Russian Journal of Nonlinear Dynamics, 2024, vol. 20, no. 1, pp. 167–178.
Full-texts are available at http://nd.ics.org.ru
DOI: 10.20537/nd240301

MATHEMATICAL PROBLEMS OF NONLINEARITY

MSC 2010: 34C45, 37D15, 37C05, 70K44

Morse – Smale 3-Diffeomorphisms with Saddles of the Same Unstable Manifold Dimension

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In this paper, we consider a class of Morse – Smale diffeomorphisms defined on a closed 3-manifold (not necessarily orientable) under the assumption that all their saddle points have the same dimension of the unstable manifolds. The simplest example of such diffeomorphisms is the well-known "source-sink" or "north pole – south pole" diffeomorphism, whose non-wandering set consists of exactly one source and one sink. As Reeb showed back in 1946, such systems can only be realized on the sphere. We generalize his result, namely, we show that diffeomorphisms from the considered class also can be defined only on the 3-sphere.

Keywords: Morse – Smale diffeomorphisms, ambient manifold topology, invariant manifolds, heteroclinic orbits, hyperbolic dynamics

Introduction and formulation of the results

The class of dynamical systems, introduced by S. Smale in 1960 [15] and known today as Morse – Smale systems, played not the least role in the formation of the modern dynamical systems theory. The study of these systems remains an important part of it because they form a class of structurally stable systems which, in addition, have zero topological entropy [3, 11, 13], which makes them in this sense "the simplest" structurally stable systems.

A close relation of the Morse – Smale diffeomorphisms (MS-diffeomorphisms) with the topology of the ambient manifold allows us to realize various topological effects in the dynamics of such systems. Systems with exactly two points of extreme Morse indices are a classical example demonstrating such a relation. In this case, it follows from the Reeb's theorem [12] that the ambient manifold is homeomorphic to the sphere.

Received September 13, 2023
Accepted February 13, 2024

The work was supported by the Russian Science Foundation (Project No. 23-71-30008).

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Another brilliant illustration of the relations under study is the decomposition of an orientable 3-manifold into a connected sum of $\mathbb{S}^2 \times \mathbb{S}^1$ whose number of summands is completely determined by a structure of the non-wandering set of an MS-diffeomorphism without heteroclinic curves defined on it. This result was obtained in papers by C. Bonatti, V. Grines, and V. Medvedev [5, 6] and is based on the breakthrough result on the existence of a tame neighborhood of a 2-sphere with one point of wildness. The ideas that the authors put into their proofs have been extremely helpful in our research.

The present paper is a straightforward generalization of the Reeb's theorem on the following class of difeomorphisms.

Let f be an MS-diffeomorphism defined on a closed connected 3-manifold M^3 and let all its saddle points have the same dimension of their unstable manifolds. Denote the class of such diffeomorphisms as $\mathcal{G}$.

Theorem 1. *Any closed connected 3-manifold M^3, admitting a diffeomorphism $f \in \mathcal{G}$, is homeomorphic to the 3-sphere.*

It is worth noting that in [8] V. Medvedev and E. Zhuzhoma considered a similar class of diffeomorphisms in the case when the ambient manifold has dimension greater than three. Actually, Theorem 1 complements their result for the last unexplored dimension.

1. Auxiliary information and facts

This section introduces basic concepts and facts from topology and dynamical systems theory.

1.1. Some topological facts

Let X, Y be topological spaces, $A \subset X$ and $B \subset Y$ be their subsets and $g\colon A \to B$ be a homeomorphism. Let $\sim$ be the minimal equivalence relation on $X \sqcup Y$ for which $a \sim g(a)$ for all $a \in A$. The factor space for this equivalence relation is said to be obtained by gluing the space Y to the space X by the map g, written $X \underset{g}{\cup} Y$.

Let X, Y be compact n-manifolds, $D_1 \subset X$, $D_2 \subset Y$ be subsets homeomorphic to $\mathbb{D}^n$, $h_1\colon \mathbb{D}^n \to D_1$, $h_2\colon \mathbb{D}^n \to D_2$ be the corresponding homeomorphisms and $g\colon \partial D_1 \to \partial D_2$ be a homeomorphism such that the map $h_2^{-1} g h_1\big|_{\partial\mathbb{D}^n} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ reverses orientation. Then the space $X \# Y = (X \setminus \operatorname{int} D_1) \underset{g}{\cup} (Y \setminus \operatorname{int} D_2)$ is called *the connected sum of X and Y*.

If $X \subset Y$, then the map $i_X\colon X \to Y$ such that $i_X(x) = x$ for all $x \in X$ is called *the inclusion map of X into Y*.

Let X and Y be C^r-manifolds. Denote by $C^r(X, Y)$ the set of all C^r-maps $\lambda\colon X \to Y$. A map $\lambda\colon X \to Y$ is said to be a C^r-embedding if it is a C^r-diffeomorphism onto the subspace $\lambda(X)$. C^0-embedding is also called *a topological embedding*.

A topological embedding $\lambda\colon X \to Y$ of an m-manifold X into an n-manifold Y ($m \leqslant n$) is said to be *locally flat at the point* $\lambda(x)$, $x \in X$, if the point $\lambda(x)$ is in the domain of such a chart (U, ψ) of the manifold Y that $\psi(U \cap \lambda(X)) = \mathbb{R}^m$, where $\mathbb{R}^m \subset \mathbb{R}^n$ is the set of points for which the last $n - m$ coordinates equal to 0 or $\psi(U \cap \lambda(X)) = \mathbb{R}^m_+$, where $\mathbb{R}^m_+ \subset \mathbb{R}^m$ is the set of points with the nonnegative last coordinate.

