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MATHEMATICAL EDUCATION OF THE DIGITAL AGE

The Work of a Mathematician As a Prefiguring of Mastering Mathematics by Students: The Role of Experiments

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Abstract—This paper considers an approach to mathematical education adequate to the task of developing mathematics and its applications in the 21th century. This approach is based on improving the efficiency of the educational process by maintaining the motivation of students of various categories. The basis for the formation of motivation is, on the one hand, independent design; invention of mathematical objects, methods of action, and models of the world around us; and the discovery of facts of mathematical reality and, on the other hand, solving of new, unexpected, and feasible tasks for the student. The student's work is similar to the work of a mathematician—researcher and programmer. The possibilities of research work in educational mathematics are significantly expanded due to computer-based intramathematic experiments. Debugging a computer program is a special kind of mathematical experiment.

Keywords: mathematical education, mathematical experiment, experiment in theoretical mathematics, student motivation, unexpected tasks, invention and discovery in mathematics, visibility in mathematical education

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1. INTRODUCTION

The need for public high-quality mathematical education and the generation and preservation of interest in mathematics is growing all over the world. However, modern mass education does not meet this need. One of the reasons for this is that the education that a child receives in school is irrelevant to what he is interested in today and what he will need tomorrow.

Simultaneously, an idea of the system of mathematical education has already been developed, in which

- the subject content meets the needs of the digital economy and the entire digital world;
- the methods of work mastered by the student from the very beginning of learning are natural for him, correspond to natural curiosity, and, at the same time, are the methods of activity of a professional mathematician and programmer;
- the educational process is accessible and motivational for most students and easily establishes individual trajectories that correspond to personal goals; and
- the main educational outcomes outside of mathematics (metasubject, personal, etc.) are also key in the world of today and that of the future.

This article, in describing the general perspective of the approach under consideration, focuses on the role of a mathematical experiment in the work of a researcher and a student; an experiment is a necessary and, perhaps, central element of this approach.

2. PROBLEMS OF MASS MATHEMATICAL EDUCATION AND WAYS TO SOLVE THEM

The system of mathematical education at different levels was created in Russia at the beginning of the 20th century, became widespread in our country in the late 1930s. and was restored after the Great Patriotic War. This system was based on the need to prepare strong public-school students (the "upper quarter of the class") to continue their education in engineering universities. The results of the rest were evaluated "by

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subtraction." Individual schoolchildren with outstanding results had the opportunity to receive individual help from a good teacher; attend a preparatory group for a university, including a correspondence schools; attend a specialized mathematical school; and then enter the university's mathematical faculty.

Today, we are faced with a paradox: the role of mathematics in society is growing and civilization is becoming digital, while interest in school mathematics in society is declining. Indirect, but clear, evidence of this is the shortage of students (at least motivated, capable ones) in the IT specialties of many universities that has persisted for decades.

The basic reason for this, we believe, is that the literal content of school mathematics is not needed in the modern world, as all this can be done by computer. The fact that it is this content that is checked in the Unified State Examination (USE) only exacerbates the situation, but the USE itself is not the main factor. Mathematics appears to be an archaic school subject out of touch with daily life.

The key slogan of the ongoing transformation is the very simple and, for many of us, obvious statement of Paul Halmos that "the only way to learn mathematics is to do mathematics" ([1], p. 7).

We believe that we have the opportunity to stop and reverse the process of declining student interest in mathematics. We can make mathematics a truly mainstream school subject, so that its content will be needed in the digital world, and educational outcomes will go far beyond just mathematics. This is due, in particular, to the achievement of goals that are already constantly proclaimed for mathematical education, but are not implemented in a public school: the development of logical thinking ("Logic"), modeling of the real world ("Modeling"), and awareness of the beauty of mathematical objects and constructions ("Aesthetics").

Along with these goals, which are becoming realistically achievable, we emphasize the importance of another, also not new, goal. This most important goal is the creation of the ability to solve completely new, unforeseen, and unexpected tasks, readiness and interest in such a solution ("Novelty"). Asmolov proposed the term "preadaptation" for this personality trait [2, 3]. We will explain this in more detail, but let us start with the fact that such a quality is an obvious quality of a professional mathematician that is necessary in his professional work. Accordingly, it is (or should be) developed in the training of professional mathematicians. The most remarkable thing is that this quality is becoming more and more necessary for a modern person in any workplace and just in everyday life.

