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Mathematical Elements of Elementary Education

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Abstract—In recent decades, several Russian schools have been implementing a world-unique education program of mathematics for elementary schools. In it, the landscape of school arithmetic is radically expanded due to the basic objects of modern mathematics and computer science. These objects and their operations are visual, making them much more comprehensible than traditional arithmetic. The range of activities also expands due to, for example, the introduction of strategies for enumeration, game winning, and algorithms (also operating in a visual environment). At the same time, the student's position changes: they independently discover and build mathematics and constantly solve personally new, but feasible tasks that are "not-known-how-to-solve." The student's resources are saved by using a computer to perform routine arithmetic operations that have already been discovered and understood. The implementation of this approach is discussed in detail and is illustrated by examples of actual tasks that are representative of the program under consideration and the whole approach.

Keywords: mathematical education, research training, elementary school, arithmetic, mathematical modeling in school, unexpected tasks, VUCA world

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1. INTRODUCTION

The foundations of the modern mathematical language were constructed from the late 19th century to the first third of the 20th century (see [1]; [2], Introduction; [3], Chap. 4). In the second half of the 20th century, this language, i.e., a set of concepts, their meanings, and applications became the basis for the construction of digital technologies and the whole digital civilization. Attempts were also made to use this language as a basis for mathematical education and as part of human culture. These attempts yielded only partial results and faced opposition from different sides [4].

In the mid-1980s, several representatives of the Russian mathematical school headed by Academician Andrei Petrovich Yershov decided to build the foundations of modern mathematical education "from an opposite end," namely, from computer science and high school. The USSR became the first country in the world where this approach was successfully and massively implemented, covering all schools of the country [5–8].

At the same time, work was begun concerning the construction of mathematical foundations for the whole school mathematics that match with tasks and possibilities of modern mathematics and the digital world. This work has been continued up to now, and its results have been reflected in teaching in tens of schools; in publishing officially recognized school textbooks [9], [10]; and in creating the Federal state educational standards [11].

This paper describes the system of mathematical concepts that has been used in the indicated approach to school mathematics and computer science over the last decades.

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2. TODAY'S LITERACY

Modern humans have to make self-conscious decisions, independently extract information, and conduct research of the environment to an increasing extent. It is concerned with nearly all kinds of activities and situations ranging from the household to subtle technological processes. Today, the ability to act in an unexpected situation and to solve unexpected tasks is much more important than the ability to strictly follow instructions and to implement without reasoning or discussing. This is caused by the following two equally important and interrelated circumstances.

• Due to the high level of development of today's technologies, nearly any activity based on predetermined rules and algorithms can be entrusted to machines or digital technologies.

• Societal needs, technology, dependence of everything on individual decisions and behavior, and constant turbulence (VUCA [12]) influence the behavior of humans and decisions they make to a large degree.

At the same time, legitimate and universally applicable scenarios of school education and the whole school life are opposite: school is still focused on reproductive, executive activity models. School in the 21st century still follows the priorities of the 19th century.

Elementary school has taught children to read, write, and count for centuries. All these skills are still considered the pillars of literacy in the computer age. However, today some elements of these skills completely lose their meaning. In most cases, these traditional skills are radically transformed if you look at them from the perspective of a modern employee, employer, or a human as an evolving personality. The results of this transformation can be described as follows.

• *Reading.* Most written sources are now available in audio format. Sometimes, a video-audio instruction is preferable to a written one, for example, in cooking recipes or instructions given in the workplace. Functional communicative literacy becomes a key point: the ability to understand others and the ability to apply and/or convey someone's understanding to another person; an information presentation, including nontextual, is a matter of pragmatism.

• *Writing.* Typing on a keyboard has replaced writing with a pen on paper everywhere, except in school. The process of writing a text or creating a message has changed as well. Short instant messages—oral and written—have transformed communicative culture. Automatic conversion of speech to a written text is being used increasingly. At the same time, probably, work with written texts, such as editing, will make sense in the coming years.

• *Calculating.* The need to count the number of objects with hand or eyes (counting in the ancient terminology) still remains, but there is no need for executing arithmetic operations (e.g., column calculations on paper); they are delegated to a computer.

3. TODAY'S MATHEMATICAL LITERACY

The educational goals in today's elementary school mathematics are as follows.

• Perform arithmetic operations accurately and quickly, thus making children to perform like an arithmometer or calculator.

• Recognize the system of relationships between numerical variables in a word problem. As a rule, relationships concern rather few situations denoted by phrases, such as together, toward each other, flow from a pipe, against the flow, into equal parts, and next day. After the necessary system of relationships has been built, the task is to design a sequence of calculations of (linear algorithm for calculating) the required quantities either arithmetically or by writing a set of algebraic equations, which is solved using a standard algorithm. In the 21st century, this can be successfully done by artificial intelligence.

In the elementary school curriculum, the word "arithmetic" was long replaced by mathematics, and in the Russian Federation 2010 standards, it was turned into the integrated field of mathematics and informatics. However, the curriculum content was affected insignificantly by these changes.

Note that mathematics opens up almost unlimited opportunities to form the ability of children to solve problems that are "not-known-how-to-solve." This ability is important not only in mathematics: the ability to solve new tasks that have not been addressed earlier by people is in demand in almost all fields of knowledge and technology. Is it possible to change the current situation and begin to develop these abilities as early as in elementary school?

Over the last three decades, a team including this article's authors has been developing elementary school courses of *Informatics* and *Mathematics and Informatics* in which the goals of traditional arithmetics are preserved, but the priority of working out mechanical skills is reduced, in particular, a calculator is allowed to be used for calculations. During this time, the courses have been used in several hundred schools. The courses rely on the following principles.

• *Discovery and invention of mathematics by students.* Instead of presenting ready rules and algorithms, conditions are created under which students develop and implement the need to discover and formulate these rules and algorithms under the guidance and with the help of a teacher.

• *Modern basic objects.* The class of basic objects used by learners in educational activities has been expanded: it includes the basic elementary objects of modern mathematics and computer science. These objects, as well as numbers, are presented in visual form, which enables all students to be involved in the learning process regardless of their initial education level and supports their engagement in the educational process.

• *Reduction in the priority and* (*motivational and time*) *costs of working out calculation skills.* Students are given the opportunity to improve their calculation skills and, at the same time, to use technological tools for calculations.

• *Novelty.* The factors listed above have opened up the opportunity to implement the important *novelty principle*, by which we mean that students are invited to solve problems of high degree of novelty and unexpectedness. Nevertheless, most students can solve such problems personally or with the help of teacher's discussion.

• *Link to everyday life* of learners and the real world.

• *Feedback*. Visualization of objects and operations over them allows students to find their mistakes independently or with the help of a teacher. When objects are used in a digital environment, the feedback possibilities are even higher.

• *Systematicity.* All the above-mentioned principles are implemented within a result-based system, including the one provided by the learning standards.

Below, the indicated principles are explained in more detail on examples.

Discovery and invention of mathematics by learners. Major elements of arithmetic can be constructed by learners on their own in mathematics classes. For example, the decimal number system can be invented by a child as a way of counting any number of objects in a box (e.g., matches or beans) if only ten digits are available. Addition and multiplication tables can be constructed by counting cells in strips and rectangles.

We encourage learners to solve particular arithmetic problems autonomously. Mental calculation is not a competition in speed, but rather the development of skills in validating and checking the correctness of a result. For example, the result $7 + 8 = 15$ can be checked in different ways: by adding two 7's and supplementing one of them to 8, expanding each term into a sum of 5 and another number, supplementing one of the terms to 10, etc. Checking and validating a numerical result via calculation in a different way is also an example of a life strategy. On the one hand, we teach children the absoluteness of truth in mathematics, but, on the other hand, we draw their attention to the fact that humans may be mistaken in assuming that they have discovered this truth. The teacher repeatedly asks the children "How did you calculate?" and helps them formulate their invented calculation methods in an appropriate mathematical language clear to their classmates. The technique can even receive the name of its creator: "Vanya's method of addition," this is especially important if the child who invented it is not the strongest mathematician in the class. It is also important to use feedback, i.e., the responses of the outside world and other people to your actions, including calculations. Today, the ability to take advantage of this reaction, to find a mistake, and to try to correct it is more important than pure correctness. This will be discussed in more detail below.

Modern basic objects. They will be described in detail in the basic part of this paper. Now we only note that these are the basic objects of mathematics and computer science; moreover,

• they are the basis for the whole mathematics;

• objects of infinite mathematics are constructed from finite objects: the properties of and reasoning about infinite objects are based on the properties of and reasoning about finite objects (of course, this "naive constructivism" is only the beginning...).

To enter the world of mathematics, children are set the task of building discrete (combinatorial) objects that satisfy prescribed conditions. These conditions can be a logical (primarily, quantifier) combination of others, simpler conditions.

Real world and feedback. The statement of a problem can be a narrative, a story about the real, or "real" world (as in word problems in a traditional course of mathematics). Such a story can be written as a short tale or a suitable fragment of a children's book can be taken. The text may describe a situation that the child encounters in the store, café, or at the station. Of course, the statement of the problem can also be a technical abstract description, but this is usually not as interesting to children. Natural and mathematical language phenomena, school schedules, and individual plans are also an important part of the real world.

A component of a problem solving technique is modeling, i.e., the construction of a mathematical model, identification of objects, processes, and relationships involved in a problem with mathematical objects, processes, and relationships. Some of the relationships and properties can be numerical, i.e., ones that can be measured or counted. Logical relationships and properties are also possible (for example, you need to ride the bus until you can get on the train, or you need to understand whether there is enough money to buy something).

The described class of problems includes, for example, constructing a height diagram for classmates and its variations over the year, calculating the number of different kinds of student's pets, the distribution of the sum of points obtained by rolling three dice, drawing up the budget of a party for classmates; scheduling tours to a neighboring city, and drawing up the budget for purchasing equipment for the school computer class.

Numbers, relationships, and dependences between them remain an essential element of a mathematical view of the world. However, a radical change in the priorities is taking place. In traditional school for mass students, the stage of modeling was not central, and a model was always created according to standard templates: “the path is the speed multiplied by time,” “the distances for pedestrians should be added.” The student only had to be able to recognize which template to choose from a small number of standard templates. In the worst case, a model will be built by the teacher, and you will only need to solve an equation and precisely calculate everything. In 21st century school, computations can be fully delegated to a computer, and the main educational load will be transferred to the modeling stage [13], [14]. In this case, the variety of modeled situations can be significantly increased, including beyond situations that are normally considered in school physics.

The construction, use, and discussion of discrete mathematics structures and their properties make it possible for learners to construct models in social humanitarian fields. Relationships on strings (finite sequences) provide the students and the teacher with a clear system of concepts, in fact, constantly used in linguistic and, for example, historic courses. Various cycles arise in the “World Around” course: seasons, moon phases, and a week cycle. Tree structures (finite graphs) are used in biology for classification and in “World Around” and historic courses for constructing genealogical trees, including family trees of students. In computer science and programming, strings describe software codes and the course of a particular computation, while bags describe possible choices.

Finally, we note that, along with mathematical models of reality, actual representations of mathematical objects are constantly used in the course. Moreover, these representations are objects of mathematical activities of students. A string of digits written on paper is a (natural) number.

Feedback. It is utterly important not to finish problem solving with a numerical answer: work of this kind is similar to the motion of a robot that has no feedback to reality when following its algorithm. The result obtained by modeling should be tested for mathematical and life likelihood, i.e., should be compared with a specific reality. A comparison of results with the problem statement, reality, and context and the ability to doubt calculations and reasoning form a set of skills important not only for mathematics. Considerations concerning the order of magnitude (whether the students in a school can make up an eight-digits number), integrality (classical “two and a half diggers”), and divisibility (if by condition all children stand in pairs, there must be an even number of them) provide the student with feedback that helps correct the wrong reasoning in problem solving and find the mistake. This makes it possible to increase the reliability of solutions in a more adequate way than traditional worked-out calculation algorithms without feedback. The teacher helps students invent and follow various strategies that help them find mistakes and discrepancies in both mathematical contexts and a wider variety of tasks.

Novelty. According to the practice of mathematical classes in the USSR and Russia, novelty and feasible difficulty of problems is an important motivating factor for various children [15].

Following Galperin [16] and his interpretation of Wolfgang Köhler’s research into cognitive behavior of animals [17], we note the following characteristic features of a personally invented solution as compared with a solution based on a standard algorithm.

- *Flexibility*. A found solution can easily be transferred to other similar situations: a monkey that got an idea of using a stick to reach a banana will try to use a stick later to solve other tasks.
- *Generalizability and modifiability.* Galperin gives an example of the same monkey that, in the absence of a stick, rolls a blanket to use it as a stick. Another monkey that did not previously hit on the idea of using a stick, but instead saw it in others, does not come up with an idea of using a blanket in this way.
- *Emotional significance.* A person is proud of his or her invention or personally found solution, remembers it, and gets pleasure from problem solving in the future. The fact that experience of a self-discovery stimulates search activity in the future was noted by other researchers, for example, in [18].

The importance (proof of concept) of the pedagogy of unexpected problems is supported by the Kangaroo Mathematics Competition [19], [20] held among 1–11 grade students. In Russia, the Competition has been held since 1994. Thirty tasks of various difficulty other than standard textbook problems are offered to children. Unusual problems often give an opportunity to a student unsuccessful in school mathematics to get a high score in the competition. This attracts children, their parents, and teachers, so the number of participants of the Competition grows from year to year: in 2022, almost 350 thousand students from 72 regions of Russia took part in the competition. More than 6 million schoolchildren participate in the International Kangaroo Competition every year [21].

Systematicity. Visual objects of discrete mathematics, judgments of them, and operations with them form the foundation of mathematics and computer science and, accordingly, the basis for our elementary school course. Nearly all introduced concepts, structures, and operations are further developed in secondary school and are used later, including professional activities of 21st century’s humans just as professionals used numbers and decimal fractions in the 19th century.

Additionally, we form:

- a system of reasoning generalizing individual examples;
- “big ideas” as orientation in the world, including general problem solving methods.

The concept of a *big idea* emerged in natural science education as an opposition to the notion of a collection of facts [22]. A *big idea* is an orientation part of the human's view of the world without which the whole representation and the behavior in the world become different. Most of the *21st century skills* make up a more ancient system inherent in an educated human than the rest of education results of the 20th century. Such skills include the abilities to learn, understand another person, set goals, analyze failures, etc. Orientation in the world is changing faster and especially rapidly now.