An embedding λ is said to be *tame* and the manifold X is said to be *tamely embedded* if λ is locally flat at every point $x \in X$. Otherwise the embedding λ is said to be *wild* and the manifold X is said to be *wildly embedded.* A point $\lambda(x)$ which is not locally flat is said to be *the point of wildness.*

Statement 1 ([4, Theorem 4]). *Let T be a two-dimensional torus tamely embedded in the manifold $\mathbb{S}^2\times\mathbb{S}^1$ in such a way that $i_{T*}(\pi_1(T))\neq 0$. Then T bounds in $\mathbb{S}^2\times\mathbb{S}^1$ a solid torus.*

Statement 2 ([5, Proposition 0.1]). *Let M^3 be a closed connected 3-manifold and let $\lambda\colon\mathbb{S}^2\to M^3$ be a topological embedding of the 2-sphere which is smooth everywhere except one point. Let $\Sigma=\lambda\left(\mathbb{S}^2\right)$. Then any neighborhood V of the sphere Σ contains a neighborhood K diffeomorphic to $\mathbb{S}^2\times[0,\,1]$.*[1]

Let us assume that the empty set and only the empty set *has a dimension* -1 ($\dim\varnothing=-1$). The separable metric space X *has a dimension* $\leqslant n$ ($\dim X\leqslant n$) if any neighborhood V_p of a point $p\in X$ contains a neighborhood U_p such that $\dim(\partial U_p)\leqslant n-1$. The space X *has a dimension* n ($\dim X=n$) if the statement $\dim X\leqslant n$ is true and the statement $\dim X\leqslant n-1$ is false.

It is said that a subset D of a connected space X *divides it* if the space $X\setminus D$ is disconnected.

Statement 3 ([1, Corollary 1, p. 48]). *Any connected n-manifold cannot be divided by a subset of dimension $\leqslant n-2$.*

1.2. Morse – Smale diffeomorphisms

Here and below, we assume that M^n is a closed connected 3-manifold with a metric d and a map $f\colon M^n\to M^n$ is a diffeomorphism.

The trajectory or *the orbit* of a point $x\in M^n$ is the set $\mathcal{O}_x=\{f^m(x),\,m\in\mathbb{Z}\}$.

A set $A\subset M^n$ is said to be f*-invariant* if $f(A)=A$, i. e. the trajectory of any point $x\in A$ belongs to A.

A compact f-invariant set $A\subset M^n$ is called *an attractor* of the diffeomorphism f if it has a compact neighborhood U_A such that $f(U_A)\subset\operatorname{int}U_A$ and $A=\bigcap\limits_{k\geqslant 0}f^k(U_A)$. The neighborhood U_A in this case is said to be *trapping.* The basin of the attractor A is the set

$$W^s_A=\left\{x\in M^n\colon \lim_{k\to+\infty}d\left(f^k(x),\,A\right)=0\right\}.$$

A repeller and its basin are defined as an attractor and its basin for f^{-1}.

A point $x\in M^n$ is said to be a *wandering point* for the diffeomorphism f if there is an open neighborhood U_x of x such that $f^k(U_x)\cap U_x=\varnothing$ for all $k\in\mathbb{N}$. Otherwise, the point x is said to be a *non-wandering point.* The set of all non-wandering points is called the *non-wandering set* and it will be denoted by Ω_f. The non-wandering set Ω_f is f-invariant, and if Ω_f is finite, then it consists only of periodic points, i. e., points $p\in M^n$ such that there exists a natural number m for which $f^m(p)=p$. If this equality is not satisfied for any natural number $k<m$, then m is called the *period of a point* p, which we denote by m_p.

For a periodic point p, let us define the sets

$$W^s_p=\left\{x\in M^n\colon \lim_{k\to+\infty}d\left(f^{km_p}(x),\,p\right)=0\right\}$$

[1]This fact was proven in [5] for an orientable manifold M^3, but the proof does not use the orientability anywhere. So we can use this result in our case too.

and

$$W^u_p = \left\{x \in M^n\colon \lim_{k\to-\infty} d\left(f^{km_p}(x),\, p\right) = 0\right\},$$

which are called *stable* and *unstable manifolds* of the point p respectively. These sets are also known as *invariant manifolds* of the point p.

A periodic point p with period m_p is said to be *hyperbolic* if the absolute value of each eigenvalue of the Jacobi matrix $\left(\frac{\partial f^{m_p}}{\partial x}\right)\Big|_p$ is not equal to 1. If the absolute values of all the eigenvalues are less than 1, then p is called an *attracting point*, a *sink point* or a *sink*; if the absolute values of all the eigenvalues are greater than 1, then p is called a *repelling point*, a *source point* or a *sink*. Attracting and repelling fixed points are called *nodes*. A hyperbolic periodic point which is not a node is called a *saddle point* or a *saddle*.

The hyperbolic structure of the periodic point p and the finiteness of the non-wandering set implies that its stable and unstable manifolds are smooth submanifolds of M^n which are diffeomorphic to $\mathbb{R}^{q_p}$ and $\mathbb{R}^{n-q_p}$ respectively, where q_p is a *Morse index of* p, i. e. the number of the eigenvalues of the Jacobi matrix whose absolute value is greater than 1.

