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Solitary Wave Interactions in the Cubic Whitham Equation

M. V. Flamarion[*,1] **and E. Pelinovsky**[**,***]

[*] *Department of Mathematics, Aeronautics Institute of Technology (ITA), São José dos Campos, SP 12228-900, Brazil*
[**] *Faculty of Informatics, Mathematics and Computer Science, HSE University, Nizhny Novgorod, 603155 Russia*
[***] *Gaponov-Grekhov Institute of Applied Physics, Nizhny Novgorod, 603122 Russia*
[1] *E-mail:* flamarion@ita.br

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Abstract — The vortical Whitham equation is modeled with quadratic and cubic nonlinearity, satisfying the unidirectional dispersion relation used to describe the propagation of nonlinear waves in the presence of a vertically sheared current of constant vorticity. In this article, we neglect the quadratic nonlinearity to numerically investigate solitary wave interactions. We show that the geometric Lax categorization is satisfied; however, an algebraic categorization based on the ratio of the initial solitary wave amplitudes is not possible. Specifically, our numerical simulations indicate that for solitary waves with large amplitudes, the interactions maintain two well-separated crests. Additionally, for solitary waves of different polarities, we find that wave-breaking may occur.

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1. INTRODUCTION

A soliton is a coherent and localized structure symmetric with respect to its crest that maintains its shape while travelling over long distances at a constant speed, exhibiting a particle-like behavior. Its significance is evident in various natural science applications, such as modeling tsunamis in water waves, signal propagation in neuroscience, optics, and plasma physics [13]. When a wave field predominantly consists of solitons, the physical system is characterized as soliton turbulence or a soliton gas, typically explored through integrable equations like the nonlinear Schrödinger equation (NLS) or the Korteweg-de Vries equation (KdV) and its variations [1]. In the context of a periodic domain with two particles, it corresponds to the highly rarefied gas scenario. An examination of the interactions between two solitons is fundamental in soliton turbulence, and has revealed the pivotal role of such processes in the statistical behavior of rarefied soliton gases within KdV-like models [2, 20, 22].

Zabusky and Kruskal [28] performed the first study of solitary wave interactions using the KdV equation. The authors observed that the solitary waves preserved their shapes and speeds after the interaction; i.e., the collision of solitary waves resulted in an elastic behavior. Solitary waves with these properties became known as solitons. This study arose interest in the properties of solitary wave collisions. A seminal work in this field was done Lax [18], who classified the types of collisions geometrically into three categories based on the number of local maxima. He also demonstrated that these categories depend solely on the ratio between two soliton amplitudes. This investigation has been extended to various frameworks, including the Euler equations [5], Whitham equation [9], the Schamel equation [11] and also in laboratory [25]. Among these models, the Euler equation is the only one that allows for an algebraic categorization, albeit with a range distinct from the one predicted by Lax.

It is broadly known that, in the shallow-water limit, the KdV equation approximates the Euler equations asymptotically [26]. However, this model does not exhibit several nonlinear properties present in the Euler equation, namely, peaking, wave-breaking, and short waves. To address these issues, Whitham [26, 27] proposed an ad-hoc model that has the unidirectional dispersion relation of the Euler equations and the KdV quadratic nonlinearity. This model presents many theoretical challenges and has been the subject of extensive mathematical investigation in recent years [3, 6–8, 12, 17, 19, 21, 24]. Extensions of the Whitham equation, including a cubic nonlinear term, have recently appeared in the study of flows over a linear vertically sheared current with constant vorticity [15, 16]. This modified equation is referred to as the vortical Whitham equation. Travelling waves to this equation were later investigated in the work of Carter et al. [4]. The inclusion of the cubic nonlinear term enriches the dynamics, possibly allowing the existence of breathers. For instance, Kalisch et al. [14] considered the cubic Whitham equation (essentially the vortical Whitham equation neglecting the quadratic term) and numerically computed breather solutions. Their results strongly indicate the existence of breathers in the cubic Whitham equation.

