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# Classification of Axiom A Diffeomorphisms with Orientable Codimension One Expanding Attractors and Contracting Repellers

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**Abstract**—Let $\mathbb{G}_k^{cod1}(M^n)$, $k\geqslant 1$, be the set of axiom A diffeomorphisms such that the nonwandering set of any $f\in\mathbb{G}_k^{cod1}(M^n)$ consists of $k$ orientable connected codimension one expanding attractors and contracting repellers where $M^n$ is a closed orientable $n$-manifold, $n\geqslant 3$. We classify the diffeomorphisms from $\mathbb{G}_k^{cod1}(M^n)$ up to the global conjugacy on nonwandering sets. In addition, we show that any $f\in\mathbb{G}_k^{cod1}(M^n)$ is $\Omega$-stable and is not structurally stable. One describes the topological structure of a supporting manifold $M^n$.

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*To our friends, Sergey Gonchenko and Vladimir Belykh*

## 1. INTRODUCTION

Axiom A diffeomorphisms (in short, A-diffeomorphisms) were introduced in hyperbolic dynamics by Smale [32]. Recall that a nonwandering set of an A-diffeomorphism has a hyperbolic structure and the nonwandering set is the topological closure of the set of periodic orbits (for the basic notation of the theory of dynamical systems, see the books [1, 9, 15, 29] and surveys [8, 32]). By Smale's spectral decomposition theorem, a nonwandering set of an A-diffeomorphism is a disjoint union of closed invariant and transitive sets called *basic sets*.

A basic set $\Lambda_a$ of an A-diffeomorphism $f:M^n\to M^n$ is called an *attractor* if there is an attracting neighborhood $U\neq M^n$ of $\Lambda_a$ such that $\cap_{i\geqslant 0}f^i(U)=\Lambda_a$. Here, $M^n$ is a smooth $n$-manifold, $n\geqslant 2$. Following [34], we call $\Lambda_a$ an *expanding attractor* provided the topological dimension of $\Lambda_a$ equals Morse's index of $\Lambda_a$, i.e., $\dim\Lambda_a=\dim W^u(x)$ where $W^u(x)$ is the unstable manifold of (any) point $x\in\Lambda_a$. A basic set $\Lambda_r$ is called a *contracting repeller* if $\Lambda_r$ is the expanding attractor for $f^{-1}$. A basic set $\Omega$ is *codimension one* provided its topological dimension $\dim\Omega$ equals $n-1$. By Theorem C in [34], a codimension one expanding attractor (contracting repeller) is locally homeomorphic to the product of a Cantor set and a Euclidean plane $\mathbb{R}^{n-1}$.

A structurally stable $A$-diffeomorphism with an orientable codimension one expanding attractor can be obtained by Smale's surgery [32, pp. 788–789] from a codimension one Anosov diffeomorphism of an torus $\mathbb{T}^n$, $n\geqslant 2$ (the orientability of an expanding attractor, roughly speaking, means the following: given any arc of a stable manifold and a codimension one unstable manifold, the index of their intersection is the same at every point of intersection; see Section 2 for details). Such diffeomorphisms are called *DA-diffeomorphisms*.

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Denote by $\mathbb{G}_k^{cod1}(M^n)$, $k \geqslant 1$, the set of A-diffeomorphisms $M^n \to M^n$ of a closed smooth connected orientable $n$-manifold $M^n$, $n \geqslant 3$, such that a nonwandering set of any $f \in \mathbb{G}_k^{cod1}(M^n)$ consists of $k$ connected codimension one orientable expanding attractors and contracting repellers. In this paper, we classify the diffeomorphisms from $\mathbb{G}_k^{cod1}(M^n)$ up to a global conjugacy on nonwandering sets (for the main definitions, see below and Section 2). In addition, we show that any $f \in \mathbb{G}_k^{cod1}(M^n)$ is $\Omega$-stable, but not structurally stable. For completeness, we describe the topological structure of the supporting manifold $M^n$.

It is not difficult to construct a diffeomorphism $f \in \mathbb{G}_2^{cod1}(M^n)$ where $M^n$ is a connected sum of two tori $\mathbb{T}^n$, $n \geqslant 2$ [30]. Indeed, take a DA-diffeomorphism $f_0 : \mathbb{T}^n \to \mathbb{T}^n$ whose nonwandering set consists of an isolated source $s_0$ and a codimension one orientable expanding attractor $\Lambda$. Then the diffeomorphism $f_0^{-1}$ has a nonwandering set consisting of a sink $s_0$ denoted by $s_1$ and a codimension one orientable contracting repeller $\Lambda$ denoted by $\Lambda_1$. We can assume that $f_0^{-1}$ is defined on a copy of $\mathbb{T}^n$. Deleting small neighborhoods of $s_0$ and $s_1$, one can construct a connected sum $M^n = \mathbb{T}^n \sharp \mathbb{T}^n$ on which $f_0$ and $f_0^{-1}$ induce $f \in \mathbb{G}_2^{cod1}(M^n)$ whose nonwandering set consists of an orientable codimension one expanding attractor $\Lambda$ and a contracting repeller $\Lambda_1$.

To formulate the main results, let us introduce some notation. Suppose for definiteness that $\Lambda$ is an orientable codimension one expanding attractor (similar notation holds for a contracting repeller) of an A-diffeomorphism $f$. Then any stable manifold $W^s(x)$, $x \in \Lambda$, is one-dimensional and $W^s(x)\backslash x$ consists of two components. Due to [5] for $n = 2$, and [14], Lemmas 1.2, 1.5 for $n \geqslant 2$, at least one component of $W^s(x)\backslash x$ intersects $\Lambda$. A point $x \in \Lambda$ is called a *boundary point* if there is a component of $W^s(x)\backslash x$ denoted by $W^s_\emptyset(x)$ which does not intersect $\Lambda$. It follows from [5, 6, 25] for $n = 2$, and [14], Lemma 1.4 for $n \geqslant 2$ (see also the books [9, 15]) that the set $B(f) \subset \Lambda$ of boundary points is nonempty, invariant and finite. Therefore, every boundary point is periodic. Let $p_1, \ldots, p_r \in B(f)$ be all boundary points such that $W^s_\emptyset(p_1)$, $\ldots$, $W^s_\emptyset(p_r)$ belong to the same component of $W^s(\Lambda) \setminus \Lambda$. The union $\cup_{i=1}^r W^u(p_i)$ denoted by $b^u$ is called a *bunch*, $r$ is called the *degree* of the bunch $b^u$, and $p_1, \ldots, p_r$ are called *associated* periodic (boundary) points. Since $\Lambda$ is orientable, each bunch of $\Lambda$ has degree two, and hence a bunch has two associated periodic points [14], Corollary 1.3.

Below, $S^l$ is homeomorphic to a standard $l$-sphere. The symbol $\sharp$ means a connected sum. Note that a connected sum for high-dimensional topological orientable manifolds was introduced in [21]. For completeness, we formulate the statement which is a partial result of [11].

**Theorem 1.** *Suppose the nonwandering set of $f \in \mathbb{G}_k^{cod1}(M^n)$, $n \geqslant 3$, consists of basic sets $\Omega_1, \ldots, \Omega_k$. Let $l_i$ be the number of bunches of $\Omega_i$, $1 \leqslant i \leqslant k$. Then $M^n$ is homeomorphic to the following connected sum:*

$$\underbrace{\mathbb{T}^n \sharp \cdots \sharp \mathbb{T}^n}_{k \geqslant 2} \sharp \underbrace{\left(S^{n-1} \times S^1\right) \sharp \cdots \sharp \left(S^{n-1} \times S^1\right)}_{r_f \geqslant 0} \tag{1.1}$$

*where* $r_f = \frac{l_1 + \cdots + l_k}{2} - k + 1$.

Let us consider now the problem of classification for the set $\mathbb{G}_k^{cod1}(M^n)$. Here, we consider the classification up to a global (topological) conjugacy for the diffeomorphisms from $\mathbb{G}_k^{cod1}(M^n)$ on their nonwandering sets. Let us recall some notation. Suppose diffeomorphisms $f, f' : M^n \to M^n$ have invariant sets $\Omega$ and $\Omega'$, respectively. We say that $f$ and $f'$ are *globally conjugate on the sets* $\Omega$ and $\Omega'$ if there is a homeomorphism $h : M^n \to M^n$ such that

$$h(\Omega) = \Omega' \quad and \quad f'|_\Omega = h \circ f \circ h^{-1}|_{\Omega'}.$$

First, we construct an invariant of global conjugacy for every $f \in \mathbb{G}_k^{cod1}(M^n)$ which is a graph $\Gamma(f)$ endowed with an additional information, and one introduces a definition of commensurability of graphs (see details below). The following result says that the graph $\Gamma(f)$ of $f \in \mathbb{G}_k^{cod1}(M^n)$ up to a commensurability is a complete invariant of global conjugacy on nonwandering sets for diffeomorphisms $\mathbb{G}_k^{cod1}(M^n)$.

