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ABSTRACT

In our previous publications, we have introduced analogs of partial derivatives on the reflection equation algebras, associated with Hecke symmetries. As a consequence, we get quantum partial derivatives on the enveloping algebras $U(gl(N))$. In the current paper, we consider the particular case $N = 2$ in detail and discuss the problem of a prolongation of these derivatives onto some central extension of the compact form $U(u(2))$ of the algebra $U(gl(2))$. Possible applications of this noncommutative geometry are discussed.

I. INTRODUCTION

Attempts to construct a noncommutative geometry related to quantum groups have been undertaken since the creation of quantum group theory. Initiated in Ref. 1, this study was developed in Refs. 2 and 3 and other papers. The authors of those papers aimed at constructing a differential calculus in which the role of the quantum function algebra is played by the algebra of quantized functions on the group $GL(N)$ (the so-called RTT algebra), while generators of the corresponding Reflection Equation (RE) algebra were treated as one-sided vector fields.[4]

In Ref. 5, we have constructed a different version of the calculus via attributing the role of function algebra to another copy of the RE algebra. By combining the generating matrices[6] of these two copies of the RE algebra, we constructed a new matrix D, whose entries ∂_i^j play the role of partial derivatives in the entries of one of the initial generating matrices.

This construction was realized in terms of the so-called quantum double. By a quantum double, we mean a couple of unital associative algebras (A, B) (hereafter, the ground field is $\mathbb{C}$) endowed with a permutation mapping $\sigma : A \otimes B \to B \otimes A$ satisfying a set of axioms (see Ref. 7 for detail). With the mapping σ, we define an associative algebra structure on the space $B \otimes A$ in such a way that the algebras A and B are proper subalgebras with respect to this new algebraic structure. In abstract algebra, analogous construction is known under the names "twisting tensor product of algebras," "factorization of algebras," etc. (see, for example, Ref. 8).

We also assume that the algebra A admits an algebra homomorphism $\varepsilon : A \to \mathbb{C}$. These ingredients enable us to define an action $\triangleright: A \otimes B \to B$ of the algebra A onto B,

$$a \triangleright b = (\mathrm{id}_B \otimes \varepsilon) \circ \sigma(a \otimes b), \quad \forall\, a \in A, \quad \forall\, b \in B.$$

Due to this reason, the algebras A and B will be, respectively, referred to as an operator subalgebra and a function subalgebra of the quantum double.

A well-known example of the quantum double is the Heisenberg double with the RTT algebra as the function subalgebra. This double enters the construction of the differential calculus developed in Refs. 2 and 3, but it does not give rise to analogs of the partial derivatives on the RTT algebra. It should be emphasized that there are very few noncommutative algebras that allow introducing analogs of partial derivatives.

One of the well-known examples is the algebra, generated by elements x_i, $1 \le i \le N$, subject to permutation relations: $x_i x_j - x_j x_i = \theta_{ij}$, $\theta_{ij} \in \mathbb{C}$. It turns out that the partial derivatives in generators of this algebra can be defined in a classical way.

The *quantum partial derivatives* (QPDs) introduced in Ref. 5 are well-defined on the RE algebra (or on its modified version), associated with a Hecke symmetry R. Recall that by Hecke symmetry, we mean a solution of the braid relation (a braiding) $R \in \mathrm{End}(V^{\otimes 2})$, V being a finite dimensional linear space $\dim_{\mathbb{C}} V = N$, which satisfies a supplementary condition,

$$R^2 = I \otimes I + (q - q^{-1})R, \quad q \in \mathbb{C}\backslash\{0, \pm 1\}.$$

Hereafter, I stands for the identity operator in $\mathrm{End}(V)$ or its matrix.

A modified RE algebra is an unital associative algebra generated by entries of the matrix $L = \|l_i^j\|_{1\le i,j\le N}$ subject to the system of relations

$$RL_1RL_1 - L_1RL_1R = h(RL_1 - L_1R), \quad L_1 = L \otimes I, \quad h \in \mathbb{C}\backslash 0. \tag{1.1}$$

At the limit $q \to 1$, this algebra tends[9] to $U(gl(N)_h)$, provided the Hecke symmetry R turns into the usual flip P [see (2.5)]. If $h = 0$, the algebra (1.1) is called the (non-modified) RE algebra. At $q \to 1$, it tends to $\mathrm{Sym}(gl(N))$.

Therefore, the RE algebras (modified or not) play the role of the function subalgebras B in the quantum doubles we are dealing with. The corresponding operator subalgebra A will be exhibited in Sec. II. Note that our QPD on the modified RE algebras turns into the usual partial derivatives on $\mathrm{Sym}(gl(N))$, as $R \to P$ and $h \to 0$.

However, the QPD introduced in such a way is well defined only on polynomials in the generators of a given modified RE algebra, whereas the usual partial derivatives are defined on any smooth function. The point is that in RE algebra, the formula of differentiation of a composed function is not valid. In the present paper, we discuss a way of extending the definition domain of the QPD in a particular case: $B = U(gl(2)_h)$.

To be more precise, we define the QPD as an extension of $U(u(2)_h)$, constructed in two steps. First, to the initial algebra $B = U(u(2)_h)$, we add some elements of its central extension $\mathcal{A}$. These elements arise from the Cayley–Hamilton identity, satisfied by the generating matrix of the algebra $U(u(2)_h)$. Second, we extend the action of our QPD to some elements of the skew-field,

$$\mathcal{B} = \mathcal{A}[\mathcal{A}^{-1}] = \{a/b \,|\, a, b \in \mathcal{A},\ b \ne 0\}.$$

The larger the final algebra, the larger the class of the usual differential operators for which we are able to exhibit their noncommutative analogs. We call the procedure of passing to the latter operators *quantization with noncommutative configuration space* (see the end of the paper). Therefore, we can hopefully construct noncommutative analogs of numerous dynamical models.