A connected component ℓ^u_p $\left(\ell^s_p\right)$ of the set $W^u_p \setminus p$ $\left(W^s_p \setminus p\right)$ is called an unstable (stable) *separatrix* of the periodic point p. For p let ν_p be $+1$ (-1) if $f^{m_p}|_{W^u_p}$ preserves (reverses) orientation and let μ_p be $+1$ (-1) if $f^{m_p}|_{W^s_p}$ preserves (reverses) orientation.

A diffeomorphism $f\colon M^3 \to M^3$ is called a *Morse – Smale diffeomorphism* $\left(f \in MS\left(M^3\right)\right)$ if

1) the non-wandering set Ω_f is finite and hyperbolic;

2) for every two distinct periodic points p, q the manifolds W^s_p, W^u_q intersect transversally.

Note that all the facts below are proved in the case when M^n is orientable, but a direct check allows us to verify the correctness of these results for non-orientable manifolds as well.

Statement 4 ([2, Theorem 2.1]). *Let* $f \in MS\left(M^3\right)$*. Then*

1) $M^n = \bigcup\limits_{p\in\Omega_f} W^u_p$,

2) W^u_p *is a smooth submanifold of the manifold* M^n *diffeomorphic to* $\mathbb{R}^{q_p}$ *for every periodic point* $p \in \Omega_f$,

3) $\mathrm{cl}\left(\ell^u_p\right) \setminus \left(\ell^u_p \cup p\right) = \bigcup\limits_{r\in\Omega_f\colon\, \ell^u_p\cap W^s_r\neq\varnothing} W^u_r$ *for every unstable separatrix* ℓ^u_p *of a periodic point* $p \in$ $\in \Omega_f$.

If σ_1, σ_2 are distinct saddle points of a diffeomorphism $f \in MS\,(M^n)$, then the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \varnothing$ is called a *heteroclinic intersection*. If $\dim\left(W^s_{\sigma_1} \cap W^u_{\sigma_2}\right) > 0$, then a connected component of the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is called a *heteroclinic manifold*, and if $\dim\left(W^s_{\sigma_1} \cap W^u_{\sigma_2}\right) = 1$, then it is called a *heteroclinic curve*. If $\dim\left(W^s_{\sigma_1} \cap W^u_{\sigma_2}\right) = 0$, then the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is countable, each point of this set is called a *heteroclinic point* and the orbit of a heteroclinic point is called a *heteroclinic orbit*.

Statement 5 ([2, Proposition 2.3]). *Let $f \in MS\,(M^n)$ and σ be a saddle point of f such that the unstable separatrix ℓ^u_σ has no heteroclinic intersections. Then*

$$\mathrm{cl}\,(\ell^u_\sigma) \setminus (\ell^u_\sigma \cup \sigma) = \{\omega\},$$

where ω is a sink point. If $q_\sigma = 1$ then $\mathrm{cl}\,(\ell^u_\sigma)$ is an arc topologically embedded into M^n, and if $q_\sigma \geqslant 2$ then $\mathrm{cl}\,(\ell^u_\sigma)$ is the sphere $\mathbb{S}^{q_\sigma}$ topologically embedded into M^n.

A diffeomorphism $f \in MS\,(M^n)$ is called a "*source-sink*" diffeomorphism if its non-wandering set consists of a unique sink and a unique source.

Statement 6 ([2, Theorem 2.5]). *If a diffeomorphism $f \in MS\,(M^n)$, $n > 1$, has no saddle points, then f is a "source-sink" diffeomorphism and the manifold M^n is homeomorphic to the n-sphere $\mathbb{S}^n$.*[2]

Statement 7 ([7, Theorem 1]). *Let $f \in MS\,(M^n)$ and Ω_A be a subset of Ω_f such that the set*

$$A = \Omega_0 \cup W^u_{\Omega_A}$$

is closed and f-invariant. Then

1) *the set A is an attractor of diffeomorphism f;*

2) $W^s_A = \bigcup\limits_{p \in (A \cap \Omega_f)} W^s_p$;

3) $\dim A = \max\limits_{p \in (A \cap \Omega_f)} \{q_p\}$.

For an orbit $\mathcal{O}_p$ of a point p, let $m_{\mathcal{O}_p} = m_p$, $q_{\mathcal{O}_p} = q_p$, $\nu_{\mathcal{O}_p} = \nu_p$, $\mu_{\mathcal{O}_p} = \mu_p$, $W^s_{\mathcal{O}_p} = \bigcup\limits_{q \in \mathcal{O}_p} W^s_q$, $W^u_{\mathcal{O}_p} = \bigcup\limits_{q \in \mathcal{O}_p} W^u_q$.

Following the classical paper by S. Smale [14], we introduce on the set of periodic orbits of $f \in MS\,(M^n)$ a partial order $\prec$:

$$\mathcal{O}_i \prec \mathcal{O}_j \iff W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} \neq \varnothing.$$

According to Szpilrajn's theorem [16], any partial order (including the Smale order) can be extended to a total order. Let us consider a special kind of such total order on the set of all periodic orbits.

