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Foundations of Mathematical Education in the Digital Age

Academician A. L. Semenov[a,b,*], A. E. Abylkassymova[c,], and S. A. Polikarpov[d,***]**

Received December 18, 2022; revised February 16, 2023; accepted March 12, 2023

Abstract—This article formulates the foundations of modern mathematical education that meet the new goals and objectives of mathematical education dictated by changes in mathematics itself over the last century, by changes in its role in the modern digital world, and by changes in the world. The basic system of concepts is described, starting from which the whole modern mathematics of the finite, as well as primary mathematical education, is built. For all levels of mathematical education and for all learners, the ability to solve problems whose solutions are not known (to the student) and to do it independently with the help and support of a motivating teacher is a fundamental skill. A significant role is played by a computer mathematical experiment and the transfer of tasks with solutions having been found by the student to the computer. This article also describes the actual state of affairs in the Russian mass education system and proposes the necessary actions to implement the formulated principles of modern mathematical education in Russia. Most of the articles presented in this issue demonstrate and detail the implementation of the formulated principles.

Keywords: foundations of modern mathematical education, mathematical literacy, digital tools for supporting mathematical discovery and proof, unexpectedness of problems

DOI: 10.1134/S1064562423700564

1. INTRODUCTION

Professional mathematicians are always concerned with issues of mathematical education, including those at primary and secondary level. They usually assume that school mathematics should be the same as at the good school they went to: it is sufficient to use good textbooks (Kiselev's century-old one is the best), find good teachers, and enroll graduates from those good schools to be trained by good mathematicians from the best universities. Mechanisms of involving learners in mathematics are discussed separately: how to get children (potential future mathematicians) interested in mathematics, how to select highly motivated students for admission to a good school, etc. The issue of training teachers for a nonelitist mass school is usually left out of discussion: for good schools, teachers are sought among graduates of strong mathematical faculties who find it attractive to work at school. Students and graduates from teacher training colleges, i.e., most teachers, are practically not considered.

In the 1960–1970s, when specialized schools were created for highly motivated children, a number of the country's most prominent mathematicians became more seriously involved in schooling issues and personally participated in the creation of textbooks and teaching techniques, including Kolmogorov [1], Gelfand [2], Faddeev [3], Kronrod [4], Dynkin [5], Bashmakov [6], and Lavrentiev [7]. Significant, internationally recognized achievements were made in this direction [8].

Nevertheless, some attempts to update mathematical component in mass education in mass education were made as well. The most significant was Kolmogorov's reform [9], [10] and the subsequent attempts to create textbooks alternative to his. However, despite these attempts, irrespective of their success, changes over the last 100 years in school mathematics concerning what and how to teach are incomparable to changes in mathematics itself; in its role in the modern digital world; and, finally, in the world itself! One exception is the introduction of algorithmics [11], [12] and probability theory into mass education.

[a] *Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia*
[b] *Berg Institute of Cybernetics and Educational Computing, Federal Research Center Computer Science and Control, Russian Academy of Sciences, Moscow, Russia*
[c] *Abay University, Almaty, Kazakhstan*
[d] *Steklov Mathematical Institute, Russian Academy of Sciences, Moscow, Russia*
**e-mail: alsemno@ya.ru*
***e-mail: aabylkassymova@mail.ru*
****e-mail: polik@mi-ras.ru*

Today it is clear that the changes can no longer be ignored: modern civilization shapes new goals and objectives of mathematical education and provides completely new opportunities for their achievement [13], [14].

Today, the necessity for a new vision of mathematical education is being achieved by our colleagues from Kazakhstan, for whom this vision includes, on the one hand, the construction of unified methodological foundations of mathematical education and, on the other, a flexible and differentiated design of educational activities for individual students and student categories [15], [16].

This special issue covers the work of a number of professional mathematicians in the field of school mathematics, including school teaching and teacher training, since the early 1990s. This work continues the productive traditions of mathematical education of previous decades and, at the same time, takes into account the abovementioned modern realities [8].