Remark 1. The problem of a prolongation of the differential calculus with the use of the Cayley–Hamilton identity was considered in Ref. 10. However, the authors of Ref. 10 considered an extension of the operator subalgebra A in the corresponding quantum double, whereas we consider an extension of the function subalgebra B.

The paper is organized as follows. In Sec. II, we define analogs of partial derivatives on the RE algebras using the quantum double construction. In a particular case of the enveloping algebra $U(gl(N)_h)$, we suggest another way to define the partial derivatives: via a coproduct, which plays the role of the Leibniz rule. In Sec. III, we consider the case $N = 2$ in detail. More precisely, we deal with the compact form $U(u(2)_h)$ of the algebra $U(gl(2)_h)$, and we exhibit the Leibniz rule in a form useful for a prolongation of the QPD on some central extension of the algebra $U(u(2)_h)$. In Sec. IV, we add to the algebra $U(u(2)_h)$ the "eigenvalues" of the generating matrix L of this algebra. Then, in Sec. V, we extend our QPD onto some elements of the corresponding skew-field, in particular those that could be useful for finding a noncommutative analog of the Dirac vector potential for the magnetic monopole field.

II. QUANTUM PARTIAL DERIVATIVES BY MEANS OF QUANTUM DOUBLE

First, we describe the quantum double construction, giving rise to QPD on the modified RE algebra, which plays the role of the function subalgebra B. The corresponding operator subalgebra A is an associative algebra generated by the entries of the $N \times N$ matrix $D = \|\partial_i^j\|$, subject to the system of quadratic relations

$$R^{-1}D_1R^{-1}D_1 = D_1R^{-1}D_1R^{-1}. \tag{2.1}$$

The permutation mapping σ is introduced as follows:

$$\sigma : D_1R_{12}L_1R_{12} \to R_{12}L_1R_{12}^{-1}D_1 + h1_BD_1R_{12} + R_{12}1_B1_A. \tag{2.2}$$

Below, we shall omit the symbols of unit elements 1_A and 1_B.

These permutation relations are consistent with the defining relations of the algebras A and B in the following sense:

$$\sigma(I_A \otimes B) \subset B \otimes I_A, \qquad \sigma(A \otimes I_B) \subset I_B \otimes A,$$

where I_A (respectively, I_B) is an ideal generated by relations in A (respectively, B), that is, by some elements in the free tensor algebra corresponding to the algebra A (to B). In addition, we assume that σ acts as the usual flip on the elements $1_A \otimes b$ and $a \otimes 1_B$ for any $a \in A$, $b \in B$.

Now, in order to convert the elements ∂_i^j into operators on the function subalgebra B, we introduce a homomorphism $\varepsilon : A \to \mathbb{C}$ by the rules

$$\varepsilon(1_A) = 1_{\mathbb{C}}, \qquad \varepsilon(\partial_i^j) = 0 \quad \forall\, i, j. \tag{2.3}$$

Then the action of an element $a \in A$ on $b \in B$ is defined as follows:

$$a \triangleright b = (\mathrm{id}_B \otimes \varepsilon) \circ \sigma(a \otimes b). \tag{2.4}$$

Here, we identify $b \otimes 1_{\mathbb{C}}$ and b; the symbol $\circ$ stands for the composition of mappings.

The relations (2.2), completed by the homomorphism ε, express a generalized Leibniz rule for the quantum derivatives. At $h = 0$ and $R = P$, the relations (2.2) turn into the classical Leibniz rules for the usual partial derivatives, while (2.1) means the mutual commutativity of all ∂_i^j.

Remark 2. Note that the problem of defining analogs of the partial derivatives on the algebra $U(g)$, where g is an arbitrary Lie algebra, was considered in Ref. 11. In that paper, partial derivatives were introduced by means of deformation quantization methods. This gave rise to a form of the Leibniz rule containing an infinite formal series in the corresponding derivatives, whereas the permutation relations (2.2) are polynomials. We do not know of other algebras $U(g)$ admitting partial derivatives with a similar property. Even the algebra $U(sl(N))$ does not admit a closed system of the permutation relations since the restriction of system (2.2) to $U(sl(N))$ with simultaneous restriction of the QPD contains an undesired term in the right hand side. However, this term disappears after applying the homomorphism ε.

Now, let us consider a particular case, $R = P$, $h \neq 0$, in more detail. In this case, the defining system (1.1) on the generators of the function subalgebra B takes the form

$$l_i^j\, l_k^s - l_k^s\, l_i^j = h(l_i^s\, \delta_k^j - l_k^j\, \delta_i^s) \quad \Leftrightarrow \quad L_1L_2 - L_2L_1 = h(L_1P_{12} - P_{12}L_1), \tag{2.5}$$

which means that $B = U(gl(N)_h)$.

The defining relations for the generators of subalgebra A and the permutation relations with the generators of B are read, respectively,

$$D_1D_2 = D_2D_1, \tag{2.6}$$

$$D_1L_2 = L_2D_1 + hD_1P_{12} + P_{12}. \tag{2.7}$$

Remark 3. There exists another system of permutation relations compatible with algebraic structures (2.5) and (2.6),

$$D_1L_2 = L_2D_1 - hP_{12}D_1 + P_{12}. \tag{2.8}$$

However, systems (2.5)–(2.7) are equivalent to systems (2.5), (2.6), and (2.8). To see this, it suffices to apply the operation of transposition to all matrices entering the first system and to replace h by −h.

On introducing the homomorphism ε (2.3), we get the action of the QPD on elements of the function subalgebra $B = U(gl(N)_h)$ in accordance with the general recipe (2.4). On the generators of $U(gl(N)_h)$, this action coincides with the classical one,

$$\partial_i^j \triangleright 1_B = 0, \qquad \partial_i^j \triangleright l_k^s = \delta_i^s \delta_k^j \quad \Leftrightarrow \quad D_1 \triangleright L_2 = P_{12}.$$

As another example, we apply the QPD to a second order monomial,

$$D_1 \triangleright (L_2L_3) = P_{12}L_3 + L_2P_{13} + hP_{12}P_{23}. \tag{2.9}$$

A general formula of the QPD action on an arbitrary monomial can be found in Ref. 12.

