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On Morse–Smale diffeomorphisms on simply connected manifolds

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ABSTRACT

We study relations between a structure of non-wandering set of a Morse–Smale diffeomorphism f and its carrying closed manifold M^n. We prove that if f has no any saddle periodic points with one-dimensional unstable manifolds, and for any periodic point σ of Morse index $(n-1)$ its unstable manifolds do not intersect stable invariant manifolds of saddle periodic points different from σ, then M^n is simply connected. This fact does not follow from Morse inequalities that give only restrictions on homology groups of M^n.

1. Introduction and statement of results

A diffeomorphism $f : M^n \to M^n$ of a closed smooth manifold M^n is called *Morse–Smale* if its non-wandering set is finite and hyperbolic, and invariant manifolds of periodic points intersect each other transversely. A number i_p equal to the dimension of unstable invariant manifold W^u_p of a hyperbolic periodic point p is called *a Morse index* of p. We will denote by k_i the number of all periodic points of f with Morse index equal to $i \in \{0, \dots, n\}$. A Morse–Smale diffeomorphism f is called *polar*, if $k_0 = k_n = 1$.

Smale proved in Ref. 1 that a gradient flow of any Morse function $\varphi : M^n \to \mathbb{R}$ can be arbitrary closely approximated by a structurally stable flow. A time-one shift along trajectories of this flow is a Morse–Smale diffeomorphism. Hence, Morse–Smale diffeomorphisms exist on all closed smooth manifolds. In Refs. 2, 3 the inequalities connecting numbers k_i with Betti numbers of M^n were obtained similar to Morse inequalities of Morse function. In particular, there was proved the following generalization of Poincare–Hopf formula:

$$\sum_{i=0}^{n}(-1)^i k_i = \chi(M^n),$$

where $\chi(M^n)$ is the Euler characteristic of M^n.

Since the Euler characteristic is a complete topological invariant for two-dimensional closed manifolds, in case $n = 2$ the formula above completely determines a topology of the manifold carrying a Morse–Smale diffeomorphism with given number of sink, source and saddle periodic points. For $n \geq 3$, this formula is not so informative, because, for instance, $\chi(M^{2k+1}) = 0$ for any manifold M^{2k+1} of odd dimension.

Some additional assumptions on the dynamics help to clarify the topology of manifolds. Denote by $G(M^n)$ a class of Morse–Smale diffeomorphisms such that for any $f \in G(M^n)$ an $(n-1)$-dimensional invariant manifold of arbitrary saddle periodic point of Morse index 1 or $(n-1)$ either do not intersect invariant manifolds of any other saddles or intersect only one-dimensional invariant manifolds. In the last case it follows from transversality condition that the intersection consists of a finite number of isolated points. If $W^u_p \cap W^s_q = \emptyset$ for any pair of saddle periodic points of a Morse–Smale diffeomorphism f, we say that *f has no heteroclinic intersections*.

In, Ref. 4 the following result is proved for $n = 3$.

Statement 1. *Let $f \in G(M^3)$ and M^3 be orientable. Then $g_f \geq 0$ and M^3 is diffeomorphic to the connected sum of the sphere S^3 and g_f copies of the direct product $S^2 \times S^1$.*

There are a lot of generalizations of this fact for $n \geq 4$ (see Ref. 5 for references). In particular, in Ref. 6, Theorem 1 the following statement is proved.

Statement 2. *Let M^n be an orientable closed manifold of dimension $n \geq 4$ and $f : M^n \to M^n$ be a Morse–Smale diffeomorphisms without heteroclinic intersections such that all saddle periodic points of f has Morse index 1. Then M^n is homeomorphic to the sphere S^n.*

It is clear that the conclusion of Statement 2 stays true for a Morse–Smale diffeomorphism $f : M^n \to M^n$ without heteroclinic intersections under the assumptions that all saddle periodic points of f has Morse index $(n-1)$ (since f^{-1} satisfies Statement 2). In Ref. 7 it is shown that the requirement of orientability of three-dimensional ambient manifold M^3 may be omitted and the statement holds for an arbitrary $f \in G(M^3)$.