In the course of practical work in schools, both those for highly motivated students and institutions of mass education, a number of principles have emerged that form a holistic system that conforms to the traditions and trends described above. Of course, the most effective approach is to implement a holistic system with all the principles taken into account. However, even individual elements of this system or combinations of several elements (principles) can also meet the needs of a range of parents, children, and teaching teams.

2. FOUNDATIONS OF MODERN MATHEMATICAL EDUCATION

We begin with the foundations underlying the content of mathematical education taking into account the specific features of primary school, which is important for the formation of primary mathematical views and activity types, and then formulate principles concerning various education levels (although, to some extent, nearly all principles are related to all education levels, from kindergarten to university):

For primary school (individual elements are already important for preschool education, but we do not single them out).

1. Basic *mathematical objects and structures:* beads (symbols), chains (finite sequences), aggregates (finite multisets), tables, geometric figures on checkered paper, graphs presented as images on paper and screen, bodily objects—manipulatives; robots that execute chains of commands and react to the outside world; and devices for measuring time, length (distance), weight, and volume in mechanical and digital versions. The properties of these objects and structures, as well as operations and actions in solving problems on them are clear.

2. Basic objects and structures serve as the basis for *all finite* (discrete) modern *mathematics* (including arithmetics) and the basis for the construction of *reality models* in this mathematics, such as patterns of language, games, human activities, and interactions of various kinds. These objects and structures are naturally generalized in the mathematics of the infinite; they are the basis for the formation of the basic ideas of all mathematics and mathematical informatics: computer science and general cognitive skills and strategies, including traditional (thoroughness, diligence, and the ability to understand and follow instructions) and, more importantly, modern (the readiness take into account feedback, correct errors, and solve problems whose solutions are unknown).

3. *Integers*—chains of numbers: taking into account arithmetic operations, this is the most important example of the object type. This type is not the simplest or most visual in terms of properties and operations: these properties and operations, in any approach (both modern and traditional), refer to the properties of the objects mentioned above (chains, sets, and geometric shapes), operations on them, measurements, etc. For every child numerical intuition ("number sense") grows individually; an important role is played by the practical work of students: counting objects, the geometric representation of numbers (for example, by the areas of polygons), measuring the size of real objects, time intervals, etc.

4. From the very beginning, a minimal, well-defined logico-algorithmic language is introduced—its use, understanding, and development are supported by visibility and a variety of tasks (see below). The repetition of a verbal cliché given by the teacher (memorizing texts "to find the subtrahend, you need ...") is not welcome; the student must invent their own formulations and, preferably, be able to explain it to others. In the logical language, "quantifier" constructions (such as "the proposition ... is true for all chains on the sheet ...," "all the propositions from the list ... are true," and "among the propositions ... there is a true one") are systematically used instead of the constructions "and," "or." This avoids some natural language ambiguities.

5. The main definitions of concepts are given on graphic examples, almost without verbal explanations. Understanding of the definition is also tested by solving visual problems.

6. Graphical definitions and wording of problems require more space than arithmetic or algebraic formulas or ordinary "text problems." This leads to the fact that paper textbooks and problem books are large, and it is another reason for the use of screen environments. In the case of on-screen objects and tasks, oral interaction of the child with the task (without the participation of the teacher) can also be used, which allows to match flexibly the level of the written language of the child.

For all levels of general education (and, with appropriate clarifications, for various areas of higher education):

7. Novelty, *unexpectedness of the task,* feasible for a particular student, is used as a powerful positive motivation for them. The source for a wide range of unexpected tasks are, to a large extent, the systematized problems of "interesting" and "entertaining" Olympiad mathematics of past centuries [17], [18]. It should be noted that many of these tasks are mathematically deeper and more similar to modern mathematics than the school material used today.

