determined by the orientation of one of fibers.

Two Seifert fibrations M, M' are *isomorphic* if there is a homeomorphism $h\colon M\to M'$ that takes fibers of one fibration to fibers of the other with preserving the orientation of fibers. In this case, the homeomorphism h is called an *isomorphism* of Seifert fibrations.

Proposition 2.1 ([7, Theorem 10.2]). *Seifert fibrations*

$$M(\Sigma,(l_1,d_1),\ldots,(l_s,d_s)),\quad M'(\Sigma',(l'_1,d'_1),\ldots,(l'_{s'},d'_{s'}))$$

are isomorphic if and only if

- Σ *is homeomorphic to* Σ',
- $s=s'$, $l_i=l'_i$, $d_i\equiv \pm d'_i \pmod{l_i}$ *for* $i\in\{1,\ldots,s\}$,
- *if the surface* Σ *is closed, then* $\sum\limits_{i=1}^{s}\frac{d_i}{l_i}=\pm\sum\limits_{i=1}^{s}\frac{d'_i}{l'_i}$.

Proposition 2.2 ([7, Theorem 2.3]). *If two small Seifert fibrations M and M' with three singular fibers are not isomorphic, then the Seifert manifolds M and M' are not homeomorphic.*

2.3. Periodic homeomorphisms. A homeomorphism $\varphi\colon S_p\to S_p$ is said to be *periodic* if there exists $n\in\mathbb{N}$ such that $\varphi^n=\mathrm{id}$. The smallest n is called a *period* of φ. We say that x_0 is a point of smaller period $n_0<n$ of a homeomorphism φ if $\varphi^{n_0}(x_0)=x_0$.

According to [8] (see also [9, 10]), for any orientation–preserving periodic homeomorphism $\varphi\colon S_p\to S_p$ the set B_φ of points of smaller period is finite and the space of orbits of the action of the homeomorphism φ on S_p is an orientable surface of genus $g\leqslant p$ (a *modular surface*).

In a neighborhood of a point x_0 of smaller period n_0, the map f^{n_0} is conjugate to a rotation by a rational angle $2\pi\frac{m_0}{l_0}$, where $l_0=n/n_0$, $0<m_0<l_0$, $(m_0,l_0)=1$. We introduce the notation: X_i, $i=1,\ldots,s$, are orbits in B_φ, n_i are their periods, and $l_i=n/n_i$. We denote by m_i/l_i the corresponding rotation number and determine the number $d_i\in\{1,\ldots,n_i-1\}$ from the condition $d_i\cdot m_i\equiv 1 \pmod{l_i}$. The set of parameters $(n,p,g,n_1,\ldots,n_s,d_1,\ldots,d_s)$ of a periodic homeomorphism φ is called the *complete characteristic*.

Proposition 2.3 ([8]). *Two periodic homeomorphisms are topologically conjugate if and only if they have the same, up to a renumbering, complete characteristics.*

Proposition 2.4 ([3, Lemma 1]). *Any diffeomorphism $f\colon S_p\to S_p\in G$ has the form $f=\zeta\varphi$, where ζ is a one-time shift of a gradient-like flow and $\varphi\colon S_p\to S_p$ is a periodic homeomorphism such that*

- $B_\varphi=\Omega_f$, $\varphi|_{B_\varphi}=f|_{\Omega_f}$ *and the saddle orbit of the diffeomorphism f has period $n/2$*;
- *the homeomorphism φ has the complete characteristic of one of the following types*:
  1) $(4p,0,p,2p,1,1,1,q,2p-q)$ *if* $0<q<2p$,
  2) $(4p,0,p,2p,1,1,1,q,6p-q)$ *if* $2p<q<4p$,
  3) $(4p+2,0,p,2p+1,2,1,1,q,2p+1-2q)$ *if* $0<q\leqslant p$,

4) $(4p+2, 0, p, 2p+1, 2, 1, 1, q, 6p+3-2q)$ *if* $p < q \leqslant 2p$;

- *the suddle orbit of the diffeomorphism* f *has period* $n/2$.

Let m/l denote the rotation number corresponding to the sink orbit of the diffeomorphism $f \in G$. Let us determine $d \in \{1, \ldots, l-1\}$ by the relation $d \cdot m \equiv 1 \pmod{l}$.