In this study, we conduct a numerical investigation into overtaking collisions of two solitary waves utilizing the vortical Whitham equation [4] while disregarding the quadratic nonlinear term, see [14]. By doing so, the modified Korteweg-de Vries (mKdV) equation emerges as an asymptotic approximation to this particular model. Our goal is to study the characteristics of overtaking collisions between solitary waves for the cubic Whitham equation. We observe that the three geometric categories described by Lax [18] are preserved. However, we demonstrate that cubic Whitham equation does not admit an algebraic Lax categorization. Moreover, when considering solitary waves of different polarities, we uncover numerical evidence indicating wave-breaking as the two solitary waves merge. This paper is structured as follows. Numerical methods to solve the cubic Whitham equation are presented in Section 2. Numerical results are discussed in Section 3 and the final remarks in Section 4.

2. THE CUBIC WHITHAM EQUATION AND NUMERICAL METHODS

We consider the cubic Whitham equation in canonical form

$$u_t - 6u_x + 6u^2u_x + K * u_x = 0. \tag{1}$$

Here, K is the nonlocal operator defined through the Fourier transform $(\widehat{K})$

$$\widehat{K}(k) = 6\sqrt{\frac{\tanh k}{k}}.$$

Notice that, for small frequencies ($k \approx 0$), we obtain the following approximation

$$\sqrt{\frac{\tanh k}{k}} \approx 1 - \frac{k^2}{6}. \tag{2}$$

Consequently, the mKdV equation

$$u_t + 6u^2u_x + u_{xxx} = 0, \tag{3}$$

approximate asymptotically Eq. (1).

In order to solve Eq. (1), we employ a Fourier pseudospectral method with an integrating factor similar to the one described by [10]. The spatial computational domain consists in a uniform grid with N points and a step size of Δx. Spatial derivatives and the operator K are computed spectrally [23]. Additionally, the time evolution is computed using the explicit fourth-order Runge–Kutta method with a time step denoted as Δt.

Solitary waves with speed c are obtained through a Newton method's type in the Fourier space. Writing Eq. (1) in Fourier space, we have

$$(-c - 6 + \widehat{K}(k))\widehat{u} + 2\widehat{u^3} = 0. \tag{4}$$

To solve this equation, the initial guess (u_0, c_0) is taken as the soliton solutions of the mKdV Eq. (3)

$$u_0(x) = A\,\mathrm{sech}(Ax) \quad \text{and} \quad c_0 = -A^2. \tag{5}$$

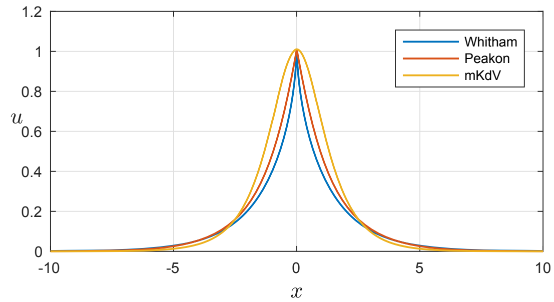

Fig. 1. A comparison of high amplitude soliton of the cubic Whitham equation. The mKdV equation and a peakon wave.

Although, in this work, we do not focus on computing the limit-wave, it is worth to mention that our numerical method can approximate a nonsmooth solutions. Figure 1 displays a comparison between a Whitham solitary wave, a mKdV soliton and the peakon wave $G(x) = ae^{-b|x|}$. As we can see these two functions are close one to another.

In order to verify the stability of the Newton method, the computed solitary waves are utilized as initial data for the Whitham Eq. (1). Simulations are conducted over extended periods (specifically, $t = 1500$), and no indications of instability are observed. Moreover, the numerical simulations are tracked by evaluating the first and second moments with machine precision and a precision of 10^{-7}, respectively.

3. RESULTS

Solitary wave interactions are investigated, as reported in [9, 11]. Initially, we compute two solitary waves, namely, S_1 and S_2 with amplitudes A_1 and A_2, respectively, where $A_1 > A_2$, for Eq. (1). Subsequently, we position them far apart and equidistantly from the origin. More precisely, we choose the initial data in the form:

$$u(x,0) = S_1(x + x_0) + S_2(x - x_0), \tag{6}$$

where x_0 represents a phase constant.

To comprehend the dynamics of solitary wave interactions, we revisit the Lax geometric and algebraic categorizations specific to the KdV equation, considering two solitons with amplitudes $A_1 > A_2$ [18].

(A) The solution of the KdV equation has two local maxima at any given time for $A_1/A_2 < (3+\sqrt{5})/2 \approx 2.62$.