8. At the same time, success in quickly and accurately solving the next "serial" problem of increasing technical complexity can also be a positive motive in individual situations, but it is by no means a universal goal. Failure is not used as a negative motivator. Taking into account the indicated motives and individual life and educational goals, as well as the possibilities of group work, the student and the teacher jointly build for the student an individual sequence of tasks of optimal novelty and complexity to achieve individual goals. Individual goals include the implementation of the educational program of the school, meet the federal standard, and can significantly supplement it.

9. The main motive in the current learning process is not the maximum compliance with some external requirements of testing papers and not the stigmatization of bad grades and not even matching the general requirements of the teacher for the whole class. As the very nature of the standard suggests, an individual's goal may be to achieve a "satisfactory" result, which is not at all a cause for disappointment or censure. Each student is focused on compliance with *individual goals,* and these goals may be honestly obtaining a satisfactory grade and reliably passing the Unified State Examination; their achievement is the basis for *satisfaction* for the student, family, teacher, and school. Naturally, if the goal is to become an engineer, programmer, or a professional in fundamental mathematical research, then the individual ways it can be achieved should be appropriate.

10. The key element of educational activity is independent creation; the invention by a student and groups of students of methods and procedures of actions, algorithms, observations; and (almost) the independent discovery of properties, laws, facts. To this is added the independent creation of definitions: the choice of suitable general concepts and terms for objects and situations.

11. A computer is used as a tool that frees up the student's time and energy for high-level creative intellectual activity. For example, when designing individual goals and ways to achieve them, the student uses the possibility (but not the obligation) of transferring operations on numbers and algebraic expressions to the calculator.

Thanks to digital technologies, fundamentally new and broad opportunities arise [19]. For example, dynamic geometry systems allow the discovery of new geometric patterns that are supported by evidence. The concepts of mathematical analysis in the digital environment acquire a clear meaning that ensures their real development and use. The data accumulated in a physical or biological experiment are visualized and can be analyzed and compared with a mathematical model. The enumeration of options programmed by the student can give the desired answer, prove its absence, or suggest the direction of the search. Computer algebra systems allow one to find exact solutions and build graphs for all school and university equations and much more. In algebra, we can trust a computer with solving equations of a certain type after the student has succeeded in inventing, with the support of the teacher, a way to solve these equations.

The results of school mathematics proclaimed today (focused primarily on engineering education in the first half of the 20th century) can be achieved by a larger proportion of students, at a higher level, and for a wider range of tasks if, as like adults, schoolchildren will be allowed to use a computer [20].

12. When training professional mathematicians, it is necessary to provide the envolving of computer tools to support mathematical discovery and proof [21]. The same should be provided when training teachers of mathematics and informatics. At the same time, as in school, the complexity of the tasks to be solved may be different for different students; it is important to master, at the individual level, general methodology and models of intellectual activity.

13. When training engineers, we should talk about gaining experience in using all the tools of computer computing, including the algebraic, analytical, and mathematics underlying modeling and design systems for the relevant engineering field. In the system of liberal education, mathematical activity can be directed, in addition to professional applications, for example, in natural language processing, medical diagnostics, or legal bases, to the formation of ideas about "how it works," the "demystification" of artificial intelligence.

14. The role of evidence. Currently in school mathematics, proofs in algebra are rudimentary and practically absent. The opportunity is not given for students to independently build a micro-"theory" in a sequence of tasks, for example, solving quadratic equations; the goal is different—as soon as possible and more reliably learn ready-made formulas or "techniques" for solving classes of equations. Precise proofs in analysis cannot be mastered by most schoolchildren: in addition to the need to master an unusual conceptual apparatus, even the process of understanding proof requires the ability to keep attention on monotonous mathematical calculations for a long time. Proofs in geometry, inheriting a venerable 2000-year-old tradition, are beyond the zone of prox-

imal development for the majority of even good and excellent students: at best, this is material for learning and reproduction with limited understanding.