Therefore, the permutation relations (2.7) together with the homomorphism (2.3) replace the classical Leibniz rule as we pass from the algebra $\mathrm{Sym}(gl(N))$ to $U(gl(N)_h)$. However, it is not clear how to use this analog of the Leibniz rule in defining the QPD action on some extensions of the algebra $U(gl(N)_h)$. Fortunately, in this case, there is a more useful form of the Leibniz rule. Namely, we introduce a homomorphism of associative algebras $\Delta : A \to A \otimes A$ setting by definition

$$\Delta(1_A) = 1_A \otimes 1_A,$$

$$\Delta(\partial_i^j) = \partial_{i\,(1)}^j \otimes \partial_{i\,(2)}^j = \partial_i^j \otimes 1_A + 1_A \otimes \partial_i^j + h\sum_{k=1}^N \partial_k^j \otimes \partial_i^k. \tag{2.10}$$

Here, in the last equality, we use the standard Sweedler's notation for the mapping $\Delta(\partial_i^j)$. On the whole, in algebra A, the mapping Δ is extended by the homomorphism property,

$$\Delta(xy) = \Delta(x)\Delta(y) = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}, \qquad \forall\, x, y \in A.$$

Proposition 4.

(i) *The homomorphism Δ is a coproduct, that is, it satisfies the coassociativity condition,*

$$(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta.$$

(ii) *The action of the QPD according to the rule*

$$\partial_i^j \triangleright (a\, b) = (\partial_{i(1)}^j \triangleright a)\, (\partial_{i(2)}^j \triangleright b), \qquad \forall\, a, b \in B, \tag{2.11}$$

coincides with the action of QPD as defined by the permutation relations (2.7) and the homomorphism (2.3).

(iii) *The action (2.11) preserves the algebraic structure of $B = U(gl(N)_h)$, which is QPD send the elements $l_k^s\, l_p^q - l_p^q\, l_k^s - h(\delta_p^s l_k^q - \delta_k^q l_p^s)$ to zero.*

Proof. The proposition is proved by straightforward calculations. ■

Remark 5. As follows from Proposition 4, the action of QPD on the algebra $U(gl(N)_h)$ can be defined via the coproduct (2.10) by the relation (2.11). Note that this coproduct endows the algebra A generated by QPD with a bi-algebra structure. In a similar way, it is possible to introduce QPD on the modified RE algebras corresponding to involutive symmetries R (i.e., such that $R^2 = I$). However, the corresponding bi-algebra structure of the algebra generated by QPD will be braided. In addition, in this case, a braiding enters the permutation relations of the generators of the QPD and those of the corresponding modified RE algebra. This property prevents constructing a prolongation of the QPD onto the extensions of the RE algebras in a way similar to that considered below. If R is a Hecke symmetry, the situation becomes much more complicated. In this case, we are not able to construct a convenient coproduct. Therefore, in general, the only possible way of introducing the QPD on a RE algebra (modified or not) is the use of permutation relations completed with the homomorphism ε.

Let us go back to the quantum double (A, B), where $B = U(gl(N)_h)$ and A is generated by the commutative entries of the matrix $D = \|\partial_i^j\|$. We perform the following change to the generating matrix D,

$$\hat{D} = D + h^{-1} I \quad \Leftrightarrow \quad \hat{\partial}_i^j = \partial_i^j + h^{-1}\delta_i^j\, 1_A. \tag{2.12}$$

In terms of the matrix $\hat{D}$, the permutation relations (2.7) take the form

$$\hat{D}_1\, L_2 = L_2\, \hat{D}_1 + h\hat{D}_1 P_{12}. \tag{2.13}$$

The action of the coproduct Δ (2.10) on the generators $\hat{\partial}_i^j$ reads

$$\Delta(\hat{\partial}_i^j) = h\sum_k \hat{\partial}_k^j \otimes \hat{\partial}_i^k \quad \Leftrightarrow \quad \Delta(\hat{D}^t) = h\hat{D}^t \dot{\otimes} \hat{D}^t, \tag{2.14}$$

where the superscript t means the matrix transposition, the notation $S \dot{\otimes} T$ stands for the matrix with entries $s_i^k \otimes t_k^j$ for any two equal size square matrices $S = \|s_i^j\|$ and $T = \|t_i^j\|$.

The coproduct (2.14) leads to the following form of the Leibniz rule:

$$\hat{D}(ab)^t = h\hat{D}(a)^t \hat{D}(b)^t, \tag{2.15}$$

where $\hat{D}(a)$ denotes the matrix with entries $\hat{\partial}_i^j \triangleright a$. Below, we use the matrix $\mathrm{D} = h\hat{D}^t$, which is multiplicative (group-like) with respect to the coproduct

$$\Delta(\mathrm{D}) = \mathrm{D} \dot{\otimes} \mathrm{D}. \tag{2.16}$$

Therefore, in terms of this matrix, the Leibniz rule takes the following form:

$$\mathrm{D}(ab) = \mathrm{D}(a)\mathrm{D}(b). \tag{2.17}$$

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III. QUANTUM PARTIAL DERIVATIVES ON $U(u(2)_h)$

Let us consider the case of $N = 2$ in more detail. Below, we use the following notations $a = l_1^1$, $b = l_1^2$, $c = l_2^1$, and $d = l_2^2$,

$$L = \begin{pmatrix} l_1^1 & l_1^2 \\ l_2^1 & l_2^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \hat{D} = \begin{pmatrix} \hat{\partial}_1^1 & \partial_1^2 \\ \partial_2^1 & \hat{\partial}_2^2 \end{pmatrix} = \begin{pmatrix} \hat{\partial}_a & \partial_c \\ \partial_b & \hat{\partial}_d \end{pmatrix}.$$

Hereafter, we use the "hat" notation only for the diagonal elements of the matrix $\hat{D}$ since the off-diagonal partial derivatives are actually not changed under the shift (2.12): $\hat{\partial}_i^j \equiv \partial_i^j$ for any $i \neq j$.