(B) The number of local maxima varies according to the law $2 \to 1 \to 2 \to 1 \to 2$. This case happens when $(3+\sqrt{5})/2 < A_1/A_2 < 3$.

(C) The number of local maxima changes as $2 \to 1 \to 2$ during the interaction. This means that, in a period of time the solitons join together to form a wave with a single local maximum. This occurs as long as $A_1/A_2 > 3$.

After the interaction ceases, the solitons undergo a phase shift. In other words, their crests are shifted from their incoming trajectories.

Numerically, it is convenient to consider the interaction in the moving frame of one of the solitary waves. For this purpose, we present the following graph in the S_2 moving frame. In this scenario, wave S_2 remains stationary while S_1 moves towards S_2. Since the cubic Whitham equation is not integrable, the collisions are not elastic. Consequently, dispersion radiation is produced during solitary wave interactions. However, the radiation is minor, and as a consequence, the solitary waves retain almost their initial shape right after the interaction. Specific instances illustrating these characteristics are provided in the accompanying Table 1. As we can see, both solitary waves almost preserve their amplitudes accompanied by phase-shifts.

Table 1. The solitary wave amplitudes before and after the interaction were examined. The amplitudes after the interaction were computed over extended periods, ensuring that the crests of S_1 and S_2 are well separated.

Initial Amplitudes		After the interaction		Phase-shift		Category
A_1	A_2	A_1	A_2	S_1	S_2	–
0.500	0.400	0.500	0.400	-9.370	12.190	**A**
0.500	0.180	0.500	0.179	-0.960	8.340	**B**
0.500	0.100	0.500	0.099	1.600	7.500	**C**

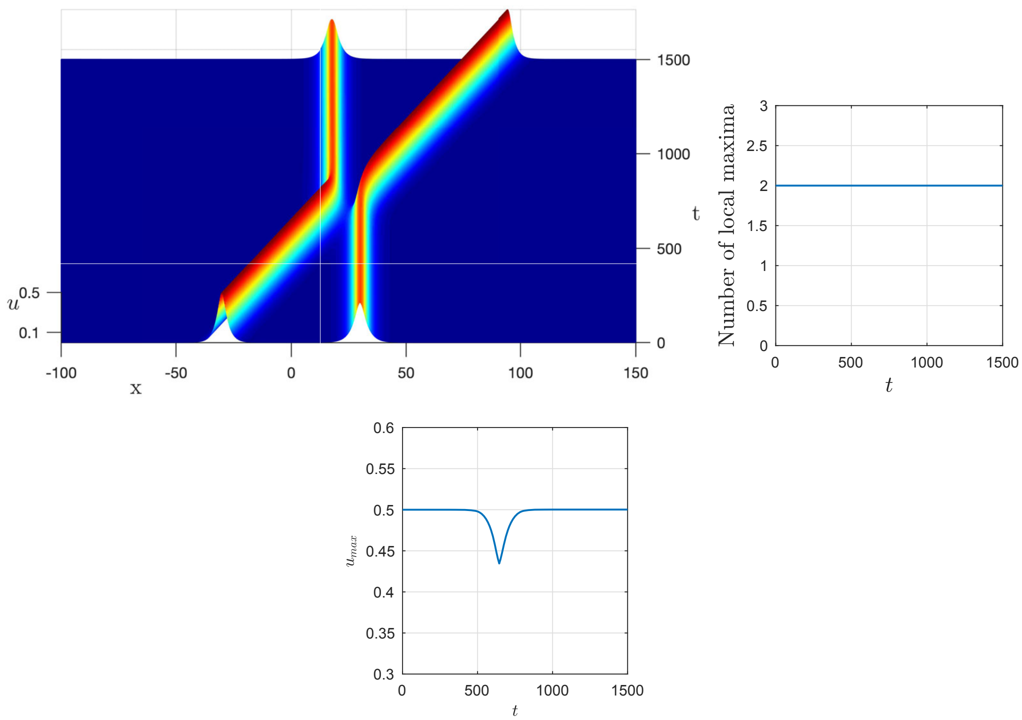

Fig. 2. Top: Solitary wave interactions for the cubic Whitham equation – category **(A)**. Bottom: (Left) The variation of the number local maxima over time. (Right) The maximum value of the interaction over time. Parameters $A_1 = 0.50$, $A_2 = 0.40$, and $x_0 = 30$.

