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(Recommended by Boris Hasselblatt)

In memoriam S. M. Natanzon

Abstract. Ribbon decomposition is a way to obtain a surface with boundary (compact, not necessarily oriented) from a collection of disks by joining them with narrow ribbons attached to segments of the boundary. Counting ribbon decompositions gives rise to a "twisted" version of the classical Hurwitz numbers (studied earlier by G. Chapuy and M. Dołęga [2] in a different context) and of the cut-and-join equation. We also provide an algebraic description of these numbers and an explicit formula for them in terms of zonal polynomials.

1. Introduction

Twisted Hurwitz numbers. A classical surgery in dimension 2 studies connected sums of spheres, that is, ways to obtain a compact surface from a collection of spheres by gluing cylinders to them. In this paper we apply similar techniques to surfaces with boundary: they are obtained from a collection of disks by gluing rectangles ("ribbons") to their boundary. Like with the classical connected sum, to glue a ribbon one is to choose the orientation of the boundary at both points of gluing, so the ribbon glued may look twisted or not.

Representation of a surface with boundary as a union of disks with the ribbons attached will be called its ribbon decomposition. See Figure 3 for examples: the upper picture is a ribbon decomposition of an annulus, the lower one, of a Moebius band.

Diagonals of ribbons form a graph embedded into the surface (a.k.a. fat graph, ribbon graph, combinatorial map, etc.), with all its vertices on the boundary. The edges adjacent to a given vertex are thus linearly ordered left to right (remember, an orientation of the boundary near every vertex is chosen); this ordering defines the embedding of the graph up to homotopy.

Fix a positive integer m and a partition $(\lambda_1 \geq \cdots \geq \lambda_s)$ of the number

$$n \overset{\text{def}}{=} |\lambda| \overset{\text{def}}{=} \lambda_1 + \cdots + \lambda_s$$

into s parts. The main object of study in this paper, the *twisted Hurwitz numbers* $h^{\sim}_{m,\lambda}$, have several definitions or models, as we call them. The first one, a topological model, uses ribbon decompositions. Denote by $\mathfrak{S}_{m,\lambda}$ the set of decompositions into m ribbons of surfaces having boundary of s components containing $\lambda_1, \ldots, \lambda_s$ vertices (endpoints of ribbon diagonals). Then the twisted Hurwitz number is defined as

$$(1.1) \qquad\qquad h^{\sim}_{m,\lambda} \overset{\text{def}}{=} \frac{1}{n!} \#\mathfrak{S}_{m,\lambda}$$

Another model for $h^{\sim}_{m,\lambda}$ is algebraic. Consider a fixed-point-free involution

$$(1.2) \qquad \tau = (1, n+1)(2, n+2)\ldots(n, 2n)$$

in the symmetric group S_{2n}. Its centralizer is isomorphic to the reflection group of type B_n. Let $\sigma_1, \ldots, \sigma_m \in S_{2n}$ be transpositions. A simple analysis (see Section 3 below) shows that the permutation

$$(1.3) \qquad u \stackrel{\text{def}}{=} \sigma_1 \ldots \sigma_m \tau \sigma_m \ldots \sigma_1 \tau \in S_{2n}$$

is decomposed into independent cycles as $u = c_1 c'_1 \ldots c_s c'_s$ where $c'_i = \tau c_i^{-1} \tau$ for every $i = 1, \ldots, s$. Denote by $\mathfrak{H}_{m,\lambda}$ the set of sequences $(\sigma_1, \ldots, \sigma_m)$ of m transpositions such that the cycles $c_1, \ldots, c_s$ of the permutation u of (1.3) have lengths $\lambda_1, \ldots, \lambda_s$. We prove (Theorem 17) that

$$(1.4) \qquad h^{\sim}_{m,\lambda} = \frac{1}{n!} \# \mathfrak{H}_{m,\lambda}.$$

The third model for $h^{\sim}_{m,\lambda}$ is algebro-geometric and is due to G. Chapuy and M. Dołęga [2], who generalized the classical notion of a branched covering to the non-orientable case. Let N denote a closed surface (compact 2-manifold without boundary, not necessarily orientable), and $p : \widehat{N} \to N$, its orientation cover. Denote by $\overline{\mathbb{H}} \stackrel{\text{def}}{=} \mathbb{C}P^1/(z \sim \overline{z}) = \mathbb{H} \cup \{\infty\}$ where $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane; its closure $\overline{\mathbb{H}}$ is homeomorphic to a disk. Let $\pi : \mathbb{C}P^1 \to \overline{\mathbb{H}}$ be the quotient map. A continuous map $f : N \to \overline{\mathbb{H}}$ is called [2] a twisted branched covering if there exists a branched covering $\widehat{f} : \widehat{N} \to \mathbb{C}P^1$ (in the classical sense, a holomorphic map) such that $\pi \circ \widehat{f} = f \circ p$, and all the critical values of $\widehat{f}$ are real. These requirements imply in particular that the ramification profile of any critical value $c \in \mathbb{R}P^1 \subset \mathbb{C}P^1$ of $\widehat{f}$ has every part repeated twice: $(\lambda_1, \lambda_1, \ldots, \lambda_s, \lambda_s)$, and $\deg \widehat{f} = 2n$ is even. In this case we say that the ramification profile of the critical value $\pi(c) \in \partial\overline{\mathbb{H}}$ of the map $f : N \to \overline{\mathbb{H}}$ is $\lambda = (\lambda_1, \ldots, \lambda_s)$.

Twisted branched coverings are split into equivalence classes via right-left equivalence. Denote by $\mathfrak{D}_{m,\lambda}$ the set of equivalence classes of twisted branched coverings having m critical values with the ramification profiles $2^1 1^{n-2}$ and one critical value ∞ with the ramification profile λ. Then

$$(1.5) \qquad h^{\sim}_{m,\lambda} = \frac{2^m}{n!} \# \mathfrak{D}_{m,\lambda}.$$

Note that we prove equations (1.4) and (1.5) differently. To prove (1.4) we establish a direct correspondence Ξ between the sets $\mathfrak{S}_{m,\lambda}$ and $\mathfrak{H}_{m,\lambda}$ and prove (Theorem 17) that it is one-to-one. To prove (1.5) we show (Theorems 23 and 25) that the generating function of the twisted Hurwitz numbers satisfies a PDE of parabolic type called twisted cut-and-join equation—just like standard Hurwitz numbers, whose generating function satisfies the "classical" cut-and-join [6]. Cardinalities of the sets $\mathfrak{D}_{m,\lambda}$ are shown in [2] to satisfy the same equation (up to a change of variables) with the same initial data, so (1.5) follows. Finding a direct correspondence between the sets $\mathfrak{D}_{m,\lambda}$ and $\mathfrak{S}_{m,\lambda}$ (or $\mathfrak{H}_{m,\lambda}$) is a challenging topic of future research.

Bibliography remarks: real Hurwitz numbers. An involution $T : \widehat{N} \to \widehat{N}$ exchanging the two preimages $a, b \in p^{-1}(c)$ of each point $c \in N$ makes $\widehat{N}$ a real curve without real points (a complex curve with an anti-holomorphic fixed-point-free involution). Now (1.5) allows us to view $h^{\sim}_{m,\lambda}$ as no-real-point real Hurwitz numbers.

Many versions of real Hurwitz numbers have been defined and studied in literature; the full review of the topic goes far beyond the scope of this article. Let us only mention the articles [11] by S. Natanzon and [7] by M. Kazarian, S. Lando and S. Natanzon; the second one contains an appendix re-explaining the results of the first. [11] addresses the

general case of real Hurwitz numbers: they are the numbers of meromorphic functions with real critical values on a real curve stratified by position of the critical points (real or not) and by the orders of the poles; for real critical points some more characteristics are taken into account. In [7] real curves with maximal (separating) real set are considered; this is the particular case of [11] opposite to the one studied here. In both articles a version of the cut-and-join equation for the generating function of the real Hurwitz numbers is proved; note though that the generating functions here and in [11, 7] are organized differently, so Theorem 25 below is not a specialization of the cut-and-join operators there.

There are also some earlier results addressing particular case of real Hurwitz numbers: for genus 0 [13, 12] and for generic meromorphic functions (i.e., $\lambda = 1^n$ in our notation) [12, 3]. Newer research also uses tropical technique, see, e.g., [10].

Structure of the paper. The paper contains three main sections in accordance with the three models described. In the first, "topological" section we study ribbon decompositions of surfaces with boundary (rigged with marked points) and the graphs (with numbered vertices and edges) formed by the diagonals of ribbons. The graphs appear to be 1-skeleta of the surface, and the surface can be retracted to them (Theorem 9); also, the graphs behave nicely under the orientation cover of the surface (Theorems 11 and 13).

Graph embeddings into oriented surfaces were studied earlier in a number of works (see [8] for the general theory without boundary, [4] for the disk and [1] for arbitrary surfaces and embeddings with a connected complement); they are in one-to-one correspondence with sequences of transpositions in the symmetric group. The cyclic structure of the product of the transpositions describes faces of the graph (i.e., connected components of its complement). The quantity of graphs with given faces is called a (classical) Hurwitz number and has been studied intensively during the last decades—the research involving dozens of authors and hundreds of works; its thorough review is far outside the scope of this paper. The algebraic model for twisted Hurwitz numbers, studied in Section 3, is a generalization of this correspondence. The section also contains an explicit formula for the cut-and-join equation (Theorem 25) and for the generating function of the twisted Hurwitz numbers (Theorem 28).

In the last section we study the notion of the twisted branched covering defined in [2] and show that they form an algebro-geometric model for twisted Hurwitz numbers.

2. Surgery: a Topological Model for Twisted Hurwitz Numbers

2.1. General definitions.

Definition 1. A *decorated-boundary surface* (DBS) is a triple $(M, (a_1, \ldots, a_n), (o_1, \ldots, o_n))$, where M is a compact surface (2-manifold) with boundary, $a_1, \ldots, a_n \in \partial M$ are marked points, and every o_i is a local orientation of ∂M (hence, of M itself, too) in the vicinity of the point a_i, such that

- every connected component of M has nonempty boundary, and
- every connected component of ∂M contains at least one point a_i.