It is fundamental that, by bringing the mathematical activity of a schoolchild closer to the activity of a professional mathematician, the listed goals can become real goals of *public* mathematical education. Of course, this does not mean that the mass of schoolchildren will discover really new mathematical results, but the public schoolchildren will gain experience in discovering new results for themselves and will be able to use this experience in later life.

When speaking about results "outside of mathematics," we mean the general, clear understanding in the mathematical community that Khinchin called "the educational effect of mathematics lessons" [4], that Firsov called "teaching mathematics" [5–7], and what in today's official texts is called "metasubject and personal results."

The most significant, in our opinion, is the possibility and necessity of switching the goals of public-school mathematics (real ones, not proclaimed ones) from *memorization* of "close to the text" algorithms, rigidly formalized heuristic (often mnemonic) rules, formulations of theorems, definitions, and proofs to a completely different system of studying mathematics. This *other* the system entails:

- independent invention and discovery, instead of memorization; experimentation, trial and error, and "debugging" (in a broader sense than debugging a program), including using a computer; and finding and exploring suitable visualization as a basis for observation and intuition;
- using error as a source of advice from the teacher and the progress of the student, and not as an indisputable basis for punishment with a bad grade;
- constantly solving nonstandard tasks that are new for the student (and in many cases for the teacher) instead of working out error-freeness and speed in solving standard problems;
- the joint search for a solution by a teacher and a student, including solutions unknown to the teacher; the teacher as a master of mathematical search, discovery, experiment, and use of error;
- solving problems similar to those already solved is by no means prohibited if it contributes to better understanding, helps in solving new creative problems, and at the same time promotes motivation: "That's what I can do quickly and accurately"; however, such a decision cannot be based on coercion or be the main element in assessing the student's results; and
- for calculations, including arithmetic and algebraic ones, solving equations, etc., which are now included in mathematics programs, the use of digital technologies is allowed: a calculator or computer algebra systems, just as happens outside the school walls.

One can express doubts about the fundamental feasibility of the described system, including the achievability of the "Novelty" goal for the majority of students. However, we possess serious arguments in favor of such achievability. This system is based on productive traditions that are intellectually much more powerful than the Soviet school system and the schools of a number of other countries that were created for the purposes of industrialization (the "industrial revolution"). This has become a stable practice of mathematical education in a number of Russian schools in recent decades. These traditions and practices include the following.

• Traditions of entertaining problems (beginning from antiquity, Alcuin's problem book, continued by Ignatiev, Perelman, et al. [8–12]).

• The International Kangaroo Olympiad, covering dozens of countries [13], was largely supported in Russia by the St. Petersburg professional mathematician Bashmakov [14], despite the negative attitude towards it of a number of influential, including administrative, educators. This Olympiad constantly attracts tens of percent of all primary-school students in the country and meets the enthusiasm of a large number of primary-school teachers. The tasks of this Olympiad are varied and completely different from the problems in elementary-school textbooks, while most students of each class successfully solve several problems; even the most difficult and, we repeat, *unexpected* tasks are accessible to many people.

• The system of circles and mathematical classes in different communities, dating back to Russian university-circle education. One of the most significant and stable is that of Konstantinov, the origins of which can be found in the Luzin group in Moscow [15].

• In parallel with the Russian tradition of modern mathematical education, traditions were also formed in other countries. In the United States, this tradition is usually associated with the name of Robert Moore, who began teaching undergraduate students at the University of Pennsylvania in 1911, which was based on what was later called "inquiry-based learning," an exploratory approach to mathematics. Moore's favorite principle was a saying attributed to Confucius: "I will hear and forget, I will see and remember, I will do and understand." In this text, we turn to one of the most prominent representatives of this tradition, Paul Halmos. This approach was limited to higher education (the journal [16] devoted to this approach is called the *Journal of Inquiry-Based Learning in Mathematics*); the contingent of students to whom this approach was addressed, in terms of mathematical level, is apparently closest to that of our high-school students from public mathematical and IT-oriented schools.

• The tasks for constructing and debugging algorithms (programming) included in textbooks on the subject "Computer Science" (formally "Fundamentals of Informatics and Computer Science" or "Computer Science and ICT"), introduced in all schools of the country in 1985–1986, have a high level of diversity and individual novelty. This line continued in the following decades and witnesses no large-scale rejection. Education in algorithmics and creative programming is developing in the world at different levels of education [17–22]; in the last decade, it has been booming under the somewhat strange name "coding."