Proposition 2.5 ([3, Theorem 1]). *Diffeomorphisms* f, $f' \in G$ *are topologically conjugate if and only if* $(l, d) = (l', d')$.

Thus, the pair (l, d) is a complete topological invariant of the diffeomorphism $f \in G$.

3 Classification of Flows in G^t

Proof of Theorem 1.1. *Necessity.* Let flows f^t, $f'^t \in G^t$ with parameters $(l, d), (l', d')$ be topologically equivalent via a homeomorphism $h\colon M_f \to M_{f'}$. Then h maps periodic orbits of the flow f^t into periodic orbits of the flow f'^t with the same type and direction of motion, i.e., $A' = h(A)$, $S' = h(S)$, $R' = h(R)$. Without loss of generality we can assume that $V' = h(V)$. Since $\gamma = \partial V \cap W^u_S$ and $\gamma' = \partial V' \cap W^u_{S'}$, we have $\gamma' = h(\gamma)$. Thus, $h|_V \colon V \to V'$ is a homeomorphism of solid tori that preserves the orientation of the generators A and A'. Then (see, for example, [11]) the homeomorphism $h|_T$ induces an isomorphism $h_* \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ given by the matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ in the generators L, M and L', M' of the tori $T = \partial V$ and $T' = \partial V'$, where $k \in \mathbb{Z}$ and

$$(l', m') = (l, m) \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}; \tag{3.1}$$

here, $\langle l, m \rangle$; $\langle l', m' \rangle$ are homotopy types of knots γ and γ' respectively. From (3.1) we immediately get

$$l' = l, \ m' \equiv m \pmod{l}. \tag{3.2}$$

By definition, the number $d' \in \{1, \ldots, [(l-1)/2]\}$ is determined by $d' \cdot m' \equiv \pm 1 \pmod{l}$. In view of (3.2), d' coincides with the number $d \in \{1, \ldots, [(l-1)/2]\}$ defined by $d \cdot m \equiv \pm 1 \pmod{l}$.

Sufficiency. Let f^t, $f'^t \in G^t$ be flows with parameters (l, d), (l', d') respectively, and let $(l, d) = (l', d')$. By Proposition 2.5, the corresponding diffeomorphisms $f, f' \in G$ are defined on the same surface S_p and there is a homeomorphism $h_0 \colon S_p \to S_p$ such that

$$h_0 f = f' h_0. \tag{3.3}$$

We introduce the homeomorphism $H\colon S_p \times \mathbb{R} \to S_p \times \mathbb{R}$ by the formula $H(s, r) = (h_0(s), r)$.

We recall that the flow ξ^t on the manifold $S_p \times \mathbb{R}$ is given by $\xi^t(s, r) = (s, r + t)$ and the diffeomorphism $g\colon S_p \times \mathbb{R} \to S_p \times \mathbb{R}$ is given by $g(s, r) = (f(s), r - 1)$, $G = \{g^k, k \in \mathbb{Z}\}$, $M_f = (S_p \times \mathbb{R})/G$, $\nu_f : S_p \times \mathbb{R} \to M_f$ is the natural projection, and the flow f^t on the manifold M_f is given by $f^t(x) = \nu_f(\xi^t(\nu_f^{-1}(x)))$. Similar notation with prime is used for the flow f'^t. It is immediately verified that $H\xi^t = \xi^t H$ and (3.3) implies $Hg = g'H$. Then the homeomorphism H is projected into the homeomorphism $h : M_f \to M_{f'}$ according to the formula $h(x) = \nu_{f'}(H(\nu_f^{-1}(x)))$; moreover, $hf^t = f'^t h$. □

4 Topology of Manifold M_f

Proof of Theorem 1.2. Let a diffeomorphism $f \in G$ have the form $f = \zeta\varphi$, where ζ is a one-time shift of a gradient-like flow and φ is a periodic homeomorphism. Thus, f and φ are homotopic. This fact implies that the manifolds M_f and M_φ are homeomorphic. By Proposition 2.4, M_φ is a Seifert manifold whose base is a sphere and with at most three singular fibers of one of the following types:

1) $M_\varphi \cong M\left(\mathbb{S}^2, (2,1), (4p, d), (4p, 2p-d)\right)$, $0 < d < 2p$, $(d, 2p) = 1$,
2) $M_\varphi \cong M\left(\mathbb{S}^2, (2,1), (4p, d), (4p, 6p-d)\right)$, $2p < d < 4p$, $(d, 2p) = 1$,
3) $M_\varphi \cong M\left(\mathbb{S}^2, (2,1), (2p+1, d), (4p+2, 2p+1-2d)\right)$, $0 < d \leqslant p$, $(d, 4p+2) = 1$,
4) $M_\varphi \cong M\left(\mathbb{S}^2, (2,1), (2p+1, d), (4p+2, 6p+3-2d)\right)$, $p < d \leqslant 2p$, $(d, 4p+2) = 1$.

Let us show that any manifold of type 2) is homeomorphic to a manifold of type 1). Indeed, let us set $d = 4p - \widetilde{d}$. Then the manifold of type 2) takes the form $M(\mathbb{S}^2, (2,1), (4p, 4p - \widetilde{d}), (4p, 2p + \widetilde{d}))$, $0 < \widetilde{d} < 2p$, $(\widetilde{d}, 2p) = 1$. Since $1 \equiv -1 \pmod 2$, $4p - \widetilde{d} \equiv -\widetilde{d} \pmod{4p}$, and $2p + \widetilde{d} \equiv -(2p - \widetilde{d}) \pmod{4p}$, from Proposition 2.1 it follows that

$$M(\mathbb{S}^2, (2,1), (4p, d), (4p, 6p-d)) \cong M(\mathbb{S}^2, (2,1), (4p, \widetilde{d}), (4p, 2p - \widetilde{d})).$$

Thus, any manifold M_f is a Seifert manifold of type 1) or 3). Expressing the parameters of the Seifert manifold in terms of the flow parameters (l, d), we obtain a list of manifolds of three types which was announced in the theorem. Moreover, the type 1 (2 or 3) corresponds to the suspension over the diffeomorphism f with fixed sink and source (fixed sink and source of period 2 or sink of period 2 and fixed source). $\square$

5 Counting the Number of Flow Equivalence Classes on a Given Manifold

Proof of Theorem 1.3. As was shown in the proof of Theorem 1.2], the supporting manifold M_f of the flow $f^t \in G^t$ can have one of the following forms:

(1) $M_f \cong M(\mathbb{S}^2, (2,1), (l, d), (l, l/2 - d))$, $l = 4p$, $p \in \mathbb{N}$,
(2) $M_f \cong M(\mathbb{S}^2, (2,1), (l, d), (l/2, \frac{l/2-d}{2}))$, $l = 4p+2$, $p \in \mathbb{N}$,
(3) $M_f \cong M(\mathbb{S}^2, (2,1), (l, d), (2l, l-2d))$, $l = 2p+1$, $p \in \mathbb{N}$.

By Proposition 2.1, the set of manifolds of type 2 coincides with the set of manifolds of type 3, and all such manifolds have the form

$$B_{p,d} = M(\mathbb{S}^2, (2,1), (4p+2, d), (2p+1, p-(d-1)/2)), \quad p \in \mathbb{N},\ d \in \{1, \dots, 2p-1\},\ (d, 4p+2) = 1.$$

By Propositions 2.1 and 2.2, $B_{p,d} \cong B_{p',d'}$ if and only if $p = p'$, $d = d'$. All manifolds of type 1 have the form

$$A_{p,d} = M(\mathbb{S}^2, (2,1), (4p, d), (4p, 2p-d)), \quad p \in \mathbb{N}, \quad d \in \{1, \dots, 2p-1\}, \quad (d, 4p) = 1,$$

and the same propositions imply that no manifold of the from $A_{p,d}$ is homeomorphic to a manifold of the form $B_{p,d}$, but $A_{p,d} \cong A_{p',d'}$ if and only if $p = p'$, $d = 2p - d'$. Thus, pairwise distinct manifolds of type 1 have the form

$$A_{p,d} = M(\mathbb{S}^2, (2,1), (4p, d), (4p, 2p-d)), \quad p \in \mathbb{N},\ d \in \{1, \dots, p-1\},\ (d, 4p) = 1.$$

The aforesaid and classification of flows of class G^t (see Theorem 1.1) lead to the following:

(1) each manifold $A_{p,d}$ admits exactly two classes of topological equivalence of flows of the class G^t, represented by flows with parameters $(4p, d)$ and $(4p, 2p - d)$,

(2) each manifold $B_{p,d}$ admits exactly two classes of topological equivalence of flows of the class G^t, represented by flows with parameters $(4p + 2, d)$ and $(2p + 1, p - (d - 1)/2)$.

