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On the number of independent and *k*-dominating sets in graphs with average vertex degree at most *k*

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Abstract. The following conjecture is formulated: if the average vertex degree in a graph is not greater than a positive integer $k \geqslant 1$, then the number of k-dominating sets in this graph does not exceed the number of its independent sets, and these numbers are equal to each other if and only if the graph is k-regular. This conjecture is proved for $k \in \{1, 2\}$.

Bibliography: 10 titles.

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§ 1. Introduction

An *independent set* in a graph is an arbitrary subset of its pairwise nonadjacent vertices. The number of independent sets in a graph G is denoted by $i(G)$. For $k \geqslant 1$ a *k-dominating set* in a graph G is a subset D_k of its vertices such that each vertex which is not in D_k is adjacent to at least k vertices in D_k. The number of k-dominating sets in a graph G is denoted by $\partial_k(G)$. A graph is said to be *k-regular* if all of its vertices have the same degree $k \geqslant 0$, and it is said to be *non-regular* if it contains two vertices of distinct degrees.

To this date there are a lot of results devoted to counting independent sets in graphs of various classes. For any $n \geqslant 1$ the star graph and the simple path are known to contain the maximum possible and the minimum possible number of independent sets, respectively, in the class of n-vertex trees (see [1]). In [2], for any $n, d \geqslant 3$ the structure of n-vertex trees with maximum degree d that contain the maximum possible number of independent sets among all trees of this kind was described. In [3]–[6] an asymptotically sharp upper estimate for the number of independent sets in an n-vertex k-regular graph was derived and the corresponding extremal graphs were described. Many other results of this kind were mentioned in the recent survey [7].

Enumerative results related to k-dominating sets are much fewer, almost all of them refer to the case $k = 1$. In the class of n-vertex trees the largest number of 1-dominating sets is featured by the star graph; at the same time there exist exponentially many trees containing the minimum possible number of 1-dominating

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sets among all graphs of this class: see [8]. In the subsequent paper [9] this result was generalized to the class of connected graphs. In the recent work [10], for any $n, k \geqslant 2$ the structure of trees containing the maximum possible or minimum possible number of k-dominating sets among all n-vertex trees is described.

For any $k \geqslant 1$ the number of independent sets in a k-regular graph coincides with the number of its k-dominating sets. Indeed, it follows from the definitions of independent sets and k-dominating sets that the complement to any independent set in a k-regular graph is a k-dominating set and vice versa. Figure 1 shows a 3-regular graph whose vertices are divided into the independent set $\{v_3, v_5\}$ (the elements of which are coloured gray) and the 3-dominating set $\{v_1, v_2, v_4, v_6\}$.

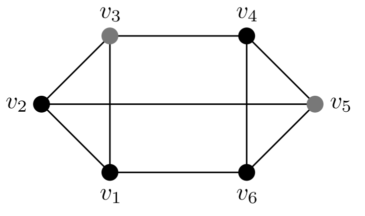

Figure 1. An example of a 3-regular graph.

The problem of establishing a relationship between the number of independent sets and the number of k-dominating sets in non-regular graphs with average vertex degree k is much more complicated. The author is confident of the following.

Proposition. *If the average vertex degree in a graph G does not exceed a positive integer $k \geqslant 1$, then $\partial_k(G) \leqslant i(G)$. Moreover, this inequality turns to equality if and only if the graph G is k-regular.*

This paper is devoted to the proof of this proposition for $k \in \{1, 2\}$. In § 2 we present some standard graph-theoretic notation. In § 3 we introduce additional notation related to independent sets and k-dominating sets in graphs, and also prove a series of auxiliary assertions. In § 4 we prove the main result of this work. Finally, in § 5 we discuss the prospect of further research in the case when $k \geqslant 3$.

§ 2. Some definitions and notation

By $N_G(v)$ we denote the *open neighbourhood* of the vertex v in the graph G, that is, the set of all vertices adjacent to v. The *closed neighbourhood* of the vertex v in G is the set $N_G[v] = N_G(v) \cup \{v\}$. Given an arbitrary set of vertices $A \subseteq V(G)$, we define its closed neighbourhood by $N_G[A] = \bigcup_{a \in A} N_G[a]$. If the graph is clear from the context, then we simplify this notation to $N(v)$, $N[v]$ and $N[A]$, respectively.

By P_n, C_n and K_n we denote the path graph, the cycle and the complete graph on n vertices, respectively. Given two adjacent vertices u and v in G, we denote the graph obtained from G by deleting the vertex u and all edges incident to it by $G \setminus u$ and the graph obtained from G by deleting the edge uv by $G - uv$. For an arbitrary subgraph H of G we denote by $G \setminus H$ the graph induced by the vertices of the set $V(G) \setminus V(H)$. The notation $K_n - e$ is used for the graph obtained from the complete graph K_n by deleting the edge e. A *triangle* in a graph is a subgraph isomorphic to K_3. Given an integer $m \geqslant 1$ and a graph G, we denote the graph containing m connected components, each of which is isomorphic to G, by mG.

The average vertex degree of a graph G is denoted by $\overline{d}(G)$. A vertex of a graph is called *isolated* (a *pendant vertex* or a *leaf*) if it has degree 0 (degree 1, respectively). A *tree* is a connected graph with no cycles. It is known that a connected graph G is a tree if and only if $\overline{d}(G) < 2$. Moreover, any tree contains at least two pendant vertices.

We denote an isomorphism between those graphs G and H by $G \cong H$. A graph H' is called a *spanning* subgraph of G if it can be obtained from G by deleting several edges. A graph H'' is called an *induced* subgraph of G if it can be obtained from G by deleting several vertices and all edges incident to these vertices.

§ 3. Preliminary results

3.1. Families of k-dominating and independent sets in a graph. Denote by $\mathfrak{D}_k(G)$ and $\mathfrak{I}(G)$ the families of all k-dominating and independent sets in a graph G, respectively, and by $\partial_k(G)$ and $i(G)$ the cardinality of each of these families, respectively. By $\mathfrak{D}_k(G, A_1^+, \dots, A_m^+, B_1^-, \dots, B_s^-)$ we denote the family of k-dominating sets in a graph G that contain all vertices from the sets $A_1, \dots, A_m$ and no vertex from $B_1, \dots, B_s$. If the set A consists of a single element a, then we use the notation $\mathfrak{D}_k(G, a^+)$ instead of $\mathfrak{D}_k(G, A^+)$. Set

$$\partial_k(G, A_1^+, \dots, A_m^+, B_1^-, \dots, B_s^-) = |\mathfrak{D}_k(G, A_1^+, \dots, A_m^+, B_1^-, \dots, B_s^-)|.$$

In a similar way we introduce the notation $\mathfrak{I}(G, A_1^+, \dots, A_m^+, B_1^-, \dots, B_s^-)$ and $i(G, A_1^+, \dots, A_m^+, B_1^-, \dots, B_s^-)$ for the family of all independent sets in the graph G and its cardinality, respectively, under the constraints formulated above.

Let us mention several simple facts.

Lemma 1. *The following relations hold for any graph G, any integer $k \geqslant 1$ and any sets $A, B, C \subseteq V(G)$:*

$$\partial_k(G, A^+, B^-) = \sum_{X \subseteq C} \partial_k(G, A^+, B^-, X^+, (C \setminus X)^-) \tag{3.1}$$

and

$$i(G, A^+, B^-) = \sum_{X \subseteq C} i(G, A^+, B^-, X^+, (C \setminus X)^-). \tag{3.2}$$

Proof. It follows from the definition of the family $\mathfrak{D}_k(G)$ that for any set $X \subseteq C$ there exists exactly $\partial_k(G, A^+, B^-, X^+, (C \setminus X)^-)$ k-dominating sets $D \in \mathfrak{D}_k(G, A^+, B^-)$ in the graph G such that $D \cap C = X$, which yields (3.1). Equality (3.2) is established in the same way.

The proof of the lemma is complete.

Remark 1. If $C = \{c\}$, then equalities (3.1) and (3.2) take the form

$$\partial_k(G, A^+, B^-) = \partial_k(G, A^+, B^-, c^+) + \partial_k(G, A^+, B^-, c^-), \tag{3.3}$$

$$i(G, A^+, B^-) = i(G, A^+, B^-, c^+) + i(G, A^+, B^-, c^-). \tag{3.4}$$

Lemma 2. *The following relations hold for any graph G, any integer $k \geqslant 1$ and any sets $A_1 \subseteq A_2 \subseteq V(G)$ and $B_1 \subseteq B_2 \subseteq V(G)$:*

$$\partial_k(G, A_1^+, B_1^-) \geqslant \partial_k(G, A_2^+, B_2^-) \tag{3.5}$$

and

$$i(G, A_1^+, B_1^-) \geqslant i(G, A_2^+, B_2^-). \tag{3.6}$$

Inequalities (3.5) and (3.6) readily follow from the inclusions

$$\mathfrak{D}_k(G, A_2^+, B_2^-) \subseteq \mathfrak{D}_k(G, A_1^+, B_1^-) \quad \text{and} \quad \mathfrak{I}(G, A_2^+, B_2^-) \subseteq \mathfrak{I}(G, A_1^+, B_1^-).$$

Lemma 3. *The following relations hold for any graph G, any integer $k \geqslant 1$ and any sets $A, B, C \subseteq V(G)$ such that $B \cap C = \varnothing$:*

$$\partial_k(G, A^+, B^-, C^-) \leqslant \partial_k(G, A^+, B^-, C^+) \tag{3.7}$$

and

$$i(G, A^+, B^-, C^+) \leqslant i(G, A^+, B^-, C^-). \tag{3.8}$$

Proof. Let us prove inequality (3.7). It follows from the definitions of a k-dominating set and the family $\mathfrak{D}_k(G)$ that for any set $D \in \mathfrak{D}_k(G, A^+, B^-, C^-)$ we have $D \cup C \in \mathfrak{D}_k(G, A^+, B^-, C^+)$, as required.

Now we prove (3.8). If either the set $A \cup C$ is not independent, or $A \cap B \neq \varnothing$, then $i(G, A^+, B^-, C^+) = 0$ and there is nothing to prove. Otherwise the following equalities hold:

$$i(G, A^+, B^-, C^+) = i(G \setminus (N[A] \cup B \cup N[C]))$$

and

$$i(G, A^+, B^-, C^-) = i(G \setminus (N[A] \cup B \cup C)).$$

Since the graph $G \setminus (N[A] \cup B \cup N[C])$ is a subgraph of $G \setminus (N[A] \cup B \cup C)$, inequality (3.8) is valid too.

The proof of the lemma is complete.