At the same time, the need to educate the mass of students in a culture of evidence was felt and was constantly formally proclaimed as an educational goal earlier; today the importance of evidence is only increasing. The opportunity to develop the mathematics of evidence today can be provided by computer and other experiments and observations that allow putting forward, testing, and refuting hypotheses for proof. This happens, including in geometry, when the independent construction of the proof by the student becomes a priority goal. Finally, the interesting problems mentioned above, as well as a wider range of problems related to computer science today, provide considerable scope for evidence-based reasoning of various forms and levels. One important example here is the inductive proof of the compliance of a program (algorithm) with a system of requirements—a specification.

15. *Programming* is becoming one of the most important sources of content and tasks for mathematical education: in the minimum algorithmic language in the appropriate learning environment or in variants of existing "industrial" programming languages. Programming combines the possibilities of experiment (including when debugging a program), often the visualization and "objectification" of errors and other defects in work. It can also solve the problem of educating accuracy, the need to follow the rules, etc., the solution of which is claimed by school mathematics [22]. We have already mentioned proofs in programming.

16. *Error* in problem solving is an important component of the educational process. This is material for developing a student's ability to look for mistakes themselves (and not only in mathematics) and to use various methods of feedback (checking). This is a subject for further work, a better understanding of the educational content. This is an important material for a conversation between a student and their teacher and the joint planning of further progress along their individual trajectory. The key here is to change the approach and change the reaction of the teacher and the educational system as a whole, not to censure and punish for a mistake, but to consider it as a reason and a source for improving the work of the student and his work, moving towards achieving the planned result. It can also be useful to consider a teacher's mistake, both made intentionally, consciously, and a real, random mistake: the teacher makes mistakes and learns like the student, and this is an important positive part of the educational process.

17. *Reality simulation:* which have to be the most important part of mathematical education and is very scarcely represented in school. In mathematics, these are "text problems," in the realities of the 19th century, with template reasoning and poor equations. The situation is somewhat better in physics, with a wider range of reality phenomena and the class of mathematical environments used (for example, basic school-level trigonometry is used). Thanks to the use of digital models and the release of time through the use of a computer for algebraic and arithmetic calculations, modeling becomes a serious, real, and extremely relevant school topic [23]. Its importance in higher education is obvious; things are somewhat better there.

18. Tasks of a *digital task book,* due to an absence of restrictions on paper volumes, form a spectrum that is much wider in terms of topics covered and deeper in terms of complexity (including the simplest tasks) than in traditional school books. The corpus of entertaining problems, olympiads, etc., accumulated in cultures over the centuries is used. A digital task book can contain an unlimited number of problems of varying complexity, illustrations (including dynamic ones), "tips," comments, etc., meeting the needs of a wide range of students and allowing them to build individual educational trajectories.

In connection with the described picture, even if we agree with it as a whole, or with its individual elements, natural questions arise:

• What efforts of the school, state, and parents will be required to move the situation in the proposed direction?

• What previous and current experience could be useful in this regard?

• What are the obstacles on the designated path?

Most of the articles in this issue are demonstrations and details of the implementation of these principles and attempts to answer these questions.

It is fundamental for us that research work, as a necessary element of the educational program, can contain elements of an "absolute" research level, that is, not just offer the student to follow a path already traveled by professional mathematicians, but really try to find mathematical proofs of new facts. In this regard, we are publishing the article "Creation of new mathematics by schoolchildren" (A.L. Semenov, S.F. Soprunov, and I.A. Ivanov-Pogodaev) related to the research work of schoolchildren in the program of the Sirius Educational Center on the theory of definability. We also include the article "Efficient search for linearly growing configurations in the tag System $\{0 \to 00, 1 \to 1101, 3\}$" (N.V. Kurilenko), where the solution to Post's long standing problem from the field of combinatorics of words was obtained with the help of a computer and a series of reviews by N.A. Vavilov: "Computers as a novel mathematical reality: 1. A personal account," "Computers as a novel mathematical reality: 2. Waring's problem," "Computers as a novel mathematical reality: 3. Mersenne numbers and divisor sums," and "Computers as a novel mathematical reality: 4. The Goldbach problem," relating to the application of the computer in the ultimate solution of classical problems in number theory. The formulation of problems and results in both cases is available to the student. Schoolchildren, on the one hand, can independently go through some specific research routes that have already been completed; on the other hand, they can get new results on related topics.