Due to Ref. 8, Theorem 1.3, the following result holds.

Statement 3. *Let M^n be a closed orientable manifold of dimension $n \geq 4$ and $f : M^n \to M^n$ be a Morse–Smale diffeomorphism without*

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heteroclinic intersections. Then if M^n is homeomorphic to sphere S^n then $k_2 = \cdots = k_{n-2} = 0$.

The next result follows from Ref. 9, Corollary 1.

Statement 4. *Let M^n be a closed manifold of dimension $n \geq 4$, and $f : M^n \to M^n$ be a Morse–Smale diffeomorphism such that $k_1 = k_{n-1} = 0$. Then f is polar and M^n is simply connected.*

Despite the fact that in the paper[9] only orientable manifolds are considered, the proof of Statement 4 does not require the orientability of the manifold M^n (see also Ref. 5, Proposition 4.1).

We show that the condition $k_1 = k_{n-1} = 0$ cannot be omitted and prove the following fact.

Proposition 1. *Let M^n be a closed manifold of dimension $n \geq 3$ and $f \in G(M^n)$ be a polar diffeomorphism. If $k_1^2 + k_{n-1}^2 \neq 0$ then M^n is not simply connected.*

The requirement $f \in G(M^n)$ above is essential, since there is a polar diffemorphisms on S^n with $k_1 = k_2 = 1, k_3 = \cdots = k_{n-1} = 0$ for any $n \geq 4$ (see, for instance, Ref. 10, Theorem 2). Due to Statement 3, wandering set of all such diffeomorphisms must contain heteroclinic intersection. It follows from transversality, that this intersection has dimension one.

Main result of the present paper is the following.

Theorem 1. *Let M^n be a closed manifold, $n \geq 4$, and $f \in G(M^n)$. If $k_1 = 0$ then M^n is simply connected.*

Let us remark that for $n = 3$, the unique simply connected manifold is sphere S^3. For $n \geq 4$, there are numerous simply connected manifolds not homeomorphic to sphere (for instance, $S^k \times S^l$, $k, l \geq 2, k+l = n$), but a complete classification of simply connected manifolds is known only for $n = 4$ due to non-trivial results of Rochlin, Freedman, Donaldson and Furuta (see Refs. 11–14 and a book[15]).

It follows from Morse inequalities that if $k_1 = 0$, then a one-dimensional homology group $H_1(M^n)$ of manifold M^n is trivial. However, it does not mean that the fundamental group $\pi_1(M^n)$ of M^n is trivial. To prove Theorem 1, we obtain the following topological version of well known smooth result that can be of independent interest:

Proposition 2. *Let Q^{n-1}, M^n be closed topological manifolds, Q^{n-1} is simply connected and locally flat in M^n. Then there exists an embedding $e : Q^{n-1} \times [-1, 1] \to M^n$ such that $e(Q^{n-1} \times \{0\}) = Q^{n-1}$.*

2. Definitions and auxiliary results

2.1. Topology

A purpose of this section is to prove Proposition 2. A sketch of proof of similar statement for smooth submanifolds is given in Ref. 16, Theorem 4. We provide a complete proof for topological manifolds and give below all necessary definitions. In fact, Proposition 2 follows from Statement 5 and Proposition 3 below.

Everywhere below $\mathbb{R}^n$ denotes the Euclidean space of dimension $n \geq 1$. For $k < n$ the space $\mathbb{R}^k$ is considered as a subset of $\mathbb{R}^n$ determined by condition $x_{k+1} = \cdots = x_n = 0$; and $\mathbb{R}^k_+$ is a subset of $\mathbb{R}^k$ determined by the inequality $x_k \geq 0$. S^{n-1}, B^n, $n \geq 1$, denote the topological $(n-1)$-dimensional sphere and the n-dimensional compact ball, that are manifolds homeomorphic to

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}, \mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\},$$

correspondingly. An annulus K^n is a manifold homeomorphic to the direct product $S^{n-1} \times [0, 1]$.