The permutation relations (2.13) have the following explicit form:

$$\begin{array}{llll} \hat{\partial}_a\, a = a\,\hat{\partial}_a + h\,\hat{\partial}_a, & \hat{\partial}_a\, b = b\,\hat{\partial}_a + h\,\partial_c, & \partial_c\, a = a\,\partial_c, & \partial_c\, b = b\,\partial_c, \\ \hat{\partial}_a\, c = c\,\hat{\partial}_a, & \hat{\partial}_a\, d = d\,\hat{\partial}_a, & \partial_c\, c = c\,\partial_c + h\,\hat{\partial}_a, & \partial_c\, d = d\,\partial_c + h\,\partial_c, \\ \partial_b\, a = a\,\partial_b + h\,\partial_b, & \partial_b\, b = b\,\partial_b + h\,\hat{\partial}_d, & \hat{\partial}_d\, a = a\,\hat{\partial}_d, & \hat{\partial}_d\, b = b\,\hat{\partial}_d, \\ \partial_b\, c = c\,\partial_b, & \partial_b\, d = d\,\partial_b, & \hat{\partial}_d\, c = c\,\hat{\partial}_d + h\,\partial_b, & \hat{\partial}_d\, d = d\,\hat{\partial}_d + h\,\hat{\partial}_d. \end{array}$$

Now, we introduce a new set of generators corresponding to the compact form $u(2)_h$ of the algebra $gl(2)_h$,

$$t = \frac{1}{2}(a + d), \qquad x = \frac{i}{2}(b + c), \qquad y = \frac{1}{2}(c - b), \qquad z = \frac{i}{2}(a - d).$$

Their Lie brackets read

$$[x, y] = hz, \qquad [y, z] = hx, \qquad [z, x] = hy, \qquad [t, x] = [t, y] = [t, z] = 0.$$

Since the change of the generators is linear, we apply the classical formula for computing the QPD in new generators: $\partial_t = \partial_a + \partial_d$, and so on. In these generators, the "shifted version" is used only for the derivative in t: $\hat{\partial}_t = \partial_t + \frac{2}{h}\,\mathrm{id}$. The permutation relations become as follows:

$$\begin{array}{llll} \hat{\partial}_t\, t - t\,\hat{\partial}_t = \frac{h}{2}\,\hat{\partial}_t, & \hat{\partial}_t\, x - x\,\hat{\partial}_t = -\frac{h}{2}\,\partial_x, & \hat{\partial}_t\, y - y\,\hat{\partial}_t = -\frac{h}{2}\,\partial_y, & \hat{\partial}_t\, z - z\,\hat{\partial}_t = -\frac{h}{2}\,\partial_z, \\ \partial_x\, t - t\,\partial_x = \frac{h}{2}\,\partial_x, & \partial_x\, x - x\,\partial_x = \frac{h}{2}\,\hat{\partial}_t, & \partial_x\, y - y\,\partial_x = \frac{h}{2}\,\partial_z, & \partial_x\, z - z\,\partial_x = -\frac{h}{2}\,\partial_y, \\ \partial_y\, t - t\,\partial_y = \frac{h}{2}\,\partial_y, & \partial_y\, x - x\,\partial_y = -\frac{h}{2}\,\partial_z, & \partial_y\, y - y\,\partial_y = \frac{h}{2}\,\hat{\partial}_t, & \partial_y\, z - z\,\partial_y = \frac{h}{2}\,\partial_x, \\ \partial_z\, t - t\,\partial_z = \frac{h}{2}\,\partial_z, & \partial_z\, x - x\,\partial_z = \frac{h}{2}\,\partial_y, & \partial_z\, y - y\,\partial_z = -\frac{h}{2}\,\partial_x, & \partial_z\, z - z\,\partial_z = \frac{h}{2}\,\hat{\partial}_t. \end{array} \tag{3.1}$$

When working with the compact form $U(u(2)_h)$ of the algebra $U(gl(2)_h)$, it is more convenient to use the larger matrix $\hat{\Theta}$ instead of the matrix D,

$$\hat{\Theta} = i\hbar \begin{pmatrix} \hat{\partial}_t & \partial_x & \partial_y & \partial_z \\ -\partial_x & \hat{\partial}_t & -\partial_z & \partial_y \\ -\partial_y & \partial_z & \hat{\partial}_t & -\partial_x \\ -\partial_z & -\partial_y & \partial_x & \hat{\partial}_t \end{pmatrix}, \tag{3.2}$$

where we introduce a rescaled parameter $\hbar = h/2i$. The normalization multiplier $i\hbar$ ensures a group-like coproduct for $\hat{\Theta}$ similar to (2.16),

$$\Delta(\hat{\Theta}) = \hat{\Theta}\,\dot{\otimes}\,\hat{\Theta}. \tag{3.3}$$

With the use of the matrix $\hat{\Theta}$, we define a linear mapping (denoted by the same symbol $\hat{\Theta}$),

$$\hat{\Theta}: \ U(u(2)_h) \to \mathrm{Mat}_2(U(u(2)_h)), \qquad a \mapsto \hat{\Theta}(a) \quad \forall\, a \in U(u(2)_h), \tag{3.4}$$

where $\mathrm{Mat}_2(U(u(2)_h)) = \mathrm{End}(\mathbb{C}^4) \otimes U(u(2)_h)$ and the symbol $\hat{\Theta}(a)$ has the same meaning as $\hat{D}(a)$ above does.

As follows from the coproduct (3.3), the mapping $\hat{\Theta}$ is actually a homomorphism of associative algebras,

$$\hat{\Theta}(ab) = \hat{\Theta}(a)\hat{\Theta}(b), \quad \forall\, a, b \in U(u(2)_h). \tag{3.5}$$

This property of the map $\hat{\Theta}$ is treated as the Leibniz rule for the QPD action on the algebra $U(u(2)_h)$.

