# On isolated periodic points of diffeomorphisms with expanding attractors of codimension 1 <br> Marina Barinova, HSE University 


#### Abstract

In the paper we consider an $\Omega$-stable 3 -diffeomorphism, chain recurrent set of which consists of isolated periodic points and expanding attractors of codimension 1, orientable or not. We estimate a minimum number of isolated periodic points using information about the structure of the attractors.


## 1 Introduction and formulation of results

Let $M^{n}$ be a closed smooth connected $n$-manifold with a metric $d$ and $f$ : $M^{n} \rightarrow M^{n}$ be a diffeomorphism. An invariant compact set $\Lambda \subset M^{n}$ is called hyperbolic if there is a continuous $D f$-invariant splitting of the tangent bundle $T_{\Lambda} M^{n}$ into stable and unstable subbundles $E_{\Lambda}^{s} \oplus E_{\Lambda}^{u}$, $\operatorname{dim} E_{x}^{s}+\operatorname{dim} E_{x}^{u}=n$ $(x \in \Lambda)$ such that for natural $k$ and for some fixed $C_{s}>0, C_{u}>0,0<\lambda<1$

$$
\begin{array}{ll}
\left\|D f^{k}(v)\right\| \leq C_{s} \lambda^{k}\|v\|, & v \in E_{\Lambda}^{s}, \\
\left\|D f^{-k}(w)\right\| \leq C_{u} \lambda^{k}\|w\|, & w \in E_{\Lambda}^{u} .
\end{array}
$$

Recall that $\varepsilon$-chain of the length $m \in \mathbb{N}$, joining points $x, y \in M^{n}$, for $f$ is called a collection of points $x=x_{0}, \ldots, x_{m}=y$ such that $d\left(f\left(x_{i-1}\right), x_{i}\right)<\varepsilon$ for $1 \leqslant i \leqslant m$. A point $x \in M^{n}$ is called chain recurrent for $f$ if for any $\varepsilon>0$ there exists $m$, depending on $\varepsilon>0$, and an $\varepsilon$-chain of length $m$, joining $x$ to itself. The set of all chain recurrent points is called chain recurrent set and is denoted by $\mathcal{R}_{f}$.

Summarizing the results in [1, [2], 3, , 4] we know that the hyperbolicity of $\mathcal{R}_{f}$ is equivalent to $\Omega$-stability of $f$, that is small perturbations of $f$ preserve the chain recurrent set (equivalently non-wandering set $N W(f)$ ) structure. Thus, by [5], $\mathcal{R}_{f}$ consists of a finite number of pairwise disjoint sets, called basic sets, each of which is compact, invariant, and topologically transitive (contains a dense orbit). If a basic set is a periodic orbit, then it is named trivial. In the opposite case, it is non-trivial. If $\operatorname{dim} \Lambda=n-1$ for some basic set $\Lambda$ then it is called a basic set of co-dimension 1 .

A stable and unstable manifolds of a point $x \in \Lambda$, where $\Lambda$ is a basic set, can be defined in the following way:

$$
\begin{aligned}
& W_{x}^{s}=\left\{y \in M^{n} \mid \lim _{k \rightarrow+\infty} d\left(f^{k}(x), f^{k}(y)\right)=0\right\} \\
& W_{x}^{u}=\left\{y \in M^{n} \mid \lim _{k \rightarrow+\infty} d\left(f^{-k}(x), f^{-k}(y)\right)=0\right\}
\end{aligned}
$$

By [5], $W_{x}^{s}$ and $W_{x}^{u}$ are injective immersions of $\mathbb{R}^{q}$ and $\mathbb{R}^{n-q}$, accordingly, for some $q \in\{0,1, \ldots, n\}$. For $r>0$ we denote by $W_{x, r}^{s}$ and $W_{x, r}^{u}$ the immersions of discs $D_{r}^{q} \subset \mathbb{R}^{q}$ and $D_{r}^{n-q} \subset \mathbb{R}^{n-q}$.