3. THE REAL SITUATION IN THE RUSSIAN MASS EDUCATION

There is a widespread belief that the state of mathematical education has deteriorated significantly in the post-Soviet period. To a certain extent, this is true and is due, among other things, to a decrease in the importance and prestige of engineering education in a country where the military industrial complex was destroyed, as well as the erosion of the authority system: the authority of the government, parents, and schools.

At the same time, the Soviet mathematical education in mass education should not be idealized either. Here are two characteristic quotes from unconditional authorities in this field.

Arnold [24] stated the following: "In the practice of teaching, this is how it is. One or more collections of tasks are taken as the basis from which the teacher chooses one or another at their own discretion. Tradition ensures to a certain extent that some certain types of "arithmetic reasoning" will be somehow represented, but what kind of types they are and whether they sufficient or, on the contrary, is there extra ballast in ordinary material, what exactly should be sought from students, there is no definite answer. It is more or less established that students should be taught to solve problems for "mixing," for "proportional division," for "joint work," for "movement," for "percentages," and for the "rule of three." If you ask about the methods of solution, then the answer is usually limited to trivial considerations about the analytical and synthetic method, about decomposing a complex problem into a number of simple ones, about the method of reduction to unity, about the method of proportions, and about problems on "assumption" ("assume that the same quantity of each variety is purchased"). Students, in one order or another, are introduced to the corresponding types of problems, and learning to solve problems often comes down to a "training" recipe, to the passive memorization by students of a small number of standard methods of solving and recognizing, by one or another sign, which of them must be applied in one case or another. The number of problems that students really solve on their own, with that strain of thought, which should be the source of the usefulness of the process of solving a problem, is negligible.

This results in the complete helplessness and inability to navigate in the simplest arithmetic situations when solving purely practical problems, as well as later, in algebra, the inability to compose and investigate equations or, in general, the inability to go beyond narrow formal schemes—in a word, what then is characterized as a "lack of mathematical development."

Khinchin [25] wrote the following: "Once I had to ask several experienced fifth grade teachers about what percentage of students actually learn to solve arithmetic problems that are not simple computational examples, i.e., those where the solution, however simple, must be found by the student himself. Of all the teachers I interviewed, only one claimed that up to 15% of students manage to learn this; all others said that only individual students mastered this art, and some even declared that it was impossible to teach it at all. Of course, having solved a number of problems of the same type, the student will easily solve a problem of exactly the same type (this explains the absence of continuous failures in exams and tests), but getting the student to independently find a solution to a problem of a new, even a very simple type, is, in the unanimous opinion of the teachers, a task that is successful only in the most exceptional cases.

Thus, that "development of ingenuity" that we like to put forward as the main goal of introducing "difficult tasks" turns out to be in no way possible even for the best teachers."

This state of affairs was caused, in part, by the goal of "industrializing" the school in various senses. On the one hand, the training of future engineers and technicians was the most important goal; on the other hand, "growing" the system of universal education, starting from the level of the elite that existed in prerevolutionary times, required "industrial" methods.

The most important event in mathematical education, but not in mainstream schools, was the creation of specialized mathematical schools and classes; this has already been discussed above.

In general school mass mathematical education, the introduction in the mid-1980s of a computer science course in all high schools of the country was important [26]. The school did not reject this course, although in many cases its algorithmic and mathematical content was partially replaced by applied computer skills. Nevertheless, most students today receive at least minimal experience in solving problems in the development of algorithms and other problems of modern mathematics that diversify the solution of typical tasks that have become traditional (logarithmic inequalities and word problems for transfusion). It is widely believed that the presence of this course at school contributed to the influx of graduates into mathematical areas of higher and secondary vocational education and to the IT industry.