Big ideas of *digital literacy* become increasingly necessary for orientation [23], [24]. The ability to use orientation, in conjunction with the ability to solve completely new problems developing in the study of mathematics and computer science, forms the basis for pre-adaptability [25].

A general method and a big idea cannot be learned; you can only accumulate situations where they are applied. A student who has mastered the general method as applied to some class of situations begins to see the solution to the next situation that is not new for him or her. Then it is desirable to go to a new material. However, if solving similar problems and demonstrating skills in such a solution remains a factor of positive motivation, such problems can nourish motivation.

In the next sections of this paper, in parallel with introducing the basic mathematical structures of elementary school (almost all of them appear in the first grade), we give examples of initial exercises and problems. These examples are intended, in particular, to illustrate the thesis of combining constant novelty with systematicity and big ideas. Examples are taken from two sets of textbooks: *Mathematics and Informatics Grades 1–4* [26], taking four or even five hours a week, and *Informatics Grades 1–4* [9], taking 1 h a week. The course *Mathematics and Informatics Grades 1–4* was coauthored by A.L. Semenov, M.A. Posicelskaya, S.E. Posicelsky, N.A. Soprunova, I.A. Khovanskaya, T.V. Mikhailova, and T.A. Rudchenko. The course *Informatics Grades 1–4* was coauthored by A.L. Semenov and T.A. Rudchenko. Although both courses were created within the same author's concept and under the general guidance of Prof. Semenov, their curricula and sizes differ. The courses were created by different teams, so the approaches and the implementation of particular topics differ. In what follows, we sometimes indicate to which course particular points apply.

4. BASIC STRUCTURES

Even before school, a child encounters ordered and unordered sets, cycles, and tables.

A chain of events is ordered: one event follows another. Words in speech go one after another: they can be separated from oral speech and transformed into writing to obtain strings of letters, which form words; strings of words form sentences, etc.

A shopping list is an example of a language string. However, the purchases lying in a cart are no ordered. They can may two loaves of bread, four bananas, a dozen eggs, and three packs of milk. When making these purchases, we most likely believe that all the loaves are identical and all the eggs in the package are identical, but an egg and a loaf are different objects.

In the child's life, the change of day and night, the weekdays, and the seasons form a long periodic chain and a closely related cycle. Finally, tables arise early in the modern children's life as a way of structuring their daily routine. They are also present in schedules of trains and buses and opening hours of clinics and shops, which children encounter in everyday life. Often tables appear in some mobile or computer applications for preschoolers.

In our elementary school courses of mathematics and informatics, all these structures are defined on visual examples: this is how the child usually learns new words. Moreover, comments are made on the examples, for instance, "a string necessarily has a beginning and an end" or "there are no first and last elements in a cycle." This creates a smooth transition from everyday thinking to scientific thinking, about which Vygotsky wrote [27].

Important objects are empty structures (empty bag, empty string, and empty cycle). There is only one empty structure of each type.

A key point is the isomorphism or identity of structures.

All objects, atomic and more complicated, can have names. The name is usually a string of characters, i.e., Russian or Latin letters and numbers.

5. ELEMENTARY (ATOMIC) OBJECTS

At school, symbols, elements of strings, bags, and cycles can be:

- material objects—beads of different shapes and colors—made of wood or plastic;
- picture cards, in particular, cards with drawn beads;
- various graphic objects on paper, namely, on separate sheets or in a notebook;
- various graphical objects on the screen, i.e., in a digital environment.

The elements can be combined into strings and bags, for example, beads cut from cardboard can be stringed together. Actual objects and graphic images on the screen are easy to move by hand or mouse. We can take something out of a bag and put something in it. A string can be moved as a whole. Something can be added to or removed from a string only at its beginning or end. A string of cards arranged in a row on the table can be treated more flexibly. For example, we can

These are **beards**:

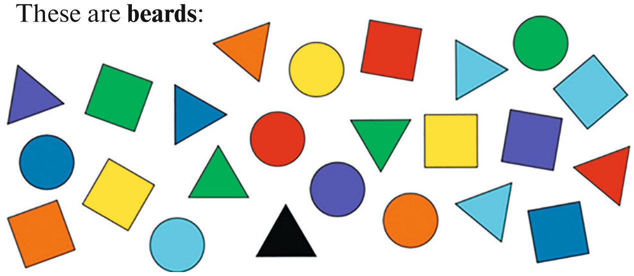


Fig. 1. Beads used in the course.

Fig. 2. Definition of identity of figures in the course *Mathematics and Informatics*. Here, all the figures are considered identical.

agree that the cards in a string can be swapped as we wish, which means that we deal with a bag of cards, rather than with a string. The same arrangements can apply to elements in a digital environment.

The elements with which children work in solving most problems of in course are beads, coins, figures (pictures), digits and symbols of different alphabets, traffic signs, etc. We call them *atomic objects*.

The beads come in eight colors (red, orange, yellow, green, light blue, dark blue, violet, and black) and three shapes: triangular, round, and square. Two beads are identical if they have the same color and shape. The beads do not differ in size (see Fig. 1).

Two coins are identical if they have the same value. Sometimes coins are depicted realistically, in a picture, and sometimes schematically, namely, as a circle with a number inside.

Figures can be different; their list is not limited. In the course *Mathematics and Informatics*, figures are considered identical if there is a plane motion transferring one figure into the other. Student can stick figures in a bag not only vertically (see Fig. 2).

In the course *Informatics*, the definition of identity of figures is narrower, but more universal. Figures are considered identical if they can be obtained from each other by a parallel translation. In this case, say, for road signs, we do not need to introduce a new definition of identity and, for letters and digits, we can use the same definition, without going into details complicated for first-graders. Within this definition, Fig. 3 depicts pairs of different figures.

An additional class of atomic objects consists of symbols: digits, letters, punctuation marks, and even hieroglyphs.

6. PROPERTIES, RELATIONS, STATEMENTS, AND ACTIONS

We believe that, for a teacher working with a course in the classroom, it is useful to have a structured system of concepts that can be used in the teacher's internal language to describe situations and processes in solving problems; in particular, these concepts can be used to discuss the student's solution with the teacher. Students can learn the meaning of these concepts gradually on examples.

At the same time, both the students and the teacher use *key concepts* introduced in the textbook consistently and explicitly, also on examples. We assume that the child's understanding of a situation appears quickly and, later, it is strengthened and broadened in solving problems.

An example of a key concept is the color of a bead—one of eight. The general concept of bead properties that beads may or may not possess refers to the teacher's internal language.

A key concept is the successor relation between the beads in a string: "this yellow bead follows that blue one": we show this to students on examples from sheets of definitions of the textbook. This relation is mastered in solving problems from this and the following lessons.

The students performs specific actions with objects, such as connecting identical figures with a green line. The general idea of the action, which is understood by the teacher, is gradually formed in the students' mind.

The general concept of a statement that can be true or false for some system of objects is first an element of the teacher's internal language. The understanding of the fact that all statements from a problem formulation hold for a given string is an element of learner's work. At some point in the course, previously trained students are given the concepts of truth and false statements. They are used to identify individual statements in a problem formulation, which are written individually, and to explicitly state the requirement that all these statements must take the specified values (mostly true, but, in some problems, false as well).

An example of a statement that can be true or false is given in Fig. 4.

In an educational situation, students can usually check quickly and reliably whether the object they created satisfy the problem formulation. As a rule, properties of objects are formulated as sets of statements. These statements may include words connected with the structure of an object and the identity of its elements. We use phrases, such as:

- there are three different beads,
- there are no three identical consecutive objects,

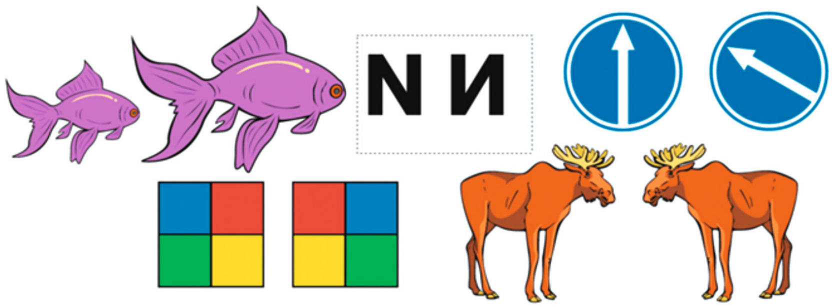


Fig. 3. Definition of identity of figures in the course *Informatics*. The figures in these pairs are considered different.

Fig. 4. Example of a statement that can be true or false.

• every blue bead is followed by either a red one or another blue bead,

• a triangular bead is preceded by a red one.

To check the fulfillment of all conditions, against each statement, it is necessary to put its value. If all values are true, the task is performed correctly.

Authors pay special attention to **quantifier words**, such as "all," "every," "there is," "exists," and others, the meanings of which corresponds to quantifiers used in mathematics: **∀, ∃**. Figure 5 gives several examples of their use.

It is possible to use a quantifier over a subset specified by a certain property (see Fig. 6).

In the case of relatively few objects, a statement with a quantifier is easy to verify (see Fig. 7).

If a false statement concerns objects from some class, it is possible to give a counterexample (see Fig. 8).

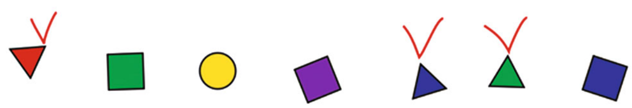


Fig. 5. Example of using quantifiers in a problem.

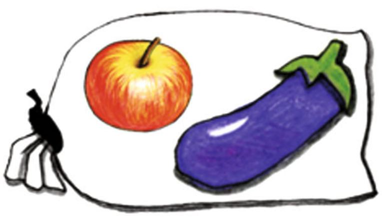
Fig. 6. Example of using quantifiers in a problem.

Fig. 7. Example of using quantifiers in a problem.

Determine whether the statements in the table are true. Give a counterexample to each false statement.

Statement	Counterexample
The sum of two one-digit numbers cannot be a two-digit number. F	$8 + 9 = 17$
The sum of a one-digit number and a three-digit number cannot be a two-digit number.	
The sum of a one-digit number and a three-digit number cannot be a three-digit number.	
The sum of a one-digit number and a three-digit number cannot be a four-digit number.	
The sum of a one-digit number and a three-digit number cannot be a five-digit number.	

Fig. 8. Counterexamples to false statements.



There is no mushroom in this bag.

There is a mushroom in this bag.

There is a mushroom in this bag.
There are two mushrooms in this bag.
There are exactly two mushrooms in this bag.
There are no three mushrooms in this bag.

There is a mushroom in this bag.
There are two mushrooms in this bag.
There are exactly three mushrooms in this bag.

Fig. 9. Examples of true statements with quantifiers.

In the Russian language, there are no articles, but there are many defaults expressed in the mathematical language by formulas with quantifiers. In our course, much attention is paid to the explication of such defaults and their discussion with children. Figure 9 gives examples of true statements about bags.

7. STRINGS (SEQUENCES)

Examples of strings in everyday life are sequence of events and their recording, i.e., a sequence of sounds of speech and their written representation, a sequence of moves in a game, and a recording of a game round.

On the one hand, a string is a recording, a model of a sequence of choices. On the other hand, the creation or drawing of a string is the result of choosing its current symbol from the bag of symbols. A string appears when objects are stringed; we try to get children to do it with their hands and material beads at least once (see Fig. 10).

A string is symbolically shown in Fig. 11.

Fig. 10. Emergence of a string of beads—bead stringing.

Fig. 11. Example of a string of beads.

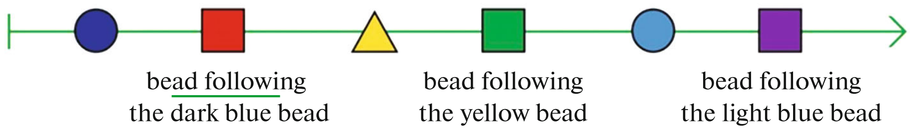


Fig. 12. Definition of a following element in a string.

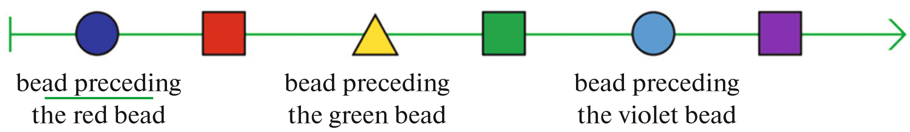


Fig. 13. Definition of a preceding element in a string.

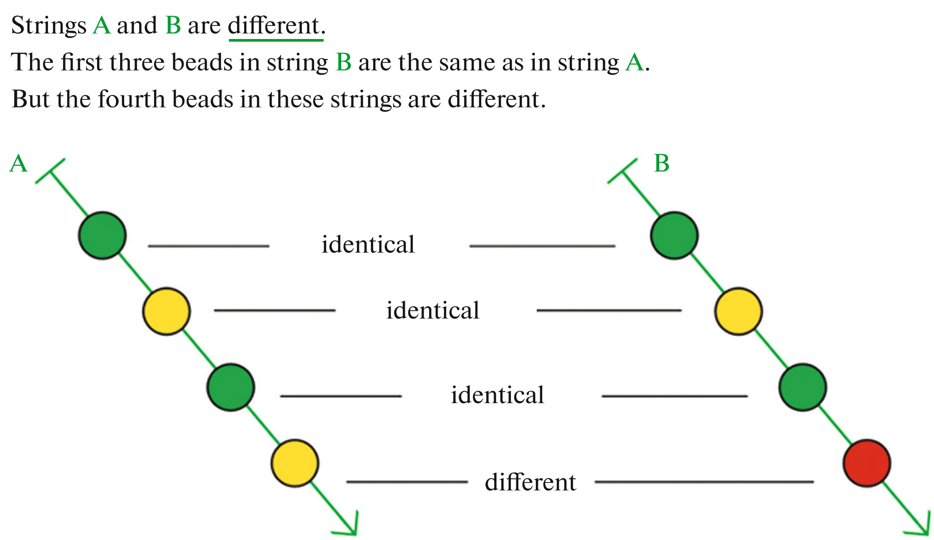


Fig. 14. Definition of different strings, an example.

A string has a beginning, which is marked with a perpendicular bar, and an end, which is marked with an arrow. For each object, except the last, there is an object following it (see Fig. 12).