The concept of orientability can be introduced for a basic set $\Lambda$ with $\operatorname{dim} W_{x}^{s}=1$ or $\operatorname{dim} W_{x}^{u}=1, x \in \Lambda$. A non-trivial basic set $\Lambda$ is called orientable if for any point $x \in \Lambda$ and any fixed numbers $\alpha>0, \beta>0$ the intersection index $]^{1} W_{x, \alpha}^{u} \cap W_{x, \beta}^{s}$ is the same at all intersection points ( +1 or -1) [7]. Otherwise, the basic set is called non-orientable.

A basic set $\Lambda$ is called an if it has a compact trapping neighborhood $U$, such that $f(U) \subset$ int $U$ and $\bigcap_{n=1}^{+\infty} f^{n}(U)=\Lambda$. Each hyperbolic attractor consists of unstable manifolds of its points by [8]. If $\operatorname{dim} \Lambda=\operatorname{dim} W_{x}^{u}, x \in \Lambda$, for a hyperbolic attractor $\Lambda$, then it is expanding.

Any co-dimension 1 expanding attractor $\Lambda$ divides its basin $W_{\Lambda}^{s}$ into a finite number of connected components. Every such a component $B$ determines $a$ bunch $b$ as the union of unstable manifolds of all periodic points from $\Lambda$ whose stable separatrix belongs to $B$. The number $k$ of such socalled boundary points is finite and it is called $a$ degree of the bunch $b$ and $b$ is called $k$-bunch with the basin $B$.

[^0]If $n \geqslant 3$ then, by [9][Theorem 2.1], any co-dimension 1 expanding attractor $\Lambda$ has 1-bunches or 2-bunches only. Moreover, the following fact takes place.

Statement 1.1. If $\Lambda$ is a hyperbolic expanding attractor of co-dimension 1 of a diffeomorphism $f: M^{n} \rightarrow M^{n}$ given on a closed smooth $n$-manifold $M^{n}$, then $\Lambda$ is non-orientable iff it has an 1-bunch.

In the paper we consider diffeomorphisms every non-trivial basic set of which is an expanding attractor of codimension 1 , and investigate the properties of such diffeomorphisms and the structure of their ambient manifolds. The main result is the following theorem.

Theorem 1. Let $f: M^{3} \rightarrow M^{3}$ be an $\Omega$-stable diffeomorphism, given on a closed 3-manifold, $\Lambda$ be a non-empty set of non-trivial basic sets of $f$. If $\Lambda$ consists of expanding attractors of codimension 1 having a total of $k_{1}$ bunches of degree 1 and $k_{2}$ bunches of degree 2, then the number of points in the set $N W(f) \backslash \Lambda$ no less then $\frac{3}{2} k_{1}+k_{2}$ and this estimate is exact.

Corollary 1. If the non-wandering set $N W(f)$ of an $\Omega$-stable diffeomorphism $f: M^{3} \rightarrow M^{3}$ consists of 2-dimensional expanding attractors with $k$ bunches in total and $k$ isolated periodic points, then

- each non-trivial attractor and $M^{3}$ are orientable;
- $\operatorname{dim} W_{p}^{u}=1$ for every isolated saddle point $p$;
- each connected component of the set $M^{3} \backslash \Lambda$ is homeomorphic to a punctured 3-sphere.

It is clear from corollary 1, that in a subclass of diffeomorphism with orientable $\Lambda$ and non-orientable $M^{3}$ the estimates from Theorem 1 can not be reached. For this case the following theorem takes place.

Theorem 2. Let an $\Omega$-stable diffeomorphism $f: M^{3} \rightarrow M^{3}$ be given on closed non-orientable manifold $M^{3}$ and a set of non-trivial basic sets consists of expanding orientable 2-dimensional attractors having a total of $k$ bunches, then the number of isolated periodic points is no less than $k+2$.