The DBS M and M' with the same number n of marked points are called equivalent if there exists a homeomorphism $h : M \to M'$ such that $h(a_i) = a'_i$ and $h_*(o_i) = o'_i$ for all $i = 1, \ldots, n$. The set of equivalence classes of DBS with n marked points will be denoted $\mathscr{DBS}_n$.
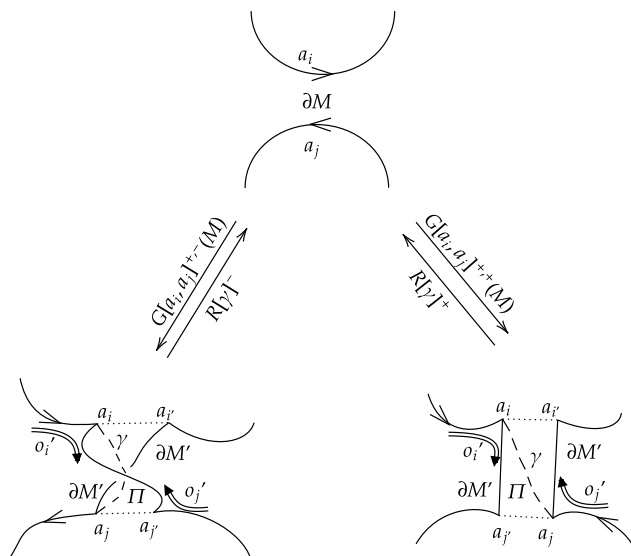
Figure 1. Gluing ribbons

Pick marked points $a_i, a_j \in \partial M$, and let $\varepsilon_i, \varepsilon_j \in \{+, -\}$. Consider points $a_i', a_j' \in \partial M$ lying near a_i, a_j and such that the boundary segment $a_i a_i'$ is directed along the orientation o_i if $\varepsilon_i = +$ and opposite to it if $\varepsilon_i = -$; the same for j. Now take a long narrow rectangle ("a ribbon" henceforth) and glue its short sides to ∂M as shown in Figure 1. The result of gluing is homeomorphic to a surface M' with the boundary $\partial M' \ni a_1, \dots, a_n$. The boundary of M' near a_i and a_j contains a segment of ∂M (the "old" part) and a segment of a long side of the ribbon glued (the "new" part); define local orientations o_i', o_j' of $\partial M'$ near a_i, a_j so that the orientations of the "old" parts would be preserved (see bold curved arrows in Figure 1); for $k \neq i, j$ take $o_k' = o_k$ by definition. Now $(M', (a_1, \dots, a_n), (o_1', \dots, o_n'))$ is a DBS, so we defined a mapping $G[i,j]^{\varepsilon_i, \varepsilon_j} : \mathcal{DBS}_n \to \mathcal{DBS}_n$ called *ribbon gluing*. The ribbon gluing $G[i,j]^{\varepsilon_i, \varepsilon_j}$ will be called twisted if $\varepsilon_i \neq \varepsilon_j$, and non-twisted otherwise; compare the left and the right picture in Figure 1.

The inverse operation is called ribbon removal. To define it, take $\varepsilon \in \{+, -\}$ and fix a smooth simple (i.e., not self-intersecting) curve γ on M joining a_i and a_j and transversal to ∂M in its endpoints. Take now a point $a_j' \in \partial M$ near a_i and $a_i' \in \partial M$ near a_j (note the subscripts!) such that the segment $a_i a_j' \subset \partial M$ is directed according to the orientation o_i if $\varepsilon = +$ and opposite to it if $\varepsilon = -$, and consider a "rectangle" Π like in Figure 1. Then $M' \overset{\text{def}}{=} M \backslash \text{int}(\Pi)$ is homeomorphic to a surface with the boundary $\partial M' \ni a_1, \dots, a_n$. A local orientation o_i' of $\partial M'$ near a_i is defined by the same rule as for the ribbon gluing: o_i and o_i' coincide on the intersection $\partial M' \cap \partial M$ near a_i; the same for o_j', and also $o_k' \overset{\text{def}}{=} o_k$ for all $k \neq i, j$. Now $(M', (a_1, \dots, a_n), (o_1', \dots, o_n'))$ is a DBS obtained from the original DBS by the operation $R[\gamma]^\varepsilon$ of ribbon removal.

Remark 2. Local orientations o_i and o_j of ∂M define orientations of the normal bundle to γ; we call γ *non-twisting* if the orientations are the same, and *twisting* otherwise. Obviously, the segment $a_j a_j'$ is directed along the orientation o_j if $\varepsilon = +$ and γ is non-twisting or $\varepsilon = -$ and γ is twisting; otherwise $a_j a_j'$ is directed opposite to o_j.

The operation $R[\gamma]^\varepsilon$ is a sort of inverse to ribbon gluing due to the following obvious statement:

Proposition 3.
(1) *Let $i, j \in \{1, \ldots, n\}$, $\varepsilon_i, \varepsilon_j \in \{+, -\}$ and γ be a diagonal of the ribbon joining a_i and a_j. Then $R[\gamma]^{\varepsilon_i} G[i, j]^{\varepsilon_i, \varepsilon_j} = \mathrm{id}_{\mathscr{DBS}_n}$.*
(2) *Let γ be a simple smooth curve on M joining a_i and a_j and transversal to the boundary, and $\varepsilon_i \in \{+, -\}$. Let $\varepsilon_j \in \{+, -\}$ be defined as $\varepsilon_j = \varepsilon_i$ if the curve γ is non-twisting and $\varepsilon_j = -\varepsilon_i$ otherwise. Then $G[i, j]^{\varepsilon_i, \varepsilon_j} R[\gamma]^{\varepsilon_i} = \mathrm{id}_{\mathscr{DBS}_n}$.*

Remark 4. Gluing a ribbon decreases the Euler characteristics of the surface by 1 and removal, increases it by 1.

2.2. Ribbon decompositions. By Definition 1 every connected component of a DBS contains a marked point. $M \in \mathscr{DBS}_n$ is called *stable* if every its connected component either contains at least two marked points or is a disk (with one marked point only).

Denote by $E_n \in \mathscr{DBS}_n$ a union of n disks with one marked point on the boundary of each.

Proposition 5. *$M \in \mathscr{DBS}_n$ is stable if and only if it can be obtained by gluing several ribbons to E_n. If M is stable then its Euler characteristics $\chi(M)$ does not exceed n, and the number of ribbons is equal to $n - \chi(M)$.*

Proof. If a surface with a ribbon glued has a component with only one marked point then the gluing left this component intact. So, gluing a ribbon to a stable DBS keeps its stability, which proves the 'only if' part of the proposition (E_n is stable by definition).

To prove the 'if' part we will need a lemma:

Lemma 6. *Let $n \geq 2$. Then for any $M \in \mathscr{DBS}_n$ which is connected and stable but is not a disk there exists a simple smooth non-separating curve γ joining two marked points.*

"Nonseparating" here means that the complement of γ is connected, too.

Proof of the lemma. M contains at least two marked points. If ∂M is not connected then take two marked points on different components of ∂M and join them with a simple smooth curve γ; such curve is always non-separating.

Let now ∂M be connected. Then M is a connected sum of a disk with several handles and/or Moebius bands. Let $S^1 \subset M$ be a circle separating the disk from a handle or from a Moebius band, and let $p, q \in S^1$ be two points. There exists a non-separating curve δ inside the handle or the Moebius band joining p and q. Now pick a curve γ_1 joining p with one marked point and γ_2 joining q with another one. Then the union $\gamma \overset{\mathrm{def}}{=} \gamma_1 \cup \delta \cup \gamma_2$ is non-separating as required. $\qquad\square$

Corollary 7. *If $M \in \mathscr{DBS}_n$ is stable and $M \neq E_n$ then there exists a curve γ on M such that $M' \overset{\mathrm{def}}{=} R[\gamma]^{\varepsilon}(M)$ is stable (regardless of ε).*

Proof of the corollary. A stable DBS different from E_n contains a component with two or more marked points. If this component is a disk then take for γ any simple curve joining these points. If it is not a disk then take for γ the non-separating curve of Lemma 6. $\quad\square$

The proposition is now proved using induction on the Euler characteristic of M. Every component of M is a manifold with nonempty bounbdary, so the 2$^{\mathrm{nd}}$ Betti number of M is zero and $\chi(M) = h_0(M) - h_1(M) \leq h_0(M) \leq n$; the equality is possible only if $M = E_n$. Let now $\chi(M) = n - m$, $m > 0$. Use Corollary 7 to obtain a curve γ in M such that $M' = R[\gamma]^+(M)$ is stable; by Remark 4 one has $\chi(M') = n - m + 1$, so by the induction hypothesis M' can be obtained from E_n by gluing $m - 1$ ribbons. By assertion 2 of Proposition 3 there exist i, j and ε such that $M = G[i, j]^{+, \varepsilon}(M')$—so, M can be obtained by gluing m ribbons. $\qquad\square$

Let now, again, $M \in \mathscr{DBS}_n$ be obtained by gluing of m ribbons to E_n:

$$(2.1) \qquad\qquad M = G[i_m, j_m]^{\varepsilon_m, \delta'_m} \dots G[i_1, j_1]^{\varepsilon_1, \delta_1} E_n$$

(that's what we will be calling a *ribbon decomposition* of M). For every ribbon, draw a diagonal joining its vertices a_{i_k} and a_{j_k}, and assign the number k to it. The union of the diagonals is a graph $\Gamma \subset M$ with m numbered edges $r_1, \dots, r_m$ and the marked points $a_1, \dots, a_n$ as vertices; we call it a *diagonal graph* of the ribbon decomposition.

Let a_i be a marked point of M, $\Gamma \subset M$ be a diagonal graph of a ribbon decomposition, and let $\ell_1, \dots, \ell_k$ be the numbers of the edges of Γ having a_i as an endpoint, listed left to right according to the orientation o_i; denote $\mathscr{P}(a_i) \overset{\text{def}}{=} (\ell_1, \dots, \ell_k)$.

Theorem 8. *The diagonal graph Γ has the following properties:*

(1) *(embedding) Γ is embedded: its edges do not intersect one another or the boundary of M except at endpoints.*

(2) *(anti-unimodality) For every vertex a_i the sequence $\mathscr{P}(a_i) = (\ell_1, \dots, \ell_k)$ is anti-unimodal: there exists $p \le k$ such that $\ell_1 > \dots > \ell_p < \dots < \ell_k$.*

(3) *(twisting rule) In the notation of the above call the edges $\ell_1, \dots, \ell_p$ negative at the endpoint a_i, and edges $\ell_p, \dots, \ell_k$, positive (note that ℓ_p is both). Then any twisting edge of Γ is positive at one of its endpoints and negative at the other one, and any non-twisting edge is either positive at both endpoints or negative at them.*

(4) *(retraction) The graph Γ is a homotopy retract of the surface M.*

Proof. Induction by the number m of ribbons; the base $m = 0$ is obvious. For any $m > 0$, let $M = G[i_m, j_m]^{\varepsilon_m \delta_m} M'$, and $\Gamma' \subset M' \subset M$ be the union of all the edges of Γ except m.

Assertion 1: the internal points of the edge m lie in the interior of the ribbon $r_m = M \setminus M'$ and thus belong neither to Γ' nor to ∂M.