Lemma 4. *If H is a subgraph of a graph G, then $\partial_k(H) \leqslant \partial_k(G)$ for any integer $k \geqslant 1$.*

Proof. It follows from the definition of a k-dominating set that adding an edge to a graph does not reduce the number of its k-dominating sets. Hence, if H is a spanning subgraph of G, then we have $\mathfrak{D}_k(H) \subseteq \mathfrak{D}_k(G)$, which gives $\partial_k(H) \leqslant \partial_k(G)$. It remains to consider the case where the set $U = V(G) \setminus V(H)$ is nonempty. It is easily verified that by the definition of k-dominating sets, for any set $D \in \mathfrak{D}_k(H)$ we have the inclusion $D \cup U \in \mathfrak{D}_k(G)$. Then (3.5) yields the double inequality $\partial_k(H) \leqslant \partial_k(G, U^+) \leqslant \partial_k(G)$.

The proof of the lemma is complete.

Remark 2. It is interesting to note that an analogue of Lemma 4 for independent sets is not true. As is known (see [7], Proposition 1), if H is a spanning subgraph of a graph G, then $i(H) \geqslant i(G)$, since deleting an edge from a graph results in a strict increase in the number of its independent sets. In turn, if H is an induced subgraph of G, then $i(H) \leqslant i(G)$, since deleting a vertex from a graph results in a strict decrease in the number of its independent sets.

Lemma 5. *Given a graph G, an integer $k \geqslant 1$ and sets $A, B \subset V(G)$ and $X = \{x_1, \dots, x_s\} \subseteq V(G) \setminus (A \cup B)$, where $s \geqslant 2$, set $X_i = X \setminus \{x_i\}$, $1 \leqslant i \leqslant s$. Then*

$$\sum_{i=1}^{s} \partial_k(G, A^+, B^-, x_i^+, X_i^-) \leqslant \sum_{i=1}^{s} \partial_k(G, A^+, B^-, x_i^-, X_i^+). \tag{3.9}$$

Proof. Relation (3.7) yields the inequality

$$\partial_k(G, A^+, B^-, x_s^+, X_s^-) \leqslant \partial_k(G, A^+, B^-, x_1^-, X_1^+).$$

Also, for any $1 \leqslant i \leqslant s-1$ we have

$$\partial_k(G, A^+, B^-, x_i^+, X_i^-) \leqslant \partial_k(G, A^+, B^-, x_{i+1}^-, X_{i+1}^+),$$

which ensures (3.9).

The proof of the lemma is complete.

Given a graph G, a subgraph G' obtained by deleting some edges and, perhaps, isolated vertices, from G, and two sets $A, B \subseteq V(G')$, we set

$$\partial_k^*(G, G', A^+, B^-) = \partial_k(G, A^+, B^-) - \partial_k(G', A^+, B^-). \tag{3.10}$$

Remark 3. Let $k = 2$, $A = \{a\}$, $B = \{b\}$, and suppose that a graph G' is obtained from G by deleting the edge ab and, perhaps, isolated vertices. In this case the quantity $\partial_2^*(G, G', a^+, b^-)$ coincides with the number of sets $D \in \mathfrak{D}_2(G, a^+, b^-)$ such that the vertex b is adjacent in G to exactly one vertex other than a in D.

3.2. The properties of independent sets. The following well-known fact (see [7], Proposition 4) is stated without proof.

Lemma 6. *For any graph G and any edge $uv \in E(G)$ the following double inequality holds*:

$$\frac{3}{4} \cdot i(G - uv) \leqslant i(G) < i(G - uv). \tag{3.11}$$

It follows from (3.8) that for any vertex v of the graph G we have $i(G, v^-) \geqslant i(G, v^+)$. To estimate the difference $i(G, v^-) - i(G, v^+)$ we need a stronger inequality.

Lemma 7. *Let the vertex v of a graph G be adjacent to vertices $v_1, v_2, \dots, v_s$ of degrees $d_1, d_2, \dots, d_s$, respectively. Then the following inequality holds*:

$$i(G, v^-) \geqslant i(G, v^+) \cdot \left(1 + \sum_{j=1}^{s} \frac{1}{2^{d_j - 1}}\right). \tag{3.12}$$

Proof. Set $V_j = N[v] \setminus v_j$, $1 \leqslant j \leqslant s$. It is easily seen that $i(G, v^+) = i(G, N[v]^-)$, and the family $\mathfrak{I}(G, v^-) \setminus \mathfrak{I}(G, N[v]^-)$ consists of precisely those independent sets G that contain at least one vertex from the neighbourhood $N(v)$. Thus,

$$i(G, v^-) - i(G, v^+) \geqslant \sum_{j=1}^{s} i(G, v_j^+, V_j^-).$$

For fixed j, $1 \leqslant j \leqslant s$, we prove that

$$i(G, v_j^+, V_j^-) \geqslant \frac{1}{2^{d_j-1}} \cdot i(G, N[v]^-).$$

The following inequalities hold:

$$i(G, v_j^+, V_j^-) = i(G \setminus (N[v] \cup N[v_j])) \quad \text{and} \quad i(G, N[v]^-) = i(G \setminus N[v]).$$

Moreover, the graph $G \setminus (N[v] \cup N[v_j])$ is an induced subgraph of $G \setminus N[v]$, which contains at least $|V(G \setminus N[v])| - d_j + 1$ vertices. We denote by G' the graph obtained by deleting from $G \setminus N[v]$ all edges not belonging to $G \setminus (N[v] \cup N[v_j])$. If follows from (3.11) that deleting an edge from a graph does not reduce the number of its independent sets. Therefore,

$$2^{d_j-1} \cdot i(G \setminus (N[v] \cup N[v_j])) \geqslant i(G') \geqslant i(G \setminus N[v]).$$

The proof of the lemma is complete.

Lemma 8. *For any $m \geqslant 1$ and any $2m$-vertex subcubic graph G the inequality $i(G) \geqslant 2^m$ holds.*

Proof. We argue by induction on m; the base case $m \leqslant 2$ is obvious. If $G \cong 2mK_1$, then $i(G) = 2^{2m}$ and there is nothing to prove. Otherwise, consider two vertices $u, v \in V(G)$ connected by an edge. Since $i(G, u^+, v^+) = 0$, we have

$$i(G) \overset{(3.2)}{=} i(G, u^-, v^-) + i(G, u^+, v^-) + i(G, u^-, v^+).$$

It is clear that $G \setminus \{u, v\}$ is a $(2m-2)$-vertex subcubic graph. Hence $i(G, u^-, v^-) \geqslant 2^{m-1}$. The graphs $G \setminus N[u]$ and $G \setminus N[v]$ are subcubic and contain at least $2m - 4$ vertices each; hence $\min\{i(G, u^-, v^+), i(G, u^+, v^-)\} \geqslant 2^{m-2}$, which proves the lemma.

3.3. A lemma on isolated vertices. The following simple fact is frequently employed in the proof of the main result of this work.

Lemma 9. *Suppose that a graph G has no pendant vertices and satisfies the condition $\overline{d}(G) \leqslant 2$. Let G' be obtained from G by deleting $p \geqslant 0$ vertices and $p+q \geqslant 1$ edges in such a way that each connected component of G' has at most one pendant vertex. Then G' contains at most q isolated vertices.*

Proof. By the hypotheses of the lemma each connected component H' of G' with at least one edge contains at most one pendant vertex; then H' is not a tree and $\overline{d}(H') \geqslant 2$. We remove from G' all of its connected components that contain edges and denote the graph obtained by G^*. Since $\overline{d}(G) \leqslant 2$, we have $|V(G)| \geqslant |E(G)|$, so that $|V(G')| \geqslant |E(G')| + q$ and $|V(G^*)| \geqslant |E(G^*)| + q$. Thus, $G^* \cong q'K_1$, where $q' \geqslant q$. Since G' contains G^* as a subgraph, the lemma is proved.

§ 4. Main result

Theorem 1. *If a graph G is not* 1*-regular and* $\overline{d}(G) \leqslant 1$, *then* $\partial_1(G) < i(G)$.

Proof. It is easily verified that the claim of the theorem holds true for all n-vertex graphs with $n \leqslant 3$. Consider a non-regular graph G with the minimum number of vertices that satisfies $\overline{d}(G) \leqslant 1$ and $i(G) \leqslant \partial_1(G)$. Clearly, G contains at least one isolated vertex. Assume that such a vertex is unique. Since $\overline{d}(G) \leqslant 1$, the graph G contains at most one vertex of degree greater than 1; moreover, in case such a vertex exists, it has degree 2. Then $G \cong aP_3 \cup bK_2 \cup K_1$ for some $a \in \{0, 1\}$ and $b \geqslant 0$, and it follows that $\partial_1(G) = 5^a \cdot 3^b < 2 \cdot 5^a \cdot 3^b = i(G)$, which is a contradiction.

Now assume that G contains at least two isolated vertices. Choose an edge $uv \in E(G)$ in such a way that the vertex u has the minimum degree among all nonisolated vertices of G (if there is no such vertex, then G contains no edges and there is nothing to prove). Consider a graph G' obtained from G by deleting two isolated vertices and the edge uv. Since $\overline{d}(G) \leqslant 1$, we have $\overline{d}(G') \leqslant 1$ and $\partial_1(G') \leqslant i(G')$ by assumption. At the same time, the edge uv has been chosen so that the vertex u is either isolated in G' or adjacent in G' to at least one vertex of degree at least 2; then the graph G' is not 1-regular and $\partial_1(G') < i(G')$. Inequality (3.11) yields the estimate $i(G) \geqslant 3 \cdot i(G')$. Let us show that $\partial_1(G) \leqslant 3 \cdot \partial_1(G')$. We have

$$\begin{aligned}
3 \cdot \partial_1(G') &\overset{(3.1)}{\geqslant} 3 \cdot (\partial_1(G', u^+, v^+) + \partial_1(G', u^-, v^-)) \\
&= 3 \cdot (\partial_1(G, u^+, v^+) + \partial_1(G, u^-, v^-)) \\
&\overset{(3.7)}{\geqslant} \partial_1(G, u^+, v^+) + \partial_1(G, u^+, v^-) + \partial_1(G, u^-, v^+) + \partial_1(G, u^-, v^-) \\
&= \partial_1(G).
\end{aligned}$$

Thus, $\partial_1(G) < i(G)$, which is a contradiction.

The proof of the theorem is complete.

Theorem 2. *If a graph G is not* 2*-regular and* $\overline{d}(G) \leqslant 2$, *then* $\partial_2(G) < i(G)$.