The theorem is proved. □

6 Homology Groups of Mapping Tori

In this section, we prove Theorem 1.4. The proof follows from a more general result on homology groups with integer coefficients for manifolds M_φ, where $\varphi = \varphi_{q,m}$ is a periodic homeomorphism defined as follows.

Let $q \in \mathbb{N}$ and Π_q be a regular $2q$-gon with the scheme $a_1 a_2 \dots a_q a_1^{-1} a_2^{-1} \dots a_q^{-1}$. We choose $m \in \{1, \dots, q-1\}$ and denote by $\overline{\varphi}\colon \Pi_q \to \Pi_q$ the rotation of the polygon around the center by the angle $\theta_m = \pi m/q$ in the positive direction. Gluing together like sides of the polygon Π_q, we obtain a closed orientable surface S_p of genus $p = [q/2]$, where the rotation $\overline{\varphi}$ induces the homeomorphism $\varphi : S_p \to S_p$ (see Figure).

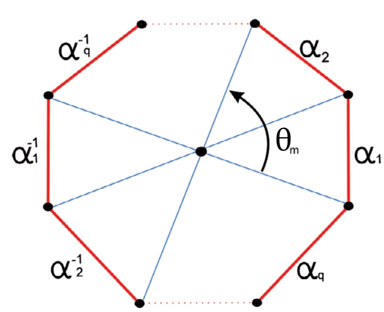

Figure.

Let $\mu\colon \Pi_q \to S_p$ be the natural projection, and let $\{i\} = i + q\mathbb{Z} \in \mathbb{Z}_q$, $i \in \mathbb{Z}$. We set $\ell = (q, m)$, $\varkappa_m = m/\ell$, $\varkappa_q = q/\ell$ and calculate the homology groups of M_φ separately in the cases of even and odd q.

Case $q = 2p$.

Lemma 6.1. *Assume that $q = 2p$ and $p \in \mathbb{N}$. Then the homology groups of the manifold $M = M_\varphi$ are isomorphic to the following groups:*

- *$H_3(M) \cong \mathbb{Z}$ and $H_2(M) \cong H_1(M) \cong \mathbb{Z}^{\ell+1}$ if $\varkappa_m$ is even,*
- *$H_3(M) \cong H_2(M) \cong \mathbb{Z}$ and $H_1(M) \cong \mathbb{Z} \times \mathbb{Z}_2^{\ell}$ if $\varkappa_m$ is odd.*

Proof. For $i = 1, \dots, q$ we set $z_{\{i\}} = \mu(a_i)$. Then $H_1(S_p) = \langle [z_{\{1\}}], \dots, [z_{\{q\}}] \rangle \cong \mathbb{Z}^q$ and the automorphism $\varphi_* \colon H_1(S_p) \to H_1(S_p)$ induced by φ is described by

$$\varphi_*([z_{\{i\}}]) = (-1)^{\left[\frac{i-1+m}{q}\right]} [z_{\{i+m\}}], \quad i = 1, \dots, q. \tag{6.1}$$

We set $N = \nu(S_p \times 0)$, where $\nu = \nu_\varphi \colon S_p \times [0,1] \to M_\varphi$ is the natural projection. To calculate the homology groups of M, we use the topological pair (M, N) and the corresponding homological sequence [12, Theorem 4.4.3]

$$\dots \longrightarrow H_n(N) \xrightarrow{\imath_*^n} H_n(M) \xrightarrow{\jmath_*^n} H_n(M,N) \xrightarrow{\partial_*^n} H_{n-1}(N) \longrightarrow \dots. \tag{6.2}$$

