On isolated periodic points of diffeomorphisms with expanding attractors of codimension 1

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Abstract

In the paper we consider an Ω -stable 3-diffeomorphism, chain recurrent set of which consists of isolated periodic points and expanding attractors of codimension 1, orientable or not. We estimate a minimum number of isolated periodic points using information about the structure of the attractors.

1 Introduction and formulation of results

Let M^n be a closed smooth connected *n*-manifold with a metric *d* and f: $M^n \to M^n$ be a diffeomorphism. An invariant compact set $\Lambda \subset M^n$ is called *hyperbolic* if there is a continuous Df-invariant splitting of the tangent bundle $T_{\Lambda}M^n$ into stable and unstable subbundles $E^s_{\Lambda} \oplus E^u_{\Lambda}$, dim E^s_x + dim $E^u_x = n$ $(x \in \Lambda)$ such that for natural k and for some fixed $C_s > 0$, $C_u > 0$, $0 < \lambda < 1$

$$\begin{aligned} \|Df^k(v)\| &\leq C_s \lambda^k \|v\|, \qquad v \in E^s_\Lambda, \\ \|Df^{-k}(w)\| &\leq C_u \lambda^k \|w\|, \qquad w \in E^u_\Lambda. \end{aligned}$$

Recall that ε -chain of the length $m \in \mathbb{N}$, joining points $x, y \in M^n$, for f is called a collection of points $x = x_0, \ldots, x_m = y$ such that $d(f(x_{i-1}), x_i) < \varepsilon$ for $1 \leq i \leq m$. A point $x \in M^n$ is called *chain recurrent* for f if for any $\varepsilon > 0$ there exists m, depending on $\varepsilon > 0$, and an ε -chain of length m, joining x to itself. The set of all chain recurrent points is called *chain recurrent set* and is denoted by \mathcal{R}_f .

Summarizing the results in [1], [2], [3], [4] we know that the hyperbolicity of \mathcal{R}_f is equivalent to Ω -stability of f, that is small perturbations of f preserve the chain recurrent set (equivalently non-wandering set NW(f)) structure. Thus, by [5], \mathcal{R}_f consists of a finite number of pairwise disjoint sets, called *basic sets*, each of which is compact, invariant, and topologically transitive (contains a dense orbit). If a basic set is a periodic orbit, then it is named *trivial*. In the opposite case, it is *non-trivial*. If dim $\Lambda = n - 1$ for some basic set Λ then it is called a *basic set of co-dimension 1*. A stable and unstable manifolds of a point $x \in \Lambda$, where Λ is a basic set, can be defined in the following way:

$$\begin{split} W^s_x &= \{ y \in M^n \mid \lim_{k \to +\infty} d(f^k(x), f^k(y)) = 0 \}, \\ W^u_x &= \{ y \in M^n \mid \lim_{k \to +\infty} d(f^{-k}(x), f^{-k}(y)) = 0 \}. \end{split}$$

By [5], W_x^s and W_x^u are injective immersions of \mathbb{R}^q and \mathbb{R}^{n-q} , accordingly, for some $q \in \{0, 1, \ldots, n\}$. For r > 0 we denote by $W_{x,r}^s$ and $W_{x,r}^u$ the immersions of discs $D_r^q \subset \mathbb{R}^q$ and $D_r^{n-q} \subset \mathbb{R}^{n-q}$.

The concept of orientability can be introduced for a basic set Λ with $\dim W_x^s = 1$ or $\dim W_x^u = 1$, $x \in \Lambda$. A non-trivial basic set Λ is called *orientable* if for any point $x \in \Lambda$ and any fixed numbers $\alpha > 0$, $\beta > 0$ the intersection index¹ $W_{x,\alpha}^u \cap W_{x,\beta}^s$ is the same at all intersection points (+1 or -1) [7]. Otherwise, the basic set is called *non-orientable*.