Proof. It is easily verified that the theorem holds true for all n-vertex graphs with $n \leqslant 3$. Consider a non-regular graph G with the minimum number of vertices such that $\overline{d}(G) \leqslant 2$ and $i(G) \leqslant \partial_2(G)$. Recall that the condition $\overline{d}(G) \leqslant 2$ is equivalent to the inequality $|V(G)| \geqslant |E(G)|$. Throughout the proof of this theorem we denote by G_m a graph obtained from G by deleting $m \geqslant 1$ vertices and the same number of edges, and by G'_m a graph obtained from G by deleting $m \geqslant 1$ vertices and $m' \geqslant m$ edges.

The proof of the theorem consists of considering the following ten cases.

Case 1: G contains at least one pendant vertex.

Case 2: G contains at least one 2-regular connected component.

Case 3: G contains at least one 3-regular connected component.

Case 4: G contains at least two adjacent vertices u and v, each of degree at least 4.

Case 5: G contains two triangles that share a common edge, and all of their vertices have degree at least 3.

Case 6: G contains a triangle with at least one vertex of degree distinct from 3.

Case 7: G contains a triangle with all vertices of degree 3, and at least two vertices adjacent to this triangle have degree at least 3.

Case 8: G contains a triangle with all vertices of degree 3, and at least two vertices adjacent to this triangle have degree 2.

Case 9: G contains a vertex of degree at least 3 which is adjacent to a vertex of degree 3 and has no common neighbours with it.

Case 10: all vertices of degree greater than 2 in G are pairwise nonadjacent.

In treating each case we assume that the graph G does not meet the assumptions of any case considered before. In particular, we make the following assumptions:

- in Cases 4–10 the graph G contains at least one isolated vertex, since otherwise it follows from the condition $\overline{d}(G) \leqslant 2$ that G contains either a pendant vertex, or a 2-regular connected component, and this situation was already considered in Cases 1 and 2;
- in Cases 7–10 the graph G contains no triangles sharing a common edge, since such a situation was already considered in Cases 5 and 6.

We turn to the cases mentioned above. In each case we prove the inequality $i(G) > \partial_2(G)$ and thereby arrive at a contradiction with the hypotheses of the theorem.

Case 1 (the graph G contains at least one pendant vertex u). Denote by v the only neighbour of the pendant vertex and consider two subcases.

Case 1a: the vertices u and v form a connected component K_2. Consider the graph $G' = (G \setminus K_2) \cup K_1$. Since $\overline{d}(G) \leqslant 2$, we have $\overline{d}(G') \leqslant 2$ and, by assumption, $\partial_2(G') < i(G')$. Moreover, $\partial_2(G) = \partial_2(G')$ and $i(G) = (3/2) \cdot i(G')$, so that $\partial_2(G) < i(G)$.

Case 1b: the vertex v is adjacent to vertices $v_1, \dots, v_m$ distinct from u. Denote by G_1 the result of deleting the vertex u from G, and by G_2' the result of deleting v from G_1. It is easily seen that $\overline{d}(G_1) \leqslant 2$ and $\overline{d}(G_2') \leqslant 2$, so that $\partial_2(G_1) \leqslant i(G_1)$ and $\partial_2(G_2') \leqslant i(G_2')$ by assumption. Since we have the equality

$$i(G) \overset{(3.4)}{=} i(G, u^-) + i(G, u^+) = i(G \setminus u) + i(G \setminus N[u]) = i(G_1) + i(G_2'),$$

it is sufficient to prove that $\partial_2(G) \leqslant \partial_2(G_1) + \partial_2(G_2')$. As the pendant vertex u is contained in each 2-dominating set of the graph G, we have

$$\begin{aligned} \partial_2(G) = \partial_2(G, u^+) &\overset{(3.3)}{=} \partial_2(G, u^+, v^+) + \partial_2(G, u^+, v^-) \\ &= \partial_2(G, v^+) + \partial_2(G \setminus \{u, v\}) = \partial_2(G_1, v^+) + \partial_2(G_2') \overset{(3.5)}{\leqslant} \partial_2(G_1) + \partial_2(G_2'). \end{aligned}$$

Since at least one of the graphs G_1 and G_2' is not 2-regular, at least one of the inequalities $\partial_2(G_1) \leqslant i(G_1)$ and $\partial_2(G_2') \leqslant i(G_2')$ is strict, which yields the strict inequality $\partial_2(G) < i(G)$.

Case 2 (the graph G contains a 2-regular connected component, that is, a cycle C_m, where $m \geqslant 3$). Since $\overline{d}(G \setminus C_m) \leqslant 2$ and the graph G is not 2-regular, the graph $G \setminus C_m$ is not 2-regular either. Since $\partial_2(C_m) = i(C_m)$ and $\partial_2(G \setminus C_m) < i(G \setminus C_m)$, we obtain $\partial_2(G) < i(G)$, as required.

Case 3 (the graph G contains a 3-regular connected component H). Suppose that H has $2m$ vertices and $3m$ edges for some $m \geqslant 2$. It is assumed that G contains no pendant vertices and $\overline{d}(G) \leqslant 2$. Then $|V(G)| \geqslant |E(G)|$, so that $|V(G \setminus H)| \geqslant |E(G \setminus H)| + m$. As the graph $G \setminus H$ still has no pendant vertices, by Lemma 9 it contains at least m isolated vertices. Denote by G' the graph obtained from G by deleting the subgraph $mK_1 \cup H$; then $|V(G')| \geqslant |E(G')|$ and $i(G') \geqslant \partial_2(G')$ by assumption. Let us show that

$$i(mK_1 \cup H) = 2^m \cdot i(H) > \partial_2(H) = \partial_2(mK_1 \cup H).$$

In fact, $i(H) \geqslant 2^m$ by Lemma 8, and the estimate $\partial_2(H) < 2^{2m}$ is trivial. Thus, $i(G) > \partial_2(G)$, as required.

Case 4 (the graph G contains at least two adjacent vertices u and v, each of degree at least 4). Consider the vertex sets $N(u) \setminus \{v\} = \{u_1, \dots, u_p\}$ and $N(v) \setminus \{u\} = \{v_1, \dots, v_q\}$. Note that these sets can have common elements, which does not affect the arguments. Denote by G_1 the graph obtained from G by deleting the edge uv and an isolated vertex. Since $\deg_{G_1}(u) \geqslant 3$, the graph G_1 is not 2-regular. As $\overline{d}(G_1) \leqslant 2$, we have $\partial_2(G_1) < i(G_1)$. It follows from (3.11) that adding an edge to a graph reduces the number of its independent sets by a factor of 4/3 at most, hence $i(G) \geqslant 2 \cdot (3/4) \cdot i(G_1)$. It remains to show that

$$\partial_2(G) \leqslant \frac{3}{2} \cdot \partial_2(G_1). \tag{4.1}$$

Since

$$\partial_2(G) \overset{(3.1)}{=} \partial_2(G, u^+, v^+) + \partial_2(G, u^+, v^-) + \partial_2(G, u^-, v^+) + \partial_2(G, u^-, v^-),$$
$$\partial_2(G, u^+, v^+) = \partial_2(G_1, u^+, v^+) \quad \text{and} \quad \partial_2(G, u^-, v^-) = \partial_2(G_1, u^-, v^-),$$

from (3.10) we obtain the relation

$$\partial_2(G) = \partial_2(G_1) + \partial_2^*(G, G_1, u^+, v^-) + \partial_2^*(G, G_1, u^-, v^+).$$

Then inequality (4.1) follows from the estimate

$$\begin{aligned}\partial_2^*(G, G_1, u^+, v^-) &+ \partial_2^*(G, G_1, u^-, v^+) \\ &\leqslant \frac{1}{2} \cdot (\partial_2(G_1, u^+, v^+) + \partial_2(G_1, u^+, v^-) + \partial_2(G_1, u^-, v^+)).\end{aligned} \tag{4.2}$$

Since

$$\max\{\partial_2(G_1, u^+, v^-), \partial_2(G_1, u^-, v^+)\} \overset{(3.7)}{\leqslant} \partial_2(G_1, u^+, v^+),$$

it is sufficient to show that

$$\partial_2^*(G, G_1, u^+, v^-) \leqslant \frac{3}{4} \cdot \partial_2(G_1, u^+, v^-) \tag{4.3}$$

and

$$\partial_2^*(G, G_1, u^-, v^+) \leqslant \frac{3}{4} \cdot \partial_2(G_1, u^-, v^+). \tag{4.4}$$

Let us prove (4.3). Consider two possible values of q.

The case $q = 3$. We have

$$\begin{aligned}
\partial_2^*(G, G_1, u^+, v^-) &= \partial_2(G, u^+, v^-, v_1^+, v_2^-, v_3^-) + \partial_2(G, u^+, v^-, v_1^-, v_2^+, v_3^-) \\
&\quad + \partial_2(G, u^+, v^-, v_1^-, v_2^-, v_3^+) \\
&\overset{(3.7)}{\leqslant} \partial_2(G, u^+, v^-, v_1^+, v_2^+, v_3^-) + \partial_2(G, u^+, v^-, v_1^-, v_2^+, v_3^+) \\
&\quad + \partial_2(G, u^+, v^-, v_1^+, v_2^-, v_3^+) \\
&= \partial_2(G_1, u^+, v^-, v_1^+, v_2^+, v_3^-) + \partial_2(G_1, u^+, v^-, v_1^-, v_2^+, v_3^+) \\
&\quad + \partial_2(G_1, u^+, v^-, v_1^+, v_2^-, v_3^+) \\
&\overset{(3.1)}{\leqslant} \partial_2(G_1, u^+, v^-) - \partial_2(G_1, u^+, v^-, v_1^+, v_2^+, v_3^+) \\
&\overset{(3.7)}{\leqslant} \frac{3}{4} \cdot \partial_2(G_1, u^+, v^-).
\end{aligned}$$

The case $q = 4$. Set

$$V_i = N_G(v) \setminus \{u, v_i\} \quad \text{and} \quad V_{i,j} = N_G(v) \setminus \{u, v_i, v_j\}$$

(here $1 \leqslant i, j \leqslant q$ and $i \neq j$). We have

$$\begin{aligned}
2 \cdot \partial_2^*(G, G_1, u^+, v^-) &= 2 \cdot \sum_{i=1}^{q} \partial_2(G, u^+, v^-, v_i^+, V_i^-) \\
&\overset{(3.7),(3.9)}{\leqslant} \Big(\partial_2(G_1, u^+, v^-, v_1^+, v_q^+, V_{1,q}^-) + \sum_{i=1}^{q-1} \partial_2(G_1, u^+, v^-, v_i^+, v_{i+1}^+, V_{i,j}^-)\Big) \\
&\quad + \sum_{i=1}^{q} \partial_2(G_1, u^+, v^-, v_i^-, V_i^+) \\
&\overset{(3.1)}{<} \partial_2(G_1, u^+, v^-).
\end{aligned}$$

Inequality (4.3) is proved. It is easily seen that (4.4) is established similarly, hence we obtain (4.2), as required.