For each object, except the first, there is an object preceding it (see Fig. 13).

An empty string is a rope with no beads, but with marked beginning and end.

Two strings are identical if they coincide term by term: the first bead in one string is the same as the first bead in the other, the second bead in one string is the same as the second bead in the other, and so on; the number of beads in the strings also has to be identical (see Fig. 14).

All empty strings are identical.

8. BAGS (SETS)

Bags are natural mathematical objects, at least, for finite mathematics. From a point of view of work with physical objects, bags are more natural than sets. One of mathematical names for a bag is a multiset, i.e., a set in which each element occurs with some multiplicity. To demonstrate a bag to children, we need to take a transparent plastic bag and put some items in it. If the

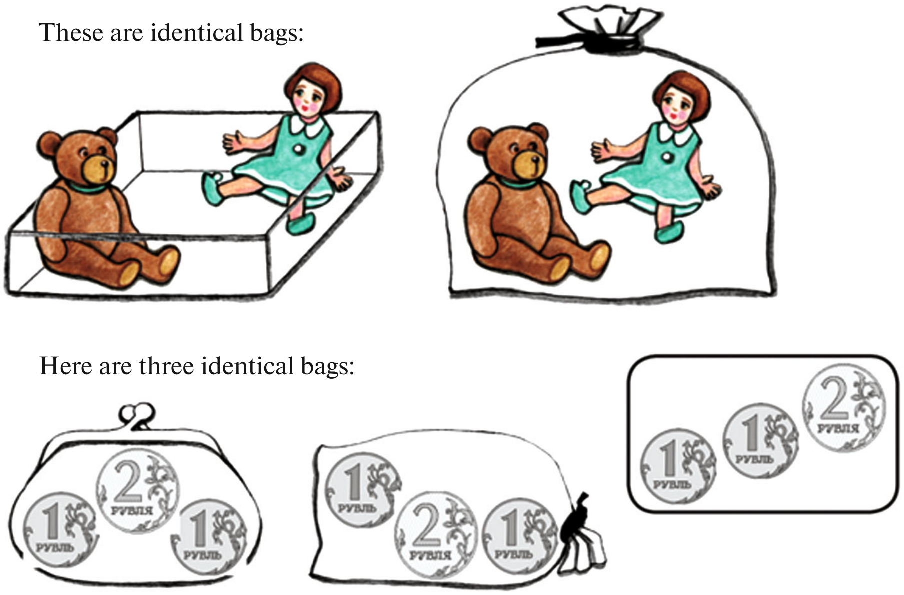


Fig. 15. Definition of identical bags.

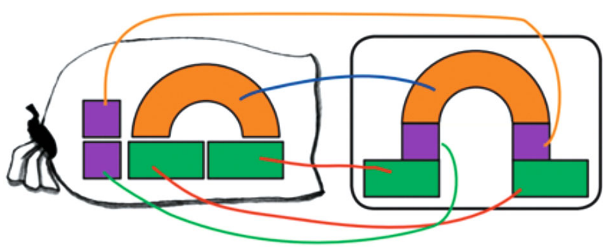
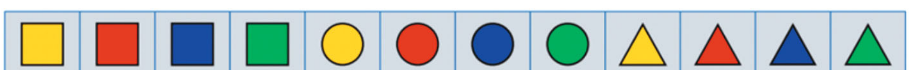


Fig. 16. Comparison of bags, an example.



0	0	1	3	2	2	1	2	3	1	0	1

Fig. 17. Example of a table for a bag.

bag is shaken, the objects inside it will move, so we know about the bag only what is inside it, but there is no order on the elements of the bag.

Sometimes bags really look like bags, but sometimes they look more like boxes. Sometimes a bag has the form of a wallet, and sometimes it is just a contour drawn around figures (see Fig. 15).

Bags are called identical if there is a one-to-one correspondence between them with identical objects in them joint in pairs. Children establish such correspondences, connecting pairs of identical figures with lines. To establish the identity of bags is a more difficult task than establishing the identity of strings: it is not clear where to start pairing and it is easier to get confused (see Fig. 16).

If a bag contains beads, it is convenient to compare the numbers of beads of each type. For this purpose, it is possible to use the so-called bag table (see Fig. 17). In such a (one-dimensional) table, the upper ("heading") line presents all types of beads present in the bag (see Fig. 17).

Of course, in the course there is a wide variety of problems that use the identity of bags: in addition to direct comparison of two bags, the course offers finding two identical among numerous ones, finding all pairs of identical bags, building two identical bags (material or drawn), completing bags so that they become identical, and so on. Complicated problems are offered in which it is necessary to change exactly one object in one bag so that two identical bags appear in the picture (see Fig. 18).

The most natural operation over bags is the sum (generalization of the sum of positive integers): "adding" two bags means joining their contents in a single bag. The other two important operations over bags are union (maximum) and intersection (minimum). Of course, bags with these operations form a lattice. It is also clear that these operations naturally apply not only to two bags, but also to an arbitrary bag of bags.

The just described operations over bags show that not only atomic objects can lie in bags, but also structures, such as bags, strings, and words (strings of letters). Working with a bag of words makes it possible to precisely formulate the standard task of inserting missing letters from the Russian language course. Possible exact wording is as follows: "write one letter in each window so as to obtain a word from the dictionary." Of course, it should be clear what dictionary is meant. The dictionary may be a standard school (paper or electronic) one created by the student himself, or the dictionary given in the problem or at the end of the textbook, etc.

In our context, complicated and meaningful problems arise of establishing the correspondence between two bags of strings, which need to be made identical (see Fig. 19).

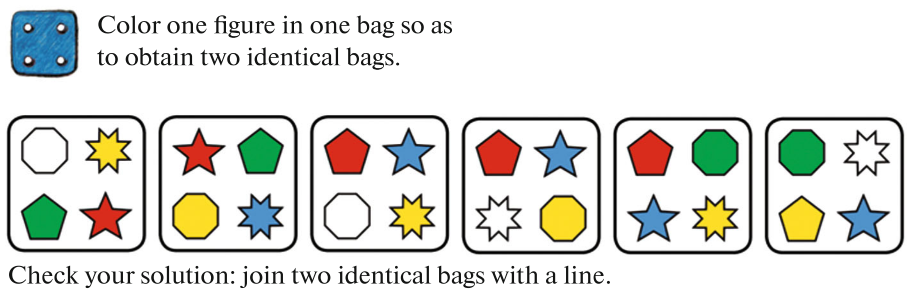


Fig. 18. Example of a complicated problem of identical bags.

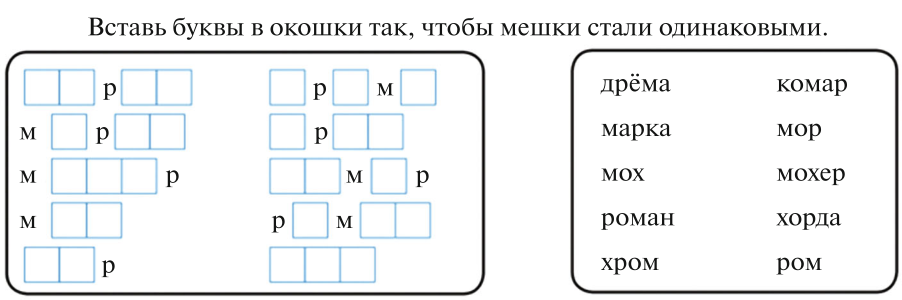


Fig. 19. Example of a problem with identical bags of words.

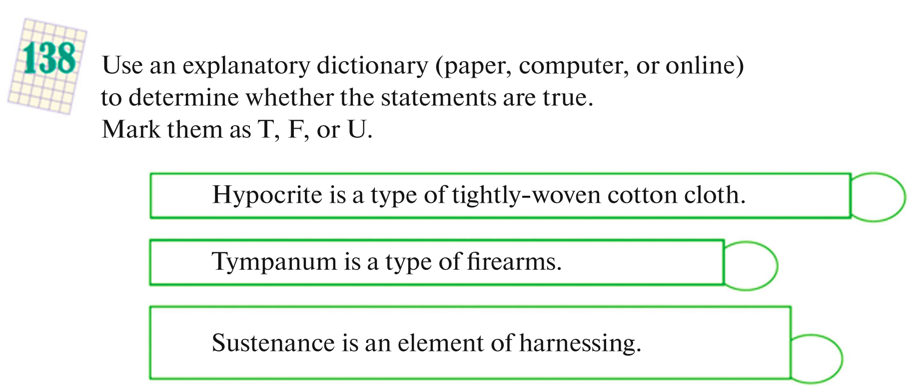


Fig. 20. Example of a problem in which an explanatory dictionary has to be used.

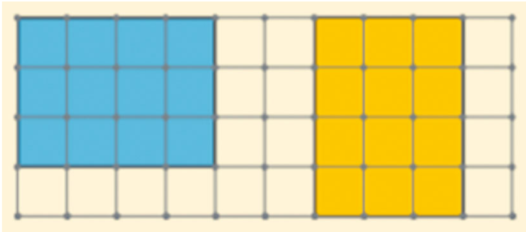


Fig. 21. Rectangles on a grid.

This problem is good because it allows simple reasoning steps and division into subproblems. For example, we can notice that there is only one four-letter word in the bag and sort out three-letter words; there are only two words beginning with r, etc. These considerations can be shared with classmates, thus mastering mathematical speech and increasing the set of heuristic techniques.

Note that such problems do not assume that children know the meaning of all words used. The problem is formal and requires only the use of a mathematical definition. It is good if someone of the students gets interested in some word and looks it up in a dictionary on their own or with the help of the teacher. In the methodological comments, we often recall such a possibility. In addition, the course contains a series of problems that require looking up specified words in an explanatory dictionary to understand their meaning and evaluate the truth of the statement given in the problem. An example of a problem is given in Fig. 20.

Of course, a bag can contain numbers (strings of digits). All numbers lying in a bag can be added or multiplied. In this case, the commutativity and associativity of these operations are put directly in the definition: the numbers lying in the bag are not ordered. On the other hand, these properties receive a meaningful interpretation. For example, the children can be asked to find the sum of a bag of four 3's and a bag of three 4's and to discuss why these sums are different. Here, it is necessary to pass to another definition of multiplication, namely, through the area of the corresponding rectangle (see Fig. 21).

In the modern world problem, the addition of several numbers is met even more often than the addition of a pair of numbers: shopping in the store, salaries, and the number of visitors rarely occur in pairs, rather large amounts of such data are added up. There are also problems in which sums have to be compared without calculating them. Working with bags of numbers allows us to easily formulate such problems in the course (see Fig. 22).

Such problems also offer a wide range of opportunities for reasoning. For example, it is possible to cross out identical numbers in bags: this operation influences the sum of each of the bags, but does not influence the difference between these sums. The remaining numbers can be compared, rather than added: if one number from the first sum is greater by 9 and another is smaller by 1, then the first sum is greater by 8. It is possible, crossing out close in value numbers in two bags, to put the difference of these numbers near the bag that contained the larger number. Of course, it is easier to perform crossing out numbers (deleting them, taking numbers out of the bag, etc.) in a digital environment than on paper.

With the use of the language of bags, it is easy to formulate arithmetic enumeration problems in which the order of terms is not important. For example, children can learn about Waring's conjecture and Lagrange's theorem (see Fig. 23).

Of course, an important and necessary application of numerical bags is modeling the factorization of numbers. It is possible to enumerate all factorizations or to specify certain constraints on the factors. An example is given in Fig. 24.

Of course, it is natural to compare a number with a bag where all numbers are prime and the product of the bag is equal to the given number: this is the bag of prime factorization of the number. Then the product of numbers is associated with the operation of the sum of two such bags, and GCD and LCM are obtained as the minimum and maximum (union and intersection) of two such bags.

9. TABLES

It is instructive to pay attention to the place of tables in traditional school. On the one hand, they are obviously used there: these are tables of addition and multiplication, various tables in Russian and foreign language courses, a great achievement of humanity—Mendeleev's table—is used in high school. A school schedule, a report card, and a school register are all tables.

However, despite the constant use of tables, they are not learned in school as an independent mathematical object. A table turns out to be something like a notebook, which does not make sense to learn in mathematics. The reason is that tables somehow do not fit into a strict sequence of understanding arithmetic. However, at the same time, the school proceeds from the fact that students somehow know about tables and are able to use them.

A table consists of a string of line names (the leftmost column), a string of column names (the top line), and cells. Mathematically speaking, a table is a map that assigns to a pair "line name, column name" the content of the cell at the intersection of this line and this column. One-dimensional tables were mentioned earlier in this paper.

For example, it is convenient to put all types of beads occurring within the course in a table (see Fig. 25).

A school schedule is a table with line names being classes and column names being weekdays.

Figure 26 gives an example of problem of drawing up a school schedule.

More life-like conditions are possible, for example: “every day there is a lesson of Russian language class,” “there is exactly one music class every week,” “certain classes go before other ones”—such requirements can be found in state sanitary regulations. There may also be restrictions related to workdays of physical education or music teachers, who do not come to school every day.

The classical use of tables concerns the solution of logical problems that involve several statements. An example is given in Fig. 27.

The task is to find out which drink is poured in which vessel. The most convenient method of structuring the given information is to put it in a table,

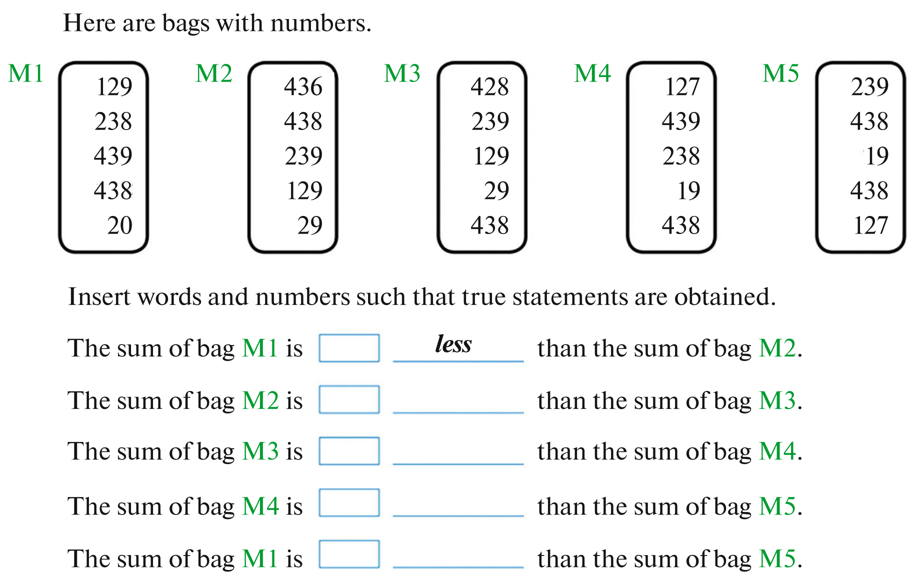


Fig. 22. Problem of bags of numbers.