Let $z^0_{\{i\}} = \nu(z_{\{i\}} \times 0)$ for $i = 1, \dots, q$. Then $H_n(N) = 0$ for $n > 2$ and

$$H_2(N) \cong \mathbb{Z}, \quad H_1(N) = \langle [z^0_{\{1\}}], \dots, [z^0_{\{q\}}] \rangle \cong \mathbb{Z}^q. \tag{6.3}$$

We calculate the groups $H_n(M,N)$. For this purpose we consider the suspension ΣS_p obtained by contracting the bases $S_p \times 0$ and $S_p \times 1$ of the cylinder $S_p \times [0,1]$ into points v_0 and v_1. Let $V = \{v_0, v_1\}$. Then $M/N = \Sigma S_p / V$. The subsets $N \subset M$ and $V \subset \Sigma S_p$ are closed. They are deformation retracts of some neighborhoods of $U_N \subset M$ and $U_V \subset \Sigma S_p$. Hence from the relations between groups of absolute and relative homology [7, Proposition 2.22] for $n > 0$ it follows that

$$H_n(M,N) \cong H_n(M/N) = H_n(\Sigma S_p/V) \cong H_n(\Sigma S_p, V). \tag{6.4}$$

According to the isomorphism of the suspension [12, Theorem 4.4.10], we have

$$H_3(\Sigma S_p) \cong H_2(S_p) \cong \mathbb{Z}, \quad H_2(\Sigma S_p) \cong H_1(S_p) \cong \mathbb{Z}^q, \quad H_1(\Sigma S_p) = 0.$$

Substituting the last formulas into the homological sequence of the pair $(\Sigma S_p, V)$, we get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\jmath_*^3} H_3(\Sigma S_p, V) \xrightarrow{\partial_*^3} 0 \xrightarrow{\imath_*^2} \mathbb{Z}^q \xrightarrow{\jmath_*^2} H_2(\Sigma S_p, V)$$
$$\xrightarrow{\partial_*^2} 0 \xrightarrow{\imath_*^1} 0 \xrightarrow{\jmath_*^1} H_1(\Sigma S_p, V) \xrightarrow{\partial_*^1} \mathbb{Z}^2 \xrightarrow{\imath_*^0} \mathbb{Z} \longrightarrow 0.$$

Consequently,

$$H_3(\Sigma S_p, V) \cong \mathbb{Z}, \quad H_2(\Sigma S_p, V) \cong \mathbb{Z}^q, \quad H_1(\Sigma S_p, V) \cong \mathbb{Z}. \tag{6.5}$$

By (6.2)–(6.5), the following sequence is exact:

$$0 \longrightarrow H_3(M) \xrightarrow{\jmath_*^3} \mathbb{Z} \xrightarrow{\partial_*^3} \mathbb{Z} \xrightarrow{\imath_*^2} H_2(M) \xrightarrow{\jmath_*^2} \mathbb{Z}^q \xrightarrow{\partial_*^2} \langle [z^0_{\{1\}}], \dots, [z^0_{\{q\}}] \rangle$$
$$\cong \mathbb{Z}^q \xrightarrow{\imath_*^1} H_1(M) \xrightarrow{\jmath_*^1} \mathbb{Z} \xrightarrow{\partial_*^1} \mathbb{Z} \xrightarrow{\imath_*^0} \mathbb{Z} \longrightarrow 0. \tag{6.6}$$

Since S_p is an orientable 2-manifold and the homeomorphism φ preserves the orientation, M is an orientable closed 3-manifold. Hence $H_3(M) \cong \mathbb{Z}$ [7, Theorem 3.26] and $\jmath_*^3$ in (6.6) is an isomorphism. However, in this case, $\partial_*^3 = 0$. On the other hand, $\imath_*^0$ is an isomorphism and, consequently, $\partial_*^1 = 0$.

Since $\nu \colon S_p \times 0 \to N$ is a homeomorphism, the formula $\varphi^0(\nu(x,0)) = \nu(\varphi(x), 0)$ defines the homeomorphism $\varphi^0 \colon N \to N$. Let $\varphi^0_* \colon H_1(N) \to H_1(N)$ be the automorphism induced by the homeomorphism φ^0.