Case 5 (the graph G contains triangles abd and bcd sharing a common edge bd, and $\deg(a), \deg(c) \geqslant 3$). Denote by H the subgraph induced by the vertices of these triangles. Two subcases are possible.

Case 5a: $ac \in E(G)$; then $H \cong K_4$. It can be assumed that only one vertex of H (say, a) can be adjacent to vertices of $G \setminus H$ (otherwise the graph H contains two adjacent vertices of degree at least 4 in G, and this case was already considered before). Denote by G_3' the graph obtained from G by deleting the vertices b, c and d. It is clear that $|V(G_3')| \geqslant |E(G_3')| - 3$ and G_3' contains no pendant vertices (except, perhaps, the vertex a). By Lemma 9 the graph G_3' contains at least three isolated vertices. Denote by G_6 the graph obtained from G_3' by deleting these vertices. By assumption $\overline{d}(G_6) \leqslant 2$ and $i(G_6) \geqslant \partial_2(G_6)$. Then

$$i(G) = 8 \cdot (i(G_6, a^+) + 4 \cdot i(G_6, a^-)) \overset{(3.8)}{\geqslant} 8 \cdot \frac{5}{2} \cdot (i(G_6, a^+) + i(G_6, a^-)) = 20 \cdot i(G_6).$$

By Lemma 4 we have $\partial_2(G \setminus H) \leqslant \partial_2(3K_1 \cup G_6) = \partial_2(G_6)$. Then we obtain

$$\partial_2(G) = \partial_2(G,a^+) + \partial_2(G,a^-) = 7 \cdot \partial_2(G_6,a^+) + 4 \cdot \partial_2(G \setminus H) \leqslant 11 \cdot \partial_2(G_6) < i(G),$$

as required.

Case 5b: $ac \notin E(G)$; then $H \cong K_4 - e$. It can be assumed that $\deg(b) \geqslant \deg(d)$. We assume that G contains no adjacent vertices of degree greater than 3. Then $\deg(d) = 3$. Since by assumption $\deg(a), \deg(c) \geqslant 3$, the graph $G \setminus d$ contains no pendant vertices. Hence it contains at least two isolated vertices by Lemma 9. Let G_3 denote the graph obtained by deleting these vertices from $G \setminus d$. Relations (3.2) and (3.8) yield the inequality $i(G_3) \leqslant 8 \cdot i(G_3, a^-, b^-, c^-)$, so that

$$\begin{aligned} i(G) &\overset{(3.4)}{=} i(G,d^-) + i(G,d^+) = i(2K_1) \cdot (i(G_3) + i(G_3,a^-,b^-,c^-)) \\ &\geqslant 4 \cdot (i(G_3) + \frac{1}{8} \cdot i(G_3)) = \frac{9}{2} \cdot i(G_3). \end{aligned}$$

Let us show that $\partial_2(G) < (9/2) \cdot \partial_2(G_3)$. We split the proof into three steps.

Step 1. We use (3.3) to obtain

$$\begin{aligned} \partial_2(G,a^+,b^+,c^+) &= \partial_2(G,a^+,b^+,c^+,d^-) + \partial_2(G,a^+,b^+,c^+,d^+) \\ &= 2 \cdot \partial_2(G_3,a^+,b^+,c^+). \end{aligned}$$

In a similar way,

$$\begin{aligned} \partial_2(G,a^+,b^-,c^+) &= \partial_2(G,a^+,b^-,c^+,d^-) + \partial_2(G,a^+,b^-,c^+,d^+) \\ &= 2 \cdot \partial_2(G_3,a^+,b^-,c^+). \end{aligned}$$

Step 2. We have

$$\begin{aligned} \partial_2(G,a^+,b^+,c^-) &\overset{(3.3)}{=} \partial_2(G,a^+,b^+,c^-,d^+) + \partial_2(G,a^+,b^+,c^-,d^-) \\ &\overset{(3.7)}{\leqslant} \partial_2(G,a^+,b^+,c^+,d^+) + \partial_2(G,a^+,b^+,c^-,d^-) \\ &\overset{(3.5)}{\leqslant} \partial_2(G_3,a^+,b^+,c^+) + \partial_2(G_3,a^+,b^+,c^-). \end{aligned}$$

In a similar way,

$$\partial_2(G,a^-,b^+,c^+) \leqslant \partial_2(G_3,a^+,b^+,c^+) + \partial_2(G_3,a^-,b^+,c^+).$$

Since each 2-independent set in the graph G that contains at most one vertex of the set $\{a,b,c\} = N_G(d)$ contains the vertex d, we have

$$\begin{aligned} \partial_2(G,a^+,b^-,c^-) &= \partial_2(G,a^+,b^-,c^-,d^+) \leqslant \partial_2(G_3,a^+,b^+,c^-), \\ \partial_2(G,a^-,b^-,c^+) &= \partial_2(G,a^-,b^-,c^+,d^+) \leqslant \partial_2(G_3,a^-,b^+,c^+), \\ \partial_2(G,a^-,b^-,c^-) &= \partial_2(G,a^-,b^-,c^-,d^+) \leqslant \partial_2(G_3,a^-,b^+,c^-). \end{aligned}$$

Step 3. Let us show that $\partial_2(G, a^-, b^+, c^-) \leqslant 2 \cdot \partial_2(G_3, a^+, b^+, c^-)$. Set $C = N_G(c) \setminus \{b, d\}$. Since $\deg_G(c) \geqslant 3$, the set C is nonempty. Then we obtain

$$\begin{aligned}\partial_2(G, a^-, b^+, c^-) &= \partial_2(G, a^-, b^+, c^-, d^+) \overset{(3.7)}{\leqslant} \partial_2(G, a^+, b^+, c^-, d^+)\\ &= \partial_2(G_3, a^+, b^+, c^-) + \partial_2(G, a^+, b^+, c^-, C^-, d^+)\\ &\overset{(3.9)}{\leqslant} \partial_2(G_3, a^+, b^+, c^-) + \partial_2(G, a^+, b^+, c^-, C^+, d^-)\\ &= \partial_2(G_3, a^+, b^+, c^-) + \partial_2(G_3, a^+, b^+, c^-, C^+)\\ &\overset{(3.5)}{\leqslant} 2 \cdot \partial_2(G_3, a^+, b^+, c^-).\end{aligned}$$

Thus, we have shown that

$$\begin{aligned}\partial_2(G) &\leqslant 4 \cdot \partial_2(G_3, a^+, b^+, c^+) + 4 \cdot \partial_2(G_3, a^+, b^+, c^-) + 2 \cdot \partial_2(G_3, a^-, b^+, c^+)\\ &\quad + 2 \cdot \partial_2(G_3, a^+, b^-, c^+) + \partial_2(G_3, a^-, b^+, c^-)\\ &< \frac{9}{2} \cdot \partial_2(G_3) \leqslant \frac{9}{2} \cdot i(G_3) \leqslant i(G).\end{aligned}$$

Case 6 (the graph G contains a triangle uvw with at least one vertex of degree distinct from 3).

Case 6a: at least two vertices of the triangle (for instance, v and w) have degree 2. We assume that G contains no 2-regular subgraphs. Then $\deg(u) \geqslant 3$. Denote by G_3 the graph obtained from G by deleting the vertices v and w and an isolated vertex. It is clear that if $\overline{d}(G) \leqslant 2$, then $\overline{d}(G_3) \leqslant 2$ too. Since $\deg_{G_3}(u) \geqslant 1$, it follows from (3.12) that $i(G_3, u^+) < i(G_3, u^-)$, so that

$$i(G) = 2 \cdot (i(G_3, u^+) + 3 \cdot i(G_3, u^-)) > 4 \cdot i(G_3).$$

By Lemma 4 we have $\partial_2(G_3) \geqslant \partial_2(G_3 \setminus u)$. Each 2-dominating set $D' \in \mathfrak{D}_2(G, u^+)$ contains at least one vertex from the set $\{v, w\}$, and each 2-dominating set $D'' \in \mathfrak{D}_2(G, u^-)$ contains both of these vertices. Thus, we obtain

$$\begin{aligned}\partial_2(G) &\overset{(3.1)}{=} \partial_2(G, u^+, v^+, w^+) + \partial_2(G, u^+, v^+, w^-)\\ &\quad + \partial_2(G, u^+, v^-, w^+) + \partial_2(G, u^-, v^+, w^+)\\ &= 3 \cdot \partial_2(G_3, u^+) + \partial_2(G_3 \setminus u) \leqslant 4 \cdot \partial_2(G_3) < i(G).\end{aligned}$$

Case 6b: exactly one vertex of the triangle (for example, the vertex w) has degree 2. Denote by G_1 the graph obtained from G by deleting the edge uv and an isolated vertex. Since $N_{G_1}(w) = \{u, v\}$, we obtain

$$i(G_1, u^-, v^-) \overset{(3.4)}{=} i(G_1, u^-, v^-, w^-) + i(G_1, u^-, v^-, w^+) \geqslant 2 \cdot i(G_1, u^+, v^+).$$

It follows from (3.8) that $\min\{i(G_1, u^-, v^+), i(G_1, u^+, v^-)\} \geqslant i(G_1, u^+, v^+)$. Then

$$4 \cdot i(G_1, u^+, v^+) \leqslant i(G_1, u^-, v^+) + i(G_1, u^+, v^-) + i(G_1, u^-, v^-). \tag{4.5}$$

Consequently,

$$\frac{i(G)}{i(G_1)} \geqslant 2 \cdot \frac{i(G_1,u^-,v^+) + i(G_1,u^+,v^-) + i(G_1,u^-,v^-)}{i(G_1,u^+,v^+) + i(G_1,u^-,v^+) + i(G_1,u^+,v^-) + i(G_1,u^-,v^-)} \geqslant \frac{8}{5}.$$

Let us show that $\partial_2(G) \leqslant (8/5) \cdot \partial_2(G_1)$. The same arguments as in Case 4 suggest that it is sufficient to prove the relation

$$\begin{aligned}&\partial_2^*(G,G_1,u^-,v^+) + \partial_2^*(G,G_1,u^+,v^-)\\ &\quad\leqslant \frac{3}{5} \cdot (\partial_2(G_1,u^+,v^+) + \partial_2(G_1,u^-,v^+) + \partial_2(G_1,u^+,v^-) + \partial_2(G_1,u^-,v^-)).\end{aligned} \tag{4.6}$$