Recall that a path-connected topological space X is called *simply connected* if its fundamental group $\pi_1(X)$ is trivial.

Let X, Y be arbitrary topological spaces. A continuous map $e : X \to Y$ is called a *topological embedding* if it homeomorphically maps X onto the subspace $e(X) \subset Y$ with a topology induced by the topology of Y. Let X be a subset of a manifold M^n. According to Ref. 17, X is called *locally flat* in M^n at a point $x \in X$ if there is an open neighborhood U_x of x in M^n and a homeomorphism $h : U_x \to \mathbb{R}^n$ such that $h(U_x \cap X) = \mathbb{R}^k$ or $h(U_x \cap X) = \mathbb{R}^k_+$ depending on whether $x \in \operatorname{int} X$ or $x \in \partial X$, respectively. If X is locally flat in M^n at all points then, X is *locally flat* in M^n. If X is not locally flat at a point $y \in X$, then y is called *a point of wildness* and X is called *wild*. By definition, the set X is locally flat in M^n if and only if it is a topological submanifold of M^n.

An $(n-1)$-dimensional topological manifold $Q^{n-1} \subset M^n$ is called *two-sided* in M^n if it has a connected neighborhood in M^n separated by Q^{n-1}. Otherwise, Q^{n-1} is one-sided in M^n.

The manifold $Q^{n-1} \subset M^n$ is called *collared* in M^n if there is an embedding $e : Q^{n-1} \times [0, 1) \to M^n$ such that $e(Q^{n-1} \times \{0\}) = Q^{n-1}$; and Q^{n-1} is called *bi-collared* in M^n if there is an embedding $e : Q^{n-1} \times (-1, 1) \to M^n$ such that $e(Q^{n-1} \times \{0\}) = Q^{n-1}$. In the first case the image $e(Q^{n-1} \times [0, 1))$ is called a *collar of Q^{n-1} in M^n*.

According to Ref. 17, Theorem 3, the following statement is true.

Statement 5. *A locally flat two-sided manifold Q^{n-1} in M^n is bi-collared.*

Hence, the proof of Proposition 2 follows from the combination of Statement 5 and the following proposition.

Proposition 3. *Let $Q^{n-1} \subset M^n$ be a simply connected locally flat closed manifold. Then Q^{n-1} is two-sided.*

To prove Proposition 3, let us recall some facts on covering maps.

Let $\widetilde{X}$, X be arbitrary topological spaces. A continuous map $p : \widetilde{X} \to X$ is called a *covering* if it satisfies the next conditions:

- for any $x \in X$ there is an open neighborhood U_x such that $p^{-1}(U_x)$ consists of disjoint union of subsets of $\widetilde{X}$;
- for any connected component V_j of the union $p^{-1}(U_x) = \bigcup_{j \in J} V_j$ the restriction $p|_{V_j} : V_j \to U_x$ is a homeomorphism.

The space $\widetilde{X}$ is called *a covering space or cover*, the space X is called *a base of the covering* p. If $\widetilde{X}$ is a simply connected, then p is called a *universal covering*.

The next statement is proved in Ref. 18, Theorem 18.1.

Statement 6. *Let $\widetilde{X}$ be a path-connected, $p : \widetilde{X} \to X$ be a covering and $p_* : \pi_1(\widetilde{X}) \to \pi_1(X)$ be a homomorphism induced by p. Then p_* is injective.*

Corollary 1. *If $p : \widetilde{X} \to X$ is a covering with simply connected base X, than $\widetilde{X}$ is either path-disconnected or simply connected*

If $p : \widetilde{X} \to X$ is a covering and $\widetilde{X}$ is path connected, then the cardinality of the set $p^{-1}(x)$ does not depend on the point $x \in X$ (see, for example, Ref. 18, Exercise 17.9 (h)). If the set $p^{-1}(x)$ is finite and consists of k points, then p is called *a k-fold covering*.

The following statement proved in Ref. 18, Theorem 17.8.