Since the early 2000s, sections of the theory of probability and mathematical statistics appeared in the mass course of mathematics. The corresponding tasks are also included in the Unified State Examination; this ensures that these sections "are done" in the vast majority of schools in the country.

The 2009 standard [27] and subsequent versions of the standards for primary education [28] very briefly list the new elements that correspond to the approach we are considering, the mathematics course as part of the single field Mathematics and Computer Science.

Corresponding to these principles (in particular, the high novelty of tasks), the Kangaroo Olympiad, which is not supported by the state, has been attracting students and teachers in many Russian primary schools for decades [29], [30].

Of course, in many educational systems, including the Russian one, the most important factor determining the content of education is the final exam; in our country it is the Unified State Examination. It reflects the changes listed in this section: there is an exam on computer science; the exam on mathematics includes tasks in probability theory and mathematical statistics. In addition, it gives an example of the massive and, obviously, highly reliable use of digital technologies and personal computers in the general educational process:

• from the very beginning of the Unified State Examination, millions of pages of the works of graduates are digitized, the digitization is transmitted to the federal data-center, and from there it is returned to the regions to experts and each graduate has access to the test results, etc.;

• for a number of years, the oral part of the foreign language exam uses a computer to reproduce and record speech;

• computers have been used in the Unified State Examination for Informatics for Grade 11 since 2021 and even earlier for Grade 9;

• the rules for conducting the Unified State Examination for Grade 9 (allowing greater diversity by region) provide for the possibility for a wider use of digital technologies, in particular, a calculator, in different subjects (but not in mathematics).

The COVID pandemic did not cancel school in Russia; the Constitution of the Russian Federation and the Law on Education continued to operate, but their implementation required the massive use of digital technologies in the educational process, including that in mathematics. At the same time, the State has not incurred any significant costs. The return to "normal" life, seen by many as the prepandemic status quo, today may prevent an adequate assessment of the experience gained during this period.

The digital transformation of mathematical learning activities for students is underway both in Russia and around the world. In hundreds of Russian schools, teachers, despite a lack of serious special additional training or moral or material incentives, have been successfully using them for decades:

• in primary school: the described content and some other elements of the described approach, implemented in textbooks, including those of the Prosveshchenie publishing house [31–33];

• in primary school: a project-research approach to the study of mathematics based on the use of the Logo and Scratch environments and the educational philosophy of constructionism [12], [34];

• in basic education: computer support for a school geometry course in the form of GeoGebra [35], Live Mathematics [36], and Mathematical Constructor 1C [37], [38], increasing the role of the experiment conducted by the student; computer support for the algebra course in the Mathematical Constructor 1C.

The Russian tradition of teaching mathematics through solving new problems, independent discovery, and experiment continues today in hundreds of mathematical classes and mathematical schools [39]; thousands of mathematical clubs (known as circles) [40]; the preparation of participants in the Olympiads; and project shifts at Sirius and other centers of mathematical education.

4. NECESSARY ACTIONS FOR IMPLEMENTING THE PRINCIPLES OF MODERN MATHEMATICAL EDUCATION IN RUSSIA

We believe that, in order to implement the proposed principles of mathematics education for a wide range of students, the following is necessary.

1. Allow students to use digital technologies in the educational process (but do not make it mandatory), including in the process of Unified State Examination.