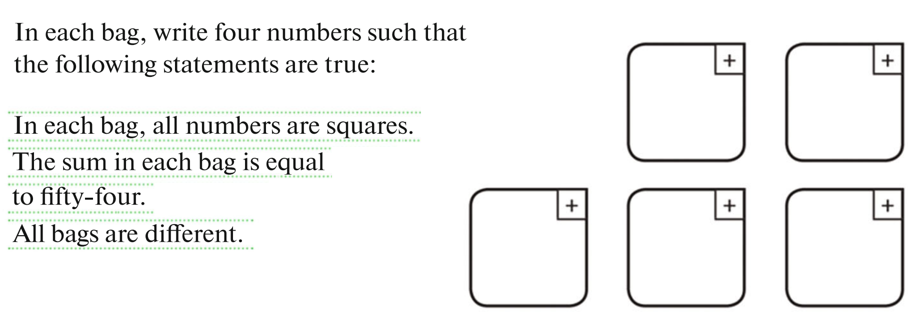


Fig. 23. Waring’s conjecture and Lagrange’s theorem in a problem of the course.

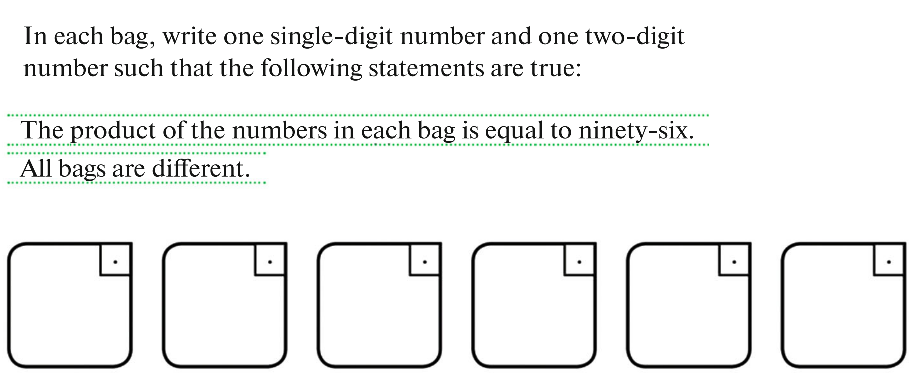


Fig. 24. Example of a number factorization problem.

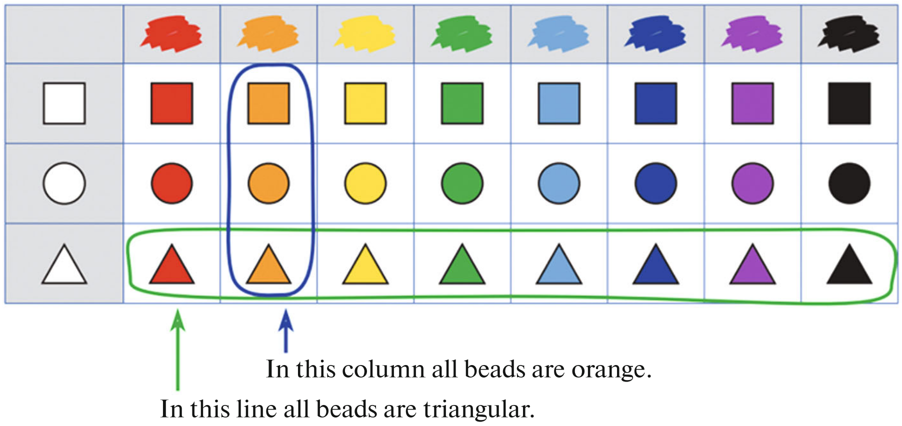


Fig. 25. Two-dimensional table of a bag of beads.

Some classes are missing in the schedule.

	Monday	Wednesday	Friday
1 class			
2 class	Mathematics		Informatics
3 class		Literature	
4 class	Informatics		Russian Language
5 class			
6 class		Music	

Write down classes in the table so that the following statements are true:

On Monday the fourth class after Reading is World Around.
On Monday Russian Language is earlier than Music.

On Wednesday History is later than Russian Language.
On Wednesday English Language is earlier than History.
On Wednesday the third class after Mathematics is English Language.

On Friday Mathematics is earlier than History.
On Friday Music is later than History.
On Friday Mathematics is earlier than History.

Fig. 26. Example of a scheduling problem.

marking the true statements about liquid in a vessel with a plus sign and crossing out false statements (see Fig. 28).

The novelty of this problem for students may be that, in addition to explicit "something in something else" statements, it also involves statements with negation and two statements combined in a single sentence. Statements about the spatial arrangement, including ones with terms like "adjacent" and "between," are good if the students have a question related to the linearity of the vessel arrangement. The difficulty in solving this problem may be associated with the statement

There are four liquids: milk, lemonade, kvass, and water.
They are in four vessels: a bottle, jar, jug, and glass. It is known that:

- the water and milk are not in the bottle;
- the lemonade vessel is between the jug and the kvass vessel;
- there is no lemonade or water in the jar;
- the glass is next to the jar and the milk vessel.

Fig. 27. Example of a traditional logical problem.

Put the information in the table: cross out cells or write a plus sign.

	milk	lemonade	kvass	water
jug				
glass				
bottle				
jar				

Which liquid is in which vessel? ______________________

Fig. 28. Example of solving a traditional logical problem by constructing a table.

This is an empty cycle. This is not a cycle.

Fig. 29. Definition of a cycle.

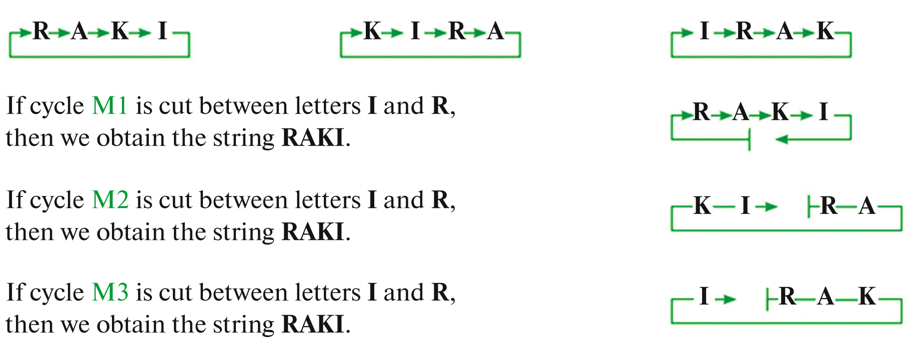


Fig. 30. Definition of identical cycles.

that “the glass is next to the jar and the milk vessel.” Is it equivalent to saying that “the glass is next to the jar and next to the milk vessel” or “the glass is next to the jar and the glass is next to the milk vessel”? Why do not we think that the glass is next to itself? Due to the difficulty of usage of “and” and “or” in the natural language, we avoid them in some variants of our course and formulate this as follows: “all statements in this bag are true” and “among the statements is this bag, there are true ones.”

Color all beads in the cycles so that the latter become identical.

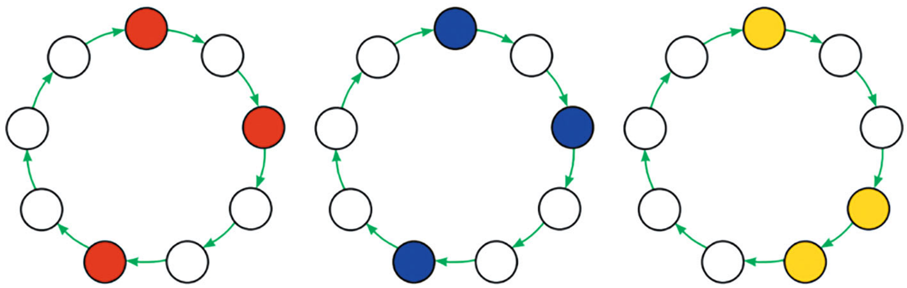
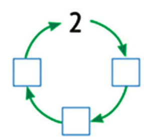


Fig. 31. Example of a complicated problem of cycles.

Fill out the table.

Number	2	2 · 2	2 · 2 · 2	2 · 2 · 2 · 2	2 · 2 · 2 · 2 · 2	2 · 2 · 2 · 2 · 2 · 2
Last digit						

Insert digits in cells such that each number in the cycle is obtained from the preceding one by multiplying by 2 and taking the last digit.



What is the last digit in the product of 10 twos?

What is the last digit in the product of 32 twos?

What is the last digit in the product of 99 twos?

Fig. 32. Problem of a periodic sequence.

Examples of using tables for finding ALL objects satisfying a system of conditions can be found, for example, in [28], [29].

10. CYCLES

In a cycle, each element has a preceding and a following one. There is no beginning or end in a cycle. The direction from the preceding element to the following one is indicated by arrows (see Fig. 29).

A cycle can be cut to obtain a string. Two cycles are identical if they can be cut so that two identical strings are obtained (see Fig. 30).

In the case of cycles, even the problem of creating two or three identical objects can be complicated and informative (see Fig. 31).

Cycles find a variety of mathematical appendices: periodic sequences, arithmetic of remainders (see Fig. 32).

The problem in Fig. 32 is an introductory one in experimental mathematics. Filling out the table (it can be supplemented by a top line with products themselves, which can be found using a calculator), we see that the last digit in the products of 2's repeats. The drawn cycle prompts the idea of the proof. In the course of solving the problem, or after its completion, the teacher can offer other problems of the same type. For example, someone of the students turns away (closes their eyes), two numbers are written on the board, and their digits, except for the last one, are covered. Can the student determine the last digit of the product? Why would it be so?

The problem becomes quite feasible if the students have invented some algorithm for multiplication of multidigit numbers, for example, the ancient Indian algorithm of "diagonal" multiplication, but discussing this issue would take us away from the main topic of this article.

11. TREES[1]

Trees are finite directed graphs with one node (root) having no incoming edges and with any other node having exactly one incoming edge. Nodes of a tree can contain any elements used previously for con-

[1] This description is relevant only to the *Informatics* course. The course *Mathematics and Informatics* is restricted to only binary trees.

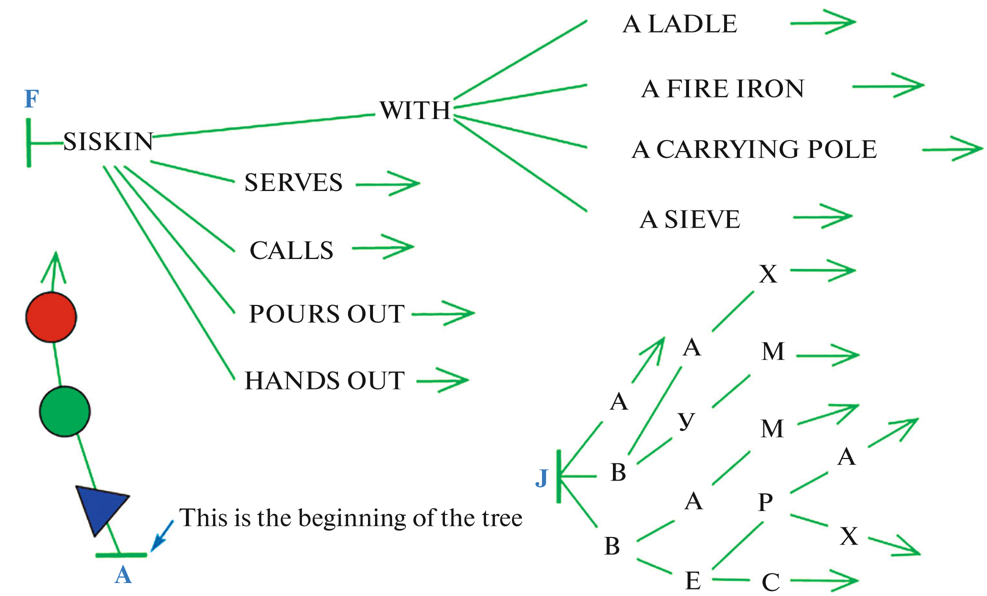
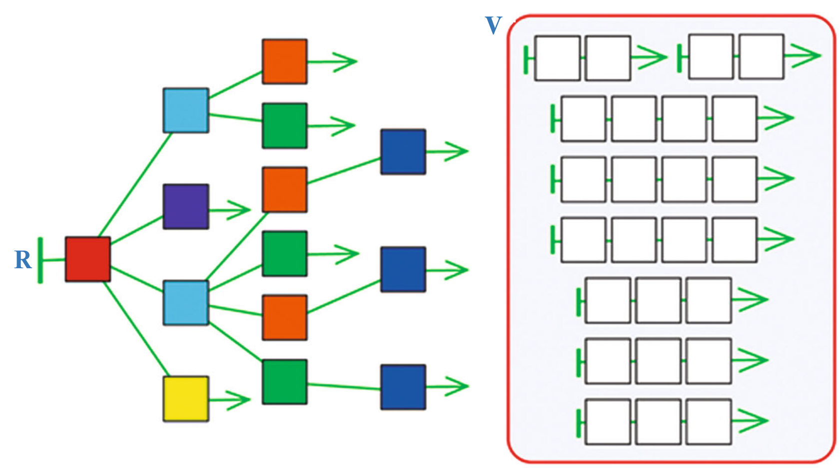


Fig. 33. Definition of a tree based on examples in the course *Informatics*.