**Theorem 2.** *Two diffeomorphisms $f, f' \in \mathbb{G}_k^{cod1}(M^n)$, $n \geqslant 3$, are globally conjugate on its nonwandering sets if and only if the graphs $\Gamma(f)$ and $\Gamma(f')$ are commensurable.*

Second, we introduce the set $\Gamma^k$, $k \geqslant 2$, of abstract graphs (see details below). To get a complete classification, we prove the following result.

**Theorem 3.** *The graph $\Gamma(f)$ of any diffeomorphism $f \in \mathbb{G}_k^{cod1}(M^n)$ belongs to the set $\Gamma^k$. Given any graph $\gamma \in \Gamma^k$, there are a closed smooth connected orientable $n$-manifold $M^n$, $n \geqslant 3$, and a diffeomorphism $f \in \mathbb{G}_k^{cod1}(M^n)$ such that $\gamma = \Gamma(f)$.*

Let us mention some results concerning the subject of the paper. There are various types of conjugacy applying to classifications of dynamical systems. We restrict ourselves to a topological conjugacy. Recall that two maps $f, g : M \to M$ are (topologically) *conjugate* provided there is a homeomorphism $h : M \to M$ such that $h \circ f = g \circ h$. It is a difficult problem to classify dynamical systems under conjugacy mappings on the whole supporting manifold. The first natural step is a classification of restrictions of dynamical systems (in particular, diffeomorphisms) on special invariant subsets. For example, Williams [34] proved that the restriction of diffeomorphism on an expanding attractor of dimension $d \geqslant 1$ is conjugate to the shift map of a generalized $d$-solenoid. The second natural step is to ask: when are two diffeomorphisms are conjugate in neighborhoods of their invariant sets? Robinson and Williams [31] constructed two diffeomorphisms $f$ and $g$ of nonhomeomorphic 5-manifolds with expanding 2-dimensional attractors $\Lambda_f$ and $\Lambda_g$, respectively, such that the restriction $f|_{\Lambda_f} : \Lambda_f \to \Lambda_f$ is conjugate to the restriction $g|_{\Lambda_g} : \Lambda_g \to \Lambda_g$, but there is not even a homeomorphism from a neighborhood of $\Lambda_f$ to a neighborhood of $\Lambda_g$ taking $\Lambda_f$ to $\Lambda_g$. For other examples, see [18], where the first type of conjugacy (i. e., a conjugacy of restrictions) is called an intrinsic conjugacy, while the second type of conjugacy (when a conjugacy map is defined in a neighborhood of an invariant set) is called a *neighborhood conjugacy*. Clearly, a neighborhood conjugacy implies an intrinsic conjugacy because the first one takes into account embedding of invariant sets in supporting manifolds. Here, we consider a global conjugacy which can be considered as an intermediate type of conjugacy.

In [22], the following four types of A-diffeomorphisms were introduced: regular, semichaotic, chaotic, and superchaotic ones (to be precise, such types were introduced for a wide class of Smale A-homeomorphisms). Basic sets of A-diffeomorphism $f$ are naturally divided into sink basic sets $\omega(f)$, source basic sets $\alpha(f)$, and saddle basic sets $\sigma(f)$. We say that $f$ is *regular* if all basic sets $\omega(f)$, $\sigma(f)$, $\alpha(f)$ are trivial, while $f$ is *semichaotic* if exactly one family from the families $\omega(f)$, $\sigma(f)$, $\alpha(f)$ consists of nontrivial basic sets, and $f$ is *chaotic* if exactly two families from the families $\omega(f)$, $\sigma(f)$, $\alpha(f)$ consist of nontrivial basic sets, and at last $f$ is *superchaotic* if all basic sets $\omega(f)$, $\sigma(f)$, $\alpha(f)$ are nontrivial. In [22], necessary and sufficient conditions of conjugacy for regular, semichaotic, and chaotic A-diffeomorphisms were formulated provided that chaotic A-diffeomorphisms have either trivial sink basic sets or trivial source basic sets. We see that the set $\mathbb{G}_k^{cod1}(M^n)$, $k \geqslant 1$, belongs to the set of chaotic A-diffeomorphisms, but every $f \in \mathbb{G}_k^{cod1}(M^n)$ has nontrivial sink and source basic sets. Thus, the main result of [22] does not cover the main results of our paper.

The structure of the paper is as follows. In Section 2, we formulate the main definitions and give some previous results. In Section 3, we prove main results (Theorems 2, 3).

## 2. BASIC DEFINITIONS AND PREVIOUS RESULTS

*A-diffeomorphisms*. Let $f$ be a diffeomorphism of a closed manifold $M^n$ endowed with some Riemannian metric $d$. $f$ is said to be an *A-diffeomorphism* if its nonwandering set $NW(f)$ is hyperbolic and periodic points are dense in $NW(f)$ [32]. The *stable manifold* $W^s(x)$ of a point $x \in NW(f)$ is defined to be the set of points $y \in M^n$ such that $d(f^i x, f^i y) \to 0$ as $i \to +\infty$. The *unstable manifold* $W^u(x)$ of $x$ is the stable manifold of $x$ for the diffeomorphism $f^{-1}$. We shall consider a stable or unstable manifold to be an immersed submanifold of $M^n$. Stable and unstable manifolds are called *invariant manifolds*. By definition, let $W^s_\varepsilon(x) \subset W^s(x)$ (resp. $W^u_\varepsilon(x) \subset W^u(x)$)

be the $\varepsilon$-neighborhood of $x$ in the intrinsic topology of the manifold $W^s(x)$ (resp. $W^u(x)$), where $\varepsilon > 0$.

The *spectral decomposition theorem* says that the nonwandering set $NW(f)$ of an $A$-diffeomorphism $f$ is a finite union of pairwise disjoint $f$-invariant closed sets $\Omega_1, \ldots, \Omega_k$ such that every restriction $f|_{\Omega_i}$ is topologically transitive. These $\Omega_i$ are called the *basic sets* of $f$. In addition, $M$ can be represented as follows:

$$M = \bigcup_{i=1}^{k} W^s(\Omega_i) = \bigcup_{i=1}^{k} W^u(\Omega_i), \text{ where } W^{s(u)}(\Omega_i) = \bigcup_{x\in\Omega_i} W^{s(u)}(x). \tag{2.1}$$

Since $f$ is transitive on each basic set $\Omega_i$, it follows that the restrictions of the bundles $E^s$, $E^u$ to $\Omega_i$ have constant dimensions. The dimension $\dim E^u_{\Omega_i} = \dim E^u_x$, $x \in \Omega_i$, is called *Morse's index* of $\Omega_i$. A dimension $\dim\Omega$ of basic set $\Omega$ means the topological dimension of $\Omega$. A basic set $\Omega$ is an *expanding attractor* if $\Omega$ is an attractor and $\dim\Omega$ equals Morse's index of $\Omega$ [34]. A basic set $\Lambda$ of an A-diffeomorphism $f$ is called a *contracting repeller* provided $\Lambda$ is an expanding attractor of $f^{-1}$.

**Lemma 1.** *Let $f : M^n \to M^n$ be an A-diffeomorphism of closed manifold $M^n$ such that the nonwandering set $NW(f)$ of any $f$ consists of $k$ attractors and repellers. Then $k \geqslant 2$, and $NW(f)$ contains at least one attractor and at least one repeller.*

*Proof.* Suppose the contrary. Then $k = 1$, and $NW(f)$ consists of either a unique attractor or unique repeller. Assume $NW(f) = \Lambda_a$ is an attractor (if $NW(f)$ is a repeller, the proof is similar). Due to (2.1), a point $x \in M^n$ belongs to an unstable manifold of some basic set. Since $\Lambda_a$ is an attractor, $W^u(\Lambda_a) = \Lambda$. Therefore, $x \in \Lambda$ because $\Lambda_a$ is a unique basic set. Hence, $\Lambda_a = M^n$. By definition, $\Lambda_a \neq M^n$. This contradiction shows that $k \geqslant 2$. Let $U$ be an attracting neighborhood of $\Lambda_a$. Suppose that $f$ has no repellers. Then any point $x \in U \setminus \Lambda_a$ belongs to some attractor. This is impossible, since $\cap_{i\geqslant 0} f^{(}U) = \Lambda_a$. We see that $f$ has at least one repeller. □

For any $x \in \Omega$, $W^u(x)$ and $W^s(x)$ are immersed submanifolds such that

$$\dim W^u(x) + \dim W^n(x) = n.$$

Moreover, $W^u(x)$ and $W^s(x)$ are homeomorphic to Euclidean space of the corresponding dimension. Therefore, both $W^u(x)$ and $W^s(x)$ are endowed with a normal and intrinsic orientation. Hence, one can define the index of intersection at each point of $W^u(x) \cap W^s(x)$ [17]. Following [4–6], we call a basic set $\Omega$ *orientable* if for any $\alpha > 0$ and $\beta > 0$ the index of $W^s_{x,\alpha} \cap W^u_{x,\beta}$ does not depend on a point of intersection. A codimension one expanding attractor of a DA-diffeomorphism is orientable. A Plykin attractor is a nonorientable expanding attractor [25].