Figure 2 displays the solitary wave interactions. The two solitary waves are initially with their crests far apart, as time elapses, S_1 moves towards S_2, resulting in a collision. As this collision unfolds, S_1 diminishes (see Fig. 2 bottom-right), while S_2 expands, leading to a complete role reversal between the two waves (see Fig. 2 in the bottom-right). It is worth noting that, at any given point in time, there exist two local maxima, indicating that the crests of the two waves never join together (refer to Fig. 2 in the bottom-left). This particular scenario corresponds to case **(A)**, as categorized by Lax. The dynamics characteristic of category **(B)** are visually represented in Figure 3. The initial solitary waves have amplitudes $A_1 = 0.50$ and $A_2 = 0.18$, respectively. In the course of their interaction, S_1 absorbs S_2, leading to the formation of a single crest. Subsequently, it bifurcates into two distinct waves, followed immediately by the resurgence of a unique local maximum. Over time, the waves gradually move apart, ultimately giving rise to two well-defined crests once again. It is noteworthy that, while not explicitly presented in this article, additional simulations were conducted, and their respective classifications can be found in Table 2.

Figure 4 displays a typical case of category **(C)**. The two solitary wave are engaged in the interaction, with S_1 exhibiting a significantly larger amplitude compared to S_2 ($A_1 = 0.50$) and $A_2 = 0.10$). During a specific time interval, the two solitary waves join together to form a single local maximum. Throughout the collision process, S_1 assimilates S_2, and subsequently they split into two waves and S_1 is later emitted again, accompanied by a phase lag in the trajectories of their crests (see Figure 4 at the bottom).

We have conducted several simulations with the aim of identifying a classification similar to that provided

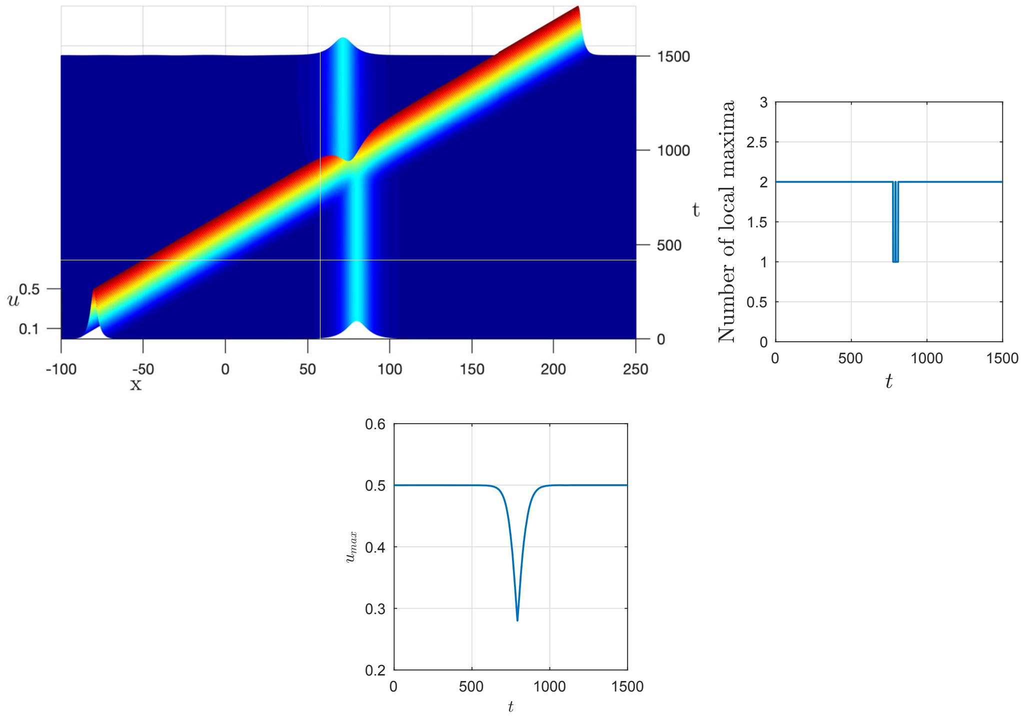

Fig. 3. Top: Solitary wave interactions for the cubic Whitham equation – category **(B)**. Bottom: (Left) The variation of the number local maxima over time. (Right) The maximum value of the interaction over time. Parameters $A_1 = 0.50$, $A_2 = 0.18$, and $x_0 = 80$.