Assertion 2: after gluing the ribbon r_m to M', the edge m is either the leftmost or the rightmost of all the edges ending at a_{i_m}. Thus, if $\mathscr{P}(a_{i_m}) = (\ell_1, \dots, \ell_k)$ then either $\ell_1 = m$ and $\ell_2, \dots, \ell_k$ is anti-unimodal by the induction hypothesis, or $\ell_k = m$ and $\ell_1, \dots, \ell_{k-1}$ is anti-unimodal. In both cases $\ell_1, \dots, \ell_k$ is anti-uninmodal.

Assertion 3 is true for the edges of $\Gamma' \subset M'$ by the induction hypothesis. Apparently, this is preserved after the ribbon r_m is glued. The edge m is the diagonal of r_m; the long sides of r_m lie in ∂M, and therefore the edge m is adjacent to ∂M at both its endpoints, from the right for one of them and from the left for the other. This proves assertion 3 for the edge m, too.

To facilitate induction for assertion 4, we make it a bit stronger: fix, for every i, a small segment $e_i \subset \partial M$, $a_i \in e_i$, and prove that there exists a homotopy retraction $f : M \to \Gamma$ such that $f(x) = a_i$ for all $x \in e_i$.

By the induction hypothesis, such f exists for M' and Γ'. W.l.o.g. the ribbon r_m containing a_i and a_j is glued to M' so that its short sides lie within the segments e_i and e_j. Thus, the induction step is reduced to the following obvious statement: there exists a homotopy retraction of a rectangle Π onto its diagonal $[ab]$ sending short sides and small neighborhoods of the points $a, b \in \partial \Pi$ to the points a and b. $\qquad\square$

Let now $M \in \mathscr{DBS}_n$ and let $\Gamma \subset M$ be an embedded loopless graph with the vertices at the marked points of M and the edges numbered $1, \dots, m$. We call Γ *properly embedded* if it satisfies all the assertions of Theorem 8: embedding, anti-unimodality, twisting rule and retraction. Connected components of the complement $M \setminus \partial M \setminus \Gamma$ will be called *faces*; connected components of $\partial M \setminus \{a_1, \dots, a_n\}$, *external edges*, and connected components of $\Gamma \setminus \{a_1, \dots, a_n\}$, *internal edges*.
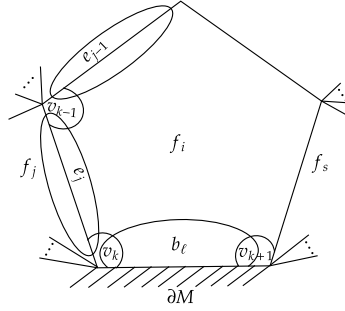
FIGURE 2. The open cover of M

Theorem 9. *Vertices, edges and faces of a properly embedded graph* Γ *form a cell decomposition of* M *(as* 0*-cells,* 1*-cells and* 2*-cells, respectively) such that every face is adjacent to exactly one external edge. The total number of faces is* n.

Proof. Let Γ have k faces $f_1, \ldots, f_k$. Cover M with the open subsets shown in Figure 2.

Sets e_i (neighborhoods of internal edges) are homeomorphic to disks, b_i (neighborhoods of external edges) and v_i (neighborhoods of vertices), to half-disks; topology of faces f_i is yet to be described. Connected components of all the nonempty intersections of the sets (including faces) are homeomorphic to disks or half-disks, too.

The nonempty intersections are:

- $2m$ connected components of $f_i \cap e_j$ for all i, j;
- n components of $f_i \cap b_j$ for all i, j;
- If δ_j is the valency of the jth vertex of the graph, then there are $\delta_j + 1$ components of $f_i \cap v_j$ for all i. The total number of components in $f_i \cap v_j$ is thus $\sum_j (\delta_j + 1) = 2m + n$;
- $2m$ components of $e_i \cap v_j$, for all i, j;
- $2n$ components of $b_i \cap v_j$, for all i, j;
- $4m$ components of $f_i \cap e_j \cap v_k$, for all i, j, k;
- $2n$ components of $f_i \cap b_j \cap v_k$.

Thus the Euler characteristics of M is

$$\chi(M) = \sum_{i=1}^{k} \chi(f_i) + m + n + n - 2m - n - (2m + n) - 2m - 2n + 4m + 2n$$

$$= \sum_{i=1}^{k} \chi(f_i) - m.$$

On the other hand, Γ is a retract of M, so $\chi(M) = \chi(\Gamma) = n - m$, hence $\sum_{i=1}^{k} \chi(f_i) = n$. Faces are connected open 2-manifolds, so $\chi(f_i) \leq 1$ for every i, and therefore $n \leq k$.

Closure of a face is a compact manifold with boundary, so it cannot retract to its boundary. It means that the boundary of any face is not a subset of the graph and must contain an external edge. The total number of external edges is n, and an external edge belongs to the boundary of one face only. This implies $n \geq k$ and therefore $n = k$ and $\chi(f_i) = 1$ for every $i = 1, \ldots, k$.

So each f_i is a disk. Its closure contains one external edge and k_i internal ones, as well as vertices, so it is an image of the map Q_i from some $(k_i + 1)$-gon to M. Every Q_i sends sides of the polygon to the edges and vertices to vertices, so collectively the Q_i, $i = 1, \ldots, k$, are characteristic maps of a cell decomposition. $\square$

Theorem 9 allows us to prove the inverse of Theorem 8:

Theorem 10. *Let $M \in \mathscr{DBS}_n$ be stable and $\Gamma \subset M$ be a properly embedded graph. Then Γ is the diagonal graph of a ribbon decomposition of M.*

Proof. Induction by the number m of edges of Γ. The base: $m = 0$ means that Γ consists of n isolated vertices. Since M is a retract of Γ, one has $M = E_n$.

Let $m > 0$. The edge e_m of Γ joins the vertices a_i and a_j (necessarily different) and separates faces f_p and f_q (which may be the same). By the anti-unimodality, e_m is adjacent to ∂M at both a_i and a_j. Using Theorem 9, consider a characteristic map Q_p of the cell f_p. It maps the side $v_0 v_1$ of the polygon to the external edge of f_p and the adjacent side $v_1 v_2$, to e_m. Let $v' \in v_0 v_1$ be a point near the vertex v_1, $a'_i \stackrel{\text{def}}{=} Q_p(v') \in \partial M$; consider the image $T_p \stackrel{\text{def}}{=} Q_p(v' v_1 v_2) \subset M$ of the triangle $v' v_1 v_2$. Then the union of T_p and a similar triangle $T_q \subset f_q$ is a ribbon H having e_m as its diagonal.

Let Γ' be the graph Γ with the edge e_m removed. Take $\varepsilon = +$ if ∂M near a_i is oriented towards a'_i, and $\varepsilon = -$ otherwise. Then Γ' is embedded into $M' \stackrel{\text{def}}{=} R[e_m]^\varepsilon(M)$; an immediate check shows that the embedding is proper, so Γ' is the diagonal graph of a ribbon decomposition of M' by the induction hypothesis. By Proposition 3 M can be obtained by gluing the ribbon H to M', which finishes the induction. $\qquad\square$

2.3. Oriented case and the orientation cover. A DBS M is called *oriented* if all the local orientations o_i are consistent with a global orientation of the surface M. For an oriented M the numbers of marked points read off the components of ∂M according to the orientation form a cyclic decomposition of some permutation $\sigma \in S_n$ called the *boundary permutation* of M (here and below we denote by S_n the permutation group). In other words, for any $k = 1, \dots, n$ the marked point adjacent to a_k in the positive direction of ∂M is $a_{\sigma(k)}$.

It is easy to see that if M is oriented and the gluing $G[i, j]^{\varepsilon_i, \varepsilon_j}$ is non-twisted ($\varepsilon_i = \varepsilon_j$) then $G[i, j]^{\varepsilon_i, \varepsilon_j}(M) \in \mathscr{DBS}_n$ is oriented, too. A ribbon decomposition

$$(2.2) \qquad\qquad M = G[i_m, j_m]^{++} \dots G[i_1, j_1]^{++} E_n$$

is called oriented; existence of such means, by obvious induction, that the surface M is oriented.

Theorem 11. *The diagonal graph Γ of the oriented ribbon decomposition (2.2) has the following properties (in addition to those granted by Theorem 8):*

  (1) *(vertex monotonicity) For every vertex a_i of Γ the sequence $\mathscr{P}(a_i) = (\ell_1, \dots, \ell_k)$ is increasing: $\ell_1 < \dots < \ell_k$.*
  (2) *(face monotonicity) For every face f_i of Γ the numbers $\ell_1, \dots, \ell_p$ of the internal edges adjacent to it are increasing if the count starts at the (only) external edge of f_i and goes counterclockwise.*
  (3) *(face separation) Every internal edge of Γ separates two different faces.*
  (4) *(boundary permutation) Let a_{i_k}, a_{j_k} be endpoints of the edge e_k of Γ, $k = 1, \dots, m$. Then the boundary permutation of M is equal to $(i_m j_m) \dots (i_1 j_1) \in S_n$.*

Proof. Vertex monotonicity is a particular case of anti-unimodality of Theorem 8.

If ℓ_j and ℓ_{j+1} are two internal edges on the boundary of f_i sharing an endpoint a then the orientation of the boundary near a is consistent with the counterclockwise orientation of f_i. Then the vertex monotonicity implies $\ell_j < \ell_{j+1}$, which proves face monotonicity. The face monotonicity implies, in its turn, the face separation: as one moves around a face, the numbers of the internal edges seen are increasing and therefore cannot repeat.

Let $a_k, a_s \in \partial M$ be neighboring vertices, that is, the endpoints of an external edge. By Theorem 9 and the face monotonicity, this is the sole external edge of a face f, its remaining sides being internal edges numbered $\ell_1 < \cdots < \ell_p$, as one moves from a_k to a_s. Consider an action of S_n on the vertices of $M \in \mathscr{DBS}_n$ by permuting their numbers; in particular, the transposition $(i_t \, j_t)$ exchanges the numbers of the vertices joined by the tth edge of the diagonal graph, leaving the other vertices intact. So, the transposition $(i_{\ell_1} \, j_{\ell_1})$ moves a_k to its neighbor at the face f; then the transposition $(i_{\ell_2} \, j_{\ell_2})$ (where $\ell_2 > \ell_1$, so it is applied *after* the first one) moves it to the next vertex of the same face, etc.; eventually, $\sigma = (i_m \, j_m) \ldots (i_1 \, j_1)$ moves a_k to $a_s = a_{\sigma(k)}$. $\qquad\square$

Every manifold M (possibly with boundary) has the orientation cover, uniquely defined up to an obvious isomorphism: it is an oriented manifold $\widehat{M}$ of the same dimension together with a fixed-point-free smooth involution $\mathscr{T} : \widehat{M} \to \widehat{M}$ reversing the orientation and such that M is diffeomorphic to its orbit space.