*Structural stability and $\Omega$-stability.* Let *Diff*$^1$ $(M^n)$ be the space of $C^1$ diffeomorphisms on $M^n$ endowed with the uniform $C^1$ topology [17]. Recall that diffeomorphisms $f, g \in Diff^1(M)$ are (topologically) *conjugate* if there is a homeomorphism $\varphi : M \to M$ such that $\varphi \circ f = g \circ \varphi$. A diffeomorphism $f \in Diff^1(M)$ is called *structurally stable* if there is a neighborhood $U(f) \subset Diff^1(M)$ of $f$ such that any $g \in U$ is conjugate to $f$.

Let $W_1$, $W_2 \subset M^n$ be two immersed submanifolds. One says that $W_1$, $W_2$ are intersected *transversally* provided that, given any point $x \in W_1 \cap W_2$, the tangent bundles $T_xW_1$, $T_xW_2$ generate the tangent bundle $T_xM^n$. In this case $\dim T_xW_1 + \dim T_xW_2 \geq \dim T_xM^n$. According to Mane [20] and Robinson [28], an A-diffeomorphism $f$ is structurally stable if and only if invariant manifolds $W^s(x)$, $W^u(y)$ are intersected transversally for any $x$, $y \in NW(f)$. The last condition is called the *strong transversality condition.*

A diffeomorphism $f \in Diff^1(M)$ is called $\Omega$-*stable* if there is a neighborhood $U(f) \subset Diff^1(M)$ of $f$ such that for any $g \in U(f)$ the restrictions $f|_{NW(f)}$, $g|_{NW(g)}$ are conjugate, i.e., there exists a homeomorphism $\varphi : NW(f) \to NW(f)$ such that $\varphi \circ f|_{NW(f)} = g \circ \varphi|_{NW(f)}$. According to Smale [33], if an A-diffeomorphism $f$ has no cycles on basic sets, then $f$ is $\Omega$-stable.

**Lemma 2.** *Every $f \in \mathbb{G}_k^{cod1}(M^n)$, $n \geqslant 3$, is $\Omega$-stable and is not structurally stable.*

*Proof.* Let $\Omega_1, \ldots, \Omega_k$ be basic sets of $f \in \mathbb{G}_k^{cod1}(M^n)$. Due to (2.1), any point $x \in M^n \setminus \big(\cup_{i=1}^k \Omega_i\big)$ belongs to a stable one-dimension manifold of a point of some attractor and an unstable one-dimensional manifold of a point of some repeller. Since $n \geqslant 3$, a strong transversality condition does not hold. Hence, $f$ is not structurally stable.

We see that a stable manifold of repeller belongs to the repeller, while an unstable manifold of attractor belongs to the attractor [24]. Therefore, a stable manifold of repeller cannot intersect an unstable manifold of attractor. It follows that the basic sets $\Omega_1, \ldots, \Omega_k$ have no cycles. According to [33], $f$ is $\Omega$-stable. □

*Characteristic spheres*. Let $\Omega$ be a codimension one orientable expanding attractor of A-diffeomorphism $f : M^n \to M^n$, $n \geqslant 3$. Plykin [26] proved that any bunch of $\Omega$ is a 2-bunch. Let $b^u_{pq} = W^u(p) \cup W^u(q)$ be a 2-bunch where $p$, $q$ are boundary periodic points of $\Omega$. It follows from [14, 26] that, given any point $x \in W^u(p) \setminus p$, there is a unique point $y \in W^u(q) \setminus q$ such that $(x, y)^s = (x, y)^s_\emptyset$, and vice versa. Therefore, one can define the mapping

$$\varphi_{pq} \stackrel{\text{def}}{=} \varphi : (W^u(p) \setminus p) \bigcup (W^u(q) \setminus q) \to (W^u(p) \setminus p) \cup (W^u(q) \setminus q)$$

where $\varphi(x) = y$, $(x, y)^s = (x, y)^s_\emptyset$. Moreover, for every $n \in \mathbb{Z}$, it holds that

$$f^{mn} \circ \varphi|_{(W^u(p)\setminus p)\cup\big(W^u(q)\setminus q\big)} = \varphi \circ f^{mn}|_{\big(W^u(p)\setminus p\big)\cup\big(W^u(q)-q\big)}, \tag{2.2}$$

where $m = m(p, q)$ is the period of the points $p$, $q$. Take a closed $(n-1)$-disk $D_p \subset W^u(p)$ bounded by a smooth $(n-2)$-sphere $S_p^{n-2} = \partial D_p$ such that $p \in \text{int}(D_p) = D_p \setminus \partial D_p$ and $D_p \subset int\big(f^m(D_p)\big)$. Then the set $C_{pq} = \cup_{x\in\partial D_p}(x, \varphi(x))^s_\emptyset$ is homeomorphic to $\Sigma^{n-2} \times (0, 1)$. Since $\varphi$ is a homeomorphism, $S_q^{n-2} = \varphi(S_p^{n-2})$ is a locally flat $(n-2)$-sphere embedded in $W^u(q)$, and hence, $S_q^{n-2}$ bounds the $(n-1)$-disk $D_q \subset W^u(q)$. As a consequence, $S_{pq} = D_p \cup D_q \cup C_{pq}$ is an $(n-1)$-sphere called a *characteristic sphere* corresponding to the bunch $b^u_{pq}$. According to [14], one can slightly deform $S_{pq}$ to $W^s(\Omega) \setminus \Omega$ to get a characteristic sphere with no intersections with $\Omega$. Note that a characteristic sphere is defined up to a small isotopy [14], Lemma 2.8. By construction, any characteristic sphere is a locally flat embedded sphere and hence, it has a neighborhood homeomorphic to $\mathbb{S}^{n-1} \times (-1; +1)$ provided $M^n$ is orientable. Similar constructions hold for $\Omega$ to be a codimension one contracting repeller. We see that there is a one-to-one correspondence between bunches and characteristic spheres.

*Attracting neighborhoods*. Recall that, if $\Omega$ is an attractor of $f : M^n \to M^n$, then there is a so-called *attracting neighborhood* $U(\Omega)$ of $\Omega$ such that $clos f(\Omega) \subset U(\Omega)$ and $\cap_{i\geqslant 0} f^i(U(\Omega)) = \Omega$. If $\Omega$ is a repeller of $f$, then $\Omega$ is an attractor for $f^{-1}$. We say that $U(\Omega)$ is an attracting neighborhood of the repeller $\Omega$ if $U(\Omega)$ is an attracting neighborhood of $\Omega$ under the diffeomorphism $f^{-1}$.

The next result follows from [14], Corollary 2.1 and Lemma 3.1 (see also Theorem 5.1). Recall that a DA-diffeomorphism $g : \mathbb{T}^n \to \mathbb{T}^n$ is an A-diffeomorphism provided the nonwandering set consists of codimension one orientable expanding attractor and finitely many isolated source periodic orbits.

**Lemma 3.** *Let $\Omega$ be a codimension one connected orientable expanding attractor of A-diffeomorphism $f : M^n \to M^n$, $n \geqslant 3$, with $l$ bunches. Then $\Omega$ has an attracting neighborhood $U(\Omega) \subset W^u(\Omega)$ whose boundary $\partial U(\Omega)$ consists of locally flat characteristic spheres $S_1, \ldots, S_l$ (with no intersections with $\Omega$), and $U(\Omega)$ is homeomorphic to an $n$-torus $\mathbb{T}^n$ with $l$ deleted closed $n$-disks, i. e., $U(\Omega) = \mathbb{T}^n \setminus \cup_{i=1}^l D_i^n$. Moreover, the restriction $f|_{U(\Omega)} : U(\Omega) \to f(U(\Omega))$ is extended to a DA-diffeomorphism $\tilde{f} : \mathbb{T}^n \to \mathbb{T}^n$. In addition, there is $r \in \mathbb{N}$ such that the spheres $S_i$, $f^r(S_i)$ bound a domain homeomorphic to $S^{n-1} \times [0; 1]$ for all $i = 1, \ldots, l$.*

Note that due to [14], Theorem 8.2, the extension $\tilde{f}$ is defined up to a conjugacy. A similar statement holds for a codimension one orientable contracting repeller.