2. Change, transform the process of training teachers of mathematics, computer science, physics, technology, including digital transformation, for which the following is necessary, in particular,

(i) declarative and normative support from the federal executive authorities;

(ii) formation of a community of university teachers who are ready for changes and who will take part in the training of teachers of mathematics, primary school teachers, and teachers of other disciplines; coordinated work of this community in studying experience, mastering technologies, organizing innovative activities in schools, creating educational and methodological materials, and working with students;

(iii) the implementation, including remote, of modules, courses, practices, and integral programs for teachers of mathematics and students (future teachers); restructuring programs for the training and retraining of teachers towards reducing mandatory material that is not directly related to school education; restructuring educational methods in the same way as school education is being transformed: independence in solving fundamentally new tasks; the use of digital technologies, while encouraging the ability to solve problems without such technologies; and the willingness to learn new things, including together with students.

3. Develop educational and methodological complexes (materials such as teaching aids, textbooks, and books for teachers) based on the discussed principles corresponding to the Federal State Educational Standards and educational programs authorized by federal executive authorities; a significant part (probably the bulk) of these teaching materials should be digital and hosted online; it is important that these teaching materials receive a stamp of approval from the Ministry of Education.

Certain results have already been achieved in all these areas. For example, during the training of mathematics teachers at the Moscow State Pedagogical University, a psychological and pedagogical module has been implemented for a number of years that involves many interactions between students and schoolchildren of different ages, starting from the first year of study [41]. The module provides an introduction not only to how mathematics and computer science are taught in middle school (grades 5–9), but also to the developmental characteristics of a student and subject mathematical material in primary school (grades 1–4). In 2016, together with an important system of mathematical circles in Moscow (little *mekhmat*), a mathematical circle started at MSGU where interested students, acting as organizers, could communicate with schoolchildren within the walls of the university.

Even earlier, the need for leveling classes to check and replenish students' knowledge of the school curriculum in mathematics was recognized. Incidentally, such classes, according to polls, are held today not only in pedagogical universities. After a break, workshops on solving problems of school mathematics for university students conducted by employees of specialized mathematical departments were restored.

In the curricula of teachers of mathematics and informatics, disciplines related to the computer teaching tools were introduced, such as systems of dynamic geometry and demonstration computer models from other sections of school mathematics. Many academic disciplines received support in the form of an electronic course. At the Faculty of Mathematics, and a little earlier at Moscow State Pedagogical University as a whole, the very concept of an electronic educational environment arose in training a teacher of mathematics and computer science.

The formulated approaches to teaching mathematics and informatics in Russia are supported at the Kazakh National Pedagogical University in the Department of Methods of Teaching Mathematics, Physics, and Informatics. Here, a number of modern digital technologies are used in the educational process, and this makes it possible to significantly improve the quality of mathematics teacher training [42]. The department strives to ensure that graduates meet modern requirements in accordance with the implementation of the Digital Kazakhstan state program adopted by Decree of the Government of the Republic of Kazakhstan of December 12, 2017.

FUNDING

Since 2020, the Russian Foundation for Basic Research has been implementing the "Fundamental Foundations of the Digital Transformation of General Education" grant program. As part of this program, the "Charter of the Digital Path of the School" program document was developed, which reflects the main principles of this work [43]. A number of mathematical teams that implement these principles work in the same program. In particular, the Foundation supported the work on projects no. 19-29-14152 mk "Fundamental Foundations for the Formation of Mathematical Literacy for a Digital Society at the Primary Level of Education"; no. 19-29-14199 mk "Fundamental Foundations of the Digital Transformation of Primary General Education"; no. 19-29-14208 MK "Analysis of the Needs and Obstacles of Digitalization in the General Education System"; no. 19-29-14217 MK "Prospective Directions and Forms of Using Computer Technologies in the School Mathematics Course"; and no. 19-29-14234 mk "Analysis of the Development, Digitalization, and Introduction of the Content of General Education Based on the Results of Fundamental Research (the fundamental core of the content of general education)." We are grateful to the Foundation for support, including the possibility of open access to papers of the this issue.

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant no. AP19680007, A.E. Abylkassymova, A.L. Semenov) and the Russian Foundation for Basic Research, project no. 19-29-14152 mk (S.A. Polikarpov).

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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