As an example, we give the explicit formulas for the images of the $U(u(2)_h)$ generators under the mapping $\hat{\Theta}$,

$$\begin{aligned} &\hat{\Theta}(1_{U(u(2)_h)}) = I, \qquad \hat{\Theta}(t) = (t + i\hbar)I, \\ &\hat{\Theta}(x) = xI + i\hbar A, \qquad \hat{\Theta}(y) = yI + i\hbar B, \qquad \hat{\Theta}(z) = zI + i\hbar C, \end{aligned} \tag{3.6}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.7}$$

The numerical matrices A, B, and C give the real matrix representation of the generators of the quaternion algebra,

$$A^2 = B^2 = C^2 = -I, \qquad AB = -BA = C, \qquad BC = -CB = A, \qquad CA = -AC = B. \tag{3.8}$$

Below, we extend the action of the QPD onto some larger algebra while preserving the property (3.5) of the mapping $\hat{\Theta}$.

IV. QUANTUM PARTIAL DERIVATIVES ON CENTRAL EXTENSION OF $U(u(2)_h)$

First, we present the Cayley–Hamilton identity for the generating matrix of the algebra $U(gl(2)_h)$,

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} t - iz & -ix - y \\ -ix + y & t + iz \end{pmatrix}.$$

This identity reads $p(L) = 0$, where the characteristic polynomial $p(\lambda)$ is of the form

$$p(\lambda) = \lambda^2 - 2(t + i\hbar)\lambda + (t^2 + x^2 + y^2 + z^2 + 2i\hbar t) I.$$

Let us denote μ_1 and μ_2 as the roots of the characteristic polynomial $p(\mu_1) = p(\mu_2) = 0$,

$$\mu_1 + \mu_2 = 2(t + i\hbar), \qquad \mu_1\mu_2 = t^2 + \text{Cas} + 2i\hbar t, \tag{4.1}$$

where $\text{Cas} = x^2 + y^2 + z^2$. Since t and Cas are central elements of $U(u(2)_h)$, then the roots μ_i belong to a central extension $\mathcal{A}$ of $U(u(2)_h)$,

$$\mathcal{A} = U(u(2)_h) \otimes \mathbb{C}[\mu_1, \mu_2]/\langle J \rangle,$$

where $\langle J \rangle$ is a two-sided ideal generated by equalities (4.1).

Now we extend the QPD action on the algebra $\mathcal{A}$ with the requirement that the multiplicative property (3.5) be valid on $\mathcal{A}$.

For this purpose, we need to compute the matrices $\hat{\Theta}(\mu_i)$, $i = 1, 2$. With the use of (3.6), it is not difficult to find their sum,

$$\hat{\Theta}(\mu_1) + \hat{\Theta}(\mu_2) = \hat{\Theta}(\mu_1 + \mu_2) = \hat{\Theta}(2t + 2i\hbar) = (2t + 4i\hbar)I. \tag{4.2}$$

Therefore, to find $\hat{\Theta}(\mu_1)$ and $\hat{\Theta}(\mu_2)$, it suffices to calculate their difference $\hat{\Theta}(\mu_1) - \hat{\Theta}(\mu_2)$.

We first compute the matrix $\hat{\Theta}(\mu^2)$, where $\mu = \mu_1 - \mu_2$. Taking into account that

$$\mu^2 = (\mu_1 + \mu_2)^2 - 4\mu_1\mu_2 = -4(\text{Cas} + \hbar^2),$$

and using (3.6), we get

$$\hat{\Theta}(\mu^2) = \hat{\Theta}(-4(x^2 + y^2 + z^2 + \hbar^2)) = (\mu^2 - 12\hbar^2)I - 8i\hbar M,$$

where $M = xA + yB + zC$.

Having calculated $\hat{\Theta}(\mu^2)$, we use the multiplicative property (3.5), which we *require* to be true on the extended algebra $\mathcal{A}$: $\hat{\Theta}(\mu^2) = \hat{\Theta}^2(\mu)$. Therefore, we need to calculate the square root of $\hat{\Theta}(\mu^2)$,

$$\hat{\Theta}(\mu_1) - \hat{\Theta}(\mu_2) = \hat{\Theta}(\mu) = \sqrt{\hat{\Theta}(\mu^2)}.$$

This can be performed with the use of the Cayley–Hamilton identities. By a straightforward calculation, one can show that the matrix $M = xA + yB + zC$ satisfies the following Cayley–Hamilton identity:

$$M^2 - 2i\hbar M + \text{Cas}\, I = (M - \lambda_1 I)(M - \lambda_2 I) = 0,$$

where the "quantum eigenvalues" λ_1 and λ_2 of M are easy to find,

$$\lambda_1 = i\hbar + \frac{\mu}{2}, \qquad \lambda_2 = i\hbar - \frac{\mu}{2}.$$

Consequently, the quantum eigenvalues of the matrix $\hat{\Theta}(\mu^2)$ are as follows:

$$\nu_1 = \mu^2 - 4\hbar^2 - 4i\hbar\,\mu = (\mu - 2i\hbar)^2, \qquad \nu_2 = \mu^2 - 4\hbar^2 + 4i\hbar\,\mu = (\mu + 2i\hbar)^2.$$

As for the matrix $\hat{\Theta}(\mu) = \sqrt{\hat{\Theta}(\mu^2)}$, it can be found via the spectral decomposition of the matrix $\hat{\Theta}(\mu^2)$,

$$\hat{\Theta}(\mu) = \frac{\hat{\Theta}(\mu^2) - \nu_2 I}{\nu_1 - \nu_2}\sqrt{\nu_1} + \frac{\hat{\Theta}(\mu^2) - \nu_1 I}{\nu_2 - \nu_1}\sqrt{\nu_2}.$$

Therefore, we have four candidates for the role of the matrix $\hat{\Theta}(\mu)$ in accordance with the possible sign choice of the roots,

$$\sqrt{\nu_1} = \epsilon_1(\mu - 2i\hbar), \qquad \sqrt{\nu_2} = \epsilon_2(\mu + 2i\hbar), \qquad \epsilon_1, \epsilon_2 \in \{\pm 1\}.$$

Immediate verification shows that only the choice $\epsilon_1 = \epsilon_2 = 1$ is compatible with the classical limit (that is, corresponding to $\hbar = 0$) for the action of the partial derivatives,

$$\partial_t \triangleright \mu = 0, \qquad \partial_x \triangleright \mu = -4\,\frac{x}{\mu}. \tag{4.3}$$

Note that the second relation in (4.3) follows from the equality $\mu^2 = -4\,\mathrm{Cas}$ valid in the classical limit.