The quotient map $\widehat{M} \to \widehat{M}/T = M$ is a 2-sheeted covering, trivial iff M is oriented. For 2-manifolds with boundary there is

Lemma 12. *The orientation covering is trivial over the boundary of a 2-manifold.*

Proof. The boundary ∂M and its cover $\partial \widehat{M}$ are unions of circles. If the covering is nontrivial over the boundary then there is a component $C \subset \partial M$ covered by a $\mathscr{T}$-invariant circle $C' \subset \partial \widehat{M}$.

A continuous map $A : S^1 \to S^1$ has at least $|\deg A - 1|$ fixed points, so the fixed-point-free map $\mathscr{T} : C' \to C'$ has degree 1 and therefore, being a covering, preserves orientation. Since $C' \subset \partial \widehat{M}$, it means that $\mathscr{T} : \widehat{M} \to \widehat{M}$ also preserves local orientation at every point $a \in C'$. But $\mathscr{T}$ is orientation-reversing everywhere—a contradiction. $\qquad\square$

Let a fixed-point-free involution $\tau \in S_{2n}$ be defined by (1.2). The notion of an orientation cover can be extended to decorated-boundary surfaces as follows: $\widehat{M} \in \mathscr{DBS}_{2n}$ with the marked points $b_1, \ldots, b_{2n}$ is called the orientation cover of $M \in \mathscr{DBS}_n$ with the marked points $a_1, \ldots, a_n$ if $\widehat{M}$ is oriented and there exists a fixed-point-free smooth involution $\mathscr{T} : \widehat{M} \to \widehat{M}$ reversing the orientation and such that $\mathscr{T}(b_k) = b_{\tau(k)}$ for all $k = 1, \ldots, 2n$, and also there exists a diffeomorphism $p : \widehat{M}/\mathscr{T} \to M$ between the orbit space and M such that $p(\{b_k, b_{\tau(k)}\}) = a_k$ for all $k = 1, \ldots, n$.

For $M \in \mathscr{DBS}_n$ the surface $\widehat{M}$ and involution $\mathscr{T} : \widehat{M} \to \widehat{M}$ are uniquely defined; the marked points are $p^{-1}(a_1) \stackrel{\text{def}}{=} \{b_i, b_{i+n}\} \subset \partial \widehat{M}$. The numbering of the two points b_i and b_{i+n} depends on the local orientation o_i of ∂M at a_i and is fixed by the following rule: the mapping $p : \partial \widehat{M} \to \partial M$ preserves the orientation at b_i and reverses it at b_{i+n}, $i = 1, \ldots, n$. Thus, for every $M \in \mathscr{DBS}_n$ an orientation cover $\widehat{M} \in \mathscr{DBS}_{2n}$ is unique.

Let $1 \le i \le n$ and $\varepsilon \in \{+, -\}$. Denote $i^\varepsilon = \begin{cases} i, & \varepsilon = +, \\ \tau(i), & \varepsilon = -. \end{cases}$

Theorem 13. *Let M be a DBS of equation* (2.1). *Then its orientation cover is*

$$(2.3) \qquad \widehat{M} = G\big[i_m^{\varepsilon_m} j_m^{\delta_m}\big]^{++} \ldots G\big[i_1^{\varepsilon_1} j_1^{\delta_1}\big]^{++} G\big[i_1^{-\varepsilon_1} j_1^{-\delta_1}\big]^{++} \ldots G\big[i_m^{-\varepsilon_m} j_m^{-\delta_m}\big]^{++} E_n.$$

The involution $\mathscr{T} : \widehat{M} \to \widehat{M}$ maps the ribbon r_ℓ to the ribbon $r_{2m+1-\ell}$ for all $\ell = 1, \ldots, 2m$.

The proof is an obvious induction. See Figure 3 for an example. The Roman numerals there mean faces, Arabic numbers, vertices, and the circled numbers mark diagonals of the ribbons.

Proof. Let a_i be a marked point of M with $\mathscr{P}(a_i) = (\ell_1, \ldots, \ell_k)$ where $\ell_1 > \cdots > \ell_p < \cdots < \ell_k$, and let $b_i, b_{\tau(i)} \in \widehat{M}$ be its preimages. Use the induction on m to prove the theorem
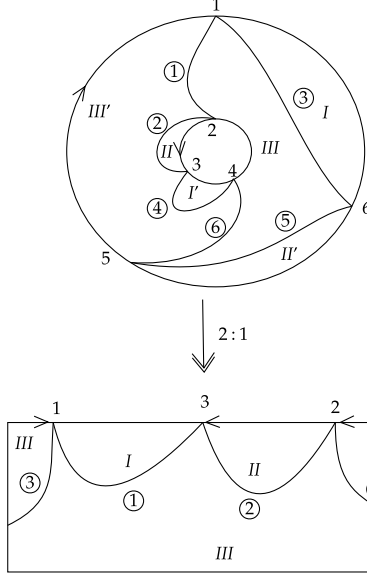
FIGURE 3. Covering of the Moebius band $M = G[1,2]^{++}G[2,3]^{++}G[1,3]^{+-}E_3$ by an annulus

showing simultaneously that $\mathscr{P}(b_i) = (m+1-\ell_1,\dots,m+1-\ell_p, m+\ell_{p+1},\dots,m+\ell_k)$ and $\mathscr{P}(b_{\tau(i)}) = (m+1-\ell_k,\dots,m+1-\ell_{p+1},\ell_p+m,\dots,\ell_1+m)$.

The base $m = 0$ is obvious. For $m > 0$ let $M = G[i,j]^{\varepsilon\delta}M'$, where i, j, ε, and δ mean i_m, j_m, ε_m, and δ_m, respectively. If $\mathscr{P}_{M'}(a_i) = (\ell_1,\dots,\ell_k)$ where $\ell_1 > \cdots > \ell_p < \cdots < \ell_k$ then $\mathscr{P}_M(a_i) = (\ell_1,\dots,\ell_k,m)$ if $\varepsilon = +$ and $\mathscr{P}_M(a_i) = (m,\ell_1,\dots,\ell_k)$ if $\varepsilon = -$; the same for a_j (depending on δ instead of ε).

Denote by $\widehat{M'}$ the orientation cover of M' and define $\widehat{M}$ by (2.3). By the induction hypothesis $\widehat{M'}$ is a subset of $\widehat{M}$ (a union of all the ribbons except r_1 and r_{2m}). Extend $\mathscr{T}: \widehat{M'} \to \widehat{M'}$ to the involution $\widehat{M} \to \widehat{M}$ sending r_1 to r_{2m} and vice versa; also extend the homeomorphism $\rho: \widehat{M'}/\mathscr{T} \to M'$ to a map $\widehat{M}/\mathscr{T} \to M$ sending r_1 and r_{2m} to the mth ribbon of M. To complete the proof we are to check that the extended $\mathscr{T}$ and ρ are continuous on the boundary of the ribbons r_1 and r_{2m}.

By the induction hypothesis, $\mathscr{P}_{\widehat{M'}}(b_i) = (m-\ell_1,\dots,m-\ell_p,\ell_{p+1}+m-1,\dots,\ell_k+m-1)$ and $\mathscr{P}_{\widehat{M'}}(b_{\tau(i)}) = (m-\ell_k,\dots,m-\ell_{p+1},\ell_p+m-1,\dots,\ell_1+m-1)$. So, if $\varepsilon = +$ then

$$\mathscr{P}_{\widehat{M}}(b_i) = (m+1-\ell_1,\dots,m+1-\ell_p,\ell_{p+1}+m,\dots,\ell_k+m,2m)$$
$$\mathscr{P}_{\widehat{M'}}(b_{\tau(i)}) = (1,m+1-\ell_k,\dots,m+1-\ell_{p+1},\ell_p+m,\dots,\ell_1+m),$$

and if $\varepsilon = -$ then

$$\mathscr{P}_{\widehat{M}}(b_i) = (1,m+1-\ell_1,\dots,m+1-\ell_p,\ell_{p+1}+m,\dots,\ell_k+m)$$
$$\mathscr{P}_{\widehat{M'}}(b_{\tau(i)}) = (m+1-\ell_k,\dots,m+1-\ell_{p+1},\ell_p+m,\dots,\ell_1+m,2m);$$

the same for b_j and $b_{\tau(j)}$, with δ instead of ε.

Thus, if $\varepsilon = +$ then the ribbon r_{2m} is adjacent to r_{ℓ_k+m} and the ribbon r_1, to $r_{m+1-\ell_k}$; the mth ribbon of $M = G[i,j]^{\varepsilon\delta}M'$ is adjacent to the ribbon numbered ℓ_k. By the induction hypothesis, $\mathscr{T}$ exchanges r_{ℓ_k+m} and $r_{m+1-\ell_k}$, so the extensions of $\mathscr{T}$ and ρ are continuous on the "long" sides of r_{2m} and r_1 containing b_i and $b_{\tau(i)}$, respectively. The proof in the case $\varepsilon = -$ is the same. A similar analysis of $\mathscr{P}(b_j)$ and $\mathscr{P}(b_{\tau(j)})$ for $\delta = +$ and $\delta = -$ shows that $\mathscr{T}$ and ρ are continuous on the other sides of r_{2m} and r_1, too. $\quad\square$

3. Algebraic Model and Twisted Cut-and-join Equation

3.1. Algebraic preliminaries and twisted Hurwitz numbers. Recall [5] that the reflection group B_n is the semidirect product $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ where S_n acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by permuting the factors; it is generated by reflections $s_{ij} \overset{\text{def}}{=} (ij) \ltimes \mathbf{1}$ and $l_i \overset{\text{def}}{=} \text{id} \ltimes \mathbf{1}_i$ where $\mathbf{1} \overset{\text{def}}{=} (1,\dots,1)$ and $\mathbf{1}_i \overset{\text{def}}{=} (1,\dots,-1,\dots,1)$ with -1 at the ith place. The group B_n can be embedded into S_{2n} as a centralizer $C(\tau)$ where τ, as above, is defined by (1.2); the isomorphism $A: C(\tau) \to B_n$ is given by $A(\sigma) = \lambda_\sigma \ltimes (\varepsilon_\sigma^{(1)},\dots,\varepsilon_\sigma^{(n)})$ where

$$\lambda_\sigma(i) = \begin{cases} \sigma(i), & \sigma(i) \le n, \\ \sigma(i) - n, & \sigma(i) \ge n+1 \end{cases} \qquad \text{and} \qquad \varepsilon_\sigma^{(i)} = \begin{cases} 1, & \sigma(i) \le n, \\ -1, & \sigma(i) \ge n+1. \end{cases}$$

Let $C^{\sim}(\tau) \overset{\text{def}}{=} \{\sigma \in S_{2n} \mid \tau\sigma = \sigma^{-1}\tau\}$ (a "twisted centralizer" of τ).