A simple structure of the orbit space of the restriction of $f$ to the set $W_{\Lambda}^{s} \backslash \Lambda$ gives us a way to obtain an $\Omega$-stable system without non-trivial basic
sets from considered one. We will describe a procedure of transition from a cascade with codimension 1 expanding attractors to a corresponding regular system in a section 2. A section 3 gives a proof of estimates from theorem 1 and theorem 2, A proof of corollary 11 is directly follows from the proof of theorem 1. Finally, in the section 4 we show that estimates are exact.

## 2 Transition to a regular system

In this section we will show how to obtain a system $\widetilde{f}: \widetilde{M^{3}} \rightarrow \widetilde{M}^{3}$ with regular dynamics from a system $f: M^{3} \rightarrow M^{3}$ with codimension 1 expanding attractors and isolated periodic points.

(a) Bunch of degree 1

(b) Bunch of degree 2

Figure 1: Components of the boundary of a trapping neighborhood near bunches of different degrees

Let $\Lambda$ be a set of non-trivial attractors of $f$ and $U_{\Lambda}$ be its trapping neighborhood. The boundary of $U_{\Lambda}$ consists of $k_{1}$ copies of $\mathbb{R} P^{2}$ and $k_{2}$ copies of $\mathbb{S}^{2}\left([10][\right.$ lemma 2.2] $)$ as in figure 1. Let $M^{3} \backslash \operatorname{int} U_{\Lambda}=M^{+} \sqcup M^{-}$, where $M^{+}$ and $M^{-}$are compact subsets of $M^{3}$ (one of them can be empty) such that $\partial M^{+}$consists of $k^{+} 2$-spheres, $\partial M^{-}$consists of $k_{1}^{-}>0$ copies of $\mathbb{R} P^{2}$ and $k_{2}^{-}$ copies of $\mathbb{S}^{2}$. Notice, that each connected component of $M^{-}$is non-orientable [11] and hence there exists a double cover $\pi: \widehat{M}^{-} \rightarrow M^{-}$[12], such that $\partial \widehat{M}^{-}$
consists of $\hat{k}^{-}=k_{1}^{-}+2 k_{2}^{-} 2$-spheres. There is the following division of $\widetilde{M^{3}}$ on disjoint closed submanifolds $\widetilde{M}^{+}$and $\widetilde{M}^{-}$:

- $\widetilde{M}^{+}=M^{+} \cup_{h^{+}}\left(D \times \mathbb{Z}_{k^{+}}\right)$, where $D=\left\{(x, y, z) \in \mathbb{R}^{3} \mid x^{2}+y^{2}+z^{2} \leqslant 1\right\}$, $h^{+}: \partial M^{+} \rightarrow \partial\left(D \times \mathbb{Z}_{k^{+}}\right)$is a diffeomorphism;
- $\widetilde{M}^{-}=\widehat{M}^{-} \cup_{h^{-}}\left(D \times \mathbb{Z}_{\hat{k}^{-}}\right), h^{-}: \partial \widehat{M}^{-} \rightarrow \partial\left(D \times \mathbb{Z}_{\hat{k}^{-}}\right)$is a diffeomorphism.

Let us introduce the following designations:

- $\mathcal{M}^{+}=\bigcup_{m=1}^{+\infty} f^{m}\left(M^{+}\right), \mathcal{M}^{-}=\bigcup_{m=1}^{+\infty} f^{m}\left(M^{-}\right)$;
- $\pi: \widehat{\mathcal{M}}^{-} \rightarrow \mathcal{M}^{-}$is a double cover of $\mathcal{M}^{-}$;
- $\widehat{\mathcal{M}}=\mathcal{M}^{+} \cup \widehat{\mathcal{M}}^{-}, k=k^{+}+\hat{k}^{-}$;
- $\widehat{f}: \widehat{\mathcal{M}} \rightarrow \widehat{\mathcal{M}}$ is a diffeomorphism such that $\left.\widehat{f}\right|_{\mathcal{M}^{+}}=\left.f\right|_{\mathcal{M}^{+}}$and $\left.\widehat{f}\right|_{\mathcal{M}^{-}}$is a lift of $\left.f\right|_{\mathcal{M}^{-}}$.