*Complete invariant of global conjugacy on a codimension one basic set.* Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be an A-diffeomorphism and $\Omega$ a codimension one basic set which is either an orientable expanding attractor or an orientable contracting repeller. Suppose $\Omega$ has $l$ bunches. Applying results by Franks [3] and Newhouse [23], it was proved in [6, 12, 13] that there are a homotopic to identity continuous map $h : \mathbb{T}^n \to \mathbb{T}^n$ and a codimension one Anosov automorphism $A : \mathbb{T}^n \to \mathbb{T}^n$ such that $h \circ f|_\Omega = A \circ h|_\Omega$. In addition, $h$ takes each pair of associated periodic points to a periodic point of $A$. Set $P = h(A(f))$ where $A(f)$ is the set of associated periodic points of $f$. Note that the cardinality $|P|$ of $P$ equals $l$. We see that $f$ corresponds to the triple $(A, P, \epsilon)$ where $\epsilon = a$ if $\Omega$ is an attractor and $\epsilon = r$ if $\Omega$ is a repeller. Suppose that an A-diffeomorphism $f' : \mathbb{T}^n \to \mathbb{T}^n$ has a codimension one basic set $\Omega'$ which is either an orientable expanding attractor or an orientable contracting repeller. Assume that $f'$ corresponds to the triple $(A', P', \epsilon')$.

Two triples $(A, P, \epsilon)$, $(A', P', \epsilon')$ are called *equivalent* provided $\epsilon = \epsilon'$, and there is an automorphism $\zeta : \mathbb{T}^n \to \mathbb{T}^n$ such that $A' = \zeta \circ A \circ \zeta^{-1}$ and $\zeta(P) = P'$. It was proved in [6, 12] that $f$, $f'$ are globally conjugate on the basic sets $\Omega$, $\Omega'$ respectively if and only if the triples $(A, P, \epsilon)$, $(A', P', \epsilon')$ are equivalent.

Below, we will sometimes identify an automorphism with its matrix.

*Construction of graph* $\Gamma(f)$. The crucial step to construct a graph $\Gamma(f)$ for $f \in \mathbb{G}_k^{cod1}(M^n)$ is the following statement (below $k \geqslant 2$).

**Lemma 4.** *Suppose $f \in \mathbb{G}_k^{cod1}(M^n)$, $n \geqslant 3$, has basic sets $\Omega_1, \ldots, \Omega_k$. Let $U(\Omega_1), \ldots, U(\Omega_k)$ be pairwise disjoint attracting neighborhoods of the basic sets $\Omega_1, \ldots, \Omega_k$, respectively, such that the boundary of every $U(\Omega_i)$ consists of characteristic spheres of $\Omega_i$, $1 \leqslant i \leqslant k$. Then any component of $M^n \setminus \left(\cup_{i=1}^k U(\Omega_i)\right)$ is homeomorphic to $\mathbb{S}^{n-1} \times [0;1]$. Moreover, one characteristic sphere that is homeomorphic to the boundary component $\mathbb{S}^{n-1} \times \{0\}$ corresponds to an attractor, while another characteristic sphere that is homeomorphic to the boundary component $\mathbb{S}^{n-1} \times \{1\}$ corresponds to a repeller.*

*Proof.* Let $N$ be a component of $M^n \setminus \left(\cup_{i=1}^k U(\Omega_i)\right)$. Since the boundary $\partial N$ consists of $(n-1)$-spheres, one can glue $n$-balls $B_1^n, \ldots, B_j^n$ to $\partial N$ to get a closed $n$-manifold $N_*$ where $j$ is the number of the boundary components of $N$. Due to Lemma 3, there is an iteration $f^m$ for sufficiently large $m$ such that the restriction $f^m|_N : N \to f^m(N)$ induces an A-diffeomorphism $f_* : N_* \to N_*$ whose nonwandering set consists of isolated nodal fixed points (sinks and sources) because $f$ has no saddle isolated periodic orbits. According to [27] (see also [10]), $N_*$ is homeomorphic to $n$-sphere $\mathbb{S}^n$, and the nonwandering set of $f_*$ consists of a unique sink and a unique source. Thus, $j = 2$ and $N = N_* \setminus (B_1^n \cup B_2^n)$. It follows from [19] (see also [2]) that $N$ is homeomorphic to $\mathbb{S}^{n-1} \times [0;1]$. $\square$

Suppose that the nonwandering set of $f \in \mathbb{G}_k^{cod1}(M^n)$ consists of a basic set $\Omega_1, \ldots, \Omega_k$ with bunches $l_1, \ldots$ and $l_k$, respectively. Due to Lemma 3, there is an attracting neighborhood $U(\Omega_i)$ that is homeomorphic to $\mathbb{T}^n \setminus \cup_{j=1}^{l_i} D_j^n$, $i = 1, \ldots, k$. Moreover, $f|_{U(\Omega_i)}$ can be extended to a DA-diffeomorphism $\mathbb{T}_i^n \to \mathbb{T}_i^n$ denoted by $f_i$ with a unique nontrivial basic set $\Omega_i$. Thus, $f_i$ corresponds to the triple $(A_i, P_i, \epsilon_i)$. We assume that a graph $\Gamma(f)$ has a group $V_i$ of vertices $v_1^i, \ldots, v_{l_i}^i$, and each $V_i$ is endowed with the triple $(A_i, P_i, \epsilon_i)$ where $P_i$ consists of the points $p_1^i, \ldots, p_{l_i}^i$ (every $p_s^i$ corresponds to $v_s^i$, and vice versa, $1 \leqslant s \leqslant l_i$). Thus, every basic set induces a group of vertices endowed with a triple in the graph $\Gamma(f)$. Due to Lemma 3, any point $p_s^i \in P_i$ corresponds to a characteristic sphere denoted by $S^{n-1}(p_s^i)$ belonging to the boundary of $U(\Omega_i)$. It follows from Lemma 4 that there is a unique basic set $\Omega_j$, $j \neq i$, such that the characteristic spheres $S^{n-1}(p_s^i)$, $S^{n-1}(p_t^j)$ bound in $M^n$ the set homeomorphic to $S^{n-1} \times [0;1]$ where the group $V_j = \{v_1^j, \ldots, v_{l_j}^j\}$ endowed with the triple $(A_j, P_j, \epsilon_j)$, $P_j = \{p_1^j, \ldots, p_{l_j}^j\}$, corresponds to $\Omega_j$. Note that $\epsilon_i \neq \epsilon_j$. In this case, we assume that the vertices $v_s^i$, $v_t^j$ are connected by an edge denoted by $L(p_s^i, p_t^j)$. Sometimes, we will say that the points $p_s^i$, $p_t^j$ are connected as well. This completes the construction of the graph $\Gamma(f)$ for $f \in \mathbb{G}_k^{cod1}(M^n)$, $n \geqslant 3$.

As a résumé, $\Gamma(f)$ is a collection of groups $V_1, \ldots, V_k$ of vertices, and each group is endowed with a triple. The degree of every vertex is equal to one, and there are no adjacent edges. Roughly speaking, $\Gamma(f)$ is a collection of pairwise disjoint segments.

*Commensurability of graphs.* Suppose $\Gamma(f)$, $\Gamma(f')$ are graphs of diffeomorphisms $f$, $f' \in \mathbb{G}_k^{cod1}(M^n)$, respectively. We say that $\Gamma(f), \Gamma(f')$ are *commensurable* if the following conditions hold: (a) there is a bijection $\psi : \Gamma(f) \to \Gamma(f')$ such that $\psi(V_i) = V_i'$ for all $i = 1, \ldots, k$, and $\psi(v_s^i) = v_s^{'i}$ for all $s = 1, \ldots, l_i$; (b) given any $1 \leqslant i \leqslant k$, the triples $(A_i, P_i, \epsilon_i)$, $(A_i', P_i', \epsilon_i')$ corresponding to the groups $V_i$, $V_i'$ are equivalent. In other words, there is a collection of automorphisms $\{\zeta_1, \ldots, \zeta_k\}$ of $\mathbb{T}^n$ such that $\zeta_i(P_i) = P_i'$, and $\zeta_i$ conjugates the automorphisms $A_i$, $A_i'$; (c) two vertices $v_s^i$, $v_t^j$ of $\Gamma(f)$ are connected by the edge $L(p_s^i, p_t^j)$ if and only if the vertices $\psi(v_s^i) = v_s^{'i}$, $\psi(v_t^j) = v_t^{'j}$ are connected by the edge $L(p_s^{'i}, p_t^{'j})$ in $\Gamma(f')$; (d) if two vertices $v_s^i$, $v_t^j$ are connected in the graph $\Gamma(f)$, then the determinants of the automorphisms $\zeta_i$, $\zeta_j$ ($\zeta_i$ conjugates $A_i$, $A_i'$, while $\zeta_j$ conjugates $A_j$, $A_j'$) have the same sign.