Lemma 14. *Let $\sigma = c_1 \dots c_m \in C^{\sim}(\tau)$ where $c_1,\dots,c_m$ are independent cycles. Then for every i either*

- *there exists $j \ne i$ such that $c_i = (u_1 \dots u_k)$ and $c_j = (\tau(u_k) \dots \tau(u_1))$; or*
- *c_i has even length $2k$ and looks like $c_i = (u_1 \dots u_k \tau(u_k) \dots \tau(u_1))$.*

In the first case we say that the cycles c_i and c_j are τ-symmetric, and in the second case the cycle c_i is τ-self-symmetric.

Proof. Let $c_i = (u_1^{(i)} \dots u_{k_i}^{(i)})$ for all $i = 1,\dots,m$. Then $\tau\sigma\tau^{-1} = c_1' \dots c_m'$ where $c_i' = (\tau(u_1^{(i)}) \dots \tau(u_{k_i}^{(i)}))$. On the other side, $\sigma^{-1} = c_1'' \dots c_m''$ where $c_i'' = (u_{k_i}^{(i)} \dots u_1^{(i)})$. Once a cycle decomposition is unique, every c_i'' must be equal to some c_j'. If $j \ne i$ then c_i and c_j are τ-symmetric, and if $j = i$ then c_i is τ-self-symmetric. $\qquad\square$

Theorem 15. *There exists a one-to-one correspondence between the following three sets:*

1. *The quotient (the set of left cosets) S_{2n}/B_n where we assume $B_n = C(\tau)$;*
2. *The set $\widetilde{B_n}$ of permutations $\sigma \in C^{\sim}(\tau)$ such that their cycle decomposition contains no τ-self-symmetric cycles.*
3. *The set of fixed-point-free involutions $\lambda \in S_{2n}$.*

The size of each set is $(2n-1)!! = 1 \times 3 \times \cdots \times (2n-1)$.

Proof. To prove the theorem we will construct injective maps $1 \to 2 \to 3 \to 1$.

$1 \to 2$: let $\sigma \in S_{2n}$ be an element of the coset $\lambda \in S_{2n}/B_n$; take $Q(\lambda) \overset{\text{def}}{=} [\sigma,\tau] \overset{\text{def}}{=} \sigma\tau\sigma^{-1}\tau$. Since τ is an involution, one has $\tau Q(\lambda)\tau = \tau\sigma\tau\sigma^{-1} = Q(\lambda)^{-1}$, so $Q(\lambda) \in C^{\sim}(\tau)$. If $\sigma' \in \lambda$ is another element of the coset then $\sigma' = \sigma\rho$ where $\rho\tau = \tau\rho$ and therefore $[\sigma',\tau] = \sigma\rho\tau\rho^{-1}\sigma^{-1}\tau = \sigma\tau\rho\rho^{-1}\sigma^{-1}\tau = Q(\lambda)$, so the map $Q: S_{2n}/B_n \to C^{\sim}(\tau)$ is well-defined. If $Q(\lambda) = Q(\lambda')$ where $\lambda,\lambda' \in S_{2n}/B_n$ are represented by σ and σ', respectively, then $\sigma\tau\sigma^{-1}\tau = \sigma'\tau(\sigma')^{-1}\tau$, which is equivalent to $(\sigma')^{-1}\sigma\tau = \tau(\sigma')^{-1}\sigma$. So $(\sigma')^{-1}\sigma \in B_n$, and $\lambda = \lambda'$, so Q is injective.

Prove that actually $Q(\lambda) \in \widetilde{B_n} \subset C^{\sim}(\tau)$. Suppose it is not the case, that is, $Q(\lambda)$ contains a τ-self-symmetric cycle $c = (u_1 \dots u_k \tau(u_k) \dots \tau(u_1))$. It implies that $\tau Q(\lambda)$ has a fixed point $u = u_k$. On the other hand, $\tau Q(\lambda) = (\tau\sigma)\tau(\tau\sigma)^{-1}$ is conjugate to τ and is therefore a product of n independent transpositions having no fixed points—a contradiction.

$2 \to 3$: the condition $\tau\sigma\tau^{-1} = \sigma^{-1}$ is equivalent to $(\sigma\tau)^2 = \text{id}$. If $\sigma = c_1 c_2 \dots c_k$ then $\sigma\tau$ sends every element of the cycle c_i to an element of its τ-symmetric cycle c_j. So if $j \ne i$ for all i then the involution $\sigma\tau$ has no fixed points. The map $\sigma \mapsto \sigma\tau$ is obviously injective.

$3 \to 1$: if ψ is a fixed-point-free involution then its cycle decomposition is a product of n independent transpositions, and therefore ψ belongs to the same conjugacy class

in S_{2n} as τ: $\psi = \sigma\tau\sigma^{-1}$ for some $\sigma \in S_{2n}$. Denote by $R(\psi) \in S_{2n}/B_n$ the left coset containing σ. The equality $\sigma_1\tau\sigma_1^{-1} = \sigma_2\tau\sigma_2^{-1}$ is equivalent to $(\sigma_1\sigma_2^{-1})\tau = \tau(\sigma_1\sigma_2^{-1})$, that is, $\sigma_1\sigma_2^{-1} \in B_n$. So the left cosets containing σ_1 and σ_2 are the same and $R(\psi) \in S_{2n}/B_n$ is well-defined. If $R(\psi_1) = R(\psi_2)$ where $\psi_i = \sigma_i\tau\sigma_i^{-1}$, $i = 1,2$, then $\sigma_1\sigma_2^{-1} \in B_n$ and therefore $\psi_1 = \psi_2$; thus, R is an injective map. □

Fix a partition $\lambda = (\lambda_1,\ldots,\lambda_s)$, $|\lambda| = n$, and denote by $B_\lambda^\sim \subset B_n^\sim$ the set of permutations whose decomposition into independent cycles consists of s pairs of τ-symmetric cycles of lengths $\lambda_1,\ldots,\lambda_s$. Apparently, $B_n^\sim = \bigsqcup_{|\lambda|=n} B_\lambda^\sim$.

Proposition 16. $B_\lambda^\sim$ *is a B_n-conjugacy class in S_{2n}.*

Proof. Let $\sigma = c_1 c_1' \ldots c_s c_s' \in B_n^\sim$ be the cycle decomposition where c_i and c_i' are τ-symmetric for all i: $c_i' = \tau c_i^{-1}\tau$. Let $x \in B_n$, that is, $x\tau = \tau x$. Then $x\sigma x^{-1} = xc_1 x^{-1} \cdot x\tau c_1^{-1}\tau x^{-1} \cdot \cdots \cdot xc_s x^{-1} \cdot x\tau c_s^{-1}\tau x^{-1}$. The permutations $\tilde{c}_i \overset{\text{def}}{=} xc_i x^{-1}$ and $\tilde{c}_i' = xc_i' x^{-1}$ are cycles of length λ_i, and they are τ-symmetric: $\tau\tilde{c}_i\tau = \tau xc_i x^{-1}\tau = x\tau c_i\tau x^{-1} = x(c_i')^{-1}x^{-1} = (\tilde{c}_i')^{-1}$. Thus, $x\sigma x^{-1} \in B_\lambda^\sim$.

On the other side, let $\tilde{\sigma} = \tilde{c}_1\tilde{c}_1'\ldots\tilde{c}_s\tilde{c}_s' \in B_\lambda^\sim$. Let $\tilde{c}_i = (v_1^{(i)}\ldots v_{\lambda_i}^{(i)})$, so $\tilde{c}_i' = (\tau(v_{\lambda_i}^{(i)})\ldots\tau(v_1^{(i)}))$. Define an element $x \in S_{2n}$ such that $x(u_s^{(i)}) = v_s^{(i)}$ and $x(\tau(u_s^{(i)})) = \tau(v_s^{(i)})$. Then $x\sigma x^{-1} = \tilde{\sigma}$ and $x\tau = \tau x$ (that is, $x \in B_n$). □

Now denote by $\mathfrak{S}_{m,\lambda}$ the set of decompositions into m ribbons of the surfaces $M \in \mathscr{DBS}_n$ such that ∂M has s components containing $\lambda_1,\ldots,\lambda_s$ marked points.

Let $\mathscr{G} \in \mathfrak{S}_{m,\lambda}$ be a ribbon decomposition of $M \in \mathscr{DBS}_n$. Denote by $\widehat{M} \in \mathscr{DBS}_{2n}$ the orientation cover of M with a ribbon decomposition given by (2.3). Now define $\Xi(\mathscr{G}) \overset{\text{def}}{=} (\sigma_1,\ldots,\sigma_m)$ where each $\sigma_k \overset{\text{def}}{=} (i_k^{\varepsilon_k}, j_k^{\delta_k}) \in S_{2n}$ is a transposition; here we are using the notation of Theorem 13. Denote

$$\mathfrak{H}_{m,\lambda} \overset{\text{def}}{=} \big\{(\sigma_1,\ldots,\sigma_m) \big| \forall s = 1,\ldots,m\; \sigma_s = (i_s j_s), j_s \neq \tau(i_s),$$
$$\sigma_1\sigma_2\ldots\sigma_m(\tau\sigma_m\tau)\ldots(\tau\sigma_1\tau) \in B_\lambda^\sim\big\}.$$

Theorem 17. *For any λ and m the map Ξ is a one-to-one correspondence between $\mathfrak{S}_{m,\lambda}$ and $\mathfrak{H}_{m,\lambda}$.*

Proof. Let $\mathscr{G} \in \mathfrak{S}_{m,\lambda}$ be a ribbon decomposition of $M \in \mathscr{DBS}_n$. By Theorem 13 the diagonal of the kth ribbon in the ribbon decomposition (2.3) of $\widehat{M}$ joins the marked points numbered $i_k^{\varepsilon_k}$ and $j_k^{\delta_k}$ if $1 \le k \le m$ and the points numbered $i_{k-m}^{-\varepsilon_{k-m}}$ and $j_{k-m}^{-\delta_{k-m}}$ if $m+1 \le k \le 2m$. Then by Property 4 of Theorem 11 the boundary permutation of $\widehat{M}$ is

$$\sigma = \sigma_1\sigma_2\ldots\sigma_m(\tau\sigma_m\tau)\ldots(\tau\sigma_1\tau);$$

it has the cyclic type $(\lambda_1,\lambda_1,\ldots,\lambda_s,\lambda_s)$ by the definition of $\mathfrak{S}_{m,\lambda}$. On the other hand, $\tau\sigma\tau = (\tau\sigma_1\tau)\ldots(\tau\sigma_m\tau)\sigma_m\ldots\sigma_1 = \sigma^{-1}$ because σ_k and $\tau\sigma_k\tau$ are involutions for all k. Thus, $\sigma \in B_n^\sim$, hence $\sigma \in B_\lambda^\sim$, and so $\Xi(\mathscr{G}) \in \mathfrak{H}_{m,\lambda}$.