We say that the numbering of the periodic orbits $\mathcal{O}_1, \ldots, \mathcal{O}_{k_f}$ of the diffeomorphism $f \in MS\,(M^n)$ is said to be *dynamical numbering* if it satisfies the following conditions:

1) $\mathcal{O}_i \prec \mathcal{O}_j \implies i \leqslant j$;

2) $q_{\mathcal{O}_i} < q_{\mathcal{O}_j} \implies i < j$.

Statement 8 ([2, Proposition 2.6]). *For any diffeomorphism $f \in MS\,(M^n)$ there is a dynamical numbering of its periodic orbits.*

[2]The second part of this statement can be known as a special case of Reeb's theorem [12].

1.3. Orbit spaces

In this section, we present concepts and facts, a detailed proof of which can be found in the monograph [2].

Let $f\colon M^n \to M^n$ be a diffeomorphism and let $X \subset M^n$ be an f-invariant set. It can be checked directly that the relation $x \sim y \iff \exists k \in \mathbb{Z}\colon y = f^k(x)$ is an equivalence relation on X. The quotient set X/f induced by this relation is called an *orbits space of the action of f on X*. Let us denote by $p_{X/f}\colon X \to X/f$ the natural projection. *A fundamental domain of the action of f on X* is a closed set $D_X \subset X$ such that there is a set $\widetilde{D}_X$ satisfying:

1) $\mathrm{cl}(\widetilde{D}_X) = D_X$;

2) $f^k(\widetilde{D}_X) \cap \widetilde{D}_X = \varnothing$ for each $k \in \mathbb{Z} \setminus \{0\}$;

3) $\bigcup\limits_{k\in\mathbb{Z}} f^k(\widetilde{D}_X) = X$.

If the projection $p_{X/f}$ is a cover and the orbits space X/f is connected then, due to the monodromy theorem (see, for example, [2, p. 60]), for a loop $\widehat{c} \subset X/f$, closed at a point $\widehat{x}$, there exists its lift $c \subset X$, which is a path joining points $x \in p^{-1}_{X/f}(\widehat{x})$ and $f^k(x)$. In this case, the map $\eta_{X/f}\colon \pi_1(X/f) \to \mathbb{Z}$, defined by the formula $\eta_{X/f}([\widehat{c}]) = k$, is a homomorphism which is called *induced by the cover $p_{X/f}$*.

Statement 9. *Let f and f' be diffeomorphisms defined on an f- and f'-invariant set X. If $\widehat{h}\colon X/f \to X/f'$ is a homeomorphism for which $\eta_{X/f} = \eta_{X/f'}\widehat{h}$, then there is a homeomorphism $h\colon X \to X$ which is a lift of $\widehat{h}$ ($p_{X/f'}h = \widehat{h}p_{X/f}$) and such that $hf = f'h$.*

Statement 10 ([2, Theorem 2.1.3]). *Let $f \in MS\,(M^n)$, A be an attractor of f, $\ell^s_A = W^s_A \setminus A$, $\widehat{\ell}^s_A = \ell^s_A/f$ and $D_{\ell^s_A}$ be a fundamental domain of the action of f on ℓ^s_A. Then the projection $p_{\widehat{\ell}^s_A}$ is a cover and the orbits space $\widehat{\ell}^s_A$ is a smooth closed n-manifold homeomorphic to $p_{\widehat{\ell}^s_A}\left(D_{\ell^s_A}\right)$. In particular, if the attractor A coincides with a sink orbit, then the manifold $\widehat{\ell}^s_A$ is homeomorphic to the following manifolds*:

- $\mathbb{S}^1$ *for* $n = 1$;
- $\mathbb{S}^{n-1}\widetilde{\times}\mathbb{S}^1$ *for* $n > 1$, $\nu_A = -1$;
- $\mathbb{S}^{n-1} \times \mathbb{S}^1$ *for* $n > 1$, $\nu_A = +1$.

2. Topology of 3-manifolds admitting $\mathcal{G}$-diffeomorphisms

Recall that $\mathcal{G}$ is a class of Morse–Smale diffeomorphisms $f\colon M^3 \to M^3$ defined on a closed connected 3-manifold M^3 (not necessarily orientable), with a non-wandering set Ω_f whose all saddle points have the same dimension of their unstable manifolds.

This section is focused on the proof of the main result of this paper.

Theorem 1. *Any closed connected 3-manifold M^3, admitting a diffeomorphism $f \in \mathcal{G}$, is homeomorphic to the 3-sphere.*

To prove the main result, let us state some auxiliary facts.

Remark 1. Further, without loss of generality, up to the power of the diffeomorphism, one may assume that Ω_f consists of fixed points only and for all $p \in \Omega_f$ the numbers ν_p and μ_p equal to $+1$. Moreover, for definiteness, we suppose that the set Ω_1 is empty.

Lemma 1. *For any diffeomorphism $f \in \mathcal{G}$, the set Ω_0 consists of a unique sink.*

Proof.
Let

$$R = W^s_{\Omega_2} \cup \Omega_3.$$

By virtue of Statement 7, the set R is a repeller of the diffeomorphism f and $\dim R = 1$. It follows from Statement 3 that $M^3 \setminus R$ is connected. On the other hand, according to Statement 4, $M^3 \setminus R = W^s_{\Omega_0}$. From the above we conclude that the set Ω_0 consists of a unique sink. □

Let us denote by ω the unique sink of the diffeomorphism $f \in \mathcal{G}$.