We prove the inequality

$$5 \cdot \partial_2^*(G,G_1,u^-,v^+) \leqslant \partial_2(G_1,u^+,v^+) + 3 \cdot \partial_2(G_1,u^-,v^+). \tag{4.7}$$

Let us show that

$$2 \cdot \partial_2^*(G,G_1,u^-,v^+) \leqslant \partial_2(G_1,u^+,v^+). \tag{4.8}$$

We introduce the notation $U = N_G(u) \setminus \{v,w\}$. Each 2-dominating set $D \in \mathfrak{D}_2(G,u^-)$ contains the vertex w. Since $\deg(w) = 2$, we have $\partial_2(G,u^+,v^+,w^+) = \partial_2(G,u^+,v^+,w^-)$, and thus

$$\begin{aligned}2 \cdot \partial_2^*(G,G_1,u^-,v^+) &= 2 \cdot \partial_2^*(G,G_1,u^-,v^+,w^+) = 2 \cdot \partial_2(G,u^-,U^-,v^+,w^+)\\ &\overset{(3.5),(3.7)}{\leqslant} 2 \cdot \partial_2(G,u^+,v^+,w^+) = \partial_2(G,u^+,v^+,w^-) + \partial_2(G,u^+,v^+,w^+)\\ &= \partial_2(G_1,u^+,v^+,w^-) + \partial_2(G_1,u^+,v^+,w^+) \overset{(3.3)}{=} \partial_2(G_1,u^+,v^+).\end{aligned}$$

Now we show that

$$\partial_2^*(G,G_1,u^-,v^+) \leqslant \partial_2(G_1,u^-,v^+). \tag{4.9}$$

We have

$$\begin{aligned}\partial_2^*(G,G_1,u^-,v^+) &= \partial_2(G,u^-,U^-,v^+,w^+) \overset{(3.7)}{\leqslant} \partial_2(G,u^-,U^+,v^+,w^+)\\ &= \partial_2(G_1,u^-,U^+,v^+,w^+) \overset{(3.5)}{\leqslant} \partial_2(G_1,u^-,v^+).\end{aligned}$$

Inequalities (4.8) and (4.9) yield (4.7). In a similar way we prove the estimate

$$5 \cdot \partial_2^*(G,G_1,u^+,v^-) \leqslant \partial_2(G_1,u^+,v^+) + 3 \cdot \partial_2(G_1,u^+,v^-),$$

which yields (4.6), as required.

Case 6c: at least one vertex of the triangle (for instance, u) has degree at least 4. Assume that $\deg(v) = \deg(w) = 3$ (all other options were covered by Cases 4, 6a and 6b). We denote by $u_1,\dots,u_m$ the vertices adjacent to u, and by v_1 and w_1 the only neighbours of v and w, respectively, outside the triangle uvw. It can be assumed that G contains no pair of triangles sharing a common edge (since this situation was considered in Cases 5 and 6b). Hence the vertices $u_1,\dots,u_m,v_1,w_1$ are pairwise distinct.

We denote by G_1 the graph obtained from G by deleting the edge uv and an isolated vertex and let us show that $i(G) \geqslant (8/5) \cdot i(G_1)$. As in Case 6b, it is sufficient to establish inequality (4.5). In the graph $G \setminus v$ the vertex w has degree 2, and all other neighbours of u have degree at most 3. Then (3.12) yields $i(G_1, u^-, v^-) \geqslant 2 \cdot i(G_1, u^+, v^-)$. Moreover, it follows from (3.8) that

$$\min\{i(G_1, u^+, v^-), i(G_1, u^-, v^+)\} \geqslant i(G_1, u^+, v^+),$$

which yields the inequality $i(G) \geqslant (8/5) \cdot i(G_1)$.

Now let us show that $\partial_2(G) \leqslant (8/5) \cdot \partial_2(G_1)$. As in Case 6b, it is sufficient to establish (4.6). We split the proof into two steps.

Step 1. We show that

$$5 \cdot \partial_2^*(G, G_1, u^-, v^+) \leqslant 3 \cdot \partial_2(G_1, u^-, v^+) + 3 \cdot \partial_2(G_1, u^-, v^-) + \frac{1}{2} \cdot \partial_2(G_1, u^+, v^+). \tag{4.10}$$

We introduce the notation $U = N(u) \setminus \{v, w\}$ and $U_i = U \setminus \{u_i\}$, $1 \leqslant i \leqslant m$. Then the inequality

$$\partial_2^*(G, G_1, u^-, v^+) \leqslant \partial_2(G_1, u^-, v^+) \tag{4.11}$$

follows from the relations

$$\begin{aligned}
\partial_2^*(G, G_1, u^-, v^+) &= \partial_2^*(G, G_1, u^-, v^+, w^+) + \partial_2^*(G, G_1, u^-, v^+, w^-) \\
&= \partial_2(G, u^-, U^-, v^+, w^+) + \sum_{i=1}^{m} \partial_2(G, u^-, u_i^+, U_i^-, v^+, w^-) \\
&\overset{(3.7),(3.9)}{\leqslant} \partial_2(G_1, u^-, U^+, v^+, w^+) + \sum_{i=1}^{m} \partial_2(G_1, u^-, u_i^-, U_i^+, v^+, w^+) \\
&\overset{(3.1)}{\leqslant} \partial_2(G_1, u^-, v^+).
\end{aligned}$$

Now let us prove that

$$\partial_2^*(G, G_1, u^-, v^+) \leqslant 2 \cdot \partial_2(G, u^-, v^-). \tag{4.12}$$

We have

$$\begin{aligned}
\partial_2^*(G, G_1, u^-, v^+) &= \partial_2(G, u^-, U^-, v^+, w^+) + \sum_{i=1}^{m} \partial_2(G, u^-, u_i^+, U_i^-, v^+, w^-) \\
&\overset{(3.3),(3.7)}{\leqslant} 2 \cdot \biggl(\partial_2(G, u^-, U^+, v^+, v_1^+, w^+) + \sum_{i=1}^{m} \partial_2(G, u^-, u_i^+, U_i^-, v^+, v_1^+, w^+)\biggr) \\
&= 2 \cdot \biggl(\partial_2(G_1, u^-, U^+, v^+, v_1^+, w^+) + \sum_{i=1}^{m} \partial_2(G_1, u^-, u_i^+, U_i^-, v^+, v_1^+, w^+)\biggr) \\
&= 2 \cdot \biggl(\partial_2(G_1, u^-, U^+, v^-, v_1^+, w^+) + \sum_{i=1}^{m} \partial_2(G_1, u^-, u_i^+, U_i^-, v^-, v_1^+, w^+)\biggr) \\
&\overset{(3.1)}{\leqslant} 2 \cdot \partial_2(G_1, u^-, v^-).
\end{aligned}$$

Finally, it follows from (3.7) that $\partial_2(G, u^-, v^+) \leqslant \partial_2(G, u^+, v^+)$. Hence we obtain

$$\partial_2^*(G, G_1, u^-, v^+) \leqslant \partial_2(G_1, u^+, v^+). \tag{4.13}$$

Inequalities (4.11)–(4.13) yields the required inequality (4.10).

Step 2. Let us show that

$$5 \cdot \partial_2^*(G, G_1, u^+, v^-) \leqslant 3 \cdot \partial_2(G_1, u^+, v^-) + \frac{5}{2} \cdot \partial_2(G_1, u^+, v^+). \tag{4.14}$$

Indeed,

$$\begin{aligned} \partial_2^*(G, G_1, u^+, v^-) &= \partial_2(G, u^+, v^-, w^+, v_1^-) + \partial_2(G, u^+, v^-, w^-, v_1^+) \\ &\overset{(3.7)}{\leqslant} \frac{2}{3} \cdot (\partial_2(G, u^+, v^+, w^+, v_1^-) + \partial_2(G, u^+, v^+, w^-, v_1^+) \\ &\qquad + \partial_2(G, u^+, v^+, w^+, v_1^+)) \\ &= \frac{2}{3} \cdot (\partial_2(G_1, u^+, v^+, w^+, v_1^-) + \partial_2(G_1, u^+, v^+, w^-, v_1^+) \\ &\qquad + \partial_2(G_1, u^+, v^+, w^+, v_1^+)) \\ &\overset{(3.1)}{\leqslant} \frac{2}{3} \cdot \partial_2(G_1, u^+, v^+). \end{aligned} \tag{4.15}$$

Moreover,

$$\begin{aligned} \partial_2^*(G, G_1, u^+, v^-) &= \partial_2(G, u^+, v^-, w^+, v_1^-) + \partial_2(G, u^+, v^-, w^-, v_1^+) \\ &\overset{(3.7)}{\leqslant} 2 \cdot \partial_2(G, u^+, v^-, w^+, v_1^+) = 2 \cdot \partial_2(G_1, u^+, v^-, w^+, v_1^+) \\ &\overset{(3.5)}{\leqslant} 2 \cdot \partial_2(G_1, u^+, v^-). \end{aligned} \tag{4.16}$$

Now (4.15) and (4.16) imply estimate (4.14). Hence we obtain $\partial_2(G) \leqslant (8/5) \cdot \partial_2(G_1)$, as required.