In the cylinder $S_p \times [0,1]$, the cycles $z_{\{i\}} \times 0$ and $z_{\{i\}} \times 1$ are homologous. In the process of constructing the manifold M, the cycle $z_{\{i\}} \times 1$ is glued to the cycle $\varphi(z_{\{i\}}) \times 0$. This means that $\varphi^0_*([z^0_{\{i\}}]) = [z^0_{\{i\}}]$ in $H_1(M)$. Then (6.1) implies that the following relations hold in $H_1(M)$:

$$[z^0_{\{i+jm\}}] = (-1)^{[\frac{i-1+jm}{q}]}[z^0_{\{i\}}], \quad i = 1, \dots, \ell, \quad j = 1, \dots, \varkappa_q. \tag{6.7}$$

Thus, the subgroup $\operatorname{im} \imath^1_* \subset H_1(M)$ is generated by the homology classes $[z^0_{\{1\}}], \dots, [z^0_{\{\ell\}}]$. For $j = q$ we have

$$\left[\frac{i-1+qm}{q}\right] = \left[\frac{i-1}{q} + \frac{qm}{q\ell}\right] = \left[\frac{i-1}{q} + \varkappa_m\right] = \varkappa_m.$$

Therefore, for $j = q$ from (6.7) for even $\varkappa_m$ the trivial relations follow: $[z^0_{\{i\}}] = [z^0_{\{i\}}]$, and for odd $\varkappa_m$ - the relations $[z^0_{\{i\}}] = -[z^0_{\{i\}}]$. In the first case, $[z^0_{\{i\}}]$ are free generators of the group $\operatorname{im} \imath^1_*$ and therefore

$$\operatorname{im} \imath^1_* = \langle [z^0_{\{1\}}], \dots, [z^0_{\{\ell\}}] \rangle \cong \mathbb{Z}^\ell. \tag{6.8}$$

In the second case, $2[z^0_{\{i\}}] = 0$ in $H_1(M)$ for all $i = 1, \dots, \ell$, and therefore,

$$\operatorname{im} \imath^1_* = \langle [z^0_{\{1\}}], \dots, [z^0_{\{\ell\}}] \parallel 2[z^0_{\{1\}}], \dots, 2[z^0_{\{\ell\}}] \rangle \cong \mathbb{Z}^\ell_2. \tag{6.9}$$

By (6.8), (6.6), and the above obtained qualities $\partial^3_* = 0$ and $\partial^1_* = 0$, we obtain the exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(M) \xrightarrow{\jmath^2_*} \mathbb{Z}^q \xrightarrow{\partial^2_*} \mathbb{Z}^q \xrightarrow{\imath^1_*} \mathbb{Z}^\ell \longrightarrow 0, \tag{6.10}$$

$$0 \longrightarrow \mathbb{Z}^\ell \longrightarrow H_1(M) \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{6.11}$$

By (6.11), we have $H_1(M) \cong \mathbb{Z}^{\ell+1}$. In the sequence (6.10), $\operatorname{im} \partial^2_* = \ker \imath^1_* \cong \mathbb{Z}^{q-\ell}$. But, in this case, $\operatorname{im} \jmath^2_* = \ker \partial^2_* \cong \mathbb{Z}^{q-(q-\ell)} = \mathbb{Z}^\ell$. Therefore, $H_2(M) \cong \mathbb{Z}^{\ell+1}$.

From (6.9) and (6.6) we obtain the exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(M) \xrightarrow{\jmath^2_*} \mathbb{Z}^q \xrightarrow{\partial^2_*} \mathbb{Z}^q \xrightarrow{\imath^1_*} \mathbb{Z}^\ell_2 \longrightarrow 0, \tag{6.12}$$

$$0 \longrightarrow \mathbb{Z}^\ell_2 \longrightarrow H_1(M) \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{6.13}$$

By (6.13), $H_1(P) \cong \mathbb{Z} \times \mathbb{Z}^\ell_2$. In (6.12), we have $\operatorname{im} \partial^2_* = \ker \imath^1_* = \mathbb{Z}^{q-\ell} \times (2\mathbb{Z})^\ell \cong \mathbb{Z}^q$. Thus, ∂^2_* is a monomorphism and, consequently, $\operatorname{im} \jmath^2_* = \ker \partial^2_* = 0$. But, in this case, $H_2(P) \cong \mathbb{Z}$. □

Case $q = 2p + 1$.