by Lax for the cubic Whitham equation. Table 2 displays the categorizations of the interaction between two solitary waves. Initially, it appears possible to apply a classification similar to the one given for the KdV Eq. [18] and the Euler Eq. [5], both based on the ratio of the initial amplitudes of the two solitary waves. Table 3 presents specific cases illustrating the impracticality of a Lax-algebraic classification relying on the initial amplitude ratio of two solitons for the cubic Whitham equation. Nevertheless, the cubic Whitham equation still captures the geometric features of the Lax categorization. These findings align with results reported by Flamarion [9]. Importantly, it is worth mentioning that the layer in which category **(B)** occurs is very thin.

Table 2. Classification of the collision for different values of A_1 and A_2.

A_1	A_2	A_1/A_2	Category
0.50	0.40	1.25	**A**
0.50	0.35	1.43	**A**
0.50	0.30	1.67	**A**
0.50	0.25	2.00	**A**
0.50	0.20	2.50	**A**
0.50	0.19	2.63	**A**
0.50	0.18	2.78	**B**
0.50	0.17	2.94	**C**
0.50	0.10	5	**C**

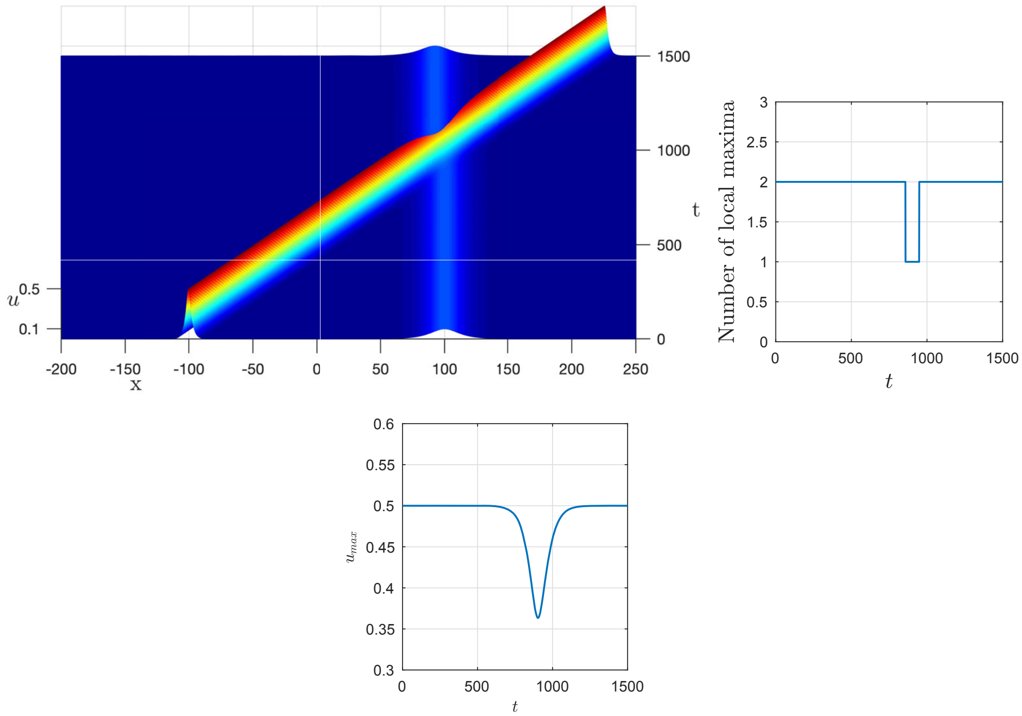

Fig. 4. Top: Solitary wave interactions for the cubic Whitham equation – category **(C)**. Bottom: (Left) The variation of the number of local maxima over time. (Right) The maximum value of the interaction over time. Parameters $A_1 = 0.50$, $A_2 = 0.10$, and $x_0 = 100$.

Table 3. Classification of the collision for different values of A_1 and A_2.