Case 7 (the graph G contains a triangle uvw with all vertices of degree 3, and at least two of the three vertices u_1, v_1 and w_1 adjacent to this triangle — for instance, u_1 and v_1 — have degree at least 3). Consider the graph G_3' obtained from the graph G by deleting the vertices u, v, and w. All the vertices of G_3', except, perhaps, w_1, are not pendant. Hence the graph G_3' contains at least three isolated vertices by Lemma 9. We denote by G_6 the graph obtained from G_3' by deleting these vertices. Then $\overline{d}(G_6) \leqslant 2$ and relation (3.2) yields the equalities

$$\begin{aligned} i(G_6) = i(G_6, u_1^+, v_1^+, w_1^+) &+ i(G_6, u_1^-, v_1^-, w_1^-) \\ &+ i(G_6, u_1^+, v_1^+, w_1^-) + i(G_6, u_1^+, v_1^-, w_1^+) + i(G_6, u_1^-, v_1^+, w_1^+) \\ &+ i(G_6, u_1^-, v_1^-, w_1^+) + i(G_6, u_1^-, v_1^+, w_1^-) + i(G_6, u_1^+, v_1^-, w_1^-) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{8} \cdot i(G) = i(G_6, u_1^+, v_1^+, w_1^+) &+ 4 \cdot i(G_6, u_1^-, v_1^-, w_1^-) \\ &+ 2 \cdot (i(G_6, u_1^+, v_1^+, w_1^-) + i(G_6, u_1^+, v_1^-, w_1^+) + i(G_6, u_1^-, v_1^+, w_1^+)) \\ &+ 3 \cdot (i(G_6, u_1^-, v_1^-, w_1^+) + i(G_6, u_1^-, v_1^+, w_1^-) + i(G_6, u_1^+, v_1^-, w_1^-)). \end{aligned}$$

From (3.8) we obtain the inequalities

$$i(G_6, u_1^+, v_1^+, w_1^+) \leqslant i(G_6, u_1^-, v_1^-, w_1^-),$$
$$i(G_6, u_1^+, v_1^+, w_1^-) + i(G_6, u_1^+, v_1^-, w_1^+) + i(G_6, u_1^-, v_1^+, w_1^+)$$
$$\leqslant i(G_6, u_1^-, v_1^+, w_1^-) + i(G_6, u_1^+, v_1^-, w_1^-) + i(G_6, u_1^-, v_1^-, w_1^+),$$

and therefore $i(G) \geqslant 20 \cdot i(G_6)$. We prove that $\partial_2(G) < 20 \cdot \partial_2(G_6)$. Consider two situations.

Case 7a: $\deg(w_1) = 2$. The proof is split into three steps.

Step 1. Since every 2-dominating sets $D \in \mathfrak{D}_2(G, u_1^+, v_1^+, w_1^+)$ contains at least one vertex from the set $\{u, v, w\}$, we have

$$\partial_2(G, u_1^+, v_1^+, w_1^+) = 7 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^+).$$

Since $\deg(w_1) = 2$, each 2-dominating sets $D' \in \mathfrak{D}_2(G, u_1^+, v_1^+, w_1^-)$ contains w, so that

$$\begin{aligned}
&\partial_2(G, u_1^+, v_1^+, w_1^-)\\
&\quad\overset{(3.1)}{=} \partial_2(G, u_1^+, v_1^+, w_1^-, u^+, v^+, w^+) + \partial_2(G, u_1^+, v_1^+, w_1^-, u^-, v^+, w^+)\\
&\qquad + \partial_2(G, u_1^+, v_1^+, w_1^-, u^+, v^-, w^+) + \partial_2(G, u_1^+, v_1^+, w_1^-, u^-, v^-, w^+)\\
&\quad\overset{(3.7)}{\leqslant} 4 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^+).
\end{aligned}$$

Step 2. Let us show that $\partial_2(G, u_1^-, v_1^+, w_1^+) \leqslant 13 \cdot \partial_2(G_6, u_1^-, v_1^+, w_1^+)$. Since every 2-dominating set $D \in \mathfrak{D}_2(G, u^-, u_1^-)$ contains the vertices v and w, we obtain

$$\begin{aligned}
\partial_2(G, u_1^-, v_1^+, w_1^+) &\overset{(3.3)}{=} \partial_2(G, u^-, v^+, w^+, u_1^-, v_1^+, w_1^+) + \partial_2(G, u^+, u_1^-, v_1^+, w_1^+)\\
&= \partial_2(G_6, u_1^-, v_1^+, w_1^+) + 4 \cdot \partial_2(G, u^+, v^+, w^+, u_1^-, v_1^+, w_1^+).
\end{aligned} \tag{4.17}$$

Consider three subcases depending on whether or not the vertex u_1 is adjacent to v_1 and w_1.

Subcase 1: the vertex u_1 is adjacent to both v_1 and w_1. Then we obviously have

$$\partial_2(G, u^+, v^+, w^+, u_1^-, v_1^+, w_1^+) = \partial_2(G_6, u_1^-, v_1^+, w_1^+).$$

Thus, it follows from (4.17) that $\partial_2(G, u_1^-, v_1^+, w_1^+) \leqslant 5 \cdot \partial_2(G_6, u_1^-, v_1^+, w_1^+)$.

Subcase 2: the vertex u_1 is adjacent to precisely one of v_1 and w_1. Set $U_1 = N(u_1) \setminus \{u, v_1, w_1\}$. Since $\deg_G(u_1) \geqslant 3$, we have $U_1 \neq \varnothing$. Then we obtain

$$\begin{aligned}
&\partial_2(G, u^+, v^+, w^+, u_1^-, v_1^+, w_1^+)\\
&\quad= \partial_2(G_6, u_1^-, v_1^+, w_1^+) + \partial_2(G, u^+, v^+, w^+, u_1^-, v_1^+, w_1^+, U_1^-)\\
&\quad\overset{(3.7)}{\leqslant} \partial_2(G_6, u_1^-, v_1^+, w_1^+) + \partial_2(G, u^+, v^+, w^+, u_1^-, v_1^+, w_1^+, U_1^+)\\
&\quad\leqslant 2 \cdot \partial_2(G_6, u_1^-, v_1^+, w_1^+).
\end{aligned}$$

Thus, it follows from (4.17) that $\partial_2(G, u_1^-, v_1^+, w_1^+) \leqslant 9 \cdot \partial_2(G_6, u_1^-, v_1^+, w_1^+)$.

Subcase 3: the vertex u_1 is adjacent to none of the vertices v_1 and w_1. Denote by $x_1, \ldots, x_s$ the neighbours of the vertex u_1 distinct from u (here $s \geqslant 2$ by assumption). Let $X = N(u_1) \setminus u$ and $X_i = X \setminus x_i$, $1 \leqslant i \leqslant s$. Then

$$\begin{aligned} &\partial_2(G, u^+, v^+, w^+, u_1^-, v_1^+, w_1^+) \\ &\quad = \partial_2(G_6, u_1^-, v_1^+, w_1^+) + \sum_{i=1}^{s} \partial_2(G, u^+, v^+, w^+, u_1^-, v_1^+, w_1^+, x_i^+, X_i^-). \end{aligned}$$

If $\deg(u_1) = s + 1 = 3$, then

$$\begin{aligned} &\sum_{i=1}^{2} \partial_2(G, u^+, v^+, w^+, u_1^-, v_1^+, w_1^+, x_i^+, X_i^-) \\ &\quad \overset{(3.7)}{\leqslant} 2 \cdot \partial_2(G, u^+, v^+, w^+, u_1^-, v_1^+, w_1^+, X^+) \\ &\quad = 2 \cdot \partial_2(G_6, u_1^-, v_1^+, w_1^+, X^+) \overset{(3.5)}{\leqslant} 2 \cdot \partial_2(G_6, u_1^-, v_1^+, w_1^+). \end{aligned}$$

If $\deg(u_1) \geqslant 4$, then

$$\begin{aligned} &\sum_{i=1}^{s} \partial_2(G, u^+, v^+, w^+, u_1^-, v_1^+, w_1^+, x_i^+, X_i^-) \\ &\quad \overset{(3.9)}{\leqslant} \sum_{i=1}^{s} \partial_2(G, u^+, v^+, w^+, u_1^-, v_1^+, w_1^+, x_i^-, X_i^+) \\ &\quad = \sum_{i=1}^{s} \partial_2(G_6, u_1^-, v_1^+, w_1^+, x_i^-, X_i^+) \overset{(3.1)}{<} \partial_2(G_6, u_1^-, v_1^+, w_1^+). \end{aligned}$$

Thus, it follows form (4.17) that

$$\partial_2(G, u_1^-, v_1^+, w_1^+) \leqslant 13 \cdot \partial_2(G_6, u_1^-, v_1^+, w_1^+).$$

The inequality $\partial_2(G, u_1^+, v_1^-, w_1^+) \leqslant 13 \cdot \partial_2(G_6, u_1^+, v_1^-, w_1^+)$ is proved in the same way.

Step 3. It is easily seen that every 2-dominating set $D \in \mathfrak{D}_2(G, u_1^+, v_1^-, w_1^-)$ contains at least two vertices of the set $\{u, v, w\}$. Then

$$\begin{aligned} &\partial_2(G, u_1^+, v_1^-, w_1^-) \\ &\quad \overset{(3.1)}{=} \partial_2(G, u_1^+, v_1^-, w_1^-, u^+, v^+, w^+) + \partial_2(G, u_1^+, v_1^-, w_1^-, u^-, v^+, w^+) \\ &\qquad + \partial_2(G, u_1^+, v_1^-, w_1^-, u^+, v^-, w^+) + \partial_2(G, u_1^+, v_1^-, w_1^-, u^+, v^+, w^-) \\ &\quad \overset{(3.7)}{\leqslant} 2 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^+) + \partial_2(G_6, u_1^+, v_1^-, w_1^+) + \partial_2(G_6, u_1^+, v_1^+, w_1^-). \end{aligned}$$

In a similar way we obtain the inequalities

$$\begin{aligned} &\partial_2(G, u_1^-, v_1^+, w_1^-) \\ &\quad \leqslant 2 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^+) + \partial_2(G_6, u_1^-, v_1^+, w_1^+) + \partial_2(G_6, u_1^+, v_1^+, w_1^-) \end{aligned}$$

and

$$\partial_2(G, u_1^-, v_1^-, w_1^+) \\ \leqslant 2 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^+) + \partial_2(G_6, u_1^-, v_1^+, w_1^+) + \partial_2(G_6, u_1^+, v_1^-, w_1^+).$$

Inequality (3.7) implies the estimate $\partial_2(G, u_1^+, v_1^-, w_1^-) \geqslant \partial_2(G, u_1^-, v_1^-, w_1^-)$. Then we obtain

$$\partial_2(G, u_1^-, v_1^-, w_1^-) \\ \leqslant 2 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^+) + \partial_2(G_6, u_1^+, v_1^-, w_1^+) + \partial_2(G_6, u_1^+, v_1^+, w_1^-).$$

Thus we have established the relation

$$\partial_2(G) \leqslant 19 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^+) + 15 \cdot \partial_2(G_6, u_1^-, v_1^+, w_1^+) \\ + 16 \cdot \partial_2(G_6, u_1^+, v_1^-, w_1^+) + 3 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^-) < 20 \cdot \partial_2(G_6).$$

Case 7b: $\deg(w_1) \geqslant 3$. We split the proof into four steps.