Let also $O$ be the centre of the disk $D$.
Theorem 3. There exists a diffeomorphism $\widetilde{f}: \widetilde{M}^{3} \rightarrow \widetilde{M^{3}}$, which has $k$ sinks at the points $O \times \mathbb{Z}_{k} \subset \widetilde{M}^{3}$ and $\left.\widetilde{f}\right|_{\widetilde{M}^{3} \backslash\left(O \times \mathbb{Z}_{k}\right)}$ is topologically conjugated with $\widehat{f}$.

Proof. Let $\mathcal{B}^{+}$and $\mathcal{B}^{-}$be sets of the bunch basins in the sets $\mathcal{M}^{+}$and $\mathcal{M}^{-}$ correspondingly. Let also $\widehat{\mathcal{B}}^{-}=\pi^{-1}\left(\mathcal{B}^{-}\right)$, and $\widehat{\mathcal{B}}=\mathcal{B}^{+} \cup \widehat{\mathcal{B}}^{-}$. Since bunch basins are periodic, then there exists a division of the set $\widehat{\mathcal{B}}$ on subsets $\widehat{\mathcal{B}}_{i}$, $i=1, \ldots, l$, each of which has a minimum natural number $m_{i}$ such that the set $\widehat{\mathcal{B}}_{i}=\bigcup_{j=1}^{m_{i}} f^{j}\left(\widehat{B}_{i}\right)$, where $\widehat{B}_{i}$ is some connected component of $\widehat{\mathcal{B}}$. Then $m_{1}+\cdots+m_{l}=k$. It follows from [10] that each $\widehat{B}_{i}$ is diffeomorphic to $\mathbb{S}^{2} \times \mathbb{R}$ and hence the orbit space of $\left.f\right|_{\widehat{\mathcal{B}}_{i}}$ is diffeomorphic to $\mathbb{S}^{2} \times \mathbb{S}^{1}$, if $\left.\widehat{f}^{m_{i}}\right|_{\widehat{B}_{i}}$ preserves orientation, or $\mathbb{S}^{2} \widetilde{\times} S^{1}$, if $\left.\widehat{f}^{m_{i}}\right|_{\widehat{B}_{i}}$ reverses one. Notice, that periodic hyperbolic sinks have the same orbit spaces in their basins.

Let $g_{i}: \mathbb{R}^{3} \times \mathbb{Z}_{m_{i}} \rightarrow \mathbb{R}^{3} \times \mathbb{Z}_{m_{i}}$ be a diffeomorphism with $m_{i}$ sinks at the origins $O \times \mathbb{Z}_{m_{i}}, g_{i}=\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, t+1 \bmod m_{i}\right)$, if $\left.\widehat{f}^{m_{i}}\right|_{\mathcal{B}_{i}}$ preserves orientation, and $g_{i}=\left(-\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, t+1 \bmod m_{i}\right)$ otherwise. Let also $h_{i}: \widehat{\mathcal{B}}_{i} \rightarrow\left(\mathbb{R}^{3} \backslash O\right) \times \mathbb{Z}_{m_{i}}$ be
diffeomorphisms, conjugated $\left.\widehat{f}\right|_{\widehat{\mathcal{B}_{i}}}$ with $\left.g_{i}\right|_{\left(\mathbb{R}^{3} \backslash O\right) \times \mathbb{Z}_{m_{i}}}$. Then diffeomorphisms $g: \mathbb{R}^{3} \times \mathbb{Z}_{k} \rightarrow \mathbb{R}^{3} \times \mathbb{Z}_{k}$ and $h: \widehat{B} \rightarrow\left(\mathbb{R}^{3} \backslash O\right) \times \mathbb{Z}_{k}$ can be composed of $g_{i}$ and $h_{i}$. Moreover, $h$ can be chosen in such a way that $h\left(U_{\Lambda}\right)=\mathbb{S}^{2} \times \mathbb{Z}_{k}$, where $\mathbb{S}^{2} \subset \mathbb{R}^{3}$ is a standard 2-sphere. Then $\widetilde{M^{3}}=\widehat{\mathcal{M}} \sqcup_{h}\left(\mathbb{R}^{3} \times \mathbb{Z}_{k}\right)$ with a natural projection $q: \widehat{\mathcal{M}} \sqcup\left(\mathbb{R}^{3} \times \mathbb{Z}_{k}\right) \rightarrow \widetilde{M^{3}}$. The desired diffeomorphism $\tilde{f}$ coincides with $q \widehat{f}\left(\left.q\right|_{\widehat{\mathcal{M}}}\right)^{-1}$ on the set $q(\widehat{\mathcal{M}})$ and with $q g\left(\left.q\right|_{\mathbb{R}^{3} \times \mathbb{Z}_{k}}\right)^{-1}$ on the sets $q\left(\mathbb{R}^{3} \times \mathbb{Z}_{k}\right)$. Notice, that by the construction $\widetilde{f}$ has $k$ sinks more then $\widehat{f}$.