A basic set Λ is called an if it has a compact trapping neighborhood U, such that $f(U) \subset int U$ and $\bigcap_{n=1}^{+\infty} f^n(U) = \Lambda$. Each hyperbolic attractor consists of unstable manifolds of its points by [8]. If dim $\Lambda = \dim W_x^u$, $x \in \Lambda$, for a hyperbolic attractor Λ , then it is *expanding*.

Any co-dimension 1 expanding attractor Λ divides its basin W^s_{Λ} into a finite number of connected components. Every such a component B determines a bunch b as the union of unstable manifolds of all periodic points from Λ whose stable separatrix belongs to B. The number k of such so-called boundary points is finite and it is called a degree of the bunch b and b is called k-bunch with the basin B.

$$\operatorname{Ind}_{x}(W^{1}, W^{2}) = \sigma(t+\delta) - \sigma(t-\delta)$$

¹Let $J^k : \mathbb{R}^k \to M^3$ be immersions, D^k be open balls of finite radii in \mathbb{R}^k , k = 1, 2. Then the restrictions $J^k : D^k \to M$ are embeddings and their images $W^k = J^k(D^k)$ are smooth embedded submanifolds of the manifold M^3 . Let U^k be a tubular neighborhood of W^k , which are images of embeddings in M^3 of spaces of (3-k)-dimensional vector bundles on W^k [6, Chapter 4, par. 5]. Since the balls D^k are contractible, then these bundles are trivial and, hence, $U^2 \setminus W^2$ consists of two connected components U^2_+ and U^2_- . It allows to define a function $\sigma : U^2_+ \cup U^2_- \to \mathbb{Z}$, such that $\sigma(x) = 1$ if $x \in U^2_+$ and $\sigma(x) = 0$ if $x \in U^2_-$. If submanifolds W^1 and W^2 intersect transversally at a point $x = J^1(t), t \in D^1$, then there exists a number $\delta > 0$ such that $J^1((t - 2\delta, t + 2\delta)) \subset U^2$. The number

is called an *intersection index* of submanifolds W^1 and W^2 in the point x. Notice, that this definition does not require orientability of the manifold M^3 .

If $n \ge 3$ then, by [9][Theorem 2.1], any co-dimension 1 expanding attractor Λ has 1-bunches or 2-bunches only. Moreover, the following fact takes place.

Statement 1.1. If Λ is a hyperbolic expanding attractor of co-dimension 1 of a diffeomorphism $f: M^n \to M^n$ given on a closed smooth n-manifold M^n , then Λ is non-orientable iff it has an 1-bunch.

In the paper we consider diffeomorphisms every non-trivial basic set of which is an expanding attractor of codimension 1, and investigate the properties of such diffeomorphisms and the structure of their ambient manifolds. The main result is the following theorem.

Theorem 1. Let $f: M^3 \to M^3$ be an Ω -stable diffeomorphism, given on a closed 3-manifold, Λ be a non-empty set of non-trivial basic sets of f. If Λ consists of expanding attractors of codimension 1 having a total of k_1 bunches of degree 1 and k_2 bunches of degree 2, then the number of points in the set $NW(f) \setminus \Lambda$ no less then $\frac{3}{2}k_1 + k_2$ and this estimate is exact.

Corollary 1. If the non-wandering set NW(f) of an Ω -stable diffeomorphism $f: M^3 \to M^3$ consists of 2-dimensional expanding attractors with k bunches in total and k isolated periodic points, then

- each non-trivial attractor and M^3 are orientable;
- dim $W_p^u = 1$ for every isolated saddle point p;
- each connected component of the set $M^3 \setminus \Lambda$ is homeomorphic to a punctured 3-sphere.

It is clear from corollary 1, that in a subclass of diffeomorphism with orientable Λ and non-orientable M^3 the estimates from Theorem 1 can not be reached. For this case the following theorem takes place.