Therefore, we come to the final formula

$$\hat{\Theta}(\mu_1) - \hat{\Theta}(\mu_2) = \hat{\Theta}(\mu) = \frac{\mu^2 - 4\hbar^2}{\mu}\,I - \frac{4i\hbar}{\mu}\,M. \tag{4.4}$$

In a similar way, we can get the action of the QPD on an arbitrary power of μ,

$$\hat{\Theta}(\mu^p) = \frac{(\mu + 2i\hbar)^{p+1} + (\mu - 2i\hbar)^{p+1}}{2\mu}\,I + \frac{(\mu - 2i\hbar)^p - (\mu + 2i\hbar)^p}{\mu}\,M. \tag{4.5}$$

At last, with the use of (4.2) and (4.4), we get

$$\hat{\Theta}(\mu_1) = \left(t + 2i\hbar + \frac{\mu^2 - 4\hbar^2}{2\mu}\right)I - \frac{2i\hbar}{\mu}\,M, \qquad \hat{\Theta}(\mu_2) = \left(t + 2i\hbar - \frac{\mu^2 - 4\hbar^2}{2\mu}\right)I + \frac{2i\hbar}{\mu}\,M.$$

Taking into account the structure of the matrix $\hat{\Theta}$ (3.2), we get the action of QPD on the quantum eigenvalues μ_i,

$$\hat{\partial}_t \triangleright \mu_1 = -\frac{i}{\hbar}\left(t + \frac{(\mu + 2i\hbar)^2}{2\mu}\right), \qquad \partial_x \triangleright \mu_1 = -\frac{2}{\mu}\,x,$$
$$\hat{\partial}_t \triangleright \mu_2 = -\frac{i}{\hbar}\left(t - \frac{(\mu - 2i\hbar)^2}{2\mu}\right), \qquad \partial_x \triangleright \mu_2 = \frac{2}{\mu}\,x,$$

and so on.

At the end of this section, we recall the notion of the quantum radius $r_\hbar$ and exhibit the result of the QPD action on it. In Ref. 12, we introduced the so-called quantum radius $r_\hbar^2 = \mathrm{Cas} + \hbar^2$. In terms of $\mu = \mu_1 - \mu_2$, it is written as $r_\hbar = \pm\mu/2i$. The sign here is not fixed because our ordering of μ_1 and μ_2 is arbitrary.

Using the relation between $r_\hbar$ and μ we get from (4.4),

$$\hat{\Theta}(r_\hbar) = \frac{r_\hbar^2 + \hbar^2}{r_\hbar}\,I + \frac{i\hbar}{r_\hbar}\,M. \tag{4.6}$$

This formula enables us to find the action of all QPDs on the quantum radius,

$$\partial_t \triangleright r_\hbar = -\frac{i\hbar}{r_\hbar}, \qquad \partial_x \triangleright r_\hbar = \frac{x}{r_\hbar}, \qquad \partial_y \triangleright r_\hbar = \frac{y}{r_\hbar}, \qquad \partial_z \triangleright r_\hbar = \frac{z}{r_\hbar}. \tag{4.7}$$

Note that at the limit $\hbar = 0$, the derivative $\partial_t \triangleright r$ vanishes, and other formulas turn into the classical ones (with $r_\hbar$ replaced by r).

Remark 6. Recall that in our treatment of the algebra $U(u(2)_h)$ as a noncommutative analog of the polynomial algebra on the Minkowski space (see Ref. 13), the generators $x, y,$ and z play the roles of spacial variables, and t plays the role of time.

Finally, we would like to shortly discuss a possible generalization of the above-mentioned results to higher dimensions. In dealing with the algebra $U(gl(N)_h)$, we use the matrix D, introduced at the end of Sec. II, instead of $\hat{\Theta}$. The generating matrix L of the algebra $U(gl(N)_h)$ also satisfies a Cayley–Hamilton identity,

$$L^N - a_1 L^{N-1} + a_2 L^{N-2} - \cdots + (-1)^N a_N I = 0,$$

where a_k, $1 \le k \le N$, are some central elements of the algebra $U(gl(N)_h)$. Let μ_i, $1 \le i \le N$, be the "roots" of the corresponding characteristic polynomial, that is,

$$\sum_{k=1}^{N} \mu_k = a_1, \qquad \sum_{1\le i_1<i_2\le N} \mu_{i_1}\mu_{i_2} = a_2, \qquad \dots \qquad \prod_{k=1}^{N} \mu_k = a_N.$$

Therefore, these roots are elements of a central extension of the algebra $U(gl(N)_h)$. We want to preserve the group-like property of map D by defining the QPD action on this central extension. Therefore, the problem of constructing such an action reduces to finding the matrices $\mathrm{D}(\mu_k)$ satisfying the system of equations

$$\sum_{k=1}^{N} \mathrm{D}(\mu_k) = \mathrm{D}(a_1), \qquad \sum_{1\le i_1<i_2\le N} \mathrm{D}(\mu_{i_1})\mathrm{D}(\mu_{i_2}) = \mathrm{D}(a_2), \qquad \dots \qquad \prod_{k=1}^{N} \mathrm{D}(\mu_k) = \mathrm{D}(a_N),$$

and having a good classical limit. This means that the QPD found from this system tends to the usual partial derivatives in the generators of the algebra $\mathrm{Sym}(gl(N))$ when $h \to 0$.