131 Construct the bag V of all strings from tree R. For this purpose, color all beads in strings in V.



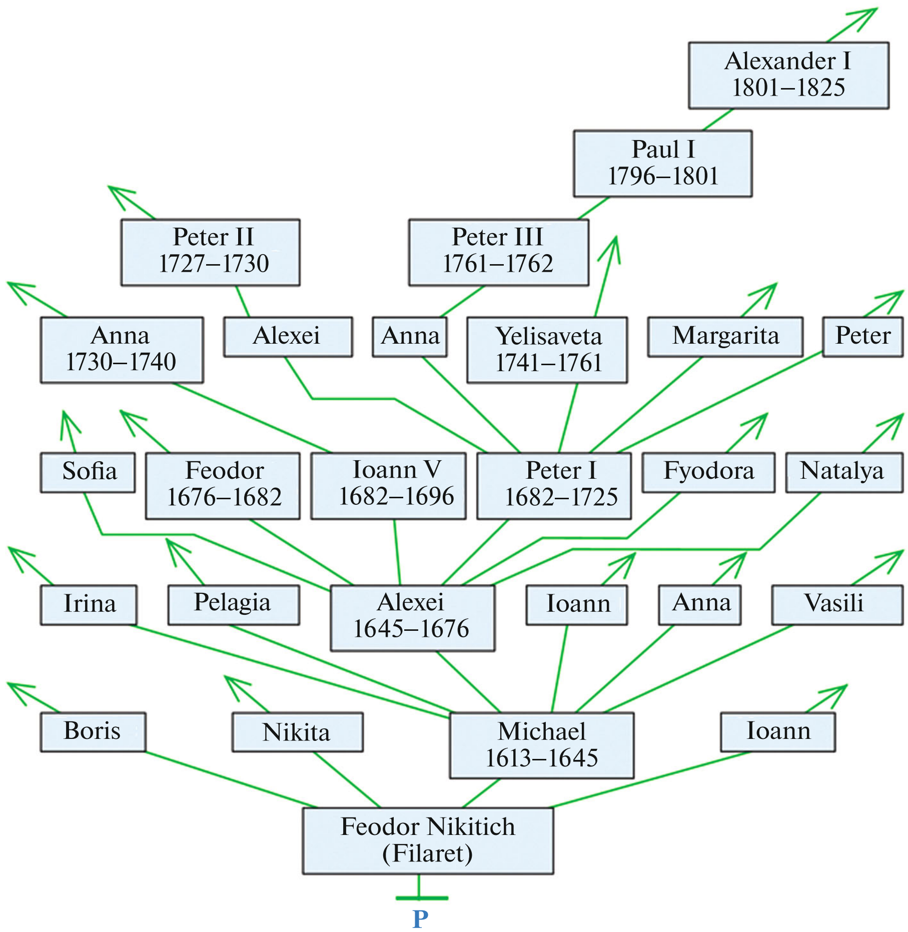
Check your solution: connect each leaf of tree R with the string from the bag that was constructed for this leaf.

Fig. 34. Problem of constructing all paths in a tree.

structing strings and bags, namely, beads, figures, letters, words, etc. By analogy with strings, the beginning (root node) of a tree is marked with a bar. Each node of a tree has exactly one preceding node (parent) (if it does not lie at the first level of the tree) and a finite number of succeeding nodes (children). If there are no children, then the node is called a leaf. Leaves of a tree are marked with outgoing arrows by analogy with the end of a string. A string is also a tree with each node (bead) other than the last having exactly one child.

Suppose that, in a tree, this is a string going from its root node to a leaf.

Examples of trees with a linguistic meaning are given in Fig. 33: following a path in the tree, we read a phrase or word.



Statement	P
Alexander I was a grandson of Peter I.	
Paul I was a son of Peter III.	
Peter I had five children.	
Anna Ioannovna was a cousin of Yelisaveta.	
Peter I reigned earlier than his brother Feodor.	
One of Sofia’s brothers had the same name as his great-grandfather.	

Fig. 35. Work with a tree of descendants.

The authors have decided that the identity of trees is too complicated and lengthy subject for elementary school, so it is not discussed in the course.

On trees, we define the operation of transition from a tree to the bag of paths consisting of all strings that can be read on the paths of the tree (see Fig. 34).

Trees can be found in various classes. The example in Fig. 35 is a tree of ancestors and descendants.

Such trees are also used in biology classes. They present various classifications and identifications based on them.

The construction of a tree helps to solve the problem of enumeration of variants; for this purpose, we construct a tree, write in all its paths, and calculate the resulting different strings (see Fig. 36).

It is also useful to construct a tree in the study of perfect information games and in the construction of a winning strategy (see Fig. 37).

A tree allows us to investigate the possible positions of Robic after performing, say, two commands (see Fig. 38).

In a sewing workshop, there are red, blue, and yellow buttons. The clown's costume should have three big buttons of three colors. How many variants of buttons are there?

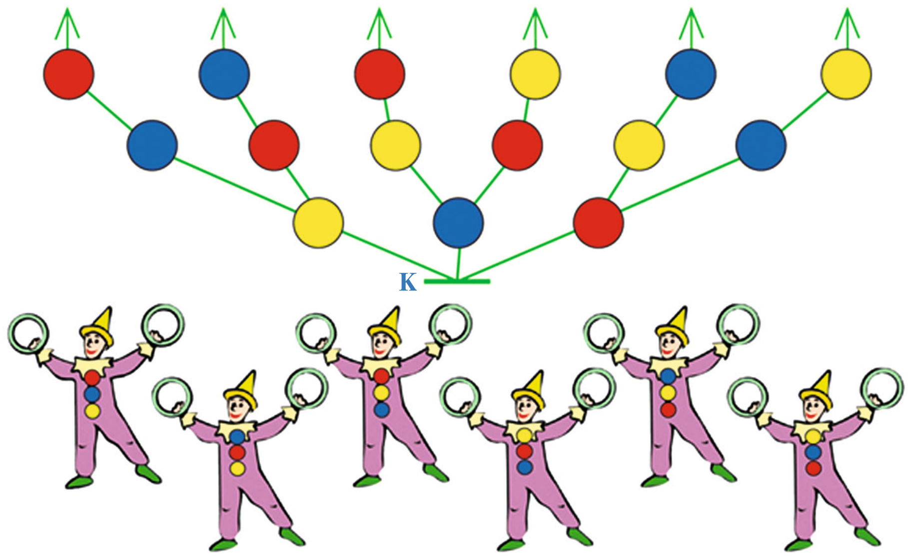


Fig. 36. Example of the problem of searching for all variants.

M is a **branch of the tree of** a tic-tac-toe game. Each string from M is a possible end of the game round from a given position, which is placed at the first level of M. All strings from M are **all possible ends** of the game round from the given position.

As in the complete tree of the game, each leaf node of the branch is an ending position.

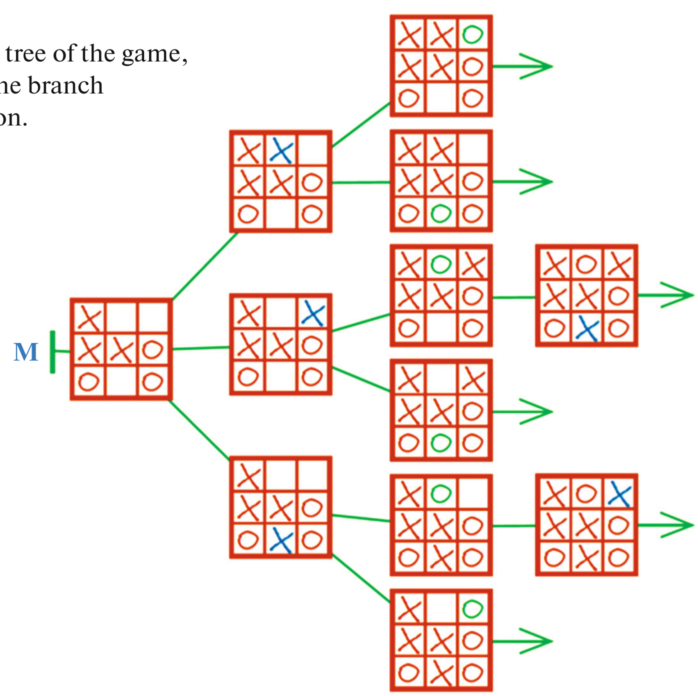


Fig. 37. Example of a branch of the tic-tac-toe tree.

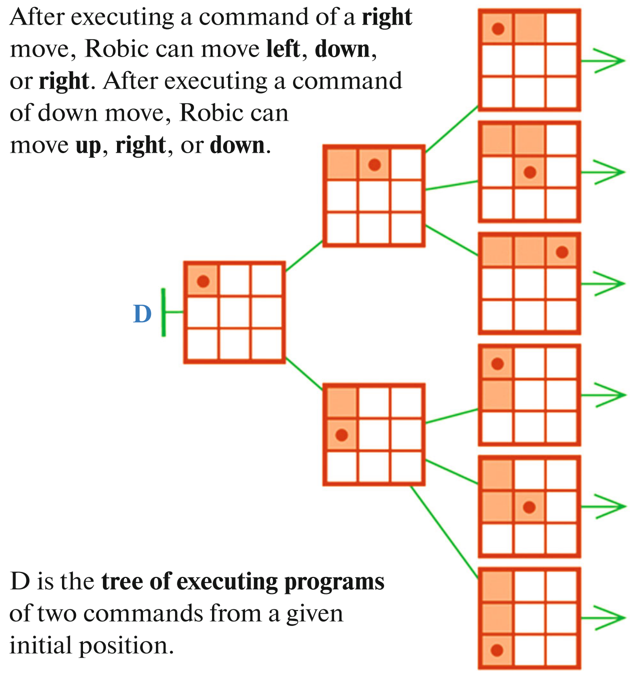


Fig. 38. Example of a tree of executing a two-command program.

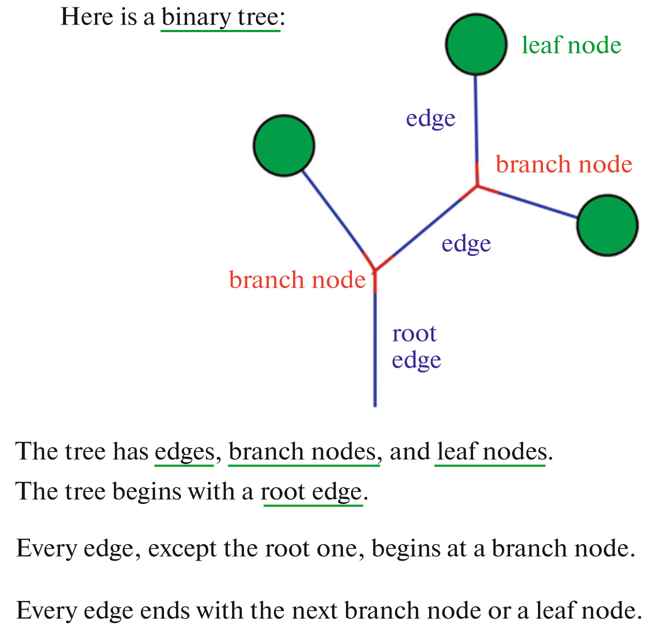


Fig. 39. Definition of a binary tree in the course *Mathematics and Informatics*.

12. BINARY TREES[2]

A variant of the tree topic in our courses is based only on binary trees. In this case, the system of definitions is somewhat different from the one considered above. A binary tree consists of a root node, edges, branch nodes, and leaf nodes (example in Fig. 39).

Exactly one edge leaves the root node. Each edge comes a branch node or a leaf node. Exactly two edges—left and right—leave each branch node.

Trees can be bent differently, but they remain the same (example in Fig. 40).

The path from the root node to a leaf can be specified as a sequence of left and right choices at all branch nodes, i.e., by a string of letters L and R. This string receives the name of the corresponding leaf node. For example, if we go from the root node to the left and, at the second branch node, again to the left and come to a leaf node, then this leaf is called LL (see Fig. 41).

Having a bag of names of all leaves, we can draw the whole tree. Several students can be invited to construct their own trees, having the same bag. Can they obtain different trees? If this happened, we propose that children exchange the figures and explain each other what they did. For comparison of trees, leaves with identical names can be colored the same color, etc. Eventually, the error will probably be found and it will become clear that there is only one correct tree.

[2] Not only binary, but also any finite trees are considered in the course *Informatics*.

The problem arises how to explain and prove that identical trees are always obtained from the same bag. In an attempt to sort out this issue, we can begin with very simple bags, for example, ones consisting of single- and two-letter names (see Fig. 42).

In constructing trees, we can try to apply the divide-and-conquer method, which is known to the children. Specifically, the bag is divided into two: one contains all names beginning with L, and the other bag, all names beginning with R. Quite quickly in the discussion, one of the children comes up with the idea of throwing away the first letter in all names (strings) in these two bags. Now we can construct a tree for each of these bags. It seems that we begin to understand why everyone gets the same tree from the same bag.

If a tree has complex geometry, it is not always easy to distinguish between its left and right parts (see Fig. 43).

In this figure, the leaf nodes of the left subtree are colored red, while the leaf nodes of the right subtree are colored blue.