Lemma 2. *In the non-wandering set of any diffeomorphism $f \in \mathcal{G}$ there exists a saddle σ such that $\ell^u_\sigma \subset \ell^s_\omega$.*

Proof.
By Lemma 1, the fixed points of the diffeomorphism f admit the following dynamical numbering:

$$\begin{gathered} \omega \prec \sigma_1 \prec \cdots \prec \sigma_k \prec \alpha_1 \prec \cdots \prec \alpha_s, \\ \text{where } \Omega_2 = \{\sigma_1, \ldots, \sigma_k\},\ \Omega_3 = \{\alpha_1, \ldots, \alpha_s\}. \end{gathered} \tag{2.1}$$

Assume that $\sigma = \sigma_1$. Then, it follows from order (2.1) that

$$\forall p \in \Omega_f \setminus \omega \implies \ell^u_\sigma \cap W^s_p = \varnothing.$$

In other words, ℓ^u_σ can only intersect with W^s_ω. By Statement 4 (1), any point $x \in \ell^u_\sigma$ has to be in the stable manifold of some fixed point. Hence, $\ell^u_\sigma \subset \ell^s_\omega$. □

Further, let the saddle $\sigma \in \Omega_2$ satisfy the conclusion of Lemma 2, and let $\Sigma_\sigma = \mathrm{cl}\,(\ell^u_\sigma)$. It follows from Statements 5 and 4 (2) that $\Sigma_\sigma = \ell^u_\sigma \cup \{\omega\} \cup \{\sigma\}$ is an embedding of the two-dimensional sphere (see Fig. 1). This embedding is smooth everywhere except perhaps the point ω. Let $\mathcal{M}_\sigma = M^3 \setminus \Sigma_\sigma$.

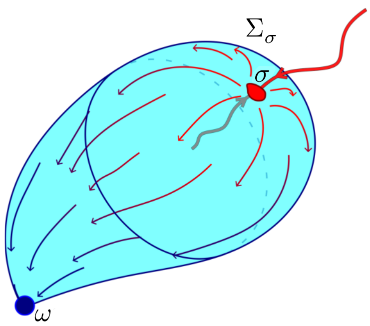

Fig. 1. Sphere Σ_σ

Lemma 3. *The manifold $\mathcal{M}_\sigma$ is disconnected.*

Proof.
Since for any manifold the notions of connectedness and path-connectedness are equivalent, they will be used interchangeably hereafter.

Step 1. First of all, let us proof that the set $\mathcal{L}_\sigma = \ell^s_\omega \setminus \ell^u_\sigma$ is disconnected. Suppose the contrary: any two distinct points $x, y \in \mathcal{L}_\sigma$ can be connected by a path in $\mathcal{L}_\sigma$ (see Fig. 2).

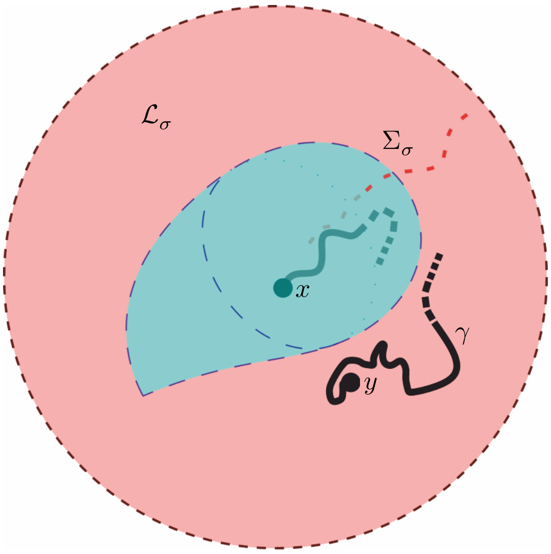

Fig. 2. The space $\mathcal{L}_\sigma$

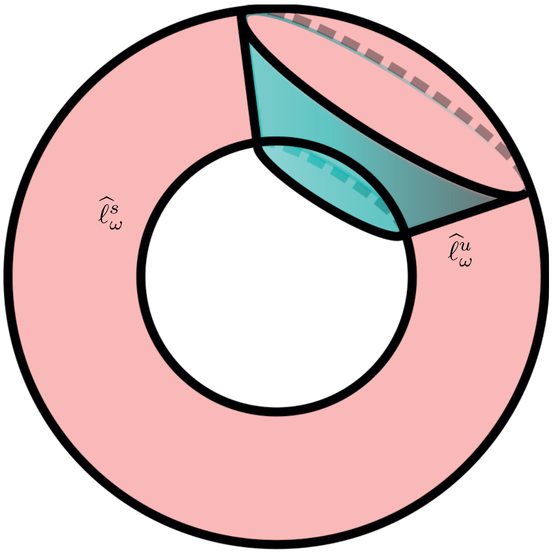

Fig. 3. The orbits space of the sink basin

Consider the orbits space $\widehat{\ell}^s_\omega = \ell^s_\omega/f$ of the sink ω and let $p_\omega = p_{\widehat{\ell}^s_\omega}\colon \ell^s_\omega \to \widehat{\ell}^s_\omega$, $\eta_\omega = \eta_{\widehat{\ell}^s_\omega}\colon \pi_1\left(\widehat{\ell}^s_\omega\right) \to \mathbb{Z}$. By Statement 10, the map p_ω is a cover, $\widehat{\ell}^s_\omega$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ and $\widehat{\ell}^u_\sigma$ is homeomorphic to the two-dimensional torus (see Fig. 3).