Statement 7. *Let $p : \widetilde{X} \to X$ be a universal covering. Then there is a one-to-one correspondence between $\pi_1(X)$ and $p^{-1}(x)$, where $x \in X$ is an arbitrary point.*

Any manifold M^n has a two-fold covering $p : \widetilde{M}^n \to M^n$ such that $p^{-1}(x)$ is a local orientation at the point $x \in M^n$ and $\widetilde{M}^n$ is orientable (see, for example[19, §3.3]). Moreover, due to Ref. 19, Proposition 3.25, a connected manifold M^n is orientable if and only of $\widetilde{M}^n$ has two connected components. If M^n is simply connected, then due to Corollary 1, $\widetilde{M}^n$ is either simply connected or disconnected. By definition, $p^{-1}(x)$ consists of two points. If $\widetilde{M}^n$ is connected then, due to Statement 7, $\pi_1(M^n)$ is not trivial. Hence $\widetilde{M}^n$ is disconnected and M^n is orientable.

Corollary 2. *Every simply connected manifold N^n is orientable.*

Proof of Proposition 3. Suppose that $Q^{n-1} \subset M^n$ is simply connected and locally flat. Then for any point $x \in Q^{n-1}$ there is a neighborhood $V_x \subset Q^{n-1}$ (open in the topology on Q^{n-1} induced by one on M^n) and a topological embedding $h_x : V_x \times [-1,1] \to M^n$ such that $h_x(V_x \times \{0\}) = V_x$. Since Q^{n-1} is compact, there is a finite number of points $x_1, \dots, x_s \in Q^{n-1}$ such that the union $V = \bigcup_{i=1}^{s} V_{x_i}$ contains Q^{n-1}. Set $U_{x_i} = h_{x_i}(V_{x_i} \times [-1,1])$, $U^+_{x_i} = h_{x_i}(V_x) \times [0,1]$, $U^-_{x_i} = h_{x_i}(V_x) \times [-1,0]$. According to Ref. 17, Lemma 4 for any V_{x_i}, V_{x_j} such that $V_{x_i} \cap V_{x_j} \neq \emptyset$ there exists a pair of topological embeddings $h^{\pm}_{i,j} : (V_{x_i} \cup V_{x_j}) \times [0,1] \to U_{x_i} \cup U_{x_j}$ such that $h^{\pm}_{i,j}((V_{x_i} \cup V_{x_j}) \times \{0\}) = V_{x_i} \cup V_{x_j}$. There are two possibilities (up to choice of i,j): either $h^+_{i,j}((V_{x_i} \cup V_{x_j}) \times [0,1]) \subset U^+_{x_i} \cup U^+_{x_j}$ or $h^+_{i,j}((V_{x_i} \cup V_{x_j}) \times [0,1]) \subset U^+_{x_i} \cup U^-_{x_j}$, In both cases the union of embeddings $h^+_{i,j}, h^-_{i,j}$ determines a bi-collared embedding of $V_i \cup V_j$ in M^n. Set $U = \bigcup_{i,j=1}^{s} h^+_{i,j}(V_{x_i} \cup V_{x_j} \times [0,1]) \cup \bigcup_{i,j=1}^{s} h^-_{i,j}(V_{x_i} \cup V_{x_j} \times [0,1])$.

By construction, U is compact and a projection $p : \partial U \to Q^{n-1}$ along one-dimensional fibers is two-fold covering. If Q^{n-1} is one-sided, then ∂U is path-connected and, due to Statement 6, is simply connected. At the same time each point $x \in Q^{n-1}$ has two preimages $p^{-1}(x) = x \times \{-1; 1\}$. Due to Statement 7 the fundamental group $\pi_1(\partial U)$ is non-trivial. The contradiction proves that Q^{n-1} is two-sided in M^n.

2.2. Dynamics

A main tool of the proof of Theorem 1 is a surgery along separatrices that we describe below. Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism. We denote by Ω_f the non-wandering set of f. W^s_p, W^u_p are the stable and unstable invariant manifolds of a periodic point $p \in \Omega_f$, correspondingly, $i_p = \dim W^u_p$, and l^u_p denotes a connected component of $W^u_p \setminus p$.