*Construction of the set* $\Gamma^k$, $k \geqslant 2$. By definition, a graph $\gamma$ belongs to the class $\Gamma^k$, $k \geqslant 2$, if it satisfies the following conditions: (1) $\gamma \in \Gamma^k$ has $k$ groups of vertices $V_i = \{v_1^i, \ldots, v_{l_i}^i\}$, $i = 1, \ldots, k$, and each group $V_i$ is endowed with a triple $(A_i, P_i, \epsilon_i)$ where $A_i : \mathbb{T}^n \to \mathbb{T}^n$ is a codimension one Anosov automorphism, $P_i = \{p_1^i, \ldots, p_{l_i}^i\}$ are finitely many periodic orbits of $A_i$, $\epsilon_i = a$ provided the stable manifolds of $A_i$ are one-dimensional, while $\epsilon_i = r$ provided the unstable manifolds of $A_i$ are one-dimensional. Moreover, there exists at least one triple $(A_i, P_i, \epsilon_i)$ with $\epsilon = a$ and at least one triple $(A_j, P_j, \epsilon_j)$ with $\epsilon = r$. In addition, there is a bijection $\psi_i : V_i \to P_i$, $p_s^i = \psi_i(v_s^i)$, for any $1 \leqslant s \leqslant |P_i| = l_i$ (note that since the set $P_i$ is invariant, the inclusion $\psi_i^{-1}(A_i^r(p_s^i)) \in V_i$ holds for every $r \in \mathbb{N}$); (2) every vertex $v_s^i$ of $\gamma \in \Gamma^k$ has degree 1; (3) suppose vertices $v_s^i = \psi_i^{-1}(p_s^i) \in V_i = \{v_1^i, \ldots, v_{l_i}^i\}$, $v_t^j = \psi_j^{-1}(p_t^j) \in V_j$ are connected by an edge $L(p_s^i, p_t^j)$. Then $\epsilon_i \neq \epsilon_j$. Moreover, if $n_s^i$ is a period of the point $p_s^i$ under $A_i$, then every vertex $\psi_i^{-1}(A_i^l(p_s^i)) \in V_i$ is connected by an edge with $\psi_j^{-1}(A_j^l(p_t^j)) \in V_j$, $l = 1, \ldots, n_s^i$; (4) given any groups $V_i$, $V_j$, $i \neq j$, there is a sequence of groups $V_{i_1}, \ldots, V_{i_r}$ with $i_1 = i$, $i_r = j$ such that any neighbor groups $V_{i_s}$, $V_{i_{s+1}}$ in the sequence contain vertices $v^{i_s} \in V_{i_s}$, $v^{i_{s+1}} \in V_{i_{s+1}}$ connected by an edge; (5) the determinants of all $A_i$, $i = 1, \ldots, k$, have the same sign.

## 3. PROOFS OF MAIN RESULTS

Below, we will use the notation introduced in Section 2. In particular, a diffeomorphism $f \in \mathbb{G}_k^{cod1}(M^n)$, $n \geqslant 3$, has the nonwandering set $NW(f)$ consisting of the basic sets $\Omega_1, \ldots, \Omega_k$. Similarly, $f' \in \mathbb{G}_k^{cod1}(M^n)$.

*Proof* (*of Theorem* 2). Necessity. Suppose $f, f' \in \mathbb{G}_k^{cod1}(M^n)$, $n \geqslant 3$, are globally conjugate on its nonwandering sets. Hence, there is a homeomorphism $\varphi : M^n \to M^n$ such that $\varphi \circ f|_{\Omega_i} = f' \circ \varphi|_{\Omega_i}$ and $\varphi(\Omega_i) = \Omega_i'$ for all $1 \leqslant i \leqslant k$. Let us show that the graphs $\Gamma(f)$, $\Gamma(f')$ are commensurable. For every $1 \leqslant i \leqslant k$, the homeomorphism $\varphi$ takes the bunches of $\Omega_i$ onto the bunches of $\Omega_i'$. Therefore, $\varphi$ induces a bijection $\psi$ between the groups $V_i = \{v_1^i, \ldots, v_{k_i}^i\} \to V_i' = \{v_1^{'i}, \ldots, v_{k_i}^{'i}\}$ and between the vertices $\psi(v_s^i) = v_s^{'i}$, $s = 1, \ldots, l_i$. This proves the condition (a) of commensurability of graphs.

Let $U(\Omega_i)$, $U(\Omega_i')$ be attracting neighborhoods of $\Omega_i$, $\Omega_i' = \varphi(\Omega_i)$, respectively, satisfying the conditions of Lemma 3. One can assume that $\varphi(U(\Omega_i)) = U(\Omega_i')$, $1 \leqslant i \leqslant k$. Set $\varphi|_{U(\Omega_i)} = \varphi_i$. Since

$\varphi$ is a global conjugacy on the nonwandering sets $NW(f)$ and $NW(f')$, $\varphi_i$ is a global conjugacy of the restrictions

$$f|_{\Omega_i} : \Omega_i \to f(\Omega_i), \quad f'|_{\Omega'_i} : \Omega'_i \to f'(\Omega'_i)$$

on the basic sets $\Omega_i$, $\Omega'_i$. It follows from [12–14, 26] that the triples $(A_i, P_i, \epsilon_i)$, $(A'_i, P'_i, \epsilon'_i)$ are equivalent. Hence, $\epsilon_i = \epsilon'_i$, and there is an automorphism $\zeta_i : \mathbb{T}^n \to \mathbb{T}^n$ such that $\zeta_i(P_i) = P'_i$, and $\zeta_i$ conjugates the automorphisms $A_i$, $A'_i$, $i = 1, \ldots, k$. We see that condition (b) holds.

Suppose the vertices $v^i_s$, $v^j_t \in \Gamma(f)$ are connected by the edge $L(p^i_s, p^j_t)$. Recall that the point $p^i_s \in P_i$ corresponds to a pair of associated periodic points of $\Omega_i$, and hence $p^i_s$ corresponds to a unique characteristic sphere $S^{n-1}(p^i_s)$ belonging to the boundary of $U(\Omega_i)$. Similarly, $p^j_t \in P_j$ corresponds to a unique characteristic sphere $S^{n-1}(p^j_t)$ of $\Omega_j$ belonging to the boundary of $U(\Omega_j)$. The existence of the edge $L(p^i_s, p^j_t)$ implies that the $(n-1)$-spheres $S^{n-1}(p^i_s)$, $S^{n-1}(p^j_t)$ bound a set $K_{ij}$ homeomorphic to the $n$-annulus $S^{n-1} \times [0, 1]$. Therefore, the neighborhoods $\varphi\big(U(\Omega_i)\big) = U(\Omega'_i)$, $\varphi\big(U(\Omega_j)\big) = U(\Omega'_j)$ are connected by the annulus $\varphi(K_{ij}) = K'_{ij}$. It follows that the vertices $\psi(v^i_s) = v'^i_s$, $\psi(v^j_t) = v'^i_t$ are connected by an edge $L(p'^i_s, p'^i_t)$ in the graph $\Gamma(f')$. And vice versa. This proves the condition (c) of commensurability of graphs.

Let $\varphi : S \to \varphi(S)$ be a homeomorphism and $S$, $\varphi(S)$ the submanifolds of $\mathbb{T}^n$. Then the orientation of $\mathbb{T}^n$ induces interior orientations on $S$ and $\varphi(S)$. One says that $\varphi : S \to \varphi(S)$ preserves orientation if $\varphi$ preserves the interior orientations of $S$ and $\varphi(S)$ [16].

To check condition (d) we need the following technical statement.

**Proposition 1.** *Let $g, g' : \mathbb{T}^n \to \mathbb{T}^n$ be DA-diffeomorphisms and $\Omega$, $\Omega'$ nontrivial basic sets of $g$, $g'$, respectively. Suppose a homeomorphism $\varphi : \mathbb{T}^n \to \mathbb{T}^n$ is a global conjugacy of $g$ and $g'$ on $\Omega$, $\Omega'$, respectively, so that the triples $(A, P, \epsilon)$, $(A', P', \epsilon')$ are equivalent, i. e., there is an automorphism $\zeta : \mathbb{T}^n \to \mathbb{T}^n$ that conjugates Anosov diffeomorphisms $A$, $A'$, and $\zeta(P) = P'$. Let $S$ be a characteristic sphere of $\Omega$. Then the restriction $\varphi|_S : S \to \varphi(S)$ preserves orientation if and only if the determinant of $\zeta$ is positive (hence, $\varphi|_S$ reverses orientation if and only if the determinant of $\zeta$ is negative).*