Since $\ell^u_\sigma \subset \ell^s_\omega$, it follows that $\widehat{\ell}^u_\sigma \subset \widehat{\ell}^s_\omega$. Moreover, $\widehat{\ell}^u_\sigma = p_\omega(\ell^u_\sigma)$, which, by Statement 4, implies that $\widehat{\ell}^u_\sigma$ is a smooth embedding of the 2-torus into $\widehat{\ell}^s_\omega$ (see Fig. 3). By Statement 10, homomorphism η_ω is nontrivial and it follows from its definition that $i_*\left(\pi_1\left(\widehat{\ell}^u_\sigma\right)\right) \neq 0$.

Then, using Statement 1, one may conclude that $\widehat{\ell}^u_\sigma$ bounds in $\widehat{\ell}^s_\omega$ a solid torus and hence it divides this orbits space into two connected components. Let us choose a point in each component and denote them by $\widehat{x}$ and $\widehat{y}$. From their preimages, we take two points $x \in p^{-1}_\omega(\widehat{x})$ and $y \in p^{-1}_\omega(\widehat{y})$. Since we assumed that $\mathcal{L}_\sigma$ is path-connected, there exists a path $\gamma\colon [0, 1] \to \mathcal{L}_\sigma$: $\gamma(0) = x$, $\gamma(1) = y$. Then, by continuity of p_ω, the map $\widehat{\gamma} = p_\omega\gamma\colon [0, 1] \to \widehat{\ell}^s_\omega \setminus \widehat{\ell}^u_\sigma$ is a path between $\widehat{x}$ and $\widehat{y}$ in $\widehat{\ell}^s_\omega \setminus \widehat{\ell}^u_\sigma$, that is a contradiction.

Thus, $\mathcal{L}_\sigma$ is disconnected.

Step 2. Let us prove that $\mathcal{M}_\sigma = M^3 \setminus \Sigma_\sigma$ is disconnected as well. Suppose the contrary. Note that $\dim \mathcal{M}_\sigma = 3$, because it is an open subset of the manifold M^3. Then, by Statement 3, $\mathcal{M}_\sigma \setminus R$ is connected. On the other hand, $\mathcal{M}_\sigma \setminus R = \left(M^3 \setminus \Sigma_\sigma\right) \setminus R = W^s_\omega \setminus \Sigma_\sigma = \mathcal{L}_\sigma$ and it contradicts the conclusion of the previous step. So, $\mathcal{M}_\sigma$ is disconnected. $\square$

Let us introduce a diffeomorphism $a\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the formula $a(x, y, z) = \left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$. It has a unique non-wandering point, a sink $O(0, 0, 0)$. Let $\ell = \mathbb{R}^3 \setminus O$.

As before, let $f \in \mathcal{G}$, let σ satisfy the conclusion of Lemma 2 and $\Sigma_\sigma = \operatorname{cl}(\ell^u_\sigma)$. By Statement 7, sphere Σ_σ is an attractor of the diffeomorphism f with its basin $W^s_{\Sigma_\sigma} = W^s_\sigma \cup W^s_\omega$. Let $\ell^s_{\Sigma_\sigma} = W^s_{\Sigma_\sigma} \setminus \Sigma_\sigma$.

Lemma 4. *The manifold $\ell^s_{\Sigma_\sigma}$ consists of two connected components ℓ_1, ℓ_2, and for each of the components ℓ_i there exists a diffeomorphism $h_i\colon \ell_i \to \ell$ conjugating $f\big|_{\ell_i}$ with $a\big|_{\ell}$.*

Proof.

By virtue of Statement 10, the orbits space $\widehat{\ell}^s_{\Sigma_\sigma} = \ell_{\Sigma^s_\sigma}/f$ is a smooth closed 3-manifold. Let us prove that $\widehat{\ell}^s_{\Sigma_\sigma} \cong \mathbb{S}^2 \times \mathbb{S}^1 \sqcup \mathbb{S}^2 \times \mathbb{S}^1$.

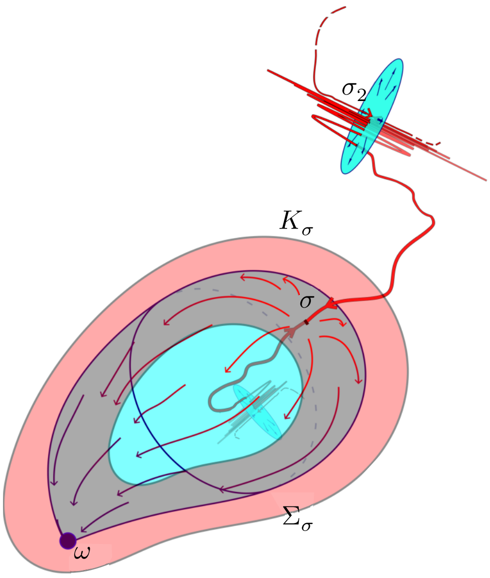

Fig. 4. The neighborhood $K_\sigma \cong \mathbb{S}^2 \times [0, 1]$ of the sphere Σ_σ