Binary trees have a deep arithmetic application. Children sometimes do not understand the meaning of taking an expression in brackets. Here is an example of calculation:

$27 + (45 : 9) =$

(i) $45 : 9 = 5$

(ii) $27 + 45 = 72$

What is the mistake? The student does not understand that the result of the first action has to be used in the second action. To resolve this difficulty, it is useful

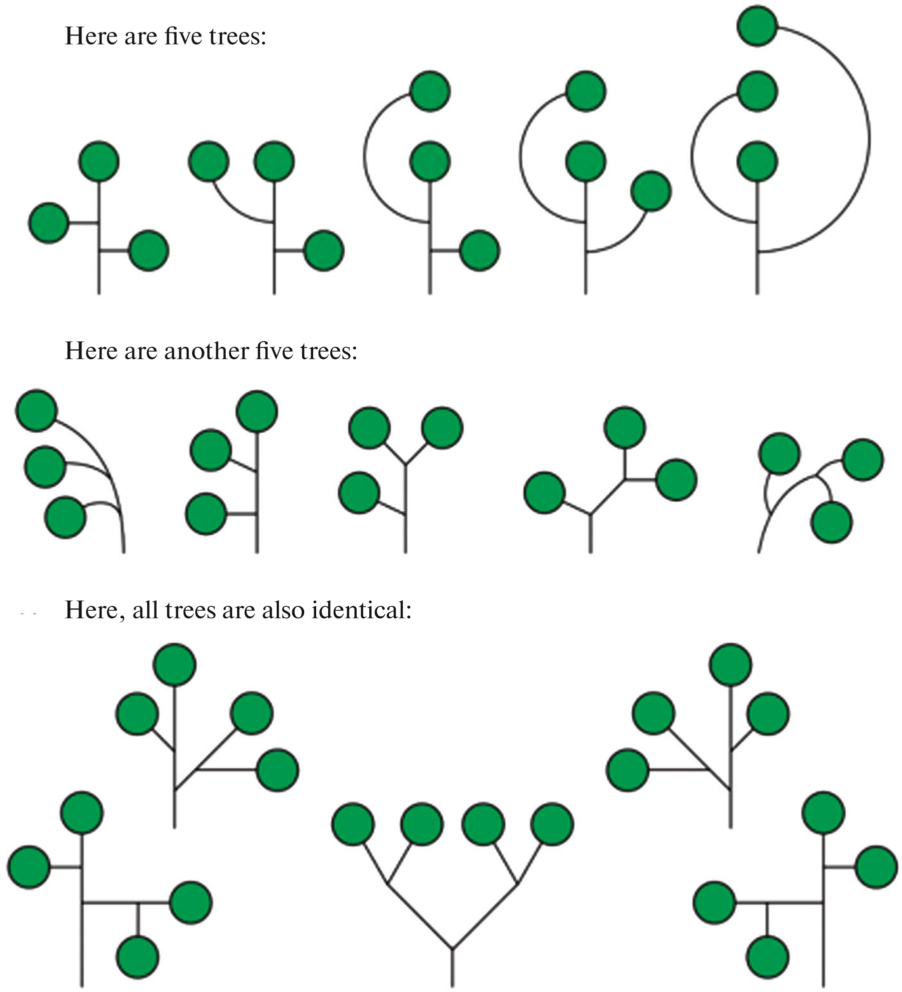


Fig. 40. Definition of identity of binary trees.

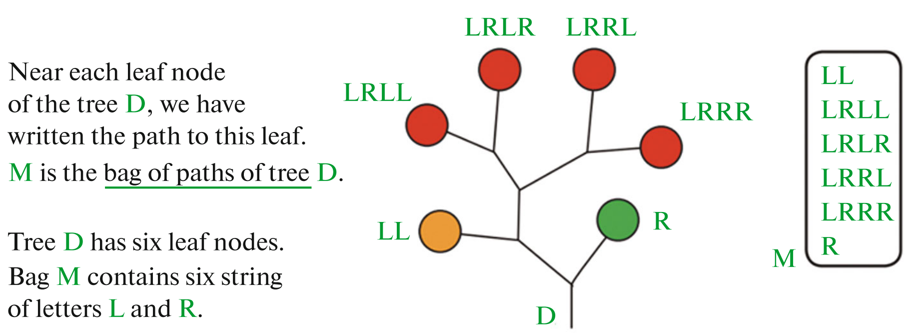


Fig. 41. Definition of the bag of paths of a binary tree.

to enclose this pair in an oval, having added brackets from above and below:

$$27 + (45 : 9) =$$

Then the oval is replaced by the result of the first action and the final result is calculated.

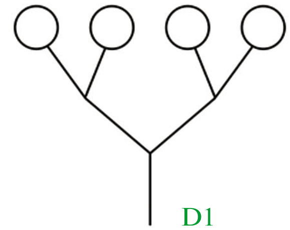
Binary trees perfectly describe this process of performing operations with two arguments. To find the value of the expression, the numbers involved in the expression are “paired” in a certain order (see Fig. 44).

Enumerating binary trees, we can simultaneously enumerate all ways of placing brackets in this numerical expression (see Fig. 45).

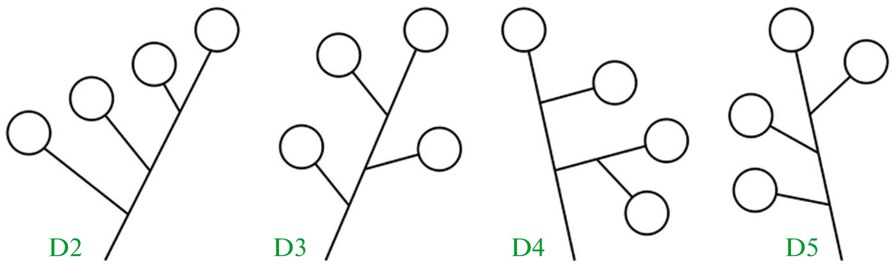
Color the leaf nodes of the trees according to the table.

Path	Color
L	red
LR	orange
LL	yellow

Path	Color
R	green
RL	blue
RR	violet



Tick the same tree as D1.



Find another pair of identical trees. Connect them to each other.

Fig. 42. Preparing to the proof of the fact that identical (binary) trees are always obtained from the same bag.

These problems are similar to ones often found in collections of recreational math problems where the task is to place brackets in an expression so that it receives a given value (see Fig. 46).

If earlier children could solve such a problem only intuitively, now they, generally speaking, can systemically enumerate all bracket arrangements and prove the possibility or impossibility of obtaining the given value of the expression.

Finally, we note that the binary property of associative and commutative operations, such as addition and multiplication, is not the natural necessity. A single bracket (oval) can contain several terms or factors; in this case, of course, it is necessary to use a nonbinary tree (see Fig. 47).

13. OPERATIONS WITH STRINGS AND BAGS

Joining bags together is the analogue of the union of sets. Two bags can be joined together to obtain a new bag (see Fig. 48).

A bag can be **divided into parts** (see Fig. 49).

In this case, some part may turn out to be empty (see Fig. 50).

The operations with bags thus introduced can be used to formulate a wide variety of problems accessible to children that may go beyond the informatics curriculum, for example, work with numbers and with language structures (letters, words) (Fig. 51).

The **operation of joining strings together** is an example of a noncommutative operation important for elementary school (different from addition and multipli-

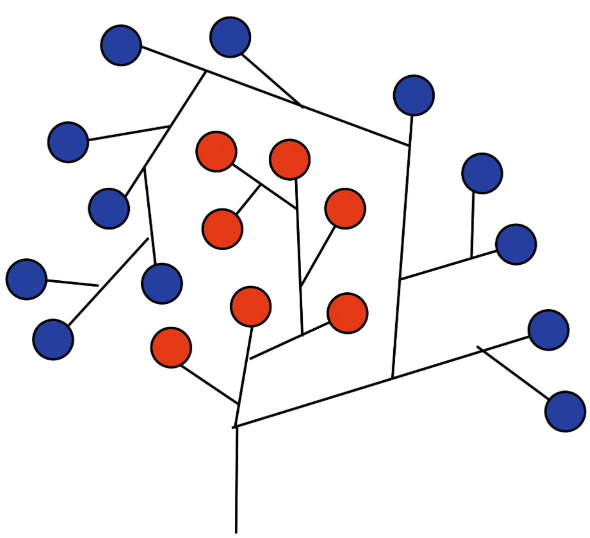


Fig. 43. Coloring of the left and right parts of the binary tree.

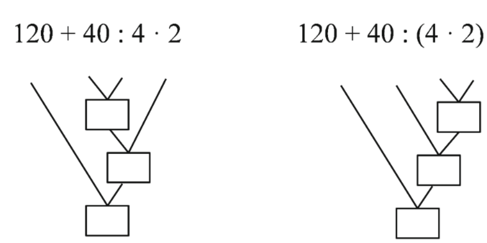


Fig. 44. Construction of a tree for calculation of an arithmetic expression.

Each window contains a numerical expression without brackets. In each window, draw a calculation tree such that all of them are different. Place brackets in the expressions so that the calculation trees correctly depict the order of operations. Solve the resulting expressions.

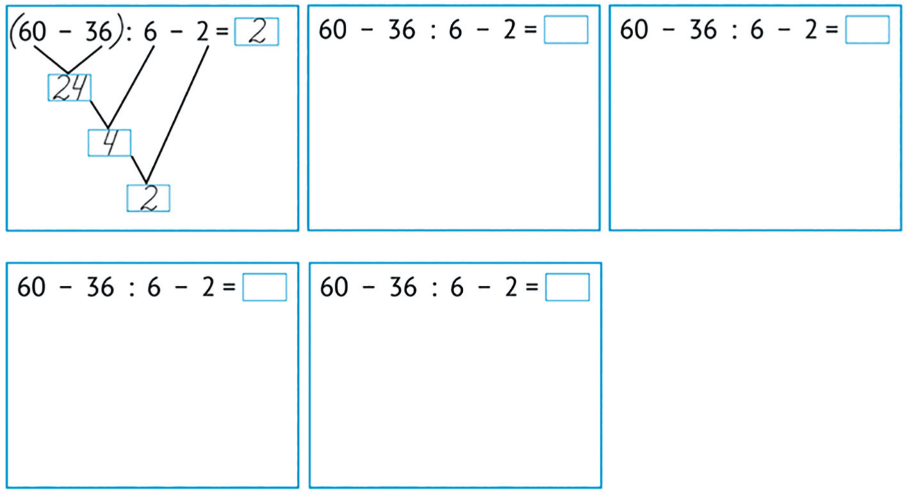


Fig. 45. Example of calculating the value of an arithmetic expression with the help of a binary tree.

Place brackets in the expression to obtain the true equality. Draw the calculation tree and check the solution.

56 : 7 + 7 · 3 · 4 · 12 – 8 : 4 : 2 = 4

Fig. 46. Traditional recreational math problem of placing brackets.

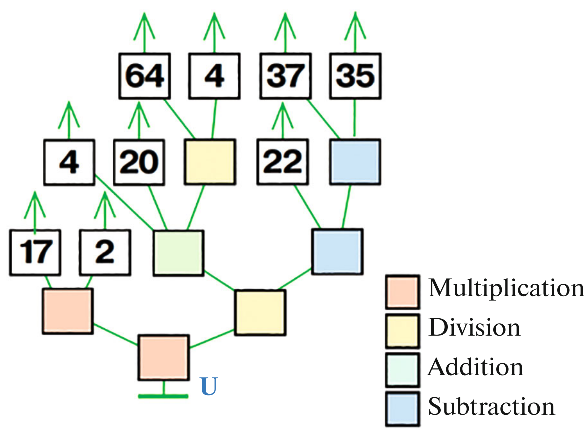


Fig. 47. Example of a nonbinary tree of calculating the value of an arithmetic expression in the *Informatics* course.

Adding all beads from bags A and B together, we obtain bag C.

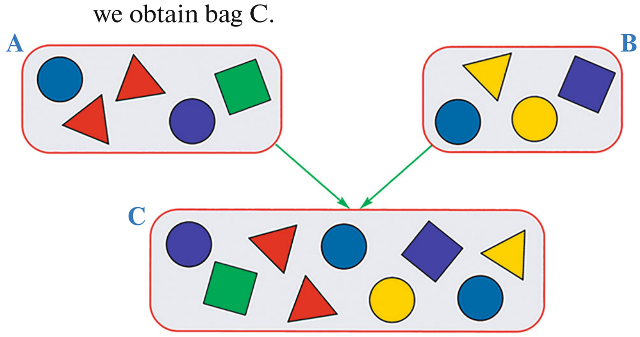


Fig. 48. Definition of the operation of joining bags together.

Partition bag Y into two parts: bags K and M.

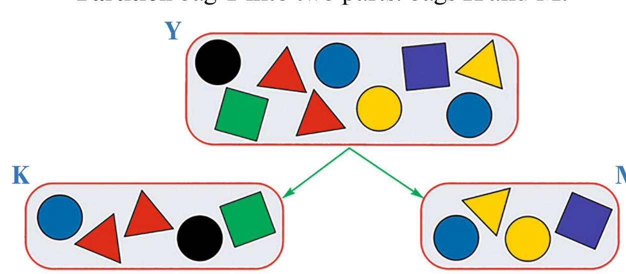


Fig. 49. Definition of the operation of partitioning a bag.

cation). Joining two strings together yields a new string. If the order of the strings in joining is changed, a different result is obtained (see Fig. 52).

If one of two strings to be joined together is empty, the joining result is the other string (see Fig. 53).

The operation of **joining bags of strings together** is useful in solving combinatorial and linguistic problems (see Fig. 54).

Due to the operation of joining bags of strings together (introduced in *Informatics 3 Grade*), the rules in Russian and foreign language courses, for example,

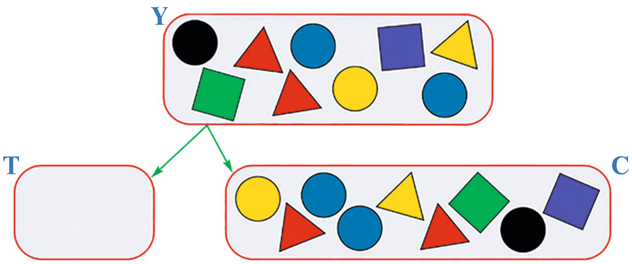


Fig. 50. In partitioning a bag, some of the resulting parts can be empty.

concerning word change and word formation can be explained in rigorous mathematical terms and can simultaneously be made visual.

An example from etymology is given in Fig. 55.

For the operation of joining bags together, it is convenient to use the table presented in Fig. 56.

14. TRANSFORMATION OF STRUCTURES INTO OTHERS

Table of a bag. The table of a bag indicates the number of objects in the bag and their type. The table can be one-dimensional (such tables were mentioned in the definition of identity of bags) or two-dimensional. Sometimes, a table is supplemented with

• another column on the right, with each of its cells containing the sum of the numbers from the preceding cells of the row,

• another row at the bottom, with each of its cells containing the sum of the numbers from the above cells of the column.

In the table, there is a corner cell added both in the row and in the column. It also contains a sum. In this case, we could obtain two sums: from the row and the column. However, they are identical! A remarkable research problem for each student is why so? (See Fig. 59).

186 Construct a partition of bag F into two parts (bags C and T) such that each number in C is less than 50 and each number in T is greater than 50. Fill out as many windows as required.

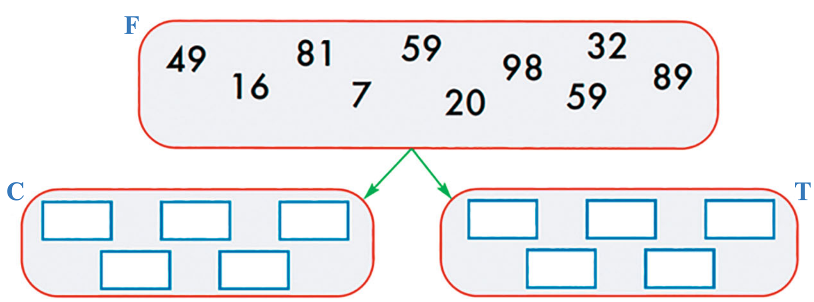


Fig. 51. Problem of partitioning a bag of numbers.