Step 1. Similarly to Case 7a we have

$$\partial_2(G, u_1^+, v_1^+, w_1^+) = 7 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^+).$$

Step 2. Since $\min\{\deg(u_1), \deg(v_1), \deg(w_1)\} = 3$, the arguments of Step 2 in Case 7a can be applied to each of u_1, v_1 and w_1. Hence

$$\partial_2(G, u_1^-, v_1^+, w_1^+) + \partial_2(G, u_1^+, v_1^-, w_1^+) + \partial_2(G, u_1^+, v_1^+, w_1^-) \\ \leqslant 13 \cdot (\partial_2(G_6, u_1^-, v_1^+, w_1^+) + \partial_2(G_6, u_1^+, v_1^-, w_1^+) + \partial_2(G_6, u_1^+, v_1^+, w_1^-)).$$

Step 3. Let us show that

$$\partial_2(G, u_1^+, v_1^-, w_1^-) + \partial_2(G, u_1^-, v_1^+, w_1^-) + \partial_2(G, u_1^-, v_1^-, w_1^+) \\ \leqslant 12 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^+).$$

Without loss of generality it can be assumed that

$$\partial_2(G, u_1^+, v_1^-, w_1^-) = \max\{\partial_2(G, u_1^+, v_1^-, w_1^-), \partial_2(G, u_1^-, v_1^+, w_1^-), \partial_2(G, u_1^-, v_1^-, w_1^+)\}.$$

Then it is sufficient to establish the inequality

$$\partial_2(G, u_1^+, v_1^-, w_1^-) \leqslant 4 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^+).$$

Since every 2-dominating sets $D \in \mathfrak{D}_2(G, u_1^+, v_1^-, w_1^-)$ contains at least two vertices from the set $\{u, v, w\}$, we obtain

$$\partial_2(G, u_1^+, v_1^-, w_1^-) \\ \overset{(3.1)}{=} \partial_2(G, u^+, v^+, w^+, u_1^+, v_1^-, w_1^-) + \partial_2(G, u^+, v^+, w^-, u_1^+, v_1^-, w_1^-) \\ + \partial_2(G, u^+, v^-, w^+, u_1^+, v_1^-, w_1^-) + \partial_2(G, u^-, v^+, w^+, u_1^+, v_1^-, w_1^-) \\ \overset{(3.7)}{\leqslant} 4 \cdot \partial_2(G, u^+, v^+, w^+, u_1^+, v_1^+, w_1^+) \leqslant 4 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^+).$$

Step 4. Since every 2-dominating set $D \in \mathfrak{D}_2(G, u_1^-, v_1^-, w_1^-)$ also contains at least two vertices from the set $\{u, v, w\}$, we obtain

$$\begin{aligned}
&\partial_2(G, u_1^-, v_1^-, w_1^-)\\
&\overset{(3.1)}{=} \partial_2(G, u^+, v^+, w^+, u_1^-, v_1^-, w_1^-) + \partial_2(G, u^+, v^+, w^-, u_1^-, v_1^-, w_1^-)\\
&\quad + \partial_2(G, u^+, v^-, w^+, u_1^-, v_1^-, w_1^-) + \partial_2(G, u^-, v^+, w^+, u_1^-, v_1^-, w_1^-)\\
&\overset{(3.7)}{\leqslant} \partial_2(G_6, u_1^+, v_1^+, w_1^+) + \partial_2(G_6, u_1^+, v_1^+, w_1^-)\\
&\quad + \partial_2(G_6, u_1^+, v_1^-, w_1^+) + \partial_2(G_6, u_1^-, v_1^+, w_1^+).
\end{aligned}$$

Thus we have established the relation

$$\begin{aligned}
\partial_2(G) &\leqslant 20 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^+) + 14 \cdot \partial_2(G_6, u_1^-, v_1^+, w_1^+)\\
&\quad + 14 \cdot \partial_2(G_6, u_1^+, v_1^-, w_1^+) + 14 \cdot \partial_2(G_6, u_1^+, v_1^+, w_1^-) < 20 \cdot \partial_2(G_6).
\end{aligned}$$

Note that the last inequality is strict, since it follows from the condition $\deg_{G_6}(u_1) \geqslant 2$ that $\partial_2(G_6, u_1^-, v_1^+, w_1^+) \geqslant 1$.

Case 8 (the graph G contains a triangle uvw all of whose vertices have degree 3 and at least two of the three vertices u_1, v_1 and w_1 adjacent to the triangle — for instance, u_1 and v_1 — have degree 2). Note that some of the vertices u_1, v_1, w_1 can be adjacent to each other, but this does not affect the lines of our reasoning. We denote by G_1 the graph obtained from G by deleting the edge uv and an isolated vertex. Then (3.11) yields the relation $i(G) \geqslant (3/2) \cdot i(G_1)$. It remains to establish (4.1). As follows from the arguments of Case 4, it is sufficient to prove (4.2). Let us show that

$$4 \cdot \partial_2^*(G, G_1, u^-, v^+) \leqslant 2 \cdot \partial_2(G_1, u^-, v^+) + \partial_2(G_1, u^+, v^+).$$

Note that every 2-dominating set $D \in \mathfrak{D}_2(G, u^-)$ contains the vertex u_1. Moreover, every 2-dominating set $D_1 \in \mathfrak{D}_2(G_1, u^-)$ contains both u_1 and w. Then we obtain

$$\begin{aligned}
\partial_2^*(G, G_1, u^-, v^+) &= \partial_2(G, u^-, v^+, w^-, u_1^+) \overset{(3.7)}{\leqslant} \partial_2(G, u^-, v^+, w^+, u_1^+)\\
&= \partial_2(G_1, u^-, v^+, w^+, u_1^+) \overset{(3.5)}{\leqslant} \partial_2(G_1, u^-, v^+).
\end{aligned}$$

Moreover,

$$\begin{aligned}
2 \cdot \partial_2^*(G, G_1, u^-, v^+) &= 2 \cdot \partial_2(G, u^-, v^+, w^-, u_1^+)\\
&\overset{(3.7)}{\leqslant} \partial_2(G, u^+, v^+, w^-, u_1^+) + \partial_2(G, u^+, v^+, w^+, u_1^+)\\
&\overset{(3.3),(3.5)}{\leqslant} \partial_2(G, u^+, v^+) = \partial_2(G_1, u^+, v^+).
\end{aligned}$$

In a similar way we establish the inequality

$$4 \cdot \partial_2^*(G, G_1, u^+, v^-) \leqslant 2 \cdot \partial_2(G_1, u^+, v^-) + \partial_2(G_1, u^+, v^+),$$

so that $\partial_2(G) \leqslant (3/2) \cdot \partial_2(G_1)$.

Note that the use of the simpler transformation from Case 8 in Case 7 does not furnish the required result. On the other hand, the transformation from Case 7 does not allow one to use Lemma 9 in Case 8.

Case 9 (the graph G contains a vertex u of degree at least 3, which is adjacent to a vertex v of degree 3 and shares no neighbours with it). We denote by $u_1, \dots, u_s$ the neighbours of the vertex u and by v_1 and v_2 the neighbours of v.

Case 9a: $\deg(u) \geqslant 4$. We denote by G_1 the graph obtained from G by deleting the edge uv and an isolated vertex. Without loss of generality it can be assumed that $\deg(u) \geqslant \max\{\deg(v_1), \deg(v_2)\}$. Let us show that

$$i(G) \geqslant \frac{5}{3} \cdot i(G_1).$$

It follows from the arguments of Case 6b that it is sufficient to prove the relation

$$5 \cdot i(G_1, u^+, v^+) \leqslant i(G_1, u^-, v^+) + i(G_1, u^+, v^-) + i(G_1, u^-, v^-). \tag{4.18}$$

We assume that all neighbours of u have degree at most 3 (all other options were considered in Case 4). If $\deg(u) \geqslant 5$, then

$$i(G_1, u^-, v^-) \overset{(3.8)}{\geqslant} i(G_1, u^-, v^+) \overset{(3.12)}{\geqslant} 2 \cdot i(G_1, u^+, v^+),$$

and we obtain (4.18).

Now let $\deg(u) = 4$. Then $\max\{\deg(v_1), \deg(v_2)\} \leqslant 4$ by assumption. In this case it follows from (3.12) that

$$i(G_1, u^-, v^+) \geqslant \frac{7}{4} \cdot i(G_1, u^+, v^+) \quad \text{and} \quad i(G_1, u^+, v^-) \geqslant \frac{5}{4} \cdot i(G_1, u^+, v^+).$$

Moreover,

$$i(G_1, u^-, v^-) \geqslant \frac{5}{4} \cdot i(G_1, u^-, v^+) > 2 \cdot i(G_1, u^+, v^+),$$

which yields (4.18) again.

It follows from the condition $\deg_G(u) \geqslant 4$ that the graph G_1 is not 2-regular. Thus, $i(G_1) > \partial_2(G_1)$ by assumption, and it is sufficient to establish the inequality

$$\partial_2(G) \leqslant \frac{5}{3} \cdot \partial_2(G_1).$$

The arguments of Case 4 show that it is sufficient to prove the inequality

$$\begin{aligned} &\partial_2^*(G, G_1, u^-, v^+) + \partial_2^*(G, G_1, u^+, v^-) \\ &\qquad \leqslant \frac{2}{3} \cdot \bigl(\partial_2(G_1, u^-, v^+) + \partial_2(G_1, u^+, v^-) + \partial_2(G_1, u^+, v^+)\bigr). \end{aligned} \tag{4.19}$$

We split the proof into two steps.