By Statement 2, the attractor Σ_σ has a neighborhood $K_\sigma \subset W^s_{\Sigma_\sigma}$ diffeomorphic to $\mathbb{S}^2 \times [0, 1]$ (see Fig. 4). Let us show that there exists a natural number N such that $f^N(x) \in \operatorname{int} K_\sigma$ for any $x \in K_\sigma$. Since $\partial K_\sigma \subset W^s_{\Sigma_\sigma}$, it follows that for all $x \in \partial K$ there exist a closed neighborhood $U_x \subset \partial K_\sigma$ and a natural number ν_x such that for any $\nu \geqslant \nu_x$ it is true that $f^\nu(U_x) \subset \operatorname{int} K_\sigma$. Due to the compactness of ∂K_σ, there exists a finite subcover of ∂K_σ in $\{U_x,\, x \in \partial K_\sigma\}$. Thus, one may choose the desired number N as the maximum of numbers ν_x corresponding to the neighborhoods of U_x in the chosen subcover. Without loss of generality, we assume the number N to be 1, then $f\left(K_{\Sigma_\sigma}\right) \subset \operatorname{int} K_\sigma$ (see Fig. 5). It follows from Lemma 3 that the sphere Σ_σ separates in K_σ the connected components of its boundary. Whence, according to [6, Theorem 3.3], $K_\sigma \setminus \operatorname{int} f(K_\sigma) \cong \mathbb{S}^2 \times [0, 1] \sqcup \mathbb{S}^2 \times [0, 1]$. It follows from the construction that the manifold $K_\sigma \setminus \operatorname{int} f(K_\sigma)$ is a fundamental domain of the action of f on the space $\ell^s_{\Sigma_\sigma}$. Then by Statement 10, $\widehat{\ell}^s_{\Sigma_\sigma} \cong \mathbb{S}^2 \times \mathbb{S}^1 \sqcup \mathbb{S}^2 \times \mathbb{S}^1$.

We denote by $\widehat{\ell}_1$, $\widehat{\ell}_2$ the connected components of the set $\widehat{\ell}^s_{\Sigma_\sigma}$ and by $\eta_{\widehat{\ell}_i}: \pi_1(\ell_i) \to \mathbb{Z}$, $i = 1, 2$ the restriction of the homomorphism $\eta_{\widehat{\ell}^s_{\Sigma_\sigma}}$ to $\pi_1(\widehat{\ell}_i)$. Let us assume $\ell_i = p^{-1}_{\widehat{\ell}^s_{\Sigma_\sigma}}(\widehat{\ell}_i)$. It follows from the definition of the homomorphism $\eta_{\widehat{\ell}_i}$ that it is an isomorphism, hence the set ℓ_i is connected. Let $\widehat{\ell} = \ell/a$. Since the point O is a sink of the three-dimensional map a, then by Statement 10, $\ell \cong \mathbb{S}^2 \times \mathbb{S}^1$ and the homomorphism $\eta_{\widehat{\ell}}\colon \pi_1(\widehat{\ell}) \to \mathbb{Z}$ is an isomorphism. Therefore, the manifolds $\widehat{\ell}_i$ and $\widehat{\ell}$ are homeomorphic smooth 3-manifolds, hence there exists a diffeomorphism $\widehat{h}_i\colon \widehat{\ell}_i \to \widehat{\ell}_i$

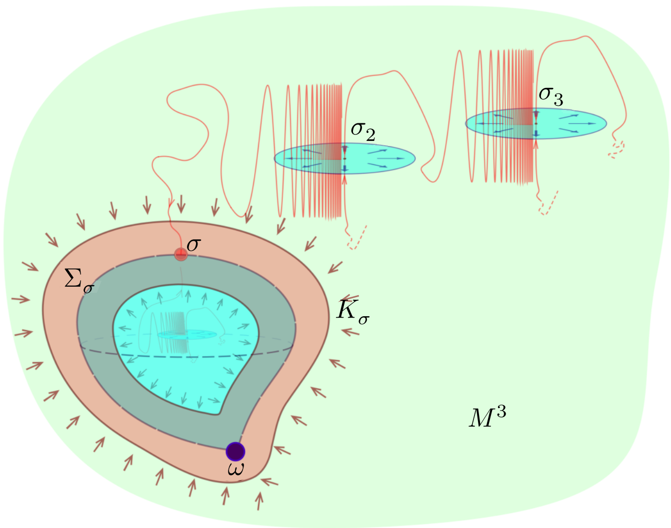

Fig. 5. The dynamics of diffeomorphism f on M^3

(see [9]). Without loss of generality, we assume that $\eta_{\widehat{\ell}}\widehat{h}_i = \eta_{\widehat{\ell}_i}$ (otherwise, one may consider its composition with a diffeomorphism $\theta\colon \mathbb{S}^2\times\mathbb{S}^1 \to \mathbb{S}^2\times\mathbb{S}^1$, given by the formula $\theta(s, r) = (s, -r)$).

By Statement 9, there exists a lift $h_i\colon \ell_i \to \ell$ of the diffeomorphism $\widehat{h}_i$, smoothly conjugating $f\big|_{\ell_i}$ with $a\big|_{\ell}$. □

Now let $\overline{M}^\sigma = \mathbb{R}^3 \sqcup \mathcal{M}_\sigma \sqcup \mathbb{R}^3$, $M^\sigma = \mathbb{R}^3 \underset{h_1}{\cup} \mathcal{M}_\sigma \underset{h_2}{\cup} \mathbb{R}^3$ and let $p_\sigma\colon \overline{M}^\sigma \to M^\sigma$ be the natural projection.