## 3 Low estimate of trivial basic sets number

In this section we will prove the estimate from theorem 1. Let $f: M^{3} \rightarrow$ $M^{3}$ be an $\Omega$-stable diffeomorphism, given on a closed connected 3-manifold. Everywhere below in this section we will assume that all isolated periodic points and also boundary periodic points are fixed, because it does not affect the lower estimates: an appropriate degree of initial system satisfies these property and has the same number of isolated periodic points. Let $R_{f}=$ $\Lambda \cup p_{1} \cup p_{2} \cup \ldots \cup p_{m}$, where $\Lambda$ is a union of expanding attractors of codimension 1 with $k_{1}$ bunches of degree 1 and $k_{2}$ bunches of degree 2 in total and $p_{i}$ is a fixed point, $i \in\{1,2, \ldots, m\}$. Below we will prove that $m \geqslant \frac{3}{2} k_{1}+k_{2}$.

Proof. Via the transition to a regular system, described in section 2, we will obtain an $\Omega$-stable diffeomorphism $\widetilde{f}: \widetilde{M^{3}} \rightarrow \widetilde{M}^{3}$ with a finite chainrecurrent set on a closed manifold $\widetilde{M}^{3}$. Notice, that all chain-recurrent points of $\widetilde{f}$ are fixed. Let $M$ be a connected component of $\mathcal{M}=M^{3} \backslash \Lambda$. Notice that $M$ is $f$-invariant. There exists a connected component $\widetilde{M}$ of $\widetilde{M}^{3}$ corresponded to $M$.

Let us denote a number of 1- and 2-bunch basins, contained in $M$, as $l_{1}$ and $l_{2}$ correspondingly. $M$, and hence $\widetilde{M}$, can be one of 2 types (see section 22: (1) $M \subset \mathcal{M}^{+}$and (2) $M \subset \mathcal{M}^{-}$. In the first case $l_{1}=0$ and $\left.\widetilde{f}\right|_{\widetilde{M}}$ has $l_{2}$ sinks more than $\left.\widehat{f}\right|_{M}$. In the second case $l_{1}>0$ and even and $\left.\widetilde{f}\right|_{\widetilde{M}}$ has $l_{1}+2 l_{2}$ sinks more than $\left.\widehat{f}\right|_{\pi^{-1}(M)}$.