Theorem 2. Let an Ω -stable diffeomorphism $f : M^3 \to M^3$ be given on closed non-orientable manifold M^3 and a set of non-trivial basic sets consists of expanding orientable 2-dimensional attractors having a total of k bunches, then the number of isolated periodic points is no less than k + 2.

A simple structure of the orbit space of the restriction of f to the set $W^s_{\Lambda} \setminus \Lambda$ gives us a way to obtain an Ω -stable system without non-trivial basic

sets from considered one. We will describe a procedure of transition from a cascade with codimension 1 expanding attractors to a corresponding regular system in a section 2. A section 3 gives a proof of estimates from theorem 1 and theorem 2. A proof of corollary 1 is directly follows from the proof of theorem 1. Finally, in the section 4 we show that estimates are exact.

2 Transition to a regular system

In this section we will show how to obtain a system $\tilde{f}: \widetilde{M}^3 \to \widetilde{M}^3$ with regular dynamics from a system $f: M^3 \to M^3$ with codimension 1 expanding attractors and isolated periodic points.

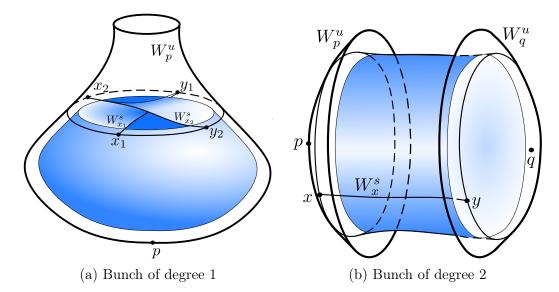


Figure 1: Components of the boundary of a trapping neighborhood near bunches of different degrees

Let Λ be a set of non-trivial attractors of f and U_{Λ} be its trapping neighborhood. The boundary of U_{Λ} consists of k_1 copies of $\mathbb{R}P^2$ and k_2 copies of \mathbb{S}^2 ([10][lemma 2.2]) as in figure 1. Let $M^3 \setminus int U_{\Lambda} = M^+ \sqcup M^-$, where M^+ and M^- are compact subsets of M^3 (one of them can be empty) such that ∂M^+ consists of k^+ 2-spheres, ∂M^- consists of $k_1^- > 0$ copies of $\mathbb{R}P^2$ and k_2^- copies of \mathbb{S}^2 . Notice, that each connected component of M^- is non-orientable [11] and hence there exists a double cover $\pi : \widehat{M}^- \to M^-$ [12], such that $\partial \widehat{M}^-$

consists of $\hat{k}^- = k_1^- + 2k_2^-$ 2-spheres. There is the following division of \widetilde{M}^3 on disjoint closed submanifolds \widetilde{M}^+ and \widetilde{M}^- :

- $\widetilde{M}^+ = M^+ \cup_{h^+} (D \times \mathbb{Z}_{k^+})$, where $D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$, $h^+ : \partial M^+ \to \partial (D \times \mathbb{Z}_{k^+})$ is a diffeomorphism;
- $\widetilde{M}^- = \widehat{M}^- \cup_{h^-} (D \times \mathbb{Z}_{\hat{k}^-}), h^- : \partial \widehat{M}^- \to \partial (D \times \mathbb{Z}_{\hat{k}^-})$ is a diffeomorphism.

Let us introduce the following designations:

- $\mathcal{M}^+ = \bigcup_{m=1}^{+\infty} f^m(M^+), \ \mathcal{M}^- = \bigcup_{m=1}^{+\infty} f^m(M^-);$
- $\pi: \widehat{\mathcal{M}}^- \to \mathcal{M}^-$ is a double cover of \mathcal{M}^- ;
- $\widehat{\mathcal{M}} = \mathcal{M}^+ \cup \widehat{\mathcal{M}}^-, \ k = k^+ + \hat{k}^-;$
- $\widehat{f}: \widehat{\mathcal{M}} \to \widehat{\mathcal{M}}$ is a diffeomorphism such that $\widehat{f}|_{\mathcal{M}^+} = f|_{\mathcal{M}^+}$ and $\widehat{f}|_{\mathcal{M}^-}$ is a lift of $f|_{\mathcal{M}^-}$.