Lemma 6.2. *Assume that* $q = 2p + 1$ *and* $p \in \mathbb{N}$. *Then the homology groups of the manifold* $M = M_\varphi$ *are isomorphic to the following groups*:

- $H_3(M) \cong \mathbb{Z}$, $H_2(M) \cong \mathbb{Z}^{\ell-1}$, *and* $H_1(M) \cong \mathbb{Z}^\ell \times \mathbb{Z}_{\varkappa_q}$ *if* m *is even,*
- $H_3(M) \cong H_2(M) \cong \mathbb{Z}$ *and* $H_1(M) \cong \mathbb{Z} \times \mathbb{Z}^{\ell-1}_2$ *if* m *is odd.*

Proof. Assume that $z_{\{i\}} = \mu(a_i + a_{i+1})$ for $i = 1, \dots, 2p$ and $z_{\{0\}} = z_{\{q\}} = \mu(a_q - a_1)$. Then $H_1(S_p) = \langle [z_{\{1\}}], \dots, [z_{\{2p\}}] \rangle \cong \mathbb{Z}^{2p}$ the automorphism $\varphi_*\colon H_1(S_p) \to H_1(S_p)$ s induced by φ satisfies (6.1). We also assume that $\nu\colon S_p \times [0,1] \to M$ is the natural projection, $N = \nu(S_p \times 0)$, and $z^0_{\{i\}} = \nu(z_{\{i\}} \times 0)$ for $i = 1, \dots, q$. As in the case $q = 2p$, the sequence (6.6) is exact. Arguing as in Lemma 6.1, we obtain the equality (6.7). Thus, the subgroup $\operatorname{im} \imath^1_* \subset H_1(M)$ is also generated by the homology classes $[z^0_{\{1\}}], \dots, [z^0_{\{\ell\}}]$. Since ℓ is odd, the numbers m and $\varkappa_m$ are simultaneously even or not. As in Lemma 6.1, for $j = q$ from (6.7) we obtain the trivial relations $[z^0_{\{i\}}] = [z^0_{\{i\}}]$ for even $\varkappa_m$ and $2[z^0_{\{i\}}] = 0$ for odd $\varkappa_m$. By the construction of $z_{\{0\}} = z_{\{q\}}$, we also have

$$[D] = \sum_{\lambda=1}^{q} (-1)^\lambda [z^0_{\{\lambda\}}] = 0.$$

We set $[\widehat{z}_{\{1+l\ell\}}] = \sum\limits_{\lambda=1}^{\ell} (-1)^\lambda [z^0_{\{\lambda+l\ell\}}]$ for $l = 0, \dots, \varkappa_q - 1$ and $[\widehat{z}_{\{i\}}] = [z^0_{\{i\}}]$ for $i = 2, \dots, \ell$. Since the matrix of transition from $[z^0_{\{1\}}], [z^0_{\{2\}}], \dots, [z^0_{\{\ell\}}]$ to $[\widehat{z}_{\{1\}}], [\widehat{z}_{\{2\}}], \dots, [\widehat{z}_{\{\ell\}}]$ is unimodular, the homology classes $[\widehat{z}_{\{1\}}], [\widehat{z}_{\{2\}}], \dots, [\widehat{z}_{\{\ell\}}]$ generate the subgroup $\operatorname{im} \imath^1_* \subset H_1(M)$. Since ℓ is odd, we have

$$[D] = \sum_{l=0}^{\varkappa_q - 1} (-1)^l [\widehat{z}_{\{1+l\ell\}}] = \sum_{l=0}^{\varkappa_q - 1} (-1)^{l\ell} [\widehat{z}_{\{1+l\ell\}}]. \tag{6.14}$$

By the construction of the group $\mathbb{Z}_q$,

$$[\widehat{z}_{\{1+jm\}}] = [\widehat{z}_{\{1+l\ell\}}], \quad l\ell = jm - \left[\frac{jm}{q}\right] = jm - \left[\frac{j\varkappa_m}{\varkappa_q}\right], \quad j = 1, \dots, q. \tag{6.15}$$