The asymptotic behavior of orbits of f is described in the following statement proved in Ref. 20, Theorem 2.3 (see also Ref. 21, Statement 2.1.1).

Statement 8. *Suppose $f : M^n \to M^n$ is a Morse–Smale diffeomorphism. Then:*

1. $M^n = \bigcup_{p \in \Omega_f} W^s_p = \bigcup_{p \in \Omega_f} W^u_p$;
2. *for any $p \in \Omega_f$ the set W^u_p is a smooth submanifold in M^n diffeomorphic to $\mathbb{R}^{i_p}$;*
3. *for any $p \in \Omega_f$ and for any separatrix l^u_p the equality $\mathrm{cl}\, l^u_p \setminus (\{p\} \cup l^u_p) = \bigcup_{q \in \Omega_f, W^s_q \cap l^u_p \neq \emptyset} W^u_q$ holds.*

Following Smale, we determine a *Smale relation* $\prec$ on the set Ω_f by the rule: $p \prec q$ if and only if $W^s_p \cap W^u_q \neq \emptyset$ or $i_p < i_q$. The next statement is well known (see, for example, Ref. 5, Statement 2.3,Ref. 21, Statement 1.2.5).

Statement 9. *Let $f : M^n \to M^n$ be a Morse–Smale diffeomorphism. Then:*

1. *for any saddle points p, q, r conditions $p \prec q$, $q \prec r$ imply $p \prec r$;*
2. *there is no set of pairwise distinct saddles $p_1, \dots, p_k$ such that for any $i \in \{1, \dots, k-1\}$ the equality $p_i \prec p_{i+1}$ holds and $p_k \prec p_1$.*

All statements below are given for a saddle periodic point σ_{n-1} of a Morse–Smale diffeomorphism $f : M^n \to M^n$, $n \geq 4$, such that $W^u_{\sigma_{n-1}}$ is $(n-1)$-dimensional and does not contain heteroclinic intersections.

Lemma 1. *There exists a unique sink point ω such that $\mathrm{cl}\, W^u_{\sigma_{n-1}} = W^u_{\sigma_{n-1}} \cup \{\omega\}$. Moreover, $\mathrm{cl}\, W^u_{\sigma_{n-1}}$ is a bi-collared $(n-1)$-dimensional sphere in M^n.*

Proof. Due to Statement 8, there is a unique sink periodic point ω such that $\mathrm{cl}\, W^u_{\sigma_{n-1}} \setminus \{\sigma_{n-1}\} \subset W^s_\omega$ and $\mathrm{cl}\, W^u_{\sigma_{n-1}} = W^u_{\sigma_{n-1}} \cup \{\omega\}$. Since W^u_{n-1} is diffeomorphic to $\mathbb{R}^{n-1}$, $\mathrm{cl}\, W^u_{\sigma_{n-1}}$ is a sphere of dimension $(n-1)$ which is locally flat in M^n at all points except ω. It follows from Ref. 22, Theorem 1 (see also Ref. 23, Corollary 3 A.6) that for $n \geq 4$ the sphere $\mathrm{cl}\, W^u_{\sigma_{n-1}}$ cannot have a unique point of wildness in M^n (in fact, the set of wildness points greater than countable). Then $\mathrm{cl}\, W^u_{\sigma_{n-1}}$ is locally flat in M^n at ω. According to Proposition 2, sphere $\mathrm{cl}\, W^u_{\sigma_{n-1}}$ is bi-collared in M^n.

Corollary 3. *There is a neighborhood $N_{\sigma_{n-1}} \subset W^s_\omega \cup W^s_{\sigma_{n-1}}$ of $\mathrm{cl}\, W^u_{\sigma_{n-1}}$ homeomorphic to $S^{n-1} \times [-1,1]$ and a number $m > 0$ such that $f^m(N_{\sigma_{n-1}}) \subset \mathrm{int}\, N_{\sigma_{n-1}}$.*

Proof. Since $\mathrm{cl}\, W^u_{\sigma_{n-1}}$ is a bi-collared sphere, there is a topological embedding $e : S^{n-1} \times (-1,1) \to M^n$ such that $e(S^{n-1} \times \{0\}) = \mathrm{cl}\, W^u_{\sigma_{n-1}}$. Set $N_{\sigma_{n-1}} = e(S^{n-1} \times [-1/2; 1/2])$.