V. EXTENSION OF QPD ON THE SKEW-FIELD $\mathcal{B} = \mathcal{A}[\mathcal{A}^{-1}]$

In this section, we extend the QPD onto some elements of the skew-field $\mathcal{B} = \mathcal{A}[\mathcal{A}^{-1}]$ while preserving the Leibniz rule in its multiplicative form (3.5). Note that this Leibniz rule can be expressed via the matrices D introduced at the end of Sec. II.

Let b be an arbitrary nontrivial element of the algebra $\mathcal{A}$. If we can extend the map D onto the element $b^{-1} \in \mathcal{B}$ while preserving the Leibniz rule, we should have

$$\mathrm{D}(b^{-1}) = \mathrm{D}(b)^{-1}.$$

Therefore, in order to compute $\mathrm{D}(b^{-1})$, we have to invert the matrix $\mathrm{D}(b)$ with non-commutative entries. In principle, this procedure can be performed by means of the Gelfand–Retakh method using the so-called quasideterminants. In order to present the entries of the matrix $\mathrm{D}(b^{-1})$ as elements of $\mathcal{B}$, we need the Ore property of the algebra $\mathcal{A}$. Hopefully, the algebra $\mathcal{A}$ possesses this property since it is so for the algebra $U(u(2)_h)$. Nevertheless, in practice, the reduction of any "left" fraction to a "right" one and vice versa in this algebra is indeed difficult.

Below, we only deal with some particular elements $b \in \mathcal{A}$ for which the computation of the matrix $\mathrm{D}(b)^{-1}$ can be performed by means of the Cayley–Hamilton identity for the matrix $\mathrm{D}(b)$. For such elements b, we succeeded in presenting the entries of the matrices $\mathrm{D}(b)^{-1}$ as elements of the skew-field $\mathcal{B}$.

First, we calculate the matrices $\mathrm{D}(b)$ for some basic $b \in \mathcal{A}$. Taking into account the explicit form of D

$$\mathrm{D} = i\hbar \begin{pmatrix} \hat{\partial}_t + i\partial_z & i\partial_x - \partial_y \\ i\partial_x + \partial_y & \hat{\partial}_t - i\partial_z \end{pmatrix}, \tag{5.1}$$

we find

$$\mathrm{D}(x) = \begin{pmatrix} x & -\hbar \\ -\hbar & x \end{pmatrix}, \qquad \mathrm{D}(y) = \begin{pmatrix} y & -i\hbar \\ i\hbar & y \end{pmatrix}, \qquad \mathrm{D}(z) = \begin{pmatrix} z-\hbar & 0 \\ 0 & z+\hbar \end{pmatrix},$$

$$\mathrm{D}(t) = (t + i\hbar)I, \qquad \mathrm{D}(r_\hbar) = \frac{1}{r_\hbar}\begin{pmatrix} r_\hbar^2 + \hbar^2 - \hbar z & -\hbar(x+iy) \\ -\hbar(x-iy) & r_\hbar^2 + \hbar^2 + \hbar z \end{pmatrix}.$$

The matrix $\mathrm{D}(r_\hbar)$ obeys the Cayley–Hamilton identity,

$$\mathrm{D}(r_\hbar)^2 - 2r_\hbar\,\mathrm{D}(r_\hbar) + (r_\hbar^2 - \hbar^2)I \equiv (\mathrm{D}(r_\hbar) - (r_\hbar + \hbar)I)(\mathrm{D}(r_\hbar) - (r_\hbar - \hbar)I) = 0. \tag{5.2}$$

Now, let b be a linear combination of generators of $U(u(2)_h)$ with numeric coefficients $\alpha_i \in \mathbb{C}$,

$$b = \alpha_0\, t + \alpha_1\, x + \alpha_2\, y + \alpha_3\, z, \qquad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \ne 0. \tag{5.3}$$

Since

$$\mathrm{D}(b) = (b + i\alpha_0)\, I - \hbar\, N, \qquad N = \begin{pmatrix} \alpha_3 & \alpha_1 + i\alpha_2 \\ \alpha_1 - i\alpha_2 & -\alpha_3 \end{pmatrix},$$

then on taking into account that $N^2 = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)I$, we find the matrix $\mathrm{D}^{-1}(b)$

$$\mathrm{D}(b)^{-1} = \frac{(b + i\alpha_0)I + \hbar N}{(b + i\alpha_0)^2 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)\hbar^2}.$$

Note that the above-mentioned fraction is not ambiguous since its numerator and denominator *commute* with each other.

Introduce the element $c = r_\hbar - b$ and impose the following restriction on the numeric coefficients:

$$\alpha_0 = 0, \qquad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1.$$

Then the Cayley–Hamilton identity for $\mathrm{D}(b)$ reads

$$(\mathrm{D}(b) - b\, I)^2 = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)\, \hbar^2\, I, \tag{5.4}$$

while for $\mathrm{D}(r_\hbar)$, it is given in (5.2). Since $r_\hbar$ is a central element, the matrices $\mathrm{D}(b)$ and $\mathrm{D}(r_\hbar)$ commute with each other due to the Leibniz rule (2.17). Therefore, it is reasonable to look for the inverse matrix $\mathrm{D}(c)^{-1}$ in the form

$$\mathrm{D}(c)^{-1} = I\, a_0 + \mathrm{D}(r_\hbar)\, a_1 + \mathrm{D}(b)\, a_2 + \mathrm{D}(r_\hbar)\mathrm{D}(b)\, a_3. \tag{5.5}$$

The requirement $\mathrm{D}(c)\mathrm{D}(c)^{-1} = I$ leads to a system of linear equations for the coefficients a_i,

$$\begin{aligned} -(r_\hbar^2 - \hbar^2)\, a_1 + (b^2 - \hbar^2)\, a_2 &= 1,\\ a_0 + 2r_\hbar\, a_1 + (b^2 - \hbar^2)\, a_3 &= 0,\\ a_0 + 2b\, a_2 + (r_\hbar^2 - \hbar^2)\, a_3 &= 0,\\ a_1 - a_2 - 2(r_\hbar - b)\, a_3 &= 0. \end{aligned}$$

Here, we used the relations (5.2) and (5.4).