*Proof.* Recall that there are homotopic to identity continuous maps $h, h' : \mathbb{T}^n \to \mathbb{T}^n$ such that $h \circ g|_\Omega = A \circ h|_\Omega$ and $h' \circ g'|_{\Omega'} = A' \circ h'|_{\Omega'}$. It was proved in [12–14, 26] that, starting with the conjugacy $\varphi$ and applying the maps $h$ and $h'$, one can construct the conjugacy map $\zeta$ between Anosov automorphisms $A$ and $A'$. And vise versa, starting with the conjugacy $\zeta$ and applying the maps $h$ and $h'$, one can construct the conjugacy $\varphi$. In particular, since both $h$ and $h'$ preserve orientation, $\varphi$ is an orientation-preserving mapping if and only if $\zeta$ preserves the orientation of $\mathbb{T}^n$. According to [16], Section 3.3, $\varphi$ preserves the orientation of $\mathbb{T}^n$ if and only if the restriction of $\varphi$ on any $n$-disk $D^n \subset \mathbb{T}^n$ preserves orientation. Note that $\mathbb{T}^n$ is an irreducible manifold, i. e., every locally flat embedded $(n-1)$-sphere $S^{n-1} \subset \mathbb{T}^n$ bounds an $n$-disk $D^n \subset \mathbb{T}^n$. As a consequence, the restriction $\varphi|_S : S \to \varphi(S)$ of $\varphi$ on any characteristic sphere $S$ of $\Omega$ is an orientation-preserving mapping if and only if $\zeta$ preserves orientation, and hence the determinant of $\zeta$ is positive. Similarly, $\varphi|_S$ reverses orientation if and only if the determinant of $\zeta$ is negative. This completes the proof of Proposition 1. □

Suppose now that vertices $v^i_s$, $v^j_t \in \Gamma(f)$ are connected by an edge. It follows from the construction of $\Gamma(f)$ that the corresponding characteristic $(n-1)$ spheres $S^{n-1}(p^i_s)$, $S^{n-1}(p^j_t)$ bound the $n$-annulus $K_{ij} \in M^n$ homeomorphic to $\mathbb{S}^{n-1} \times [0; 1]$. Hence, the restrictions $\varphi|_{S^{n-1}(p^i_s)}$, $\varphi|_{S^{n-1}(p^j_t)}$ of $\varphi : K_{ij} \to \varphi(K_{ij})$ on the boundary components $S^{n-1}(p^i_s)$, $S^{n-1}(p^j_t)$ of $K_{ij}$ are isotopic. According to [7], Theorem 14.5, the restrictions $\varphi|_{S^{n-1}(p^i_s)}$, $\varphi|_{S^{n-1}(p^j_t)}$ either both preserve orientation or both reverse orientation. In any case, due to Proposition 1, the determinants of the automorphisms $\zeta_i$, $\zeta_j$ have the same sign, $\det \zeta_i = \det \zeta_j$. This proves the condition (c) of commensurability of the graphs $\Gamma(f)$, $\Gamma(f')$.

Sufficiency. Suppose now that the graphs $\Gamma(f)$, $\Gamma(f')$ are commensurable. It follows from the condition (a) of commensurability that there is a bijection $\psi:\Gamma(f)\to\Gamma(f')$ which induces a bijection of groups of vertices. Without loss of generality, one can assume that $\psi(v_s^i)=v_s^{'i}$ for all $i=1,\dots,k$ and $s=1,\dots,l_i$ where $V_i=\{v_1^i,\dots,v_{l_i}^i\}$ are groups of the vertices of $\Gamma(f)$, and $V_i'=\{v_1^{'i},\dots,v_{l_i}^{'i}\}$ are groups of vertices of $\Gamma(f')$, $i=1,\dots,k$. Recall that every group of vertices corresponds to a unique basic set. Therefore, $\psi$ induces a one-to-one bijection $\Omega_i\Longleftrightarrow\Omega_i'$, $i=1,\dots,k$, between the basic sets of $f$, $f'$.

According to condition (b), given any $1\leqslant i\leqslant k$, the triples $(A_i,P_i,\epsilon_i)$, $(A_i',P_i',\epsilon_i')$ corresponding to the groups $V_i=\{v_1^i,\dots,v_{l_i}^i\}$, $V_i'=\{v_1^{'i},\dots,v_{l_i}^{'i}\}$ are equivalent. In other words, there is a collection of automorphisms $\{\zeta_1,\dots,\zeta_k\}$ of $\mathbb{T}^n$ such that $\zeta_i(P_i)=P_i'$, and $\zeta_i$ conjugates the automorphisms $A_i$, $A_i'$, $i=1,\dots,k$. It follows from [12–14, 26] that there is a homeomorphism $\varphi_i:U(\Omega_i)\to U(\Omega_i')$ such that $\varphi_i\circ f|_{\Omega_i}=f'\circ\varphi_i|_{\Omega_i}$. We have to prove that the homeomorphisms $\varphi_i:U(\Omega_i)\to U(\Omega_i')$, $i=1,\dots,k$, can be extended to a common homeomorphism $\varphi:M^n\to M^n$.

Due to Lemma 4, the set $M^n\setminus\left(\cup_{i=1}^k U(\Omega_i)\right)$ is a union of $n$-annuli, each homeomorphic to $\mathbb{S}^{n-1}\times[0;1]$. It follows from the description of $\Gamma(f)$ that every $n$-annulus corresponds to an edge in $\Gamma(f)$. Suppose the vertices $v_s^i$, $v_t^j\in\Gamma(f)$ are connected by the edge $L(p_s^i,p_t^j)$ where $p_s^i\in P_i$, $p_t^j\in P_j$. It follows from Lemma 4 that the associated $(n-1)$-spheres $S^{n-1}(p_s^i)$, $S^{n-1}(p_t^j)$ bound an $n$-annulus $K_{ij}\subset M^n$. Due to condition (c), the vertices $\psi(v_s^i)=v_s^{'i}$, $\psi(v_t^j)=v_t^{'j}$ are connected by the edge $L(p_s^{'i},p_t^{'j})$ in $\Gamma(f')$. Hence, the associated $(n-1)$-spheres $S^{n-1}(p_s^{'i})$, $S^{n-1}(p_t^{'j})$ bound an $n$-annulus $K_{ij}'\subset M^n$ homeomorphic to $S^{n-1}\times[0;1]$. The condition (d) of commensurability implies that the determinants of the automorphisms $\zeta_i$, $\zeta_j$ have the same sign, i. e., $\det\zeta_i\times\det\zeta_j>0$. According to Proposition 1, the restrictions $\varphi_i|_{S^{n-1}(p_s^i)}$, $\varphi_j|_{S^{n-1}(p_t^j)}$ either both preserve orientation or reverse orientation. Due to [7], Theorem 14.5, these restrictions are isotopic. Therefore, the homeomorphisms $\varphi_i$, $\varphi_j$ can be extended to a homeomorphism $\varphi_{ij}:K_{ij}\to K_{ij}'$. Continuing in a similar way for all $n$-annuli of the set $M^n\setminus\left(\cup_{i=1}^k U(\Omega_i)\right)$, we get a homeomorphism $\varphi:M^n\to M^n$ that is an extension of the homeomorphisms $\varphi_i:U(\Omega_i)\to U(\Omega_i')$, $i=1,\dots,k$. Thus, $f$ and $f'$ are globally conjugate on its nonwandering sets. □

*Proof* (*of Theorem* 3). First, let us show that a graph $\Gamma(f)$ of $f\in\mathbb{G}_k^{cod1}(M^n)$ belongs to the set $\Gamma^k$. It follows from the construction of $\Gamma(f)$ that any basic set $\Omega_i$ with $l_i\geqslant 1$ bunches corresponds to a group $V_i=\{v_1^i,\dots,v_{l_i}^i\}$ of vertices endowed with a triple $(A_i,P_i,\epsilon_i)$ where $A_i:\mathbb{T}^n\to\mathbb{T}^n$ is a codimension one Anosov automorphism and $P_i=\{p_1^i,\dots,p_{l_i}^i\}$ is an invariant set of $A_i$ consisting of finitely many periodic points. Moreover, $\epsilon_i=a$ provided the stable manifolds of $A_i$ is one-dimensional, while $\epsilon_i=r$ provided the unstable manifolds of $A_i$ are one-dimensional [12–14, 26]. It follows from Lemma 1 that there exists at least one triple $(A_i,P_i,\epsilon_i)$ with $\epsilon=a$ and at least one triple $(A_j,P_j,\epsilon_j)$ with $\epsilon=r$. Since $|P_i|=l_i$, there is a one-to-one correspondence between points of $P_i$ and vertices of $V_i$. Without loss of generality, one can assume that there is a bijection $\psi_i:V_i\to P_i$ such that $p_s^i=\psi_i(v_s^i)$, $i=1,\dots,k$, $s=1,\dots,k_i$. We see that the condition (1) of the description of the set $\Gamma^k$ holds.