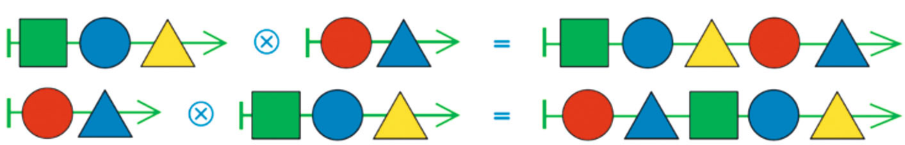


Fig. 52. Definition of the operation of joining together (concatenation of) strings. Noncommutativity.

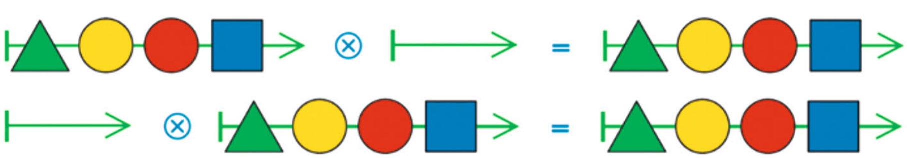


Fig. 53. Definition of the operation of joining together (concatenation of) strings. Joining an empty string.

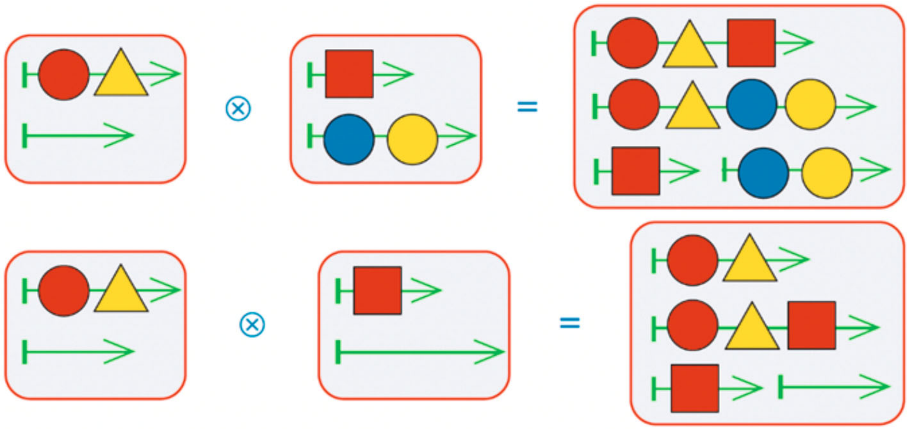


Fig. 54. Definition of the operation of joining together (concatenation of) bags of strings.

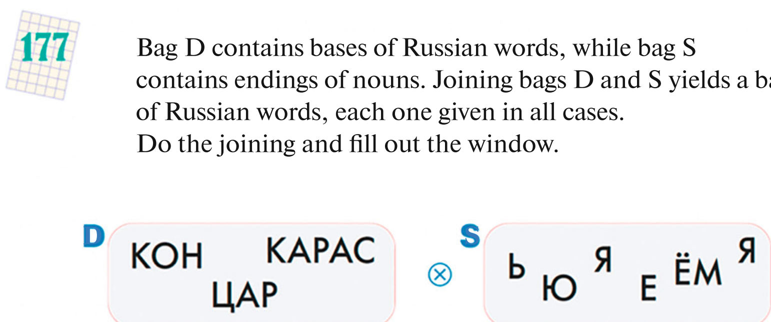


Fig. 55. Example of a problem of gluing bags of words.

A bag is uniquely determined by its table if the string of row and column names is specified. Bags are identical if and only if their tables are identically filled (we mean that the tables are identical, i.e., they have the same strings of row and column names).

Interesting problems are obtained if some of the added cells and some of the original are filled out (Fig. 58).

Students also can think up problems that involve filling out a table (see, e.g., Table 1) using numerical information on objects. Possibly, they will face situations when the problem has no solutions or, on the contrary, there are several solutions. That is where real mathematics begins.

On the counter, there are Christmas pastries arranged in 10 rows with 9 items in each: different cakes and 70 biscuits. There are only 10 ginger cakes left... Pity! But there are honey ones, not only cakes, but also biscuits. A total of 60. How many honey biscuits are there on the counter?

Work with tables is the first step in understanding and applying spreadsheets.

Joining a string in a cycle can be performed in only one way (see Fig. 59).

Cutting a cycle to obtain a string can be done along any arrow (see Fig. 60).

Sometimes it is not easy to reconstruct the string from the cycle from which it was obtained by cutting (see Fig. 61).

The number of different strings obtained cutting a given cycle does not exceed the number of elements in the cycle, but can be smaller if the cycle has symmetries.

Cutting a cycle of length 7 yields either one or seven different strings. For any divisor m of the cycle length N, it is possible to think up a cycle of length N that when cut yields exactly m different strings.

Bag of beads of a string or a cycle. From all beads of a bag, it is possible to construct numerous different strings (see Fig. 62).

Table 1

	Honey	Ginger	Total
Biscuits			
Cakes			
Total			

200

If necessary, it is possible to join three bags. The result will be a bag of all strings produced by joining a string from the first bag, a string from the second bag, and a string from the third one. For this joining operation, it is convenient to draw a tree. Using the tree W, build the bag V ⊗ S ⊗ F and fill out the window.

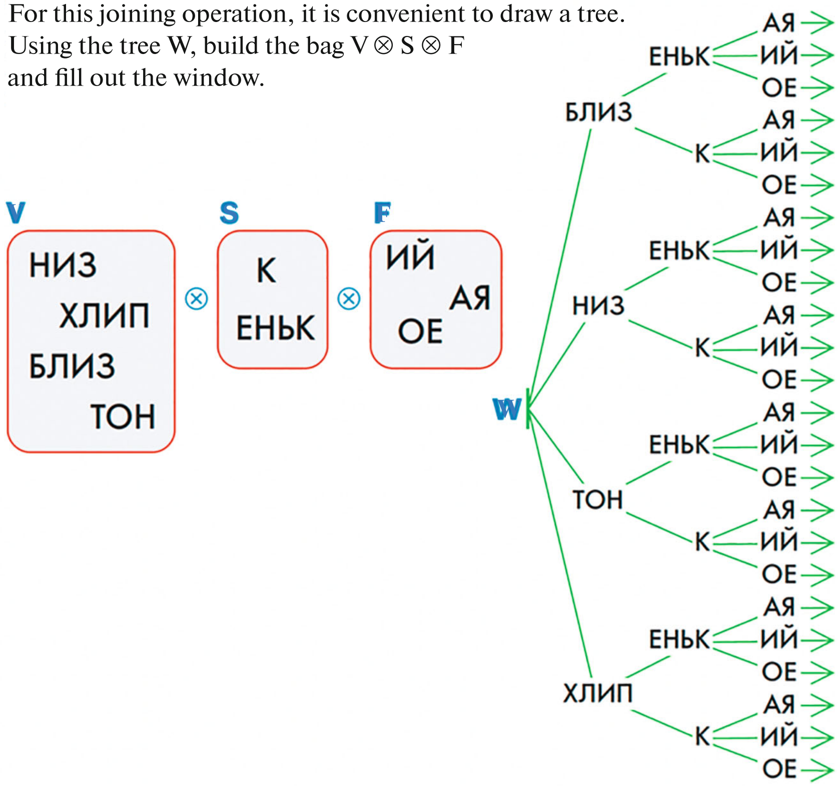


Fig. 56. Example of a problem of gluing three bags: roots, suffixes, and endings.

202

Fill out the windows so that the equality is true and the bag X ⊗ Z contains all Russian names of numbers from 13 to 19.

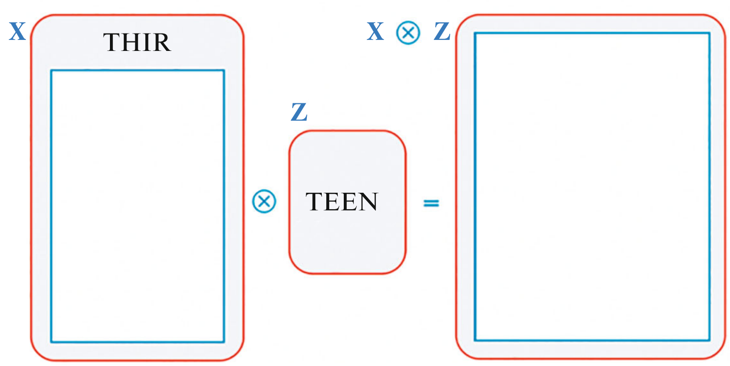


Fig. 57. Example of a problem of joining two bags together: generation of numbers of the second ten.

However, a given string has only one bag of beads.

How many different cycles can be made from a given bag of beads? If all beads are identical, then only one. And what if some of them are different or some of them are identical? An exhaustive search of variants is discussed in more detail in M.A. Posicelskaya's paper

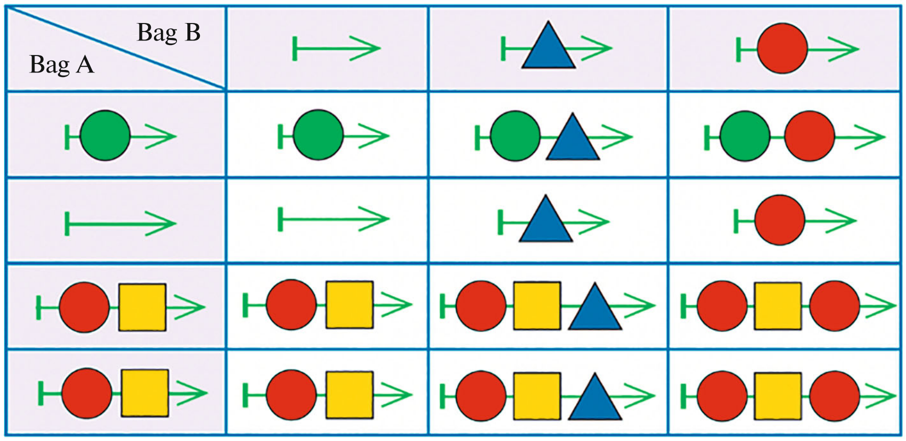
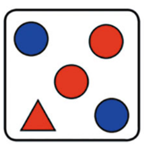


Fig. 58. Use of a two-dimensional table for joining two bags of strings together.



			Total
△	1	0	1
○	2	2	4
Total	3	2	5

Fig. 59. Example of the table of a bag with added total column and row.

Vasya saw a bag with figures and began filing out its table. Without looking at Vasya's bag, complete the table.

	apples	pears	plums	total
red	2	5		10
yellow			0	
green		1		3
total	6	6	5	

Fig. 60. Example of a problem of filling out the table of a bag with added sum cells.

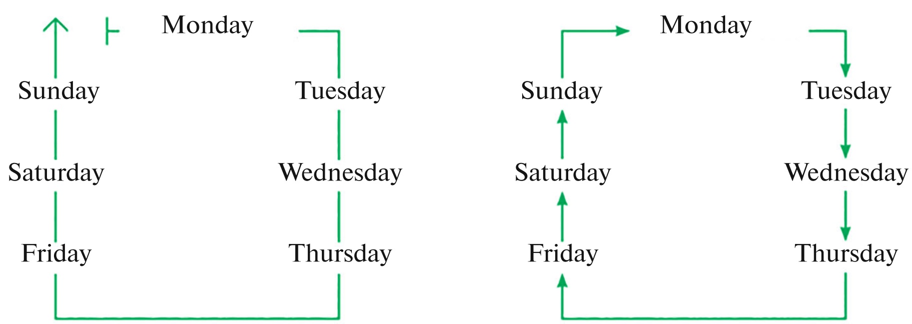


Fig. 61. Joining a string into a cycle.

"Constructive combinatorics in elementary school mathematics" published in this issue.

15. PROCESSES

Processes developing in a visual environment and their specification with the help of software programs and game rules provide important meaningful classes of problems. These problems clearly contribute to the achievement of the goals of modern mathematical education in elementary school, to the formation of computational thinking, and to the preparation for further education and life in the digital world. This was stated by Ershov with his slogan "Programming is the second literacy" [6], [7].

We only mention this range of problems without providing all necessary definitions. The interested reader is referred to the Russian informatics courses

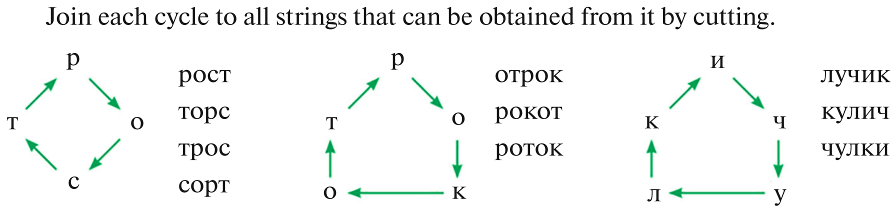


Fig. 62. Problem of cutting cycles.

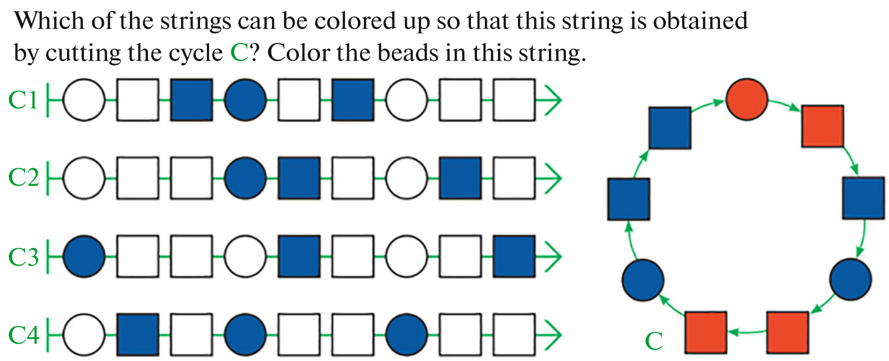


Fig. 63. Problem of reconstructing a string from a cycle.