Let also O be the centre of the disk D.

Theorem 3. There exists a diffeomorphism $\tilde{f} : \widetilde{M}^3 \to \widetilde{M}^3$, which has k sinks at the points $O \times \mathbb{Z}_k \subset \widetilde{M}^3$ and $\tilde{f}|_{\widetilde{M}^3 \setminus (O \times \mathbb{Z}_k)}$ is topologically conjugated with \hat{f} .

Proof. Let \mathcal{B}^+ and \mathcal{B}^- be sets of the bunch basins in the sets \mathcal{M}^+ and $\mathcal{M}^$ correspondingly. Let also $\widehat{\mathcal{B}}^- = \pi^{-1}(\mathcal{B}^-)$, and $\widehat{\mathcal{B}} = \mathcal{B}^+ \cup \widehat{\mathcal{B}}^-$. Since bunch basins are periodic, then there exists a division of the set $\widehat{\mathcal{B}}$ on subsets $\widehat{\mathcal{B}}_i$, $i = 1, \ldots, l$, each of which has a minimum natural number m_i such that the set $\widehat{\mathcal{B}}_i = \bigcup_{j=1}^{m_i} f^j(\widehat{B}_i)$, where \widehat{B}_i is some connected component of $\widehat{\mathcal{B}}$. Then $m_1 + \cdots + m_l = k$. It follows from [10] that each \widehat{B}_i is diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$ and hence the orbit space of $f|_{\widehat{\mathcal{B}}_i}$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, if $\widehat{f}^{m_i}|_{\widehat{B}_i}$ preserves orientation, or $\mathbb{S}^2 \times S^1$, if $\widehat{f}^{m_i}|_{\widehat{B}_i}$ reverses one. Notice, that periodic hyperbolic sinks have the same orbit spaces in their basins.

Let $g_i : \mathbb{R}^3 \times \mathbb{Z}_{m_i} \to \mathbb{R}^3 \times \mathbb{Z}_{m_i}$ be a diffeomorphism with m_i sinks at the origins $O \times \mathbb{Z}_{m_i}, g_i = (\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, t+1 \mod m_i)$, if $\widehat{f}^{m_i}|_{\widehat{\mathcal{B}}_i}$ preserves orientation, and $g_i = (-\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, t+1 \mod m_i)$ otherwise. Let also $h_i : \widehat{\mathcal{B}}_i \to (\mathbb{R}^3 \setminus O) \times \mathbb{Z}_{m_i}$ be

diffeomorphisms, conjugated $\widehat{f}|_{\widehat{\mathcal{B}}_i}$ with $g_i|_{(\mathbb{R}^3\setminus O)\times\mathbb{Z}_{m_i}}$. Then diffeomorphisms $g: \mathbb{R}^3 \times \mathbb{Z}_k \to \mathbb{R}^3 \times \mathbb{Z}_k$ and $h: \widehat{B} \to (\mathbb{R}^3 \setminus O) \times \mathbb{Z}_k$ can be composed of g_i and h_i . Moreover, h can be chosen in such a way that $h(U_\Lambda) = \mathbb{S}^2 \times \mathbb{Z}_k$, where $\mathbb{S}^2 \subset \mathbb{R}^3$ is a standard 2-sphere. Then $\widetilde{M}^3 = \widehat{\mathcal{M}} \sqcup_h (\mathbb{R}^3 \times \mathbb{Z}_k)$ with a natural projection $q: \widehat{\mathcal{M}} \sqcup (\mathbb{R}^3 \times \mathbb{Z}_k) \to \widetilde{M}^3$. The desired diffeomorphism \widetilde{f} coincides with $q\widehat{f}(q|_{\widehat{\mathcal{M}}})^{-1}$ on the set $q(\widehat{\mathcal{M}})$ and with $qg(q|_{\mathbb{R}^3 \times \mathbb{Z}_k})^{-1}$ on the sets $q(\mathbb{R}^3 \times \mathbb{Z}_k)$. Notice, that by the construction \widetilde{f} has k sinks more then \widehat{f} .