Without loss of generality we suppose that all points of $\partial N_{\sigma_{n-1}}$ belong to the union $W^s_\sigma \cup W^s_\omega$ (otherwise we take $N_{\sigma_{n-1}}$ as the image of $S^{n-1} \times [-\varepsilon, \varepsilon]$ for sufficiently small $\varepsilon > 0$). Then for any point $x \in \partial N_{\sigma_{n-1}}$ there is $m_x > 0$ such that $f^{m_x}(x) \subset \mathrm{int}\, N_{\sigma_{n-1}}$. Since f is a homeomorphism, for any $x \in \partial N_{\sigma_{n-1}}$ there is a neighborhood $u_x \subset \partial N_{\sigma_{n-1}}$ such that $f^{m_x}(y) \subset \mathrm{int}\, N_{\sigma_{n-1}}$ for any $y \in u_x$. Since $\partial N_{\sigma_{n-1}}$ is compact, the set of neighborhoods $\{u_x, x \in \partial N_{\sigma_{n-1}}\}$ contains a finite subset $\{u_{x_i}, x_i \in \partial U_{\sigma^1_{n-1}}, i \in \{1, \dots, M\}\}$, covering $\partial N_{\sigma^1_{n-1}}$. Set $m = max\{m_{x_i}, i \in \{1, \dots, M\}\}$. Then $f^m(\partial N_{\sigma_{n-1}}) \subset \mathrm{int}\, N_{\sigma_{n-1}}$.

Remark 1. In the case $n = 3$ Corollary 3 is true but the closure $\mathrm{cl} W^u_{\sigma_2}$ can be a wild sphere in M^3, so, the proof of the existence of its neighborhood is a rather difficult problem. This proof is given in Ref. 4 (see also Ref. 21, Section 6.1.1).

Set $S_{\sigma_{n-1}} = \mathrm{cl}\, W^u_{\sigma_{n-1}} = W^u_{\sigma_{n-1}} \cup \{\omega\}$. Suppose that σ_{n-1}, ω are fixed and $f(U_{\sigma_{n-1}}) \subset \mathrm{int}\, U_{\sigma_{n-1}}$ (otherwise consider the diffeomorphism f^m for an enough big $m \in \mathbb{N}$). It follows from Lemmas 1 and 3, that the set $(W^s_\omega \cup W^s_{\sigma_{n-1}}) \setminus S_{\sigma_{n-1}}$ consists of two f-invariant connected components U_+, U_-.

Proposition 4. *There is a homeomorphism $h_\pm : U_\pm \to \mathbb{R}^n \setminus \{O\}$ such that*

$$f|_{U_\pm} = h_\pm^{-1} a h_\pm|_{U_\pm},$$

where $a : \mathbb{R}^n \to \mathbb{R}^n$ is a linear automorphism defined by $a(x_1, \dots, x_n) = \left(\frac{1}{2}x_1, \dots, \frac{1}{2}x_n\right)$.

Proof. Set $K = N_{\sigma_{n-1}} \setminus \mathrm{int}\, f(N_{\sigma_{n-1}})$. Since K belongs to an open annulus $S_{\sigma_{n-1}} \times (-1,1)$, it also can be embedded in $\mathbb{R}^n \setminus \{O\}$. Due to Annulus theorem (see Ref. 5, Theorem 14.3 for references), K is a union of two disjoint closed annuli K_+, K_-. Suppose that $K_+ \subset U_+$. Then $\bigcup_{n \in \mathbb{Z}} f^n(K_+) = U_+$ and for any $x \in (W^s_\omega \cup W^s_{\sigma_{n-1}}) \setminus S_{\sigma_{n-1}}$ there exists $n_x \in \mathbb{Z}$ such that $f^{n_x}(x) \in K_+$.