Every vertex $v_s^i\in V_i$ corresponds to a unique bunch of the basic set $\Omega_i$. Due to Lemma 3, $v_s^i\in V_i$ corresponds to a characteristic sphere $S^{n-1}(v_s^i)$ which belongs to the boundary of an attracting neighborhood $U(\Omega_i)$ of $\Omega_i$. It follows from Lemma 4 that there is a basic set $\Omega_j$ with an attracting neighborhood $U(\Omega_j)$ which contains a boundary component $S$ such that the $(n-1)$-spheres $S^{n-1}(v_s^i)$, $S$ bound an $n$-annulus $K\subset M^n$ homeomorphic to $S^{n-1}\times[0;1]$. This implies that the vertex $v_s^i$ has degree 1 according to the description of the graph $\Gamma(f)$. Thus, condition (2) holds.

Suppose vertices $v_s^i=\psi_i^{-1}(p_s^i)\in V_i=\{v_1^i,\dots,v_{l_i}^i\}$, $v_t^j=\psi_j^{-1}(p_t^j)\in V_j$ are connected by an edge $L(p_s^i,p_t^j)$. Due to Lemma 4, $\epsilon_i\neq\epsilon_j$. The existence of the edge $L(p_s^i,p_t^j)$ implies the existence of an $n$-annulus $K_{ij}$ bounded by characteristic spheres $S^{n-1}(p_s^i)$, $S^{n-1}(p_t^j)$. Clearly, any iteration $f^l(K_{ij})$

of $K_{ij}$ is an $n$-annulus homeomorphic to $S^{n-1}\times[0;1]$. Since the set of bunches is invariant, every vertex $\psi_i^{-1}\big(A_i^r(p_s^i)\big)\in V_i$ is connected by a unique edge with the vertex $\psi_j^{-1}(A_j^r(p_t^j))\in V_j$ for all $r=1,\dots,n_s^i$ where $n_s^i$ is a period of the point $p_s^i$. We see that condition (3) holds. Since $M^n$ is a connected manifold, condition (4) holds as well.

Suppose vertices $v_s^i$, $v_t^j\in\Gamma(f)$ are connected by an edge $L(p_s^i,p_t^j)$. Recall that the vertices $v_s^i$, $v_t^j$ belong to some groups endowed with the triples $(A_i,P_i,\epsilon_i)$, $(A_j,P_j,\epsilon_j)$, respectively, so that $p_s^i\in P_i$, $p_t^j\in P_j$. It follows from the description of $\Gamma(f)$ that the characteristic spheres $S^{n-1}(p_s^i)$, $S^{n-1}(p_t^j)$ bound an $n$-annulus $K_{ij}\subset M^n$ where $S^{n-1}(p_s^i)$, $S^{n-1}(p_t^j)$ are components of the boundary of attracting neighborhoods $U(\Omega_i)$, $U(\Omega_j)$ of basic sets $\Omega_i$, $\Omega_j$, respectively. Since $S^{n-1}(p_s^i)$, $S^{n-1}(p_t^j)$ bound the annulus $K_{ij}$ that is homeomorphic to $S^{n-1}\times[0;1]$, the restrictions $f|_{S^{n-1}(p_s^i)}$, $f|_{S^{n-1}(p_t^j)}$ are isotopic. According to [7], Theorem 14.5, these restrictions $f|_{S^{n-1}(p_s^i)}$, $f|_{S^{n-1}(p_t^j)}$ either both preserve orientation or both reverse orientation. According to Lemma 3, the diffeomorphisms $f_i$, $f_j$ are extended to DA-diffeomorphisms $\tilde f_i$, $\tilde f_j:\mathbb{T}^n\to\mathbb{T}^n$, respectively. Moreover, both $S^{n-1}(p_s^i)$ and $S^{n-1}(p_t^j)$ bound an $n$-ball in $\mathbb{T}^n$. Since $f|_{S^{n-1}(p_s^i)}$, $f|_{S^{n-1}(p_t^j)}$ are isotopic, the diffeomorphisms $\tilde f_i$, $\tilde f_j$ either both preserve orientation or both reverse orientation. It follows from the relations $A_i=(\tilde f_i)_*$, $A_j=(\tilde f_j)_*$ that the determinants of $A_i$, $A_j$ have the same sign. Now condition (5) follows from (4).

Take now an abstract graph $\gamma\in\Gamma^k$. We will show that there are a closed orientable $n$-manifold $M^n$, $n\geqslant 3$, and a diffeomorphism $f\in\mathbb{G}_k^{cod1}(M^n)$ such that $\gamma=\Gamma(f)$. By the description of $\Gamma^k$, every group of vertices $V_i=\{v_1^i,\dots,v_{k_i}^i\}$ is endowed with a triple $(A_i,P_i,\epsilon_i)$ where $A_i:\mathbb{T}^n\to\mathbb{T}^n$ is Anosov automorphism with a finite invariant set of periodic points $P_i$, and $\epsilon_i=a$ provided the stable manifolds of $A_i$ is one-dimensional, while $\epsilon_i=r$ provided the unstable manifolds of $A_i$ is one-dimensional. Using Smale's surgery operation [32] (see also [14, 26]), one can construct a DA-diffeomorphism $f_i:\mathbb{T}^n\to\mathbb{T}^n$ with a codimension one orientable connected basic set $\Omega_i$ containing $|P_i|=k_i$ bunches. Moreover, if $\epsilon_i=a$ then $\Omega_i$ is an expanding attractor, and if $\epsilon_i=r$ then $\Omega_i$ is a contracting repeller. In addition, every bunch corresponds to some point $p_s^i\in P_i$ and vertex $v_s^i=\psi^{-1}(p_s^i)\in V_i$. According to [12–14, 26], the triple $(A_i,P_i,\epsilon_i)$ is a complete invariant of conjugacy for the diffeomorphism $f_i$. Recall that every component of $\mathbb{T}^n\setminus\Omega_i$ contains a unique isolated node periodic point surrounded by a characteristic sphere of the corresponding bunch.

Let us take $k$ copies $\mathbb{T}_1^n,\dots,\mathbb{T}_k^n$ of $\mathbb{T}^n$. It is convenient to consider $f_i:\mathbb{T}_i^n\to\mathbb{T}_i^n$ defined on $\mathbb{T}_i^n$, $i=1,\dots,k$. Due to condition (2) of the description of $\Gamma^k$, every vertex $v_s^i=\psi^{-1}(p_s^i)$ is connected with a unique vertex $v_t^j=\psi_j^{-1}(p_t^j)$ by an edge $L(p_s^i,p_t^j)\subset\gamma$ where $i\neq j$. According to condition (3), the vertex $v_t^j$ belongs to a group $V_j$ endowed with a triple $(A_j,P_j,\epsilon_j)$ where $p_t^j=\psi_j(v_t^j)\in P_j$, $\epsilon_i\neq\epsilon_j$. For definiteness, assume that $\epsilon_i=a$ and $\epsilon_j=r$, i. e., $\Omega_i$ is an attractor and $\Omega_j$ is a repeller.

It follows from Lemma 3 that there is an attracting neighborhood $U(\Omega_i)\subset W^s(\Omega_i)$ of $\Omega_i$ such that the set $\mathbb{T}_i^n\setminus U(\Omega_i)$ is the union of pairwise disjoint $n$-disks $B_1^i,\dots,B_{k_i}^i$, and the boundary $\partial U(\Omega_i)$ is the union of characteristic spheres $\widehat S_1^i=\partial B_1^i,\dots,\widehat S_{k_i}^i=\partial B_{k_i}^i$, $s=1,\dots,k_i$. Since $f_i$ is a DA-diffeomorphism, every $n$-disk $B_m^i$ contains a unique source periodic point. One of them, denoted by $q_s^i$, corresponds to the vertex $v_s^i$. Without loss of generality, one can assume that $q_s^i\in B_1^i$. Since $f_i\left(U(\Omega_i)\right)\subset U(\Omega_i)$, $B_1^i\subset f_i^{r_s^i}(B_1^i)$ where $r_s^i\in\mathbb{N}$ is a period of the point $q_s^i$ under $f_i$. Clearly, the orbit $O(q_s^i)$ of $q_s^i$ does not belong to $U(\Omega_i)$. Similarly, there is an attracting neighborhood $U(\Omega_j)\subset W^u(\Omega_j)$ of the basic set $\Omega_j$ such that the set $\mathbb{T}_j^n\setminus U(\Omega_j)$ is the union of pairwise disjoint $n$-disks $B_1^j,\dots,B_{k_j}^j$, and the boundary $\partial U(\Omega_j)$ is the union of characteristic spheres $\widehat S_1^j,\dots,\widehat S_{k_j}^j$. Every $n$-disk $B_m^j$ contains a unique sink periodic point. One of them, denoted by $q_t^j$, corresponds to the vertex $v_t^j$. Without loss of generality, one can assume that $q_t^j\in B_1^j$. It follows from the existence of the edge $L(p^i_s, p^j_t)$ and condition (3) that there exist the following $r^i_s$ edges:

$$L\left(\psi_i^{-1}\big(A_i^m(p_s^i)\big), \psi_j^{-1}\big(A_j^m(p_t^j)\big)\right), \quad m = 1, \ldots, r_s^i,$$

which connect the points $\psi_i^{-1}\big(A_i^m(p_s^i)\big)$, $\psi_j^{-1}\big(A_j^m(p_t^j)\big)$, $m = 1, \ldots, r_s^i$, in $\gamma$. Recall that condition (2) means that any vertex of $\gamma \in \Gamma^k$ has degree 1. Therefore, the period of the point $p^j_t \in P_j$ equals $r^i_s$ under the automorphism $A_j$. As a consequence, the period of the point $q^j_t$ also equals $r^i_s$ under $f_j$.