3 Low estimate of trivial basic sets number

In this section we will prove the estimate from theorem 1. Let $f: M^3 \to M^3$ be an Ω -stable diffeomorphism, given on a closed connected 3-manifold. Everywhere below in this section we will assume that all isolated periodic points and also boundary periodic points are fixed, because it does not affect the lower estimates: an appropriate degree of initial system satisfies these property and has the same number of isolated periodic points. Let $R_f = \Lambda \cup p_1 \cup p_2 \cup \ldots \cup p_m$, where Λ is a union of expanding attractors of codimension 1 with k_1 bunches of degree 1 and k_2 bunches of degree 2 in total and p_i is a fixed point, $i \in \{1, 2, \ldots, m\}$. Below we will prove that $m \ge \frac{3}{2}k_1 + k_2$.

Proof. Via the transition to a regular system, described in section 2, we will obtain an Ω -stable diffeomorphism $\tilde{f}: \widetilde{M}^3 \to \widetilde{M}^3$ with a finite chain-recurrent set on a closed manifold \widetilde{M}^3 . Notice, that all chain-recurrent points of \tilde{f} are fixed. Let M be a connected component of $\mathcal{M} = M^3 \setminus \Lambda$. Notice that M is f-invariant. There exists a connected component \widetilde{M} of \widetilde{M}^3 corresponded to M.

Let us denote a number of 1- and 2-bunch basins, contained in M, as l_1 and l_2 correspondingly. M, and hence \widetilde{M} , can be one of 2 types (see section 2): (1) $M \subset \mathcal{M}^+$ and (2) $M \subset \mathcal{M}^-$. In the first case $l_1 = 0$ and $\widetilde{f}|_{\widetilde{M}}$ has l_2 sinks more than $\widehat{f}|_M$. In the second case $l_1 > 0$ and even and $\widetilde{f}|_{\widetilde{M}}$ has $l_1 + 2l_2$ sinks more than $\widehat{f}|_{\pi^{-1}(M)}$.

Let C_j , j = 0, 1, 2, 3, be a number of fixed points p of $\tilde{f}|_{\widetilde{M}}$ with dim $W_p^u = j$, for example, C_0 be a number of sinks. Also $\tilde{f}|_{\widetilde{M}}$ has at least 1 source, since it is Ω -stable. Then by the Lefschetz formula the alternating sum of C_j is

equal to 0:

$$C_3 - C_2 + C_1 - C_0 = 0.$$

At the same time since \widetilde{M} is connected, then $C_1 - C_0 + 1 \ge 0$ [13]. If \widetilde{M} of the type (1) then $C_0 \ge l_2 > 0$ and there is no additional restrictions. The finding of the minimum of the sum $C_0 + C_1 + C_2 + C_3$ is a linear programming problem, it can be solved by a simplex method. Then the minimum of fixed points of $\widetilde{f}|_{\widetilde{M}}$ can be reached if $C_3 = 1$, $C_2 = 0$, $C_1 = l_2 - 1$, $C_0 = l_2$. Therefore $f|_M$ has at least l_2 isolated fixed points if \widetilde{M} of the type (1). It follows from [14] that \widetilde{M} is homeomorphic to S^3 in this case.