Let $\psi_0 : S_+ \to \mathbb{S}^{n-1}$ be an arbitrary homeomorphism, where S_+ is a connected component of ∂K that belongs to K_+. Define a homeomorphism $\psi_1 : f(S_+) \to a(\mathbb{S}^{n-1})$ by $\psi_1 = a\psi_0 f^{-1}$. It follows from Ref. 24, Ref. 25, Proposition 6, that there exists a homeomorphism $\psi : K_+ \to \mathbb{K}^n$ such that $\psi|_{S_+} = \psi_0$, $\psi|_{f(S_+)} = \psi_1$. Then the desired homeomorphism $h_+ : U_+ \to \mathbb{R}^n \setminus \{O\}$ is defined by $h_+(x) = a^{-n_x}(\psi(f^{n_x}(x)))$, where $x \in U_+$ and $f^{n_x}(x) \subset K_+$. The homeomorphism $h_- : U_- \to \mathbb{R}^n \setminus \{O\}$ can be constructed in similar way.

For points $x \in U_\pm, y \in \mathbb{R}^n \times \mathbb{Z}_2$ set $x \sim y$ if $y = h_\pm(x)$ and denote by M' a factor-space of $(M^n \setminus S_{\sigma_{n-1}}) \cup (\mathbb{R}^n \times \mathbb{Z}_2)/_\sim$. The natural projection $p : (M^n \setminus S_{\sigma_{n-1}}) \cup (\mathbb{R}^n \times \mathbb{Z}_2) \to M'$ induces on M' a structure of a smooth manifold. A map $f' : M' \to M'$ that coincides with pf on $p(M^n \setminus S_{\sigma_{n-1}})$ and with pa on each connected component of $p(\mathbb{R}^n \times \mathbb{Z}_2)$ is a Morse–Smale diffeomorphisms. We will say that the pair $\{M', f'\}$ is obtained from $\{M^n, f\}$ by *surgery along* $W^u_{\sigma_{n-1}}$. The following result immediately follows from the definition of surgery and from Ref. 26, Lemma 7.

Proposition 5. *Let $\{M', f'\}$ be obtained from $\{M^n, f\}$ by surgery along $W^u_{\sigma_{n-1}}$. Then*

1. $k'_0 = k_0 + 1, k'_{n-1} = k_{n-1} - 1; k'_i = k_i$ *for all* $i \in \{1, 2, \ldots, n-2, n\}$;
2. *if M' has two connected components M_+, M_-, then M^n is homeomorphic to a connected sum of M_+ and M_-;*
3. *if M^n is orientable and M' is connected, then M^n is homeomorphic to $M'\#(S^{n-1} \times S^1)$.*

3. Proof of main results

3.1. Proof of Theorem 1

Let M^n be a connected manifold of dimension $n \geq 3$ and $f \in G(M^n)$, that is $f : M^n \to M^n$ be a Morse–Smale diffeomorphism such all $(n-1)$-dimensional unstable manifolds of its saddle periodic points either do not intersect any invariant manifolds of other saddles or intersect only one-dimensional invariant manifolds. We prove that M^n is simply connected. We suppose that $k_1 = 0$ and prove that M^n is simply connected. For $k_{n-1} = 0$ the theorem follows from Statement 4. Suppose $k_{n-1} \neq 0$. Since we are interested only in topology of the manifold M^n, we suppose without loss of generality that all periodic points of f are fixed (that is 1-periodic, in the opposite case we may consider a homeomorphism f^N for sufficiently large N defined on the same manifold).

Due to Statement 8, M^n is the union of stable manifolds of all fixed points of f. Since the union X of stable manifolds of all source and saddle fixed points of f have dimension less than $(n-1)$, it does not divide M^n. Hence, $M^n \setminus X$ is connected. But $M^n \setminus X$ coincides with the union of pairwise disjoint stable manifolds of all sink fixed points. Consequently, the number k_0 of sinks of f equals one.