• An innovative computer-science course for elementary schools, integrated with mathematics or studied separately, has been successfully studied in various variants [23–25] by tens of thousands of students in a number of elementary schools in the Russian Federation for decades, which convinces us that it is possible to achieve results in working with each student. The course provides for the possibility of optimal selection of tasks for each student with a combination of novelty effects with the effect of "reliability and confidence."

The spread of the traditions of training professional mathematicians in the best universities and working with highly motivated children in specialized mathematical schools and mathematical circles to public mathematical education is dictated by the needs of digital civilization and becomes possible thanks to the digital technologies of the modern world.

The following characteristics are essential for our construction of the content of mathematical education.

1. Content relevant to the digital age:

• in the professional and private life of a person, the need for technical computing skills has disappeared, as computers help, and routine calculations and other standard elements of educational work are also allowed to be performed using a computer at school;

• at the school level, the general foundations of modern mathematics and computer science are systematically visualized, significantly expanding traditional arithmetic with its four actions; and

• the content of the mathematics course is integrated with a computer-science course and is the basis for understanding "how" digital technologies and artificial intelligence work.

2. Mathematical experiment and discovery is an essential part of learning activity. Time for this is freed up as a result of reducing the amount of training in manual calculations.

3. A high level of novelty, "nonstandard" tasks is provided that is individually selected for each student at the optimal level of complexity.

One of the main (perhaps the most important) of the projected and actually achieved effects is the growth of interest in mathematics among children.

A computer is explicitly mentioned in characteristic 1; at the same time, it is necessary in most of the experiments of schoolchildren (characteristic 2) and ensures the reality and effectiveness of the personalization of mass education (characteristic 3): the number of various tasks that can be stored in a digital environment is fundamentally greater than in a paper problem book; individual goals, degrees, and ways to achieve them in the digital environment are built naturally and comfortably for the student and teacher.

We summarize the results of the changes for the public-school student:

• *acquisitions:* development of mathematical and general intellectual abilities; interest in mathematics and learning; the ability to apply digital technologies in solving a wide class of problems; mastering the elements of modern mathematics; and experience in independent discovery and proof of mathematical statements, in particular, (a small number of) geometrical ones and

• *losses:* the ability to perform fluent and reliable arithmetic and algebraic calculations without a computer, which was useful 50 years ago, and knowledge close to the text of some formulations and proofs of geometrical theorems in a volume close to that that would have existed 100 years ago.

It is the use of digital technologies, in particular, computer experiments, that makes it possible to make serious mathematical education public.

At the end of this section, let us recall that the symbol of mathematical education in our country has become N.P. Bogdanov-Belsky's (1868–1945) 1895 painting *Verbal Counting. In the Folk School of S.A. Rachinsky* [26]. Note that in this painting, students are not "asked" to mentally multiply two five-digit numbers at speed or to solve a problem about diggers. On the contrary: each student is offered to try to solve an unexpected problem, which is clearly not similar to what he had seen before, and at the same time at an individual speed in an individual discussion with the teacher (forcibly, not in a digital environment).

3. THE GENERAL MATHEMATICAL PERSPECTIVE

The last decades of the 19th century and the first decades of the 20th century were a period of *metamathematical* understanding and modeling of human mathematical activity. The works of Frege, Cantor, Hilbert, Gödel, Turing, and Tarski offered a mathematical description of what the language of mathematics is and what mathematical proof, mathematical definition, and mathematical calculation are. The ideas that were developed had a unambivalent influence on mathematical education. The natural inclination of mathematicians to apply this influence in the practice of the public-education system led to a number of large-scale dramas (New Math, "Kolmogorov's reform"). Among the reasons for the failures in these realizations were moving away from the world of the child, rather than approaching it; ignoring tradition; weakness in working with the teachers of today and those of tomorrow. It is important for us that, instead of activities with real, tangible (in one sense or another) objects, children were presented with work with abstract definitions, which, unfortunately, often degenerated into memorization. Today, these reasons are supplemented by a lack of connection with digital reality. However, the analysis of these causes is not the subject of this work. Possible problems with the implementation of our approach and ways to solve these problems are discussed will be discussed in a special section.