Let $C_{j}, j=0,1,2,3$, be a number of fixed points $p$ of $\left.\widetilde{f}\right|_{\widetilde{M}}$ with $\operatorname{dim} W_{p}^{u}=$ $j$, for example, $C_{0}$ be a number of sinks. Also $\left.\widetilde{f}\right|_{\widetilde{M}}$ has at least 1 source, since it is $\Omega$-stable. Then by the Lefschetz formula the alternating sum of $C_{j}$ is
equal to 0 :

$$
C_{3}-C_{2}+C_{1}-C_{0}=0
$$

At the same time since $\widetilde{M}$ is connected, then $C_{1}-C_{0}+1 \geqslant 0$ [13]. If $\widetilde{M}$ of the type (1) then $C_{0} \geqslant l_{2}>0$ and there is no additional restrictions. The finding of the minimum of the sum $C_{0}+C_{1}+C_{2}+C_{3}$ is a linear programming problem, it can be solved by a simplex method. Then the minimum of fixed points of $\left.\widetilde{f}\right|_{\widetilde{M}}$ can be reached if $C_{3}=1, C_{2}=0, C_{1}=l_{2}-1, C_{0}=l_{2}$. Therefore $\left.f\right|_{M}$ has at least $l_{2}$ isolated fixed points if $\widetilde{M}$ of the type (1). It follows from [14] that $\widetilde{M}$ is homeomorphic to $S^{3}$ in this case.

If $\widetilde{M}$ of the type (2) then $C_{1}, C_{2}$, and $C_{3}$ are even, because isolated periodic points of $\left.f\right|_{M}$ is doubled in this case. Also $C_{0} \geqslant l_{1}+2 l_{2}>0$. Without loss of generality we suppose that 1-dimensional separatrices of saddles do not intersect ${ }^{2}$. Then we can arrange points in the non-wandering set of $\left.\widetilde{f}\right|_{\widetilde{M}}$ agreed with Smale relation ${ }^{3}$ Moreover, the order can be chosen in such a way that each saddle of index 1 comes before all saddles of index 2 . Thus we have $\omega_{1} \prec \ldots \prec \omega_{C_{0}} \prec \sigma_{1} \prec \ldots \prec \sigma_{C_{1}} \prec \beta_{1} \prec \ldots \prec \beta_{C_{2}} \prec \alpha_{1} \prec \ldots \prec \alpha_{C_{3}}$, where each $\omega_{i}$ is a sink, each $\sigma_{i}$ is a saddle of index 1 , each $\beta_{i}$ is a saddle of index 2 , and each $\alpha_{i}$ is a source.

It follows from the paper [15] that a set $\mathcal{A}=\bigcup_{i=1}^{c_{1}} c l\left(W_{\sigma_{i}}^{u}\right)$ is 1-dimensional and connected. The double cover $\pi$ induces an involution $\varphi$ on the set $\mathcal{A} \backslash\left(\omega_{1} \cup \ldots \cup \omega_{C_{0}}\right)$ and can be extended by continuity on the whole $\mathcal{A}$. Moreover, a set of fixed points of the extended involution $\varphi$ coincides with the set of sinks corresponded to 1-bunches.

Let $\mathcal{A}^{*}=\mathcal{A} / \varphi$. Since a natural projection is a continuous map, then connectedness of $\mathcal{A}$ implies the connectedness of $\mathcal{A}^{*}$. $\mathcal{A}^{*}$ contains $\left(C_{0}+l_{1}\right) / 2$ sinks and hence it is needed at least $\left(C_{0}+l_{1}\right) / 2-1$ saddles of index 1. Therefore $\mathcal{A}$ contains at least $\left(C_{0}+l_{1}-2\right)$ saddles of index 1, i.e. $C_{1} \geqslant$ $C_{0}+l_{1}-2$.

Let us solve a linear programming task for this case:

[^1]\[

$$
\begin{aligned}
& C_{3}-C_{2}+C_{1}-C_{0}=0 \\
& C_{0} \geqslant l_{1}+2 l_{2} \\
& C_{1}-C_{0} \geqslant l_{1}-2 \\
& C_{3} \geqslant 2
\end{aligned}
$$
\]

The optimal values are: $C_{0}=l_{1}+2 l_{2}, C_{1}=2 l_{1}+2 l_{2}-2, C_{2}=l_{1}$, and $C_{3}=2$. Then there are at least $l_{1}+l_{2}-1$ saddles of index $1, l_{1} / 2$ saddles of index 2 , and 1 source at the component $M$.