It follows from the definition of the class $G(M^n)$ and Statement 9, that there is a smallest with respect Smale relation $\prec$ saddle fixed point $\sigma_{n-1} \in \Omega_f$. Then $W^u_{\sigma_{n-1}}$ does not intersect stable manifold of any saddle fixed point different from σ_{n-1}. Due Lemma 1, there exist a sink ω such that $\mathrm{cl}\, W^u_{\sigma_{n-1}} = W^u_{\sigma_{n-1}} \cup W^u_\omega$, and the set $\mathrm{cl}\, W^u_{\sigma_{n-1}}$ is a locally flat $(n-1)$-dimensional sphere in M^n. Applying the surgery operation along $W^u_{\sigma_{n-1}}$, we obtain a pair $\{f_1, M_1\}$ of a closed manifold M_1 and a Morse–Smale diffeomorphism $f_1 : M_1 \to M_1$. If $k_{n-1} = 1$, then non-wandering set of f_1 does not contain saddle fixed points of indices 1 and $(n-1)$ and contain exactly two sinks. It follows from Statement 4, that M_1 has two connected components M_+, M_-, each of which is simply-connected. Hence, M^n is a connected sum of M_+, M_-. It follows from Van Kampen Theorem (see, for instance, Ref. 19, Theorem 1.20) that M^n is simply connected.

If $k_{n-1} > 1$, then restriction of f' on at least one component M_+, M_- satisfies the conditions of Theorem 1 and we repeat the surgery operation and all arguments above. After k_{n-1} steps we obtain a manifold $M_{k_{n-1}}$ consisting of $k_{n-1}+1$ connected components and a Morse–Smale diffeomorphism $f_{k_{n-1}} : M_{k_{n-1}} \to M_{k_{n-1}}$ such that the restriction $f_{k_{n-1}}$ on each connected component is polar. Then each connected component of $M_{k_{n-1}}$ is simply connected and so is M^n.

3.2. Proof of Proposition 1

Let M^n be a closed connected manifold of dimension $n \geq 3$, and $f \in G(M^n)$ be a polar flow such that $k_1^2 + k_{n-1}^2 \neq 0$. We prove that M^n is not simply connected. Suppose the contrary. Hence, due to Corollary 2, M^n is orientable.

We may assume that $k_{n-1} \neq 0$ (otherwise consider the diffeomorphism f^{-1}) and that all periodic points of f are fixed. It follows from the definition of the class $G(M^n)$ and Statement 9, that there is a smallest with respect Smale relation $\prec$ saddle fixed point $\sigma_{n-1} \in \Omega_f$. Then $W^u_{\sigma_{n-1}}$ does not intersect stable manifold of any saddle fixed point different from σ_{n-1}. Due Lemma 1, there exist a sink ω such that $\mathrm{cl}\, W^u_{\sigma_{n-1}} = W^u_{\sigma_{n-1}} \cup W^u_\omega$, and the set $\mathrm{cl}\, W^u_{\sigma_{n-1}}$ is a locally flat $(n-1)$-dimensional sphere in M^n. Let us prove that $M^n \setminus \mathrm{cl}\, W^u_{\sigma_{n-1}}$ is connected. Suppose the contrary. Then each connected component of $M^n \setminus \mathrm{cl}\, W^u_{\sigma_{n-1}}$ contain a connected components of $W^s_\sigma \setminus \sigma$, consequently, in virtue of Statement 8, contain a source. But by assumption f is polar, so its non-wandering set contains exactly one source. Hence, $M^n \setminus \mathrm{cl}\, W^u_{\sigma_{n-1}}$ is connected.

Applying surgery along $W^u_{\sigma_{n-1}}$ we obtain a pair $\{f', M'\}$ of a closed connected manifold M' and a Morse–Smale diffeomorphism $f' : M' \to M'$. Due to Proposition 5, M^n is a connected sum of M' and $S^{n-1} \times S^1$. Hence, according to Van Kampen Theorem, the fundamental group $\pi_1(M^n)$ of M^n is a free product of $\pi_1(M')$ and $\pi_1(S^{n-1} \times S^1)$. The last group is isomorphic to $\mathbb{Z}$, so $\pi_1(M^n)$ is non-trivial.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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