One hundred years after the advent of *meta*mathematics, the final decades of the 20th century and the first decades of the 21st became the period when the indicated mathematical understanding spread beyond mathematics: mathematical models of human language, thinking, and activity in various spheres of life already beyond mathematics arose. It is no less significant that these models were implemented in the form of computers ("hardware") and microprocessors ("chips"), as well as software ("codes"), the individually designed integral elements and complexes of which were several orders of magnitude larger than any mathematical or literary works created by man before. The same can be said about processors and chips—computer hardware—in comparison with mechanical devices. This software and hardware today control the physical processes that take place in the world around us: transport, energy, manufacturing, medicine, trade, social processes, etc.

The 21st century has been marked by accelerating changes in our ideas about the human person, in particular, about what it means that a person—including a student at school—knows and is able to do something. As Plato once noted, followed by Lev Vygotsky, the emergence of writing led to the expansion of a person's personality: a person's memory expanded due to writing (from Homer and Socrates, who remembered their works in the cells of their brains, to Tolstoy and Kant, who remembered them on paper), as did computational abilities due to calculations on accounts or paper [27, 28]. Today, a person remembers what he needs (for example, the phones of friends) in a piece of his extended personality—a mobile phone—and this same mobile phone instantly connects a person with the memory of all humanity on the Internet.

Along with models of rationality in the human psyche, models of intuition have emerged, for example, pattern recognition based on machine learning.

4. EXPERIMENTS IN MATHEMATICS FROM AN IMAGINARY EXPERIMENT TO A REAL ONE. EXPERIMENTAL MATHEMATICS AND THE COMPUTER

In his paper "On Teaching Mathematics" [29], written on the basis speaking at a discussion on the teaching of mathematics at the Palais de Découverte in Paris on March 7, 1997, Vladimir Igorevich Arnold says, "Mathematics is part of physics. Physics is an experimental, natural science, a part of natural science. Mathematics is that part of physics where experiments are cheap." The deliberately paradoxical nature of this statement by the great mathematician emphasizes for us the role of the mathematical experiment.

Another great mathematician of the last century, Halmos, whom have already quoted, said,

> "Mathematics is not a deductive science, as it is presented in the common cliché. When you are trying to prove a theorem, you do not just write down hypotheses and then start reasoning. What you are doing is trial and error, experimentation, guesswork. You want to find out what the facts are, and what you are doing is like the work of an experimenter or a laboratory assistant ... The joy of suddenly discovering a hitherto unknown truth ... accompanied by a flash of enlightenment, an almost incredible improvement in vision, ecstasy and euphoria of liberation and release" [30].

In previous centuries, this experiment could have taken place in the brain of a mathematician or on paper. The history of one of the most important trends in mathematics began when Euclid proved the infinity of the set of primes by setting up a thought experiment that consisted in their finiteness and led to the conclusion that there are still prime numbers beyond this finiteness. Carl Friedrich Gauss went to a university (Charles College) in Braunschweig at the age of 15 and became interested in the question of how many primes are contained in the initial segments of the natural series and based on experiments, now on paper, he suggested that $\pi(x)$ is the number of primes that do not exceed the number x, asymptotically comes to $\pi(x) \approx \frac{x}{\ln x}$. It is true that, like the other findings of his student days, he did not make this observation public for a long time. The proof of this fact was obtained only 100 years after its experimental discovery by Gauss. More accurate hypotheses about the behavior of this function are equivalent to one of the main problems of modern mathematics—the Riemann Hypothesis—which has been confirmed by numerous thought experiments regarding the consequences derived from it. Today, Yurii Matiyasevich is trying to conduct an experimental study of the behavior of series related to the Riemann function, already with the help of modern computers [31]. The largest prime numbers today exist in the framework of a massive computer experiment (see below).

The concept of a thought experiment (Gedankenexperiment) was introduced into scientific research by Albert Einstein. An example quote is that "I was sitting on a chair in my patent office in Bern. Suddenly the thought dawned on me: if a person fell freely, he would not feel his own weight. I was stunned: a simple thought experiment made a deep impression on me. This led me to the theory of gravity" [32, 33].

Today, in the expanded personality of the researcher and the student as a researcher, the thought experiment is easily transferred to the computer screen. This consideration is key to us.

Models of mental activity realized outside the brain began to return to mathematics, helping mathematicians to set up experiments, observe mathematical reality, perform numerical and symbolic calculations, and enumerate options. One of the famous early examples is Appel and Haken's solution to the four-color problem, which was put into a format that mathematicians can trust by Georges Gontier [34].