If \widetilde{M} of the type (2) then C_1 , C_2 , and C_3 are even, because isolated periodic points of $f|_M$ is doubled in this case. Also $C_0 \ge l_1 + 2l_2 > 0$. Without loss of generality we suppose that 1-dimensional separatrices of saddles do not intersect². Then we can arrange points in the non-wandering set of $\widetilde{f}|_{\widetilde{M}}$ agreed with Smale relation³. Moreover, the order can be chosen in such a way that each saddle of index 1 comes before all saddles of index 2. Thus we have $\omega_1 \prec \ldots \prec \omega_{C_0} \prec \sigma_1 \prec \ldots \prec \sigma_{C_1} \prec \beta_1 \prec \ldots \prec \beta_{C_2} \prec \alpha_1 \prec \ldots \prec \alpha_{C_3}$, where each ω_i is a sink, each σ_i is a saddle of index 1, each β_i is a saddle of index 2, and each α_i is a source.

It follows from the paper [15] that a set $\mathcal{A} = \bigcup_{i=1}^{c_1} cl(W^u_{\sigma_i})$ is 1-dimensional and connected. The double cover π induces an involution φ on the set $\mathcal{A} \setminus (\omega_1 \cup \ldots \cup \omega_{c_0})$ and can be extended by continuity on the whole \mathcal{A} . Moreover, a set of fixed points of the extended involution φ coincides with the set of sinks corresponded to 1-bunches.

Let $\mathcal{A}^* = \mathcal{A}/_{\varphi}$. Since a natural projection is a continuous map, then connectedness of \mathcal{A} implies the connectedness of \mathcal{A}^* . \mathcal{A}^* contains $(C_0 + l_1)/2$ sinks and hence it is needed at least $(C_0 + l_1)/2 - 1$ saddles of index 1. Therefore \mathcal{A} contains at least $(C_0 + l_1 - 2)$ saddles of index 1, i.e. $C_1 \ge C_0 + l_1 - 2$.

Let us solve a linear programming task for this case:

²Each Ω -stable diffeomorphism with finite chain-recurrent set has ε -close Morse-Smale diffeomorphism with the same amount of chain-recurrent points. Therefore we can consider this Morse-Smale diffeomorphism instead of initial one to calculate desired estimates.

³Let Λ_1 and Λ_2 be basic sets of an Ω -stable diffeomorphism $f: M \to M$. $\Lambda_1 \prec \Lambda_2$ if $W^s_{\Lambda_1} \cap W^u_{\Lambda_2} \neq \emptyset$.

$$\begin{array}{l} C_3 - C_2 + C_1 - C_0 = 0, \\ C_0 \geqslant l_1 + 2l_2, \\ C_1 - C_0 \geqslant l_1 - 2, \\ C_3 \geqslant 2. \end{array}$$

The optimal values are: $C_0 = l_1 + 2l_2$, $C_1 = 2l_1 + 2l_2 - 2$, $C_2 = l_1$, and $C_3 = 2$. Then there are at least $l_1 + l_2 - 1$ saddles of index 1, $l_1/2$ saddles of index 2, and 1 source at the component M.

Summing over all connected components of \mathcal{M} we obtain that f has at least $\frac{3}{2}k_1 + k_2$ isolated periodic points: at least s sources, $(k_1 + k_2 - s)$ saddles of index 1, and $k_1/2$ saddles of index 2, where s is a number of connected components of \mathcal{M} .

Below we will prove theorem 2.

Proof. If M^3 is non-orientable, but Λ contains only orientable attractors, then by [9] W^s_{Λ} is homeomorphic to a punctured 3-torus, $\mathcal{M}^- = \emptyset$, and there exists a non-orientable connected component M of the set \mathcal{M}^+ . Then corresponded manifold \widetilde{M} is also non-orientable, and $\widetilde{f}|_{\widetilde{M}}$ has saddles of different indices [14], that is $C_2 > 0$ and $C_1 > 0$. There are two optimal possibilities: 1 source, 1 saddle of index 2, l_2 saddles of index 1, and l_2 sinks or 2 source, 1 saddle of index 2, $l_2 - 1$ saddles of index 1, and l_2 sinks, — for the both possibilities a total number of points in non-wandering set of $\widetilde{f}|_{\widetilde{M}}$ is $2l_2 + 2$, so $f|_M$ has at least $l_2 + 2$ isolated periodic points.