The map Ξ is obviously one-to-one. □

Corollary 18. *The boundary permutation of the orientation cover $\widehat{M} \in \mathscr{DBS}_{2n}$ of any $M \in \mathscr{DBS}_n$ belongs to $B_n^\sim$.*

Corollary 19. *If the twisted Hurwitz number $h_{m,\lambda}^\sim$ is defined by equation (1.1) then equality (1.4) takes place.*

Example 20. For $m = 1$, any $n = |\lambda|$ and any transposition $\sigma \neq (i, i+n)$ the permutation $\mu = \sigma\tau\sigma\tau$ belongs to $B_{n,2^1 1^{n-2}}$. Now $\#\mathfrak{H}_{1,2^1 1^{n-2}}$ is the total number of all transpositions

$\sigma \in S_{2n}$ except $(i, i+n)$, which is $\frac{1}{2}(2n)(2n-1) - n = 2n(n-1)$. So, $h_{1,2^1 1^{n-2}} = \frac{2n(n-1)}{n!} = \frac{2}{(n-2)!}$ and $h_{1,\lambda} = 0$ for all other λ.

Let $m = 2$, $n = |\lambda| = 2$; here $\tau = (13)(24) \in S_4$. The set $B_2^{\sim} \subset B_2 = C(\tau) \subset S_4$ is a union of two conjugacy classes, $B_{[2]}^{\sim} = \{(12)(34), (14)(23)\}$ and $B_{[1,1]}^{\sim} = \{e\}$.

Consider the permutation $\mu \overset{\text{def}}{=} \sigma_1 \sigma_2 \tau \sigma_2 \sigma_1 \tau$ where $\sigma_1, \sigma_2 \in \{(12), (14), (23), (34)\}$; totally, there are 16 of them. It is easy to see that $\mu = e \in B_{[1,1]}^{\sim}$ if and only if $\sigma_2 = \sigma_1$ or $\sigma_2 = \tau \sigma_1 \tau$; the remaining 8 pairs of transpositions (σ_1, σ_2) give $\mu \in B_{[2]}^{\sim}$. This gives $h_{2,[1,1]}^{\sim} = h_{2,[2]}^{\sim} = \frac{8}{2!} = 4$.

For $n = 3$, $m = 2$ the calculations (in S_6) are similar but more cumbersome, giving eventually $h_{2,[1,1,1]}^{\sim} = h_{2,[2,1]}^{\sim} = 4$ and $h_{2,[3]}^{\sim} = 16$.

3.2. Twisted cut-and-join operator. Now denote

$$(3.1) \qquad \mathscr{C}_{\lambda}^{\sim} \overset{\text{def}}{=} \sum_{\sigma \in B_{\lambda}^{\sim}} \sigma \in \mathbb{C}[B_n^{\sim}].$$

(a conjugacy class sum). Also, call the set

$$\mathscr{Z}(B_n^{\sim}) \overset{\text{def}}{=} \left\{ y \in \mathbb{C}[B_n^{\sim}] \,\middle|\, x y x^{-1} = y \,\forall x \in B_n \right\}$$

a *twisted center* of B_n. It is clear that $\mathscr{C}_{\lambda}^{\sim}$ belong to $\mathscr{Z}(B_n^{\sim})$ and form a basis in it.

Let $\mathbb{C}[p]$ be a space of polynomials of the countable set of variables $p = (p_1, p_2, \ldots)$. Assume $\deg p_k = k$ for all k and denote by $\mathbb{C}[p]_n$ the space of homogeneous polynomials of degree n. A linear map $\Psi : \mathscr{Z}(B_n^{\sim}) \to \mathbb{C}[p]_n$ defined by

$$(3.2) \qquad \Psi(\mathscr{C}_{\lambda}^{\sim}) = p_{\lambda} \overset{\text{def}}{=} p_{\lambda_1} \ldots p_{\lambda_s}$$

is obviously an isomorphism of vector spaces.

Define an operator $\mathfrak{CJ}^{\sim} : \mathscr{Z}(B_n^{\sim}) \to \mathscr{Z}(B_n^{\sim})$ by

$$\mathfrak{CJ}^{\sim}(\sigma) = \sum_{\substack{1 \le i < j \le 2n \\ j \neq \tau(i)}} (ij)\sigma(\tau(i)\tau(j)).$$

Definition 21. The *twisted cut-and-join operator* is a linear map $\mathscr{CJ}^{\sim} : \mathbb{C}[p]_n \to \mathbb{C}[p]_n$ making the following diagram commutative:

$$(3.3) \qquad \begin{array}{ccc} \mathscr{Z}[B_n^{\sim}] & \xrightarrow{\mathfrak{CJ}^{\sim}} & \mathscr{Z}[B_n^{\sim}] \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Psi} \\ \mathbb{C}[p]_n & \xrightarrow[\mathscr{CJ}^{\sim}]{} & \mathbb{C}[p]_n \end{array}$$

Let λ, μ be partitions such that $|\lambda| = |\mu| = n$. Take an element $\sigma \in B_{\lambda}^{\sim}$ and consider a set

$$S(\sigma; \mu) \overset{\text{def}}{=} \left\{ (i,j) \,\middle|\, 1 \le i, j \le 2n, j \neq i, \tau(i), (ij)\sigma(\tau(i)\tau(j)) \in B_{\mu}^{\sim} \right\}.$$

Proposition 16 implies that for every $x \in B_n$ and $\sigma \in B_{\lambda}^{\sim}$ the map $(i, j) \mapsto (x(i), x(j))$ is a bijection between $S(x\sigma x^{-1}, \mu)$ and $S(\sigma, \mu)$. So, the size of the set $S(\sigma, \mu)$ for $\sigma \in B_{\lambda}^{\sim}$ depends on λ and μ only.

We will be using "physical" notation for matrix elements of a linear operator: $\mathfrak{CJ}^{\sim}(\mathscr{C}_{\lambda}^{\sim}) = \sum_{\mu} \langle \lambda | \mathfrak{CJ}^{\sim} | \mu \rangle \mathscr{C}_{\mu}^{\sim}$.

Theorem 22. $\langle \lambda | \mathfrak{CJ}^{\sim} | \mu \rangle = \frac{1}{2} \#S(\sigma, \mu)$ *for any* $\sigma \in B_{\lambda}^{\sim}$.

Proof. By definition,

$$(3.4) \qquad \mathfrak{CJ}^{\sim}(\mathscr{C}_{\lambda}^{\sim}) = \sum_{\sigma \in B_{\lambda}^{\sim}} \mathfrak{CJ}^{\sim}(\sigma) = \sum_{\sigma \in B_{\lambda}^{\sim}} \sum_{\substack{1 \le i < j \le 2n \\ j \neq \tau(i)}} (ij)\sigma(\tau(i)\tau(j)).$$

As it was noted above, (3.4) is a sum of identical summands, so

$$\mathfrak{CJ}^{\sim}(\mathscr{C}_{\lambda}^{\sim}) = \#B_{\lambda}^{\sim} \sum_{\mu} \sum_{\substack{1 \le i < j \le 2n \\ j \ne \tau(i) \\ (ij)\sigma(\tau(i)\tau(j)) \in B_{\mu}^{\sim}}} (ij)\sigma(\tau(i)\tau(j)).$$

for any fixed $\sigma \in B_{\lambda}^{\sim}$, and therefore

$$\mathfrak{CJ}^{\sim}(\mathscr{C}_{\lambda}^{\sim}) = \sum_{\mu} \sum_{\substack{1 \le i < j \le 2n \\ j \ne \tau(i) \\ (ij)\sigma(\tau(i)\tau(j)) \in B_{\mu}^{\sim}}} \sum_{\tau \in B_{\mu}^{\sim}} \tau$$

$$= \frac{1}{2} \sum_{\mu} \# \left\{ (i,j) \,\middle|\, j \ne i, \tau(i), (ij)\sigma(\tau(i)\tau(j)) \in B_{\mu}^{\sim} \right\} \mathscr{C}_{\mu}^{\sim}. \qquad \square$$

Consider the generating function $\mathscr{H}^{\sim}(\beta, p)$ of the twisted Hurwitz numbers defined as follows:

$$(3.5) \qquad \mathscr{H}^{\sim}(\beta, p) = \sum_{m \ge 0} \sum_{\lambda} \frac{h_{m,\lambda}^{\sim}}{m!} p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_s} \beta^m.$$

Theorem 23. $\mathscr{H}^{\sim}$ *satisfies the cut-and-join equation* $\frac{\partial \mathscr{H}^{\sim}}{\partial \beta} = \mathscr{C}\mathscr{J}^{\sim}(\mathscr{H}^{\sim})$.

Proof. Fix a positive integer n and denote by $\mathscr{H}_n^{\sim}$ a degree n homogeneous component of $\mathscr{H}^{\sim}$. The twisted cut-and-join operator preserves the degree, so $\mathscr{H}^{\sim}$ satisfies the cut-and-join equation if and only if $\mathscr{H}_n^{\sim}$ does (for each n).

Let

$$\mathscr{G}_n \stackrel{\text{def}}{=} \sum_{m \ge 0} \sum_{\lambda : |\lambda| = n} \frac{n! h_{m,\lambda}^{\sim}}{m!} \mathscr{C}_{\lambda}^{\sim} \beta^m \in \mathbb{C}[S_{2n}],$$

where $\mathscr{C}_{\lambda}^{\sim}$ is defined by (3.1). An elementary combinatorial reasoning gives

$$\mathscr{G}_n = \sum_{m \ge 0} \frac{\beta^m}{m!} (\mathfrak{CJ}^{\sim})^m (e_{2n}),$$

where $e_{2n} \in S_{2n}$ is the unit element. Clearly

$$\mathfrak{CJ}^{\sim}(\mathscr{G}_n) = \sum_{m \ge 0} \frac{\beta^m}{m!} (\mathfrak{CJ}^{\sim})^{m+1}(e_{2n}) = \sum_{m \ge 1} \frac{\beta^{m-1}}{(m-1)!} (\mathfrak{CJ}^{\sim})^m (e_{2n}) = \frac{\partial \mathscr{G}_n}{\partial \beta}.$$

Applying Ψ one obtains $\Psi \mathfrak{CJ}^{\sim}(\mathscr{G}_n) = \Psi(\frac{\partial \mathscr{G}_n}{\partial \beta}) = \frac{\partial}{\partial \beta} \Psi(\mathscr{G}_n)$. By (3.2), $\Psi(\mathscr{G}_n) = \mathscr{H}_n^{\sim}$, hence $\frac{\partial}{\partial \beta} \Psi(\mathscr{G}_n) = \frac{\partial \mathscr{H}_n^{\sim}}{\partial \beta}$. By the definition of the twisted cut-and-join operator, $\Psi \mathfrak{CJ}^{\sim}(\mathscr{G}_n) = \mathscr{C}\mathscr{J}^{\sim}(\Psi(\mathscr{G}_n)) = \mathscr{C}\mathscr{J}^{\sim}(\mathscr{H}_n^{\sim})$, and the equality $\frac{\partial \mathscr{H}_n^{\sim}}{\partial \beta} = \mathscr{C}\mathscr{J}^{\sim}(\mathscr{H}_n^{\sim})$ follows. $\qquad \square$

Corollary 24. $\mathscr{H}^{\sim}(\beta, p) = \exp(\beta \mathscr{C}\mathscr{J}^{\sim}) \exp(p_1)$.