Step 1. We prove the inequality

$$3 \cdot \partial_2^*(G, G_1, u^-, v^+) \leqslant 2 \cdot \partial_2(G_1, u^-, v^+) + \frac{1}{2} \cdot \partial_2(G_1, u^+, v^+). \tag{4.20}$$

Let us show that $\partial_2^*(G, G_1, u^-, v^+) \leqslant \partial_2(G_1, u^-, v^+)$. Set $U_i = N(u) \setminus \{u_i, v\}$, $1 \leqslant i \leqslant s$. Then we have

$$\begin{aligned}\partial_2^*(G, G_1, u^-, v^+) &= \sum_{i=1}^{k} \partial_2(G, u^-, v^+, U_i^-, u_i^+) \overset{(3.9)}{\leqslant} \sum_{i=1}^{k} \partial_2(G, u^-, v^+, U_i^+, u_i^-) \\ &= \sum_{i=1}^{k} \partial_2(G_1, u^-, v^+, U_i^+, u_i^-) \overset{(3.1)}{<} \partial_2(G_1, u^-, v^+).\end{aligned}$$

Now let us show that $2 \cdot \partial_2^*(G, G_1, u^-, v^+) \leqslant \partial_2(G_1, u^+, v^+)$:

$$\begin{aligned}2 \cdot \partial_2^*(G, G_1, u^-, v^+) &= 2 \cdot \sum_{i=1}^{k} \partial_2(G, u^-, v^+, U_i^-, u_i^+) \\ &\overset{(3.7)}{\leqslant} 2 \cdot \sum_{i=1}^{k} \partial_2(G, u^+, v^+, U_i^-, u_i^+) \\ &\overset{(3.9)}{\leqslant} \sum_{i=1}^{k} \partial_2(G_1, u^+, v^+, U_i^-, u_i^+) + \sum_{i=1}^{k} \partial_2(G_1, u^+, v^+, U_i^+, u_i^-) \\ &\overset{(3.1)}{<} \partial_2(G_1, u^+, v^+).\end{aligned}$$

Step 2. We prove the inequality

$$3 \cdot \partial_2^*(G, G_1, u^+, v^-) \leqslant 2 \cdot \partial_2(G_1, u^+, v^-) + \frac{4}{3} \cdot \partial_2(G_1, u^+, v^+). \tag{4.21}$$

Let us show that $2 \cdot \partial_2^*(G, G_1, u^+, v^-) \leqslant (4/3) \cdot \partial_2(G_1, u^+, v^+)$:

$$\begin{aligned}\partial_2^*(G, G_1, u^+, v^-) &= \partial_2(G, u^+, v^-, v_1^+, v_2^-) + \partial_2(G, u^+, v^-, v_1^-, v_2^+) \\ &\overset{(3.7)}{\leqslant} \frac{2}{3} \cdot \big(\partial_2(G, u^+, v^+, v_1^-, v_2^+) + \partial_2(G, u^+, v^+, v_1^+, v_2^-) \\ &\qquad + \partial_2(G, u^+, v^+, v_1^+, v_2^+)\big) \\ &\overset{(3.1)}{\leqslant} \frac{2}{3} \cdot \partial_2(G, u^+, v^+) = \frac{2}{3} \cdot \partial_2(G_1, u^+, v^+).\end{aligned}$$

Next we show that $\partial_2^*(G, G_1, u^+, v^-) \leqslant 2 \cdot \partial_2(G_1, u^+, v^-)$:

$$\begin{aligned}\partial_2^*(G, G_1, u^+, v^-) &= \partial_2(G, u^+, v^-, v_1^+, v_2^-) + \partial_2(G, u^+, v^-, v_1^-, v_2^+) \\ &\overset{(3.7)}{\leqslant} 2 \cdot \partial_2(G, u^+, v^-, v_1^+, v_2^+) = 2 \cdot \partial_2(G_1, u^+, v^-, v_1^+, v_2^+) \\ &\overset{(3.5)}{\leqslant} 2 \cdot \partial_2(G_1, u^+, v^-).\end{aligned}$$

Inequalities (4.20) and (4.21) yield (4.19), as required.

Case 9b: $\deg(u) = 3$. It can be assumed that the connected component H that contains u and v is not 3-regular and contains no pendant vertices (all other options were covered by Cases 1 and 3). Moreover, we assume that each vertex of degree higher than 3 in H is adjacent only to vertices of degree 2 (all other options were considered in Cases 4 and 9a). At the same time, since H is not 3-regular, it contains at least one vertex w of degree 2. Consider a shortest path from the vertex v to w. It is clear that this path goes through at least one vertex of degree 3 adjacent to a vertex of degree 3 and a vertex of degree 2. We rename the vertices of H so that the vertices u and v are adjacent to each other, have degree 3, and at least one vertex adjacent to v (for instance, v_2) is of degree 2.

We denote by G_1 the graph obtained from G by deleting the edge uv and an isolated vertex. It follows from the conditions $\max\{\deg(u_1), \deg(u_2), \deg(v_1)\} \leqslant 3$ and $\deg(v_2) = 2$ that

$$i(G_1, u^+, v^-) \overset{(3.12)}{\geqslant} \frac{7}{4} \cdot i(G_1, u^+, v^+) \quad \text{and} \quad i(G_1, u^-, v^+) \overset{(3.12)}{\geqslant} \frac{3}{2} \cdot i(G_1, u^+, v^+).$$

Inequality (3.12) implies that $i(G_1, u^-, v^-) > i(G_1, u^+, v^-)$. Then we obtain

$$5 \cdot i(G_1, u^+, v^+) < i(G_1, u^-, v^+) + i(G_1, u^+, v^-) + i(G_1, u^-, v^-). \tag{4.22}$$

Now, as in Case 9a, inequality (4.22) implies that $i(G) > (5/3) \cdot i(G_1)$.

It remains to prove that $\partial_2(G) \leqslant (5/3) \cdot \partial_2(G_1)$. As in Case 9a, it is sufficient to establish relation (4.19). We split the proof into two steps.

Step 1. We prove that

$$3 \cdot \partial_2^*(G, G_1, u^-, v^+) \leqslant 2 \cdot \partial_2(G_1, u^-, v^+) + \frac{4}{3} \cdot \partial_2(G_1, u^+, v^+). \tag{4.23}$$

Note that every 2-dominating set $D \in \mathfrak{D}_2(G, u^-, v^+)$ contains at least one of the vertices u_1 and u_2. On the other hand, every 2-dominating set $D' \in \mathfrak{D}_2(G_1, u^-, v^+)$ contains both of them. Then we obtain the relations

$$\begin{aligned}
\partial_2^*(G, G_1, u^-, v^+) &= \partial_2(G, u^-, u_1^+, u_2^-, v^+) + \partial_2(G, u^-, u_1^-, u_2^+, v^+) \\
&\overset{(3.7)}{\leqslant} 2 \cdot \partial_2(G, u^-, u_1^+, u_2^+, v^+) = 2 \cdot \partial_2(G_1, u^-, u_1^+, u_2^+, v^+) \\
&\overset{(3.5)}{\leqslant} 2 \cdot \partial_2(G_1, u^-, v^+)
\end{aligned}$$

and

$$\begin{aligned}
2 \cdot \partial_2^*(G, G_1, u^-, v^+) &= 2 \cdot (\partial_2(G, u^-, u_1^+, u_2^-, v^+) + \partial_2(G, u^-, u_1^-, u_2^+, v^+)) \\
&\overset{(3.7)}{\leqslant} \frac{4}{3} \cdot \big(\partial_2(G_1, u^+, u_1^+, u_2^-, v^+) + \partial_2(G_1, u^+, u_1^-, u_2^+, v^+) \\
&\qquad + \partial_2(G_1, u^+, u_1^+, u_2^+, v^+)\big) \\
&\overset{(3.1)}{\leqslant} \frac{4}{3} \cdot \partial_2(G_1, u^+, v^+),
\end{aligned}$$

which, in turn, imply (4.23).

Step 2. We prove that

$$3 \cdot \partial_2^*(G, G_1, u^+, v^-) \leqslant 2 \cdot \partial_2(G_1, u^+, v^-) + \frac{1}{2} \cdot \partial_2(G_1, u^+, v^+). \tag{4.24}$$

Since $\deg(v_2) = 2$, every 2-dominating set $D \in \mathfrak{D}_2(G, v^-)$ contains v_2. Thus, we have the relations

$$\partial_2^*(G, G_1, u^+, v^-) = \partial_2(G, u^+, v^-, v_1^-, v_2^+) \overset{(3.7)}{\leqslant} \partial_2(G, u^+, v^-, v_1^+, v_2^+)$$
$$= \partial_2(G_1, u^+, v^-, v_1^+, v_2^+) \overset{(3.5)}{\leqslant} \partial_2(G_1, u^+, v^-)$$

and

$$2 \cdot \partial_2^*(G, G_1, u^+, v^-) = 2 \cdot \partial_2(G, u^+, v^-, v_1^-, v_2^+)$$
$$\overset{(3.7)}{\leqslant} \partial_2(G, u^+, v^+, v_1^-, v_2^+) + \partial_2(G, u^+, v^+, v_1^+, v_2^+)$$
$$\overset{(3.3),(3.5)}{\leqslant} \partial_2(G, u^+, v^+) = \partial_2(G_1, u^+, v^+),$$

which, in turn, yield (4.24). Inequalities (4.23) and (4.24) imply (4.19), as required.

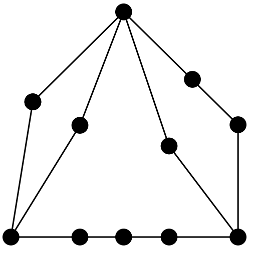

Figure 2. An example of the graph G^* in Case 10.

Case 10 (all vertices of degree higher than 2 in G are pairwise nonadjacent). We denote by G^* the graph obtained from G by deleting all isolated vertices (Figure 2). Consider an arbitrary set $D \in \mathfrak{D}_2(G^*)$. It is clear that at least one endpoint of each edge in G^* belongs to D, for otherwise the graph G^* contains a vertex of degree 2 which does not belong to D and has at least one neighbour that does not belong to D either, which is impossible. Then $V(G^*) \setminus D \in \mathfrak{I}(G^*)$ and $F(D) = V(G^*) \setminus D$ is an injective map from $\mathfrak{D}_2(G^*)$ to $\mathfrak{I}(G^*)$. Thus, we obtain $\partial_2(G) = \partial_2(G^*) \leqslant i(G^*) < i(G)$, as required.

The proof of Theorem 2 is complete.

§ 5. Conclusion

The methods that we have employed to investigate the case $k = 2$ are not very efficient in the case when $k \geqslant 3$ for two reasons.

First, in contrast to 2-regular graphs, for $k \geqslant 3$ k-regular graphs have a nontrivial structure. Moreover, any non-regular graph with average vertex degree at most 2 contains necessarily at least one pendant or isolated vertex; for graphs with average vertex degree at most k, where $k \geqslant 3$, a similar assertion is not true. Thus, Lemma 9 on isolated vertices, which is very important for the case $k = 2$, is inapplicable to $k \geqslant 3$.

Second, most transformations employed in the case $k = 2$ consist in deleting equally many vertices and edges from the graph. In the case when $k \geqslant 3$ such transformations have a significantly more complicated structure. For example, for $k = 3$ deleting two vertices must be accompanied by deleting at least three edges (for otherwise the average vertex degree in the graph increases), and it seems unreasonable to delete one vertex and two edges.

In the author's opinion, the most promising direction of further research consists in considering some important classes of graphs in which the average vertex degree obeys some natural constraints (such as the classes of subcubic, outerplanar and planar graphs). The structure of such graphs is significantly simpler than the structure of generic graphs with the same average vertex degree. For example, each maximal outerplanar graph is Hamiltonian, and every inner edge in such a graph divides it into two maximal outerplanar subgraphs of smaller size. It is probable that by using structural properties of this type one can develop new approaches to the solution of the problem under investigation.

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