Summing over all connected components of $\mathcal{M}$ we obtain that $f$ has at least $\frac{3}{2} k_{1}+k_{2}$ isolated periodic points: at least $s$ sources, $\left(k_{1}+k_{2}-s\right)$ saddles of index 1 , and $k_{1} / 2$ saddles of index 2 , where $s$ is a number of connected components of $\mathcal{M}$.

Below we will prove theorem 2 ,
Proof. If $M^{3}$ is non-orientable, but $\Lambda$ contains only orientable attractors, then by [9] $W_{\Lambda}^{s}$ is homeomorphic to a punctured 3 -torus, $\mathcal{M}^{-}=\varnothing$, and there exists a non-orientable connected component $M$ of the set $\mathcal{M}^{+}$. Then corresponded manifold $\widetilde{M}$ is also non-orientable, and $\left.\widetilde{f}\right|_{\widetilde{M}}$ has saddles of different indices [14, that is $C_{2}>0$ and $C_{1}>0$. There are two optimal possibilities: 1 source, 1 saddle of index $2, l_{2}$ saddles of index 1 , and $l_{2}$ sinks or 2 source, 1 saddle of index $2, l_{2}-1$ saddles of index 1 , and $l_{2}$ sinks, $-\underset{\sim}{f}$ the both possibilities a total number of points in non-wandering set of $\left.\widetilde{f}\right|_{\widetilde{M}}$ is $2 l_{2}+2$, so $\left.f\right|_{M}$ has at least $l_{2}+2$ isolated periodic points.

## 4 Achievability of the estimates

Realizations of diffeomorphisms with a minimum number of trivial basic sets are given in this section, i.e. we will prove the second part of theorems 1 and 2. First of all, we will answer on a question: how to obtain an $\Omega$-stable cascade $f: M^{3} \rightarrow M^{3}$ with a set of expanding attractors of codimension 1 $\Lambda$ with $k_{1} \geqslant 0$ bunches of degree 1 and $k_{2} \geqslant 0$ bunches of degree 2 in total $\left(k_{1}+k_{2}>0\right)$ and $\frac{3}{2} k_{1}+k_{2}$ periodic points outside of $\Lambda$.

Let $f$ be a diffeomorphism of considered class with the following properties:

- all bunches and isolated periodic points are fixed;
- if $k_{2}>0$, than $M^{+}$is connected and has $k_{2}$ boundary components, otherwise $M^{+}$is empty;
- if $k_{1}>0$, than each non-trivial attractor has 1-bunches and $M^{-}$has $k_{1} / 2$ connected components, each of which is homeomorphic to $\mathbb{R} P^{2} \times$ $[-1,1]$.

Corresponding regular system $\left.\widetilde{f}\right|_{\widetilde{M}^{+}}$for the set $M^{+}$realizing the minimum can be as in figure 2. It has $k_{2}$ sinks, $k_{2}-1$ saddles, and 1 source.


Figure 2: Morse-Smale system for $\mathcal{M}^{+}$, realizing low estimates
If $k_{1}>0$, all bunches of degree 1 are divided into pairs in such a way that after gluing the cylinders $\mathbb{R} P^{2} \times[-1,1]$ to a trapping neighborhood of $\Lambda$ we will obtain a connected manifold $M^{3}$. Let the restriction $\left.f\right|_{M}$ of the desired diffeomorphism $f$ on each connected component $M$ of $\mathcal{M}^{-}$is topologically conjugated to a diffeomorphism $\left(g_{1} \times g_{2}\right)$, where $g_{1}: \mathbb{R} P^{2} \rightarrow \mathbb{R} P^{2}$ is on figure 3 and $g_{2}: \mathbb{R} \rightarrow \mathbb{R}$ such that $g_{2}(x)=2 x$.