On the other hand, in view of (6.7), we have

$$[\widehat{z}_{\{1+jm\}}] = (-1)^{[\frac{jm}{q}]} [\widehat{z}_{\{1\}}] = (-1)^{[\frac{j\varkappa_m}{\varkappa_q}]} [\widehat{z}_{\{1\}}]. \tag{6.16}$$

From (6.14) and (6.15) it follows that

$$[D] = \sum_{j=1}^{\varkappa_q} (-1)^{jm - [\frac{j\varkappa_m}{\varkappa_q}]} [\widehat{z}_{\{1+jm\}}] = \sum_{j=1}^{\varkappa_q} (-1)^{jm - [\frac{j\varkappa_m}{\varkappa_q}] + [\frac{j\varkappa_m}{\varkappa_q}]} [\widehat{z}_{\{1\}}] = [\widehat{z}_{\{1\}}] \sum_{j=1}^{\varkappa_q} (-1)^{jm}.$$

By the last formula, $[D] = \varkappa_q [\widehat{z}_{\{1\}}] = 0$ for even m and $[D] = -[\widehat{z}_{\{1\}}] = 0$ for odd m. Thus,

$$\operatorname{im} \imath^1_* = \langle [\widehat{z}_{\{1\}}], \dots, [\widehat{z}_{\{\ell\}}] \parallel \varkappa_q [\widehat{z}_{\{1\}}] \rangle \cong \mathbb{Z}^{\ell-1} \times \mathbb{Z}_{\varkappa_q} \tag{6.17}$$

for even m and

$$\operatorname{im} \imath^1_* = \langle [\widehat{z}_{\{2\}}], \dots, [\widehat{z}_{\{\ell\}}] \parallel 2[z^0_{\{2\}}], \dots, 2[z^0_{\{\ell\}}] \rangle \cong \mathbb{Z}_2^{\ell-1} \tag{6.18}$$

for odd m. From (6.17), (6.6), and the equalities $\partial^3_* = 0$ and $\partial^1_* = 0$ we obtain the exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(M) \xrightarrow{\jmath^2_*} \mathbb{Z}^{2p} \xrightarrow{\partial^2_*} \mathbb{Z}^{2p} \xrightarrow{\imath^1_*} \mathbb{Z}^{\ell-1} \times \mathbb{Z}_{\varkappa_q} \longrightarrow 0, \tag{6.19}$$

$$0 \longrightarrow \mathbb{Z}^{\ell-1} \times \mathbb{Z}_{\varkappa_q} \longrightarrow H_1(M) \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{6.20}$$

By (6.20), $H_1(P) \cong \mathbb{Z}^\ell \times \mathbb{Z}_{\varkappa_q}$. In (6.19), $\operatorname{im} \partial^2_* = \ker \imath^1_* = \mathbb{Z}^{2p-\ell+1} \times \varkappa_q \mathbb{Z} \cong \mathbb{Z}^{2p-\ell+2}$. But, in this case, $\operatorname{im} \jmath^2_* = \ker \partial^2_* \cong \mathbb{Z}^{2p-(2p-\ell+2)} = \mathbb{Z}^{\ell-2}$. Therefore, $H_2(P) \cong \mathbb{Z}^{\ell-1}$. By (6.18) and (6.6), we have the exact sequences

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(M) \xrightarrow{\jmath^2_*} \mathbb{Z}^{2p} \xrightarrow{\partial^2_*} \mathbb{Z}^{2p} \xrightarrow{\imath^1_*} \mathbb{Z}_2^{\ell-1} \longrightarrow 0, \tag{6.21}$$

$$0 \longrightarrow \mathbb{Z}_2^{\ell-1} \longrightarrow H_1(M) \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{6.22}$$

By (6.22), $H_1(P) \cong \mathbb{Z} \times \mathbb{Z}_2^{\ell-1}$. In (6.21), $\operatorname{im} \partial_*^2 = \ker \imath_*^1 = \mathbb{Z}^{2p-\ell+1} \times (2\mathbb{Z})^{\ell-1} \cong \mathbb{Z}^{2p}$. Thus, ∂_*^2 is a monomorphism and, consequently, $\operatorname{im} \jmath_*^2 = \ker \partial_*^2 = 0$. But, in this case, $H_2(P) \cong \mathbb{Z}$. $\square$

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