The coefficients of the system above commute with each other, so the solution can be found by the standard Cramer's rules of classical linear algebra

$$a_0 = \frac{2(r_\hbar^2 + b^2 - 3br_\hbar - \hbar^2)}{(r_\hbar - b)((r_\hbar - b)^2 - 4\hbar^2)}, \qquad a_1 = \frac{3b - r_\hbar}{(r_\hbar - b)((r_\hbar - b)^2 - 4\hbar^2)},$$
$$a_2 = \frac{3r_\hbar - b}{(r - b)((r_\hbar - b)^2 - 4\hbar^2)}, \qquad a_3 = \frac{-2}{(r_\hbar - b)((r_\hbar - b)^2 - 4\hbar^2)}.$$

Now, we are able to extend the action of the QPD on the element $c^{-1} = (r_\hbar - b)^{-1}$. Namely, from (5.1), we get

$$(\mathrm{D}_1^1 + \mathrm{D}_2^2)(c^{-1}) = 2i\hbar\, \hat{\partial}_t c^{-1}, \qquad (\mathrm{D}_1^2 + \mathrm{D}_2^1)(c^{-1}) = -2\hbar\, \partial_x c^{-1}, \quad \text{etc.}$$

To write down the general answer, it is convenient to introduce the vectors

$$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3), \qquad \vec{r_\hbar} = (x, y, z), \qquad \vec{\nabla} = (\partial_x, \partial_y, \partial_z).$$

Then, by using the relation $\mathrm{D}(c^{-1}) = \mathrm{D}(c)^{-1}$, we present the final result,

$$\partial_t\Big(\frac{1}{r_\hbar - b}\Big) = \frac{-i\hbar}{r_\hbar((r_\hbar - b)^2 - 4\hbar^2)},$$
$$\vec{\nabla}\Big(\frac{1}{r_\hbar - b}\Big) = \frac{r_\hbar\vec{\alpha} - \vec{r_\hbar}}{r_\hbar}\, \frac{1}{((r_\hbar - b)^2 - 4\hbar^2)} - (\hbar\, \vec{\alpha} + i[\vec{r_\hbar} \times \vec{\alpha}])\, \frac{2\hbar}{r_\hbar(r_\hbar - b)((r_\hbar - b)^2 - 4\hbar^2)},$$

where $[\cdot \times \cdot]$ stands for the vector product of two vectors. We point out that the components of the vector $\vec{r_\hbar}$ do *not commute* with the element $r_\hbar - b$ entering the denominator. Therefore, we have to explicitly fix the order of factors in the formula for $\vec{\nabla}(r_\hbar - b)^{-1}$.

In addition, note that in the limit $\hbar \to 0$, we recover the classical result

$$\vec{\nabla}\left(\frac{1}{r-b}\right) = \frac{r\vec{\alpha}-\vec{r}}{r(r-b)^2},$$

where $r = |\vec{r}|$ and $b = \vec{\alpha}\cdot\vec{r}$ is the Euclidean scalar product.

Therefore, by applying the QPD to elements of the form ac^k, $a \in \mathcal{A}$, $k \in \mathbb{Z}$, we can represent the results as elements of $\mathcal{B}$.

The interest in calculating the QPD action on the element $(r_\hbar - b)^{-1}$ is motivated by our wish to construct a noncommutative analog of the Dirac potential **A**. Let us recall that in the classical setting, this potential is a solution of the differential equation rot **A** = **H**, where the magnetic field **H** is a stationary solution of Maxwell's equations with zero electric field,

$$\operatorname{div}\mathbf{H} = 0, \qquad \operatorname{rot}\mathbf{H} = 4g\pi\delta(r). \tag{5.6}$$

Here, g is a nontrivial constant, and the notations div and rot stand for the divergence and the rotor of a given vector field, respectively. The symbol $\delta(r)$ denotes the delta-function.

Dirac found a solution to system (5.6) in the form $\mathbf{H} = g\,\vec{r}/r^3$. Besides, he constructed the following family of vector potentials for this model:

$$\mathbf{A} = \frac{g}{r}\frac{[\vec{r}\times\vec{\alpha}]}{(r-\vec{r}\cdot\vec{\alpha})}, \tag{5.7}$$

where $\vec{\alpha}$ is a unit vector and $\vec{r}\cdot\vec{\alpha}$ is the Euclidean scalar product. Each of these vector potentials is singular on a half-line.

In Ref. 13, we have found a noncommutative counterpart of the Dirac monopole [a solution of (5.6)],

$$\mathbf{H} = \frac{g\,\vec{r_\hbar}}{r_\hbar(r_\hbar^2-\hbar^2)}. \tag{5.8}$$

Unfortunately, we have not succeeded in finding the corresponding noncommutative potential **A**, which would solve the equation rot **A** = **H** for **H** defined by (5.8). We would like to emphasize that the problem of finding such a potential in the framework of our noncommutative setting is much more complicated than in the classical case. It suffices to note that in looking for a solution to the equation rot **A** = **H** in a form similar to (5.7), we have to take into account that the element $(r_\hbar - b)^{-1}$ does not commute with the components of the vector $\vec{r_\hbar}$.

In general, in the framework of quantization with a noncommutative configuration space, we have to replace the usual partial derivatives in the generators of the algebra $\mathrm{Sym}(gl(N))$ by the QPD acting on $U(gl(N)_\hbar)$ and to replace the coefficients at the partial derivatives by their noncommutative analogs. However, in general, we can construct such noncommutative analogs for the coefficients, which are *polynomials* in the generators of $\mathrm{Sym}(gl(N))$. If it is not so, we have additional difficulties in constructing the noncommutative analogs of non-polynomial functions.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Dimitry Gurevich: Writing – original draft (equal). **Pavel Saponov**: Writing – original draft (equal).

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Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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