First, we consider the case $r^i_s = 1$, i. e., the points $q^i_s$, $q^j_t$ are fixed points. Then $B^i_1 \subset f_i(B^i_1)$, $f_j(B^j_1) \subset B^j_1$. Let us delete the disks $B^i_1$, $f_j(B^j_1)$ from $\mathbb{T}^n_i$, $\mathbb{T}^n_j$, respectively. Take an $n$-annulus $K_{ij}$, and glue its boundary component to $\partial B^i_1$, $\partial f_j(B^j_1)$ so that the set

$$\left(\mathbb{T}_i^n \setminus B_1^i\right) \bigcup \left(\mathbb{T}_j^n \setminus f_j(B_1^j)\right) \bigcup K_{ij} = M_{ij}^n$$

becomes a smooth closed orientable manifold. Since $B^i_1 \subset f_i(B^i_1)$ and $f_j(B^j_1) \subset B^j_1$, the topological closures $K_i$, $K_j$ of the sets $f_i(B^i_1) \setminus B^i_1$, $B^j_1 \setminus f_j(B^j_1)$, respectively, are $n$-annuli. Therefore, $K_{ij} \cup K_i$ is an $n$-annulus with two boundary components $\partial f_i(B^i_1)$, $\partial f_j(B^j_1)$, while the union $K_{ij} \cup K_j$ is an $n$-annulus with the boundary components $\partial B^i_1$, $\partial B^j_1$. By condition (5), the determinants of $A_i$, $A_j$ have the same sign. This implies that the restrictions $f_i|_{\partial B^i_1}$, $f_j|_{\partial B^j_1}$ either both preserve orientation or both reverse orientation. Due to [7], Theorem 14.5, these restrictions are isotopic. Therefore, there is a mapping

$$\varphi_{ij} : K_{ij} \cup K_j \to K_{ij} \cup K_i \ \text{ such that } \ \varphi_{ij}|_{\partial B_1^i} = f_i|_{\partial B_1^i},\ \varphi_{ij}|_{\partial B_1^j} = f_j|_{\partial B_1^j}.$$

Since $K_{ij} \cup K_j$ is an $n$-annulus, we can define $\varphi_{ij}$ so that all points on $K_{ij} \cup K_j \cup K_i$ move from $\partial B^j_1$ to $\partial B^i_1$ under positive iteration of $\varphi_{ij}$. Moreover, $\varphi_{ij}$ can be made to agree with the restrictions $f_i|_{\partial B^i_1}$, $f_j|_{\partial B^j_1}$ near $\partial B^i_1$, $\partial B^j_1$, respectively, so that a mapping

$$f_{ij}|_{\mathbb{T}_i^n \setminus B_1^i} = f_i|_{\mathbb{T}_i^n \setminus B_1^i},\ f_{ij}|_{\mathbb{T}_j^n \setminus B_1^j} = f_j|_{\mathbb{T}_j^n \setminus B_1^j},\ f_{ij}|_{K_{ij} \cup K_j} = \varphi_{ij}$$

becomes a diffeomorphism $f_{ij} : M^n_{ij} \to M^n_{ij}$. Keeping in mind the property of $\varphi_{ij}|_{K_{ij} \cup K_j}$, we see that $f_{ij}$ has no nonwandering points on $K_{ij} \cup K_j \cup K_i$. Hence, $f_{ij}$ is an A-diffeomorphism whose nonwandering set consists of the orientable connected codimension basic sets $\Omega_i$, $\Omega_j$ and trivial basic sets of the diffeomorphisms $f_i$, $f_j$ without the points $q^i_s$, $q^j_t$. In a sense, the $n$-annulus $K_{ij}$ corresponds to the edge $L(p^i_s, p^j_t)$.

Now let us consider the case $r^i_s \geqslant 2$. It follows from the inclusion $B^i_1 \cap \Omega_i = \emptyset$ that the pairwise disjoint disks $B^i_1$, $f_i(B^i_1)$, $\ldots$, $f^{r^i_s - 1}(B^i_1)$ have no intersections with $\Omega_i$. Similarly, the pairwise disjoint disks $B^j_1$, $f_j(B^j_1)$, $\ldots$, $f_j^{r^i_s - 1}(B^j_1)$ have no intersections with $\Omega_j$. Since $B^i_1 \subset f_i^{r^i_s}(B^i_1)$ and $f_j^{r^i_s}(B^j_1) \in B^j_1$, the topological closure $K_i$, $K_j$ of $f_i^{r^i_s}(B^i_1) \setminus B^i_1$, $B^j_1 \setminus f_j^{r^i_s}(B^j_1)$, respectively, are $n$-annuli. Let us delete the disks $B^i_1$, $f_j^{r^i_s}(B^j_1)$ from $\mathbb{T}^n_i$, $\mathbb{T}^n_j$ respectively, and glue $\partial B^i_1$, $\partial f_j^{r^i_s}(B^j_1)$ with two boundary components of $K^{(0)}_{ij}$. Similarly, for every $l = 1, \ldots, r^i_s - 1$, let us delete the disks $f^l_i(B^i_1)$, $f^l_j(B^j_1)$ from $\mathbb{T}^n_i$, $\mathbb{T}^n_j$, respectively, and glue $\partial f^l_i(B^i_1)$, $\partial f^l_j(B^j_1)$ with two boundary components of $K^{(l)}_{ij}$ so that

$$\bigcup_{l=0}^{r_s^i - 1} K_{ij}^{(l)} \bigcup_{l=0}^{r_s^i - 1} \left(\mathbb{T}_i^n \setminus f_i^l(B_1^i)\right) \bigcup_{l=1}^{r_s^i} \left(\mathbb{T}_j^n \setminus f_j^l(B_1^j)\right) = M_{ij}^n$$

becomes a smooth closed orientable manifold. Again, by condition (5), the determinants of $A_i$, $A_j$ have the same sign. Similarly to the case $r^i_s = 1$ above, one can introduce the following diffeomorphisms:

$$\varphi_{ij}^{(1)} : K_{ij}^{(0)} \cup K_j \to K_{ij}^{(1)} \text{ such that } \varphi_{ij}^{(1)}|_{\partial B_1^i} = f_i|_{\partial B_1^i},\ \varphi_{ij}^{(1)}|_{\partial B_1^j} = f_j|_{\partial B_1^j},$$

$$\varphi_{ij}^{(l)} : K_{ij}^{(l-1)} \to K_{ij}^{(l)} \text{ such that } \varphi_{ij}^{(l)}|_{\partial f_i^{l-1}(B_1^i)} = f_i|_{\partial f_i^{l-1}(B_1^i)},$$

$$\varphi_{ij}^{(l)}|_{\partial f_j^{l-1}(B_1^j)} = f_j|_{\partial f_j^{l-1}(B_1^j)},\ l = 2, \ldots, r_s^i - 1, \text{ provided } r_s^i \geqslant 3,$$

$$\varphi_{ij}^{(r_s^i)} : K_{ij}^{(r_s^i-1)} \to K_{ij}^{(0)} \cup K_j \text{ such that } \varphi_{ij}^{r_s^i-1}|_{\partial f_i^{r_s^i-1}(B_1^i)} = f_i|_{\partial f_i^{r_s^i-1}(B_1^i)},$$

$$\varphi_{ij}^{r_s^i-1}|_{\partial f_j^{r_s^i-1}(B_1^j)} = f_j|_{\partial f_j^{r_s^i-1}(B_1^j)}, \text{ provided } r_s^i \geqslant 2$$

which generate together with the restrictions $f_i|_{\mathbb{T}_i^n \setminus \cup_{l=0}^{r_s^i-1} f_i^l(B_1^i)}$, $f_j|_{\mathbb{T}_j^n \setminus \cup_{l=0}^{r_s^i-1} f_j^l(B_1^j)}$ an A-diffeomorphism $f_{ij} : M_{ij}^n \to M_{ij}^n$ whose nonwandering set consists of the orientable connected codimension basic sets $\Omega_i$, $\Omega_j$ and trivial basic sets of the diffeomorphisms $f_i$, $f_j$ without the orbits of the points $q_s^i$, $q_t^j$. Continuing in this way for other vertices and edges of the graph $\gamma$, one gets a manifold $M^n$ and a diffeomorphism $f \in \mathbb{G}_k^{cod1}(M^n)$ as desired. □

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## CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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