Achievability of the estimate from theorem 2 we will show with a diffeomorphism $f: M^{3} \rightarrow M^{3}$ with the following properties:

- $f$ has only 1 non-trivial attractor $\Lambda$, which is connected and has $k_{2}$ bunches of degree 2;
- all bunches and isolated periodic points of $f$ is fixed;
- a set $M^{+}$consists of $k_{2}$ connected components.


Figure 3: Morse-Smale system on $\mathbb{R} P^{2}$

Let a corresponded regular system $\tilde{f}: \widetilde{M^{3}} \rightarrow \widetilde{M^{3}}$ be given on $\mathbb{S}^{3} \times \mathbb{Z}_{k_{2}-1} \sqcup$ $\mathbb{S}^{2} \widetilde{\times} \mathbb{S}^{1}$, the dynamics on each 3 -sphere be "sink-source" and on the $\mathbb{S}^{2} \widetilde{\times} \mathbb{S}^{1}$ be as on the figure 4. Therefore $f: M^{3} \rightarrow M^{3}$ has exactly $k_{2}+2: k_{2}$ sources and 2 saddles of different indices, - isolated chain recurrent points and $M^{3}$ is non-orientable.


Figure 4: Morse-Smale system on $\mathbb{S}^{2} \widetilde{\times} \mathbb{S}^{1}$
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[^0]:    ${ }^{1}$ Let $J^{k}: \mathbb{R}^{k} \rightarrow M^{3}$ be immersions, $D^{k}$ be open balls of finite radii in $\mathbb{R}^{k}, k=1,2$. Then the restrictions $J^{k}: D^{k} \rightarrow M$ are embeddings and their images $W^{k}=J^{k}\left(D^{k}\right)$ are smooth embedded submanifolds of the manifold $M^{3}$. Let $U^{k}$ be a tubular neighborhood of $W^{k}$, which are images of embeddings in $M^{3}$ of spaces of $(3-k)$-dimensional vector bundles on $W^{k}$ [6, Chapter 4, par. 5]. Since the balls $D^{k}$ are contractible, then these bundles are trivial and, hence, $U^{2} \backslash W^{2}$ consists of two connected components $U_{+}^{2}$ and $U_{-}^{2}$. It allows to define a function $\sigma: U_{+}^{2} \cup U_{-}^{2} \rightarrow \mathbb{Z}$, such that $\sigma(x)=1$ if $x \in U_{+}^{2}$ and $\sigma(x)=0$ if $x \in U_{-}^{2}$. If submanifolds $W^{1}$ and $W^{2}$ intersect transversally at a point $x=J^{1}(t), t \in D^{1}$, then there exists a number $\delta>0$ such that $J^{1}((t-2 \delta, t+2 \delta)) \subset U^{2}$. The number

    $$
    \operatorname{Ind}_{x}\left(W^{1}, W^{2}\right)=\sigma(t+\delta)-\sigma(t-\delta)
    $$

    is called an intersection index of submanifolds $W^{1}$ and $W^{2}$ in the point $x$. Notice, that this definition does not require orientability of the manifold $M^{3}$.

[^1]:    ${ }^{2}$ Each $\Omega$-stable diffeomorphism with finite chain-recurrent set has $\varepsilon$-close Morse-Smale diffeomorphism with the same amount of chain-recurrent points. Therefore we can consider this Morse-Smale diffeomorphism instead of initial one to calculate desired estimates.
    ${ }^{3}$ Let $\Lambda_{1}$ and $\Lambda_{2}$ be basic sets of an $\Omega$-stable diffeomorphism $f: M \rightarrow M . \Lambda_{1} \prec \Lambda_{2}$ if $W_{\Lambda_{1}}^{s} \cap W_{\Lambda_{2}}^{u} \neq \varnothing$.