A_1	A_2	A_1/A_2	category
0.75	0.255	2.94	**A**
0.75	0.27	2.78	**A**
0.75	0.30	2.50	**A**

Our simulations reveal that interactions involving solitary waves of large amplitude consistently maintain two local maxima (category **(A)**), albeit with increased dispersion in the interaction. Figure 5 displays the collision of two solitary waves of amplitudes $A_1 = 0.75$ and $A_2 = 0.30$ at different times. As we can see, a visible dispersion tails appears right after the collision and the solitary waves no longer conserve energy.

The characteristics of pair soliton interactions of different polarities is depicted in Fig. 6. In the lower part of this figure (right plot), the evolution of the wave field maximum is illustrated. It is evident that, prior to the interaction, this maximum value is equivalent to the amplitude of the larger soliton. During the interaction, it surges to 0.570, and upon separation, it reverts to its initial value, persisting at the level of 0.400. This observation implies that the energy is nearly conserved post-collision, with deviations up to the order of $\mathcal{O}(10^{-3})$. Notably, there is an absence of radiation that dampens both solitons. These interaction characteristics are highly dependent on the amplitudes of the solitons. Another instance of pair soliton interactions reveals that the maximum wave field value surpasses the sum of the amplitudes of the solitary waves. Specifically, it was observed that for solitary waves of larger amplitude, the wave field becomes sharper, leading to code break (see Fig. 7). These observations indicate the onset of wave breaking in bipolar interactions. Further investigation using a more suitable numerical method is warranted to explore this phenomenon in greater detail. It is important to mention that these results were obtained using different numbers of Fourier modes and the results were the same.

4. CONCLUSIONS

In this work, we have investigated the properties of solitary wave interactions within the framework of the cubic Whitham equation. We have demonstrated that the collision is nearly elastic for solitary waves

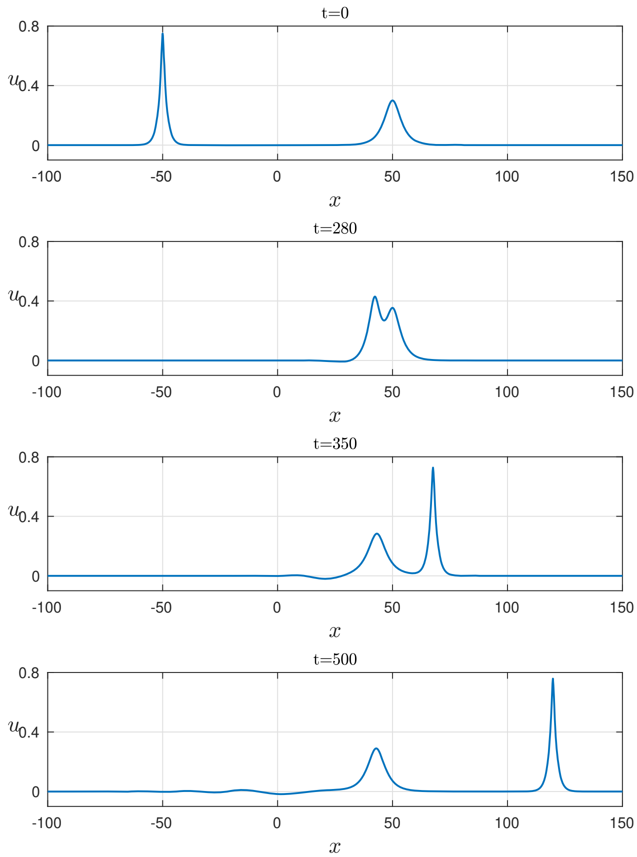

Fig. 5. A series of snapshots of the interaction of the solitary waves during the collision–category **(A)**. Parameters $A_1 = 0.75$, $A_2 = 0.30$, and $x_0 = 50$.

with moderate amplitudes, fitting into the geometric Lax categorization. Additionally, we have shown that an algebraic categorization based on the ratio of the initial solitary wave amplitudes is not possible. Furthermore, in the case of bipolar collisions, there appears to be an initiation of wave-breaking when two solitary waves coalesce, even for solitary waves of modest amplitude.