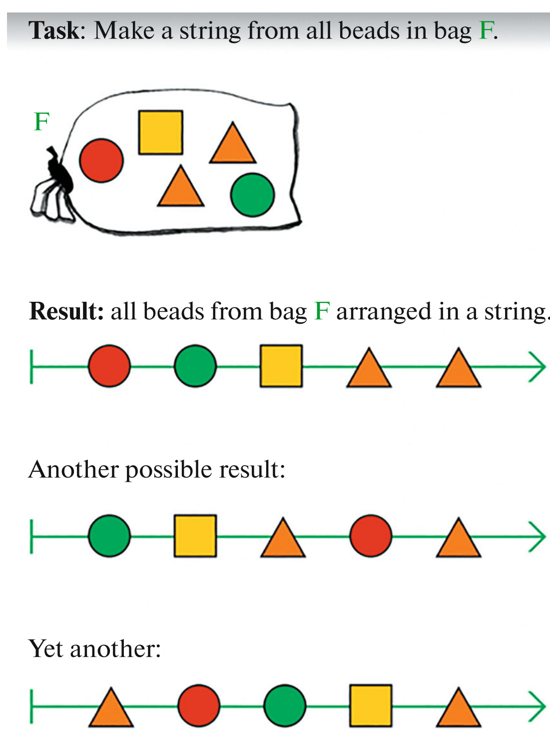
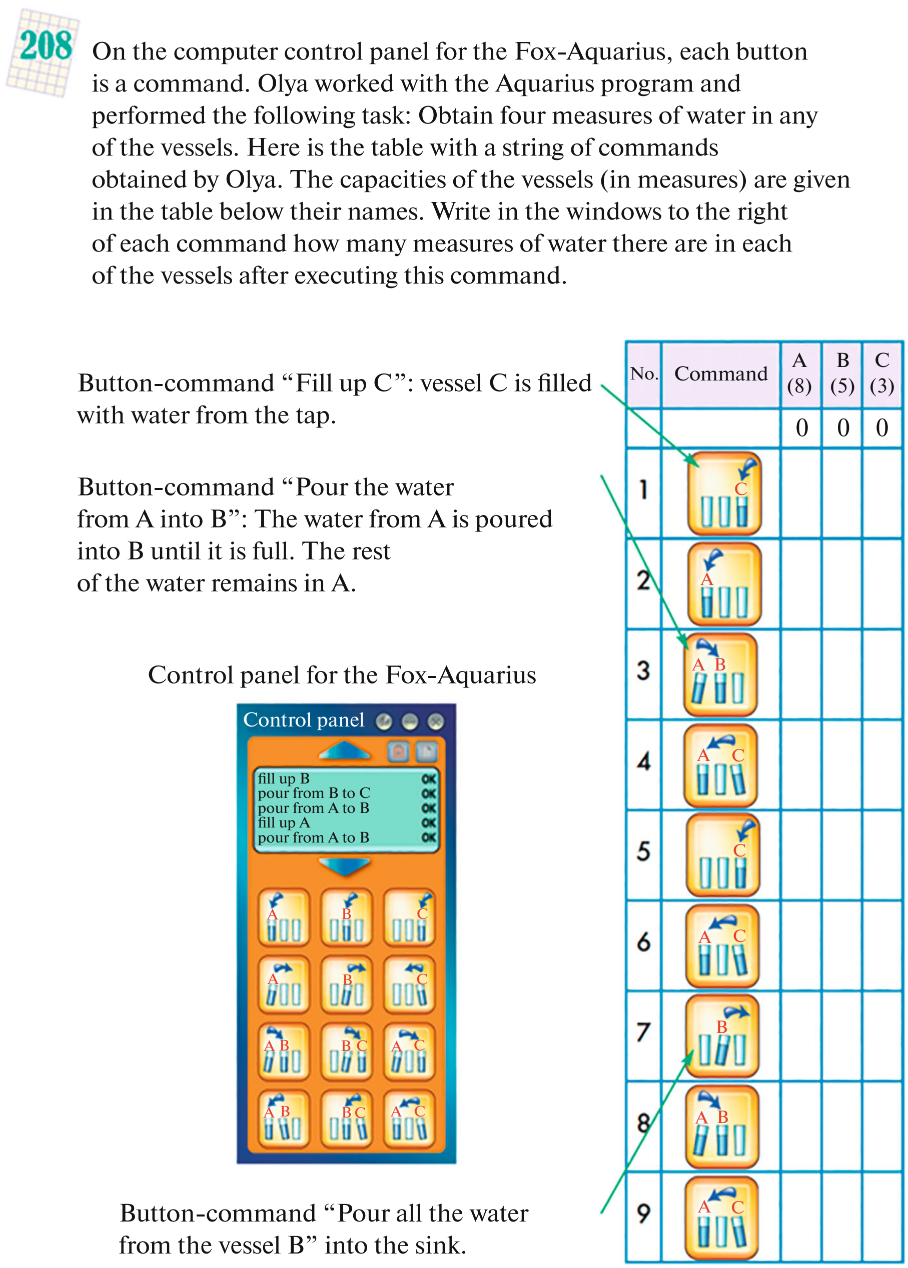


Fig. 64. Making a string from all beads in a bag.

created over the last decades we the participation of this paper's authors [9], [10], [30].

The **Virtual robot Aquarius** (implemented as a stand-alone computer program) simplifies the formulation of water pouring problems for teachers, while, for second-graders, it helps to make numerous problem solving attempts and, when an answer is obtained, it demonstrates the path that led to it, namely, a string



208 On the computer control panel for the Fox-Aquarius, each button is a command. Olya worked with the Aquarius program and performed the following task: Obtain four measures of water in any of the vessels. Here is the table with a string of commands obtained by Olya. The capacities of the vessels (in measures) are given in the table below their names. Write in the windows to the right of each command how many measures of water there are in each of the vessels after executing this command.

Button-command “Fill up C”: vessel C is filled with water from the tap.

Button-command “Pour the water from A into B”: The water from A is poured into B until it is full. The rest of the water remains in A.

Control panel for the Fox-Aquarius

Button-command “Pour all the water from the vessel B” into the sink.

No.	Command	A (8)	B (5)	C (3)
		0	0	0
1				
2				
3				
4				
5				
6				
7				
8				
9				

Fig. 65. Problem in a paper textbook about the virtual robot Aquarius.

of commands. To solve such a problem on paper, it is necessary to fill out a table of states, which, of course, takes much more time than in a screen environment (see Fig. 63).

Problems with the **Robic performer** are propaedeutics preparing for work with a Robot performer in middle school. Robic works on a grid of squares. It can perform four commands (up, down, right, and left) and automatically colors the square that has been passed through. In addition to simple linear programs, a repeat structure (cycle) is introduced into the course (see Fig. 64).

Perfect information game is one in which, after every move, all players know all previous game positions and all positions that can be obtained after the player’s current move. The course involves several two-player games with simple rules: stones, tic-tac-toe, sliders, and Sim. A game round is a string of game positions (see Fig. 65).

For simple games with a small number of possible positions, one can construct a complete game tree. It is a convenient tool for investigating all positions and constructing a winning strategy, if any. For example, Fig. 66 shows the complete tree of a stone game with

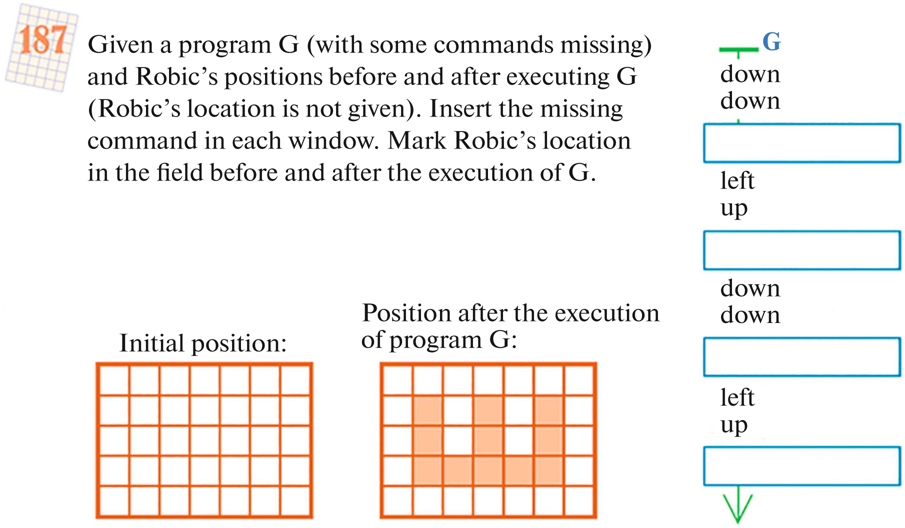


Fig. 66. Problem in a paper textbook about the Robic performer.

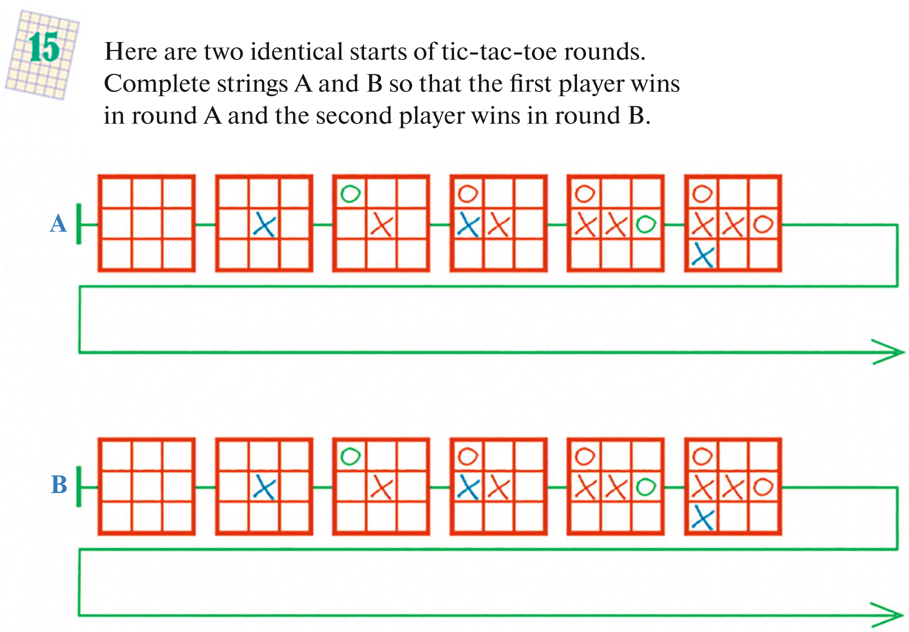


Fig. 67. Problem of constructing tic-tac-toe rounds with an indicated winner in each round.

seven stones in the initial position and with one, three, or four stones allowed to be taken every move. For each position, the number of remaining stones is indicated. The losing and winning positions (from the point of view of the player whose turn it is) are colored in blue and red, respectively (see Fig. 66).

Final positions (tree leaves) are always losing (the game is up, and the player whose turn it is has lost). Next, moving from the leaves to the root node, we color a position red if there at least one blue position among the ones succeeding it. As a result, we find that the initial position of the game is blue, so the first player has no winning strategy, while the second player can win if each of his or her moves leaves a losing position to the opponent.

16. CONCLUSIONS

Today, looking back at the 35-year experience in creating teaching materials, textbooks, programs, and standards and their use by hundreds of teachers and tens of thousands of students, we can say that the pro-

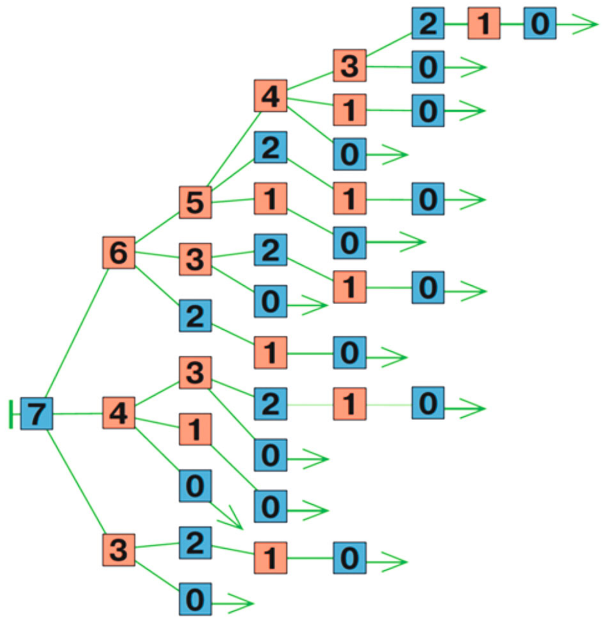


Fig. 68. Complete tree of a stone game with seven stones in the initial position and with 1, 3, or 4 stones allowed to be taken every move.

posed approach is sustainable and accessible to teachers who are ready to look anew at school mathematics, student's work, and the environment of this work. Among the courses we created, there were ones that successfully combined the main material discussed above with more traditional topics of school arithmetic. It seems that such a combination can also be productive, effectively helping the traditional component. It also seems obvious that this new mathematical framework is relevant for the development of computational thinking [31], [32] and a mathematical digital competency [33].

It seems that our approach is free from obvious shortcomings of New Math. The only essential obstacle to its further spread is intellectual inertia and natural resistance to everything new and "different."

Concluding this discussion, let us once again dwell on the obstacles that arise in the way of implementing our approach, both from the point of view of the system of basic objects and types of tasks, and from the point of view of methodology. We discussed the need to have tasks of varying degrees of difficulty, in particular task sequences where each increment of difficulty would be optimal for each student. Thus, the total number of tasks is increased compared to the option for one student. Another problem is related to the following. A traditional arithmetic example or text problem takes up little space on the page in a problem book or textbook. We manage to place our tasks on the page, usually no more than five, and sometimes one task takes more than a page. If both circumstances are taken into account—the need to have more tasks and use more space on the page for each—then we get an increase in the volume of the problem book several times compared to the traditional one. Naturally, this is reflected in the cost of publication, especially considering the use of color printing. Nevertheless, our paper manuals are used in dozens of public and private schools. But our experience shows that the cost of producing a textbook is a significant barrier. The way out of this situation is obvious today: this is a textbook (task book) on a digital medium—on the screen of a tablet.

We continue our work and consider it important to introduce it to teachers, parents, mathematicians and the wider community.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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