4 Achievability of the estimates

Realizations of diffeomorphisms with a minimum number of trivial basic sets are given in this section, i.e. we will prove the second part of theorems 1 and 2. First of all, we will answer on a question: how to obtain an Ω -stable cascade $f: M^3 \to M^3$ with a set of expanding attractors of codimension 1 Λ with $k_1 \ge 0$ bunches of degree 1 and $k_2 \ge 0$ bunches of degree 2 in total $(k_1 + k_2 > 0)$ and $\frac{3}{2}k_1 + k_2$ periodic points outside of Λ .

Let f be a diffeomorphism of considered class with the following properties:

• all bunches and isolated periodic points are fixed;

- if $k_2 > 0$, than M^+ is connected and has k_2 boundary components, otherwise M^+ is empty;
- if $k_1 > 0$, than each non-trivial attractor has 1-bunches and M^- has $k_1/2$ connected components, each of which is homeomorphic to $\mathbb{R}P^2 \times [-1, 1]$.

Corresponding regular system $\tilde{f}|_{\tilde{M}^+}$ for the set M^+ realizing the minimum can be as in figure 2. It has k_2 sinks, $k_2 - 1$ saddles, and 1 source.

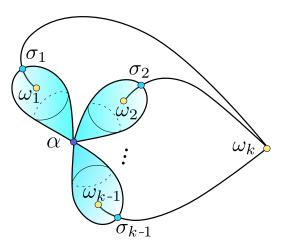


Figure 2: Morse-Smale system for \mathcal{M}^+ , realizing low estimates

If $k_1 > 0$, all bunches of degree 1 are divided into pairs in such a way that after gluing the cylinders $\mathbb{R}P^2 \times [-1, 1]$ to a trapping neighborhood of Λ we will obtain a connected manifold M^3 . Let the restriction $f|_M$ of the desired diffeomorphism f on each connected component M of \mathcal{M}^- is topologically conjugated to a diffeomorphism $(g_1 \times g_2)$, where $g_1 : \mathbb{R}P^2 \to \mathbb{R}P^2$ is on figure 3 and $g_2 : \mathbb{R} \to \mathbb{R}$ such that $g_2(x) = 2x$.

Achievability of the estimate from theorem 2 we will show with a diffeomorphism $f: M^3 \to M^3$ with the following properties:

- f has only 1 non-trivial attractor Λ , which is connected and has k_2 bunches of degree 2;
- all bunches and isolated periodic points of f is fixed;
- a set M^+ consists of k_2 connected components.

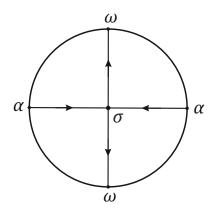


Figure 3: Morse-Smale system on $\mathbb{R}P^2$

Let a corresponded regular system $\tilde{f}: \widetilde{M}^3 \to \widetilde{M}^3$ be given on $\mathbb{S}^3 \times \mathbb{Z}_{k_2-1} \sqcup \mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$, the dynamics on each 3-sphere be "sink-source" and on the $\mathbb{S}^2 \widetilde{\times} \mathbb{S}^1$ be as on the figure 4. Therefore $f: M^3 \to M^3$ has exactly $k_2 + 2$: k_2 sources and 2 saddles of different indices, - isolated chain recurrent points and M^3 is non-orientable.

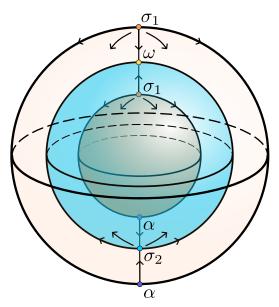


Figure 4: Morse-Smale system on $\mathbb{S}^2 \times \mathbb{S}^1$

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