Lemma 5. *The space M^σ consists of two connected components M_1^σ, M_2^σ, with both of them being a closed smooth 3-manifold such that*

$$M^3 = M_1^\sigma \# M_2^\sigma.$$

Moreover, the manifold M_i^σ, $i = 1, 2$, admits a diffeomorhism $f_i\colon M_i^\sigma \to M_i^\sigma$ belonging to the class $\mathcal{G}$ and having less saddle points than f.

Proof.
It follows from Lemma 4 that the manifold $\mathcal{M}_\sigma$ is a disjoint union of two manifolds, and hence the space M^σ has exactly the same number of connected components, which we denote as M_1^σ and M_2^σ. Since h_i glues open subsets of 3-manifolds, then the projection p_σ induces the structure of a smooth 3-manifold on M^σ. Since the glued manifolds have no boundary, the manifold M^σ has no boundary as well. Due to the compactness of M^3, the manifold M^σ is closed. Moreover, it follows directly from the definition of the connected sum that $M^3 = M_1^\sigma \# M_2^\sigma$.

According to [10, Theorem 18.3 (the pasting lemma)], the map $f_\sigma\colon M^\sigma \to M^\sigma$, defined by the formula

$$f_\sigma(x) = \begin{cases} p_\sigma\left(f\left(p_\sigma^{-1}(x)\right)\right), & \text{if } x \in p_\sigma\left(\mathcal{M}^\sigma\right), \\ p_\sigma(a(p_\sigma(x))), & \text{if } x \in p_\sigma\left(\overline{M^\sigma} \setminus \mathcal{M}_\sigma\right), \end{cases}$$

is a diffeomorphism. Let $f_i = f_\sigma|_{M_i^\sigma}$ (see Fig. 6, 7). By the construction, the diffeomorphism f_σ is smoothly conjugated with f on $p_\sigma(\mathcal{M}^\sigma)$ and with a on $p_\sigma(\overline{M^\sigma} \setminus \mathcal{M}^\sigma)$ (O_i $(i = 1, 2)$ is a point

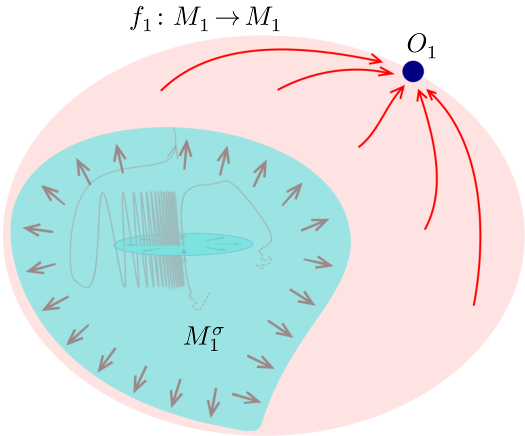

Fig. 6. The dynamics of f_1 on M_1^σ

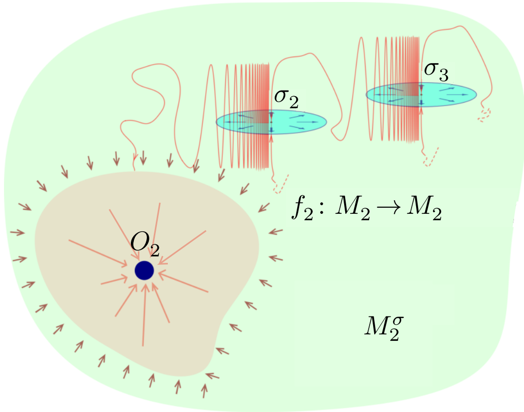

Fig. 7. The dynamics of f_2 on M_2^σ

conjugated with the fixed sink $O(0,0,0)$ of a). Hence, $f_\sigma \in \mathcal{G}$ and its non-wandering set have one saddle point less than the non-wandering set of the diffeomorphism f. □

Now let us prove the main result of this paper.

Theorem 1. *Any closed connected 3-manifold M^3, admitting a diffeomorphism $f \in \mathcal{G}$, is homeomorphic to the 3-sphere.*

Proof.

Let $f\colon M^3 \to M^3$ be in $\mathcal{G}$. Moreover, we assume that f satisfies the Remark. We prove Theorem 1 by the induction on the number k of the saddle points of the diffeomorphism f.

Base case. $k = 0$.

It follows from Statement 6 that M^3 is homeomorphic to the 3-sphere.

Step of induction. $k > 0$.

Inductive hypotheses. *Any diffeomorphism from the class $\mathcal{G}$ in which the number of saddle points is less than some natural number k can be defined only on a manifold homeomorphic to the 3-sphere.*

The diffeomorphism $f\colon M^3 \to M^3$ lies in $\mathcal{G}$ and has exactly k saddle points. By Lemma 2, there exists a saddle σ whose unstable manifold has no heteroclinic intersections. This saddle was chosen according to the order 2.1. By Lemma 5, $M^3 = M_1^\sigma \# M_2^\sigma$ and the manifold M_i^σ, $i = 1, 2$, admits a diffeomorphism $f_i\colon M_i^\sigma \to M_i^\sigma$ from the class $\mathcal{G}$ which has less saddle points than f.

In this case, it follows from the inductive hypotheses that $M_i^\sigma \cong \mathbb{S}^3$. Thus, M^3 is a connected sum of 3-spheres and hence $M^3 \cong \mathbb{S}^3$. □

Conflict of interest

The authors declare that they have no conflicts of interest.

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