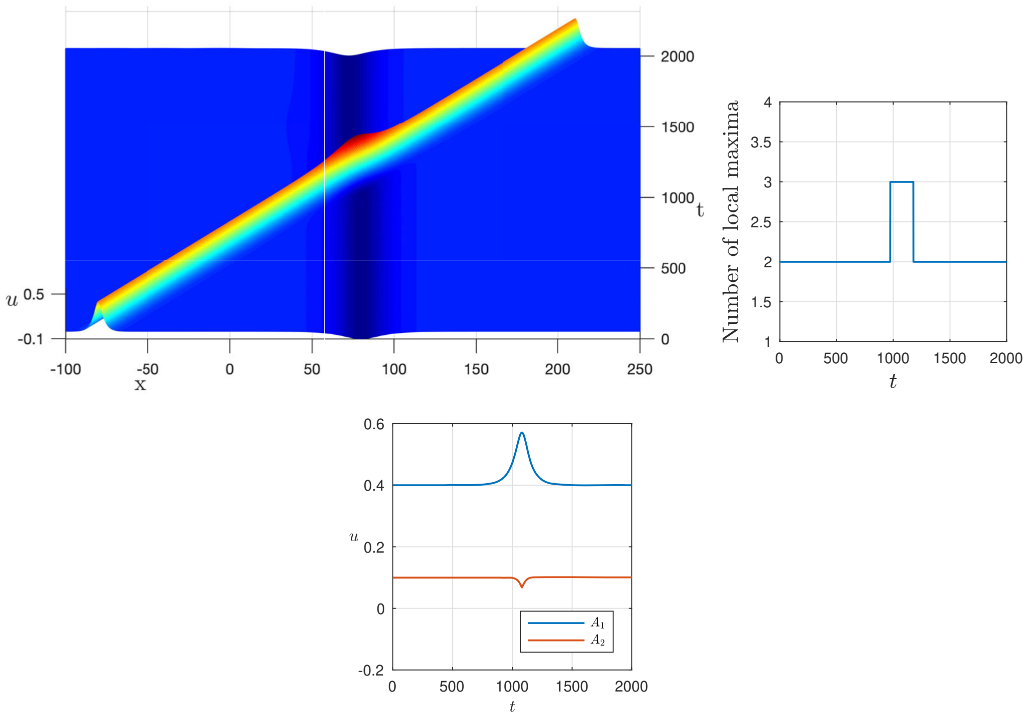

Fig. 6. Top: Bipolar of two solitons for the cubic Whitham equation. Bottom: (Left) The number of local maxima as a function of time. (Right) The maximum and minimum amplitude as a function of time. The parameters are $A_1 = 0.40$, $A_2 = -0.10$, and $x_0 = 80$.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

DATA AVAILABILITY

Data sharing is not applicable to this article as all parameters used in the numerical experiments are informed in this paper.

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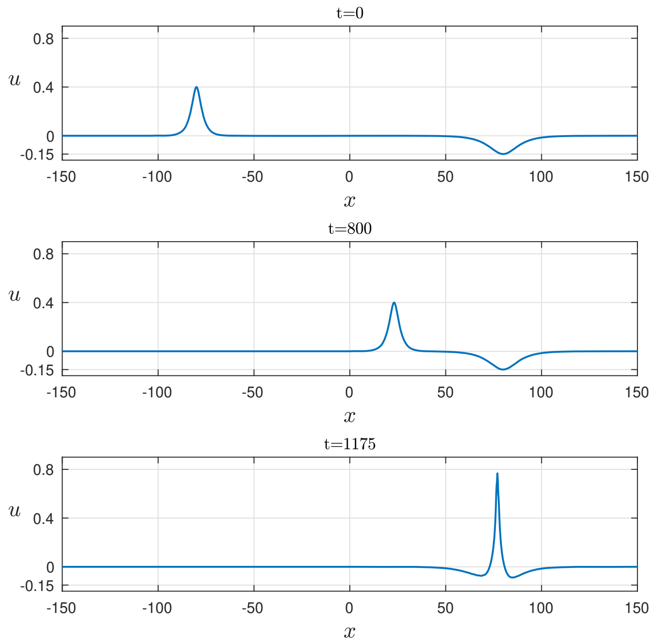
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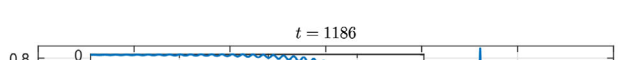
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