Proof. It follows from (1.1) that $h_{0,\lambda} = \frac{1}{n!}$ if $\lambda = 1^n$ and $h_{0,\lambda} = 0$ otherwise. Thus, $\mathscr{H}^{\sim}(0, p) = \exp(p_1)$, and the formula follows from Theorem 23. $\qquad \square$

3.3. Explicit formulas. In this section we prove explicit formulas for the cut-and-join operator (Theorem 25) and for twisted Hurwitz numbers (Theorem 28).

Theorem 25. *The twisted cut-an-join operator is given by*

$$\mathscr{C}\mathscr{J}^{\sim} = \sum_{i,j \ge 1} (i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} + 2ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} + \sum_{k \ge 1} k(k-1) p_k \frac{\partial}{\partial p_k}$$

$$= \sum_{i,j \ge 1} (i+j)(p_i p_j + p_{i+j}) \frac{\partial}{\partial p_{i+j}} + 2ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j}.$$
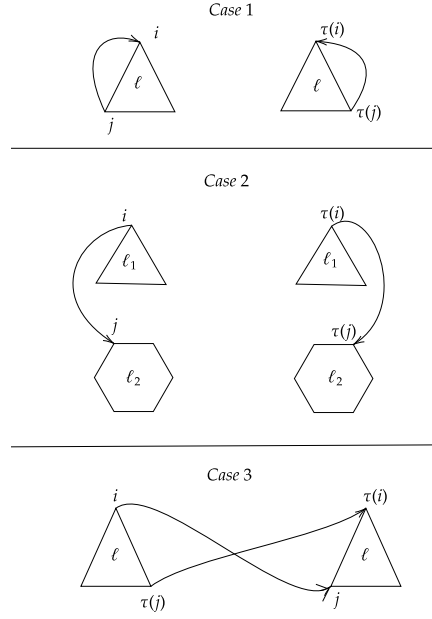
FIGURE 4. Terms of $\mathfrak{C}\mathfrak{J}^{\sim}$

(The two formulas are equivalent because there are $k-1$ pairs (i, j) such that $i, j \geq 1$ and $i + j = k$.)

To prove Theorem 25 we calculate explicitly the matrix elements $\langle \lambda \,|\, \mathfrak{C}\mathfrak{J}^{\sim} \,|\, \mu \rangle$ for all possible λ, μ.

Let $\sigma \in S_n$ and $1 \leq i < j \leq n$. The cycle structure of the product $\sigma' = (i\,j)\sigma$ depends on the cycle structure of σ and on i and j as follows: if i and j belong to the same cycle $(x_1, \ldots, x_\ell)$ of σ (where $i = x_1$, $j = x_m$), then σ' contains two cycles $(x_1, \ldots, x_{m-1})$ and $(x_m, \ldots, x_\ell)$ instead ("a cut"). If i and j are in different cycles $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_k)$ (where $i = x_1$ and $j = y_1$) then σ' contains the cycle $(x_1, \ldots, x_m, y_1, \ldots, y_k)$ instead ("a join").

Let now $\sigma \in B_\lambda^{\sim} \subset B_n^{\sim}$ where $\lambda = 1^{a_1} 2^{a_2} \ldots n^{a_n}$ (in other words, the element $\sigma \in S_{2n}$ contains a_s pairs of τ-symmetric cycles of length s for $s = 1, \ldots, n$). Let $1 \leq i < j \leq 2n$, $j \neq \tau(i)$ and $\sigma' \overset{\text{def}}{=} (i\,j)\sigma(\tau(i)\tau(j)) \in B_\mu^{\sim}$. The cyclic structure of σ' depends on the positions of i, j, $\tau(i)$, $\tau(j)$ and on the cycles of σ; there are three possible cases shown in Figure 4.

Case 1. Here μ is obtained from λ by a cut:

$$\mu = 1^{a_1} \ldots m^{a_m+1} \ldots k^{a_k+1} \ldots \ell^{a_\ell-1} \ldots n^{a_n}, \quad m + k = \ell, m < k,$$

or

$$\mu = 1^{a_1} \ldots m^{a_m+2} \ldots \ell^{a_\ell-1} \ldots n^{a_n}, \qquad m = \ell/2.$$

For a fixed $\sigma \in B_\lambda^{\sim}$ look for i, j such that $\sigma' \overset{\text{def}}{=} (i\,j)\sigma(\tau(i)\tau(j)) \in B_\mu^{\sim}$. The element σ contains $2a_\ell$ cycles of length ℓ, so there are $2\ell a_\ell$ possible positions for i. In σ' the elements i and j are in different cycles; if $m < k$ then m may be the length of either. So if $m < k$ then j should be in the same cycle in σ as i, and the distance between them is either m or k. So there are two possible positions for j once i is chosen, and $\langle \mu \,|\, \mathfrak{C}\mathfrak{J}^{\sim} \,|\, \lambda \rangle = \frac{1}{2}\#S(\sigma, \mu) = 2\ell a_\ell$. If $m = k = \ell/2$ then the position for j is unique and $\langle \mu \,|\, \mathfrak{C}\mathfrak{J}^{\sim} \,|\, \lambda \rangle = \ell a_\ell$.

Similar calculations are used in Case 2.

Case 2. Here μ is obtained from λ by a join:

$$\mu = 1^{a_1}\ldots m^{a_m-1}\ldots k^{a_k-1}\ldots \ell^{a_\ell+1}\ldots n^{a_n}, \quad m+k=\ell, m<k,$$

or

$$\mu = 1^{a_1}\ldots m^{a_m-2}\ldots \ell^{a_\ell+1}\ldots n^{a_n}, \qquad m=\ell/2$$

If $m < k$ then i may belong to the cycle of either length. If i belongs to the cycle of length m then there are $2ma_m$ possible positions for it (cf. Case 1) and $2ka_k$ positions for j; vice versa if i belongs to the cycle of length k. The matrix element is then $\langle \mu | \mathfrak{CJ}^\sim | \lambda \rangle = 4mka_ma_k$. If $m = k = \ell/2$ then i and j belong to cycles of the same length m; the cycle containing j contains neither i nor $\tau(i)$. Hence there are $4a_m(a_m-1)$ possibilities for choosing a pair of cycles to contain i and j, and m^2 possible positions for i and j in them. Therefore $\langle \mu | \mathfrak{CJ}^\sim | \lambda \rangle = 2m^2 a_m(a_m-1)$.

Case 3. Here $\mu = \lambda$. Like in the previous cases we have $2\ell a_\ell$ possible positions for i and $\ell-1$ positions for $j \neq \tau(i)$ (in the cycle τ-symmetric to the one containing i) once i is fixed. Thus, $\langle \mu | \mathfrak{CJ}^\sim | \lambda \rangle = \sum_\ell 2\ell(\ell-1)a_\ell$.

Proof of Theorem 25. It follows from Theorem 22 and Definition 21 that $\mathscr{CJ}^\sim p_\lambda = \sum_\mu \langle \lambda | \mathfrak{CJ}^\sim | \mu \rangle p_\mu$.

For a given λ there are three types of μ such that $\langle \lambda | \mathfrak{CJ}^\sim | \mu \rangle \neq 0$ listed above. Hence $\mathscr{CJ}^\sim$ is a sum of three terms.

Suppose μ is like in Case 1 with $m < k$. The monomial p_λ contains $p_m^{a_m} p_k^{a_k} p_\ell^{a_\ell}$ and the monomial p_μ contains $p_m^{a_m+1} p_k^{a_k+1} p_\ell^{a_\ell-1}$; the other factors are the same. So the term in (25) acting on p_λ and giving p_μ is $2\ell p_m p_k \frac{\partial}{\partial p_\ell} p_\lambda = 2\ell a_\ell p_\mu = \langle \mu | \mathfrak{CJ}^\sim | \lambda \rangle p_\mu$ (actually there are two equal terms in the sum: $i=m, j=k$ or vice versa, hence the factor 2).

If μ is like in Case 1 with $m=\ell/2$ then p_λ contains $p_{\ell/2}^{a_{\ell/2}} p_\ell^{a_\ell}$ and μ contains $p_{\ell/2}^{a_{\ell/2}+2} p_\ell^{a_\ell-1}$. So the only term in (25) acting on p_λ and giving p_μ is

$$\ell p_{\ell/2}^2 \frac{\partial}{\partial p_\ell} p_\lambda = \ell a_\ell p_\mu = \langle \mu | \mathfrak{CJ}^\sim | \lambda \rangle p_\mu.$$

The calculations for the two remaining cases are similar. $\square$

By Theorem 25, $\mathscr{CJ}^\sim = \mathscr{CJ}_0 + \mathscr{R}$ where

$$\mathscr{CJ}_0 = \sum_{i,j \geq 1} (i+j) p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j}$$

is the classical cut-and-join, and

$$\mathscr{R} = \sum_{i,j \geq 1} p_{i+j} \left((i+j) \frac{\partial}{\partial p_{i+j}} + ij \frac{\partial^2}{\partial p_i \partial p_j} \right).$$

The one-parametric family

$$\Delta_\alpha \overset{\text{def}}{=} \mathscr{CJ}_0 + (\alpha-1)\mathscr{R} = \sum_{i,j \geq 1} (i+j)\big(p_i p_j + (\alpha-1)p_{i+j}\big) \frac{\partial}{\partial p_{i+j}} + \alpha ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j}$$

is called [9] the Laplace–Beltrami operator; in particular, $\Delta_1 = \mathscr{CJ}_0$ is the classical cut-and-join and $\Delta_2 = \mathscr{CJ}^\sim$ the twisted cut-and-join. By the classical results of [9, p. 376 and after], the eigenvalues (and eigenvectors) of Δ_α are indexed by partitions. The eigenvalue corresponding to $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ is equal to

$$e(\lambda, \alpha) = \sum_{i=1}^s \lambda_i(\alpha \lambda_i + 2 - 2i - \alpha).$$

The corresponding eigenvector is a polynomial $J_\lambda^{(\alpha)}(p)$ of degree $|\lambda| \stackrel{\text{def}}{=} \lambda_1 + \cdots + \lambda_s$ called Jack polynomial; it is normalized so that the coefficient at p_1^n in it is 1. Polynomials $Z_\lambda \stackrel{\text{def}}{=} J_\lambda^{(2)}$ are called zonal.

Theorem 26 ([9])**.** *We have*

$$\sum_\lambda \frac{J_\lambda^{(\alpha)}(p) J_\lambda^{(\alpha)}(q)}{H_\lambda(\alpha) H'_\lambda(\alpha)} = \exp\left(\sum_{k \geq 1} \frac{p_k q_k}{k\alpha}\right),$$

where $H_\lambda(\alpha) \stackrel{\text{def}}{=} \prod_{(i,j) \in Y(\lambda)} (\alpha a(i,j) + \ell(i,j) + 1)$ and $H'_\lambda(\alpha) \stackrel{\text{def}}{=} \prod_{(i,j) \in Y(\lambda)} (\alpha a(i,j) + \ell(i,j) + \alpha)$. Here $Y(\lambda)$ is the Young diagram of the partition λ, and $a(i,j)$ and $\ell(i,j)$ are the arm and the leg, respectively, of the cell $(i,j) \in Y(\lambda)$.

Substituting $q_1 = \alpha, q_2 = q_3 = \cdots = 0$ and taking into account the normalization of the Jack polynomials one obtains

Corollary 27. $\sum_\lambda \dfrac{\alpha^{|\lambda|} J_\lambda^{(\alpha)}(p)}{H_\lambda(\alpha) H'_\lambda(\alpha)} = \exp(p_1).$

Taking now $\alpha = 2$ and substituting the formula of Corollary 27 into Corollary 24, one obtains

Theorem 28. $\widetilde{\mathscr{H}}(\beta, p) = \sum_\lambda \exp\left(2\beta \sum_i \lambda_i(\lambda_i - i)\right) \dfrac{2^{|\lambda|} Z_\lambda(p)}{H_\lambda(2) H'_\lambda(2)}.$

This is a twisted analog of the formula expressing the usual Hurwitz numbers via Schur polynomials, see [6].

Example 29. The zonal polynomials Z_λ for small λ are

$$Z_{[1]} = p_1, \text{ with } H_{[1]}(2) H'_{[1]}(2) = 2$$
$$Z_{[1,1]} = p_1^2 - p_2 \text{ with } H_{[1,1]}(2) H'_{[1,1]}(2) = 12$$
$$Z_{[2]} = p_1^2 + 2p_2 \text{ with } H_{[2]}(2) H'_{[2]}(2) = 24$$
$$Z_{[1,1,1]} = p_1^3 - 3p_2 p_1 + 2p_3 \text{ with } H_{[1,1,1]}(2) H'_{[1,1,1]}(2) = 144$$
$$Z_{[2,1]} = p_1^3 + p_2 p_1 - 2p_3 \text{ with with } H_{[2,1]}(2) H'_{[2,1]}(2) = 80$$
$$Z_{[3]} = p_1^3 + 6p_2 p_1 + 8p_3 \text{ with } H_{[3]}(2) H'_{[3]}(2) = 720$$

This gives us the first few terms in the expansion of $\widetilde{\mathscr{H}}(\beta, p)$:

$$\widetilde{\mathscr{H}}(\beta, p) = p_1 + \frac{p_1^2}{6}(2e^{-2\beta} + e^{4\beta}) + \frac{p_2}{3}(-e^{-2\beta} + e^{4\beta}) + \frac{p_1^3}{90}(9e^{2\beta} + e^{12\beta} + 5e^{-6\beta})$$
$$+ \frac{p_2 p_1}{30}(2e^{12\beta} + 3e^{2\beta} - 5e^{-6\beta}) + \frac{p_3}{45}(4e^{12\beta} - 9e^{2\beta} + 5e^{-6\beta}) + \dots$$
$$= \left(p_1 + \frac{p_1^2}{2} + \frac{p_1^3}{6} + \dots\right) + \beta(2p_1 + 2p_1 p_2 + \dots)$$
$$+ \beta^2(2p_1^2 + 2p_2 + 2p_1^3 + 2p_1 p_2 + 8p_3 + \dots) + \dots;$$

they agree with (3.5) and Example 20.

4. Algebro-geometric Model: Twisted Branched Coverings

A classical notion of the branched covering was extended to the non-orientable case by G. Chapuy and M. Dołęga in [2]. Let N denote a closed surface (compact 2-manifold without boundary, not necessarily orientable), and $p : \widehat{N} \to N$, its orientation cover. As above, denote by $\mathscr{T} : \widehat{N} \to \widehat{N}$ an orientation-reversing involution without fixed points such that $p \circ \mathscr{T} = p$. Also denote by $\mathscr{J} : \mathbb{C}P^1 \to \mathbb{C}P^1$ the complex conjugation, and let $\overline{\mathbb{H}} \stackrel{\text{def}}{=} \mathbb{C}P^1 / (z \sim \mathscr{J}(z)) = \mathbb{H} \cup \{\infty\}$ where $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane; $\overline{\mathbb{H}}$ is homeomorphic to a disk. Denote by $\pi : \mathbb{C}P^1 \to \overline{\mathbb{H}}$ the quotient map.

Definition 30 ([2])**.** A continuous map $f : N \to \overline{\mathbb{H}}$ is called a twisted branched covering if there exists a branched covering $\widehat{f} : \widehat{N} \to \mathbb{C}P^1$ such that

    (1) $\pi \circ \widehat{f} = f \circ p$, and
    (2) all the critical values of $\widehat{f}$ are real.

Property (1) is equivalent to saying that $\widehat{f}$ is a real map with respect to $\mathscr{T}$, that is, $\widehat{f} \circ \mathscr{T} = \mathscr{J} \circ \widehat{f}$. The involution $\mathscr{T}$ has no fixed points, so the critical points of $\widehat{f}$ come in pairs $(a, \mathscr{T}(a))$, the ramification profile of every critical value $c \in \mathbb{R}P^1 \subset \mathbb{C}P^1$ of $\widehat{f}$ has every part repeated twice: $(\lambda_1, \lambda_1, \dots, \lambda_s, \lambda_s)$, and $\deg \widehat{f} = 2n$ is even. In this case we say that the ramification profile of the critical value $\pi(c) \in \partial\overline{\mathbb{H}}$ of the map $f : N \to \overline{\mathbb{H}}$ is $\lambda = (\lambda_1, \dots, \lambda_s)$.

The twisted branched covering f is called simple if all its critical values, except possibly $\infty \in \overline{\mathbb{H}}$, have the ramification profile $2^1 1^{n-2}$. (Equivalently, each critical value of $\widehat{f}$ has 2 simple critical points and $2n-4$ regular points as preimages.) Let $u \in \partial\overline{\mathbb{H}}$ be a regular (not critical) value of f; then the preimage $f^{-1}(u) \subset N$ consists of n points. Fix a bijection $\nu : f^{-1}(u) \to \{1, \dots, n\}$ (a labeling); then the triple (f, u, ν) is called a labeled simple twisted branched covering.

Labeled simple twisted branched coverings are split into equivalence classes via right-left equivalence: $(f_1, u_1, \nu_1) \sim (f_2, u_2, \nu_2)$ if there exist orientation-preserving diffeomorphisms $D_1 : \widehat{N} \to \widehat{N}$ and $D_2 : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that

    • (f_1 transforms to f_2) $\widehat{f_1} \circ D_1 = D_2 \circ \widehat{f_2}$,
    • (D_1 and D_2 are equivariant) $\mathscr{T} \circ D_1 = D_1 \circ \mathscr{T}$ and $D_2 \circ \mathscr{J} = \mathscr{J} \circ D_2$,
    • (D_1, D_2 preserve labeling) $D_2(\pi^{-1}(u_1)) = \pi^{-1}(u_2)$ and $\nu_2 \circ D_1 = \nu_1$.

For an integer $m \geq 0$ and a partition λ denote by $\mathfrak{D}_{m,\lambda}$ the set of equivalence classes of labeled simple twisted branched coverings having m simple critical values and such that the ramification profile of ∞ is λ.

Theorem 31. $\#\mathfrak{D}_{m,\lambda} = \frac{1}{2^m}\#\mathfrak{S}_{m,\lambda} = \frac{1}{2^m}\#\mathfrak{H}_{m,\lambda} = \frac{n!}{2^m}h^{\sim}_{m,\lambda}$.

Proof. The generating function $\mathscr{D}(\beta, p) \overset{\text{def}}{=\joinrel=} \sum_{m \geq 0} \sum_{\lambda} \frac{\#\mathfrak{D}_{m,\lambda}}{n!m!} p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_s} \beta^m$ is shown in [2, Theorem 6.5 for $b = 1$] to satisfy the twisted cut-and-join equation

$$\frac{\partial \mathscr{D}}{\partial \beta} = \frac{1}{2}\mathscr{C}\mathscr{J}^{\sim}(\mathscr{D}),$$

where $\mathscr{C}\mathscr{J}^{\sim}$ is given by equation (25).

Let $m = 0$, so the branched covering $\widehat{f} \in \mathfrak{D}_{0,\lambda}$ is unramified except possibly over ∞. Denote by $N_0 \subset \widehat{N}$ any connected component of $\widehat{N}$, by n_0, the degree of $\widehat{f}|_{N_0}$, and $k \overset{\text{def}}{=} \#\widehat{f}|_{N_0}^{-1}(\infty)$. Then the Euler characteristic $\chi(\mathbb{C}P^1 \setminus \{\infty\}) = 1$ and therefore $\chi(N_0 \setminus \widehat{f}^{-1}(\infty)) = \chi(N_0) - k = n_0$. The set N_0 is a smooth compact 2-manifold, so $2 \geq \chi(N_0) = n_0 + k$, implying $n_0 = k = 1$. It means that $\widehat{f}$ is unramified over ∞, too, so $\lambda = 1^n$ and $\widehat{f}$ is a collection of n orientation-preserving diffeomorphisms of spheres. Obviously, $\widehat{f}$ is unique up to the right-left equivalence described above. Thus, $\#\mathfrak{D}_{0,1^n} = 1$ and $\#\mathfrak{D}_{0,\lambda} = 0$ for other λ. So, $\mathscr{D}(0, p) = \exp(p_1)$, and Corollary 24 implies that

$$\mathscr{D}(\beta, p) \equiv \mathscr{H}^{\sim}(\beta/2, p),$$

proving the theorem. $\square$

Remark. Note that, unlike Theorem 17, we do not know any "natural" map between the sets $\mathfrak{D}_{m,\lambda}$ and $\mathfrak{S}_{m,\lambda}$ (or $\mathfrak{H}^{\sim}_{m,\lambda}$). Finding one is a challenging topic of future research.

In [2], a one-parametric generalization of Hurwitz numbers is defined by counting twisted branched coverings with parameter-dependent weights. The parameter value

$b = 0$ gives classical Hurwitz numbers, and $b = 1$, twisted Hurwitz numbers. A natural one-to-one correspondence between $\mathfrak{D}_{m,\lambda}$ and $\mathfrak{S}_{m,\lambda}$ would allow to transfer these weights to ribbon decompositions and to define parametric Hurwitz numbers using them.

Note also that in [2], more general two-part Hurwitz numbers were studied; currently we do not know other models for them.

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