While observing the emerging computing practice, such new phenomena as the Appel and Haken precedent and a number of others, Michael Atiyah published a wonderful paper back in 1984 called "Mathematics and the Computer Revolution" [35], which was published in Russian 32 years later in *Izvestiya AN* [36]. (We do not believe that it took the Russian mathematical community these 32 years to realize the importance of the topic.) Today, this paper sounds more than modern, and we will continually return to it.

In describing the perspective associated with the "experiments and computers in mathematics," M. Atya writes, "In mathematics, as in the natural sciences, discovery consists of several stages, and formal proof is only the last stage of it. The very first stage is to identify the essential facts, arrange them into meaningful structures, and extract some kind of plausible law or formula. Next comes the turn of checking this proposed formula for compliance with new experimental facts, and only then is the question of proof considered" ([35], p. 10).

Of course, what is here called a "plausible law or formula" may turn out to be a ratio of segments or numbers in a problem, or an element of strategy in a game, etc.

It is essential that the scope of experimentation is fundamentally expanded by means of a the computer. Atyah writes,

> "At each of the early stages, computers can play some role, particularly when large or complex systems are being considered. For example, interesting questions in number theory may involve very large prime numbers, and some of the deepest hypotheses currently being studied have been based on extensive computer calculations. Similarly, the problems of the theory of differential equations, which involve the evolution over a very long time of some systems (for example, fluid flow), have been very strongly influenced by experimental facts found on computers.
>
> ... the computer turns out to be practically very useful for mathematicians at all stages of their work, but, perhaps, primarily at the stage of research or experiment. The great mathematicians of the past, such as Euler or Gauss, did a

lot of tedious manual calculations in order to provide themselves with primary material, 'raw materials' from which they could guess some general law or discover some wonderful example. As mathematical research gets deeper and we get more ambitious, the raw material becomes correspondingly more messy and complex. The computer can help us analyze this material and point the way to further progress and understanding" ([35], p. 10).

Further developments confirm Atiyah's observations and predictions. Moreover, in the case of the four-color hypothesis and in a number of other cases, it turns out to be possible to construct an "exhaustive" mathematical experiment that "closes" an important problem. An example from number theory is, in particular, Goldbach's ternary problem on the possibility of representing any odd number, starting from 7, as the sum of three primes. Ivan Matveevich Vinogradov in 1937 proved this possibility for all sufficiently large odd numbers. However, the final solution to the problem—for all odd numbers—was obtained only in 2013 by Harald Helfgott using modern computers [37].

From a purely mathematical, applied, technological, social, and educational perspective, interest is presented by the "folk" search for Mersenne primes—the largest known primes, that is, primes of the form $M_p = 2^p - 1$, where p is a prime. Until 1914, 12 Mersenne numbers were found. The largest of them contained 39 digits. Further progress had to wait for the advent of computers: in 1952–2018, the next 39 numbers were found. Since 1996, they have been found by "ordinary people" conducting an experiment on many thousands of computers as part of the "Great Internet Mersenne Prime Search" project [38]. The largest number was found by the programmer Patrick Laroche on a regular (in his case, at a local church) personal computer for $p = 82589933$; in this number, there are 24862048 decimal places [39]. The experiment here is to test for simplicity: if it succeeds, we obtain proof simplicity or complexity [40].

Many mathematicians are prejudiced against proofs that use a computer. They rejoice when a computer proof is followed by "real," "manual," or "paper" evidence that a person can check. Sometimes (but not often), it is. The statement that McHale attributes to Halmos, "computers are important, but not in mathematics," are characteristic [41]. Of course, this was said (if it was) more than 20 years ago, but it was said by a mathematician who fully shares the general approach of this article (see the quote above).

Another point of view is also possible: that a proof constructed and/or verified using a computer deserves more confidence. Vladimir Voevodsky came to the program of using a computer to create mathematics from this very end. Having discovered and corrected errors in his proofs of results that were important for other mathematicians, Voevodsky decided that, in some cases, automating the proof was the only way to guarantee the correctness of complex proofs [42]. Moreover, he began the construction of mathematics on new (so-called "univalent") foundations, supported by a number of other mathematicians, to some extent revising, to some extent using the ideas of the foundation of mathematics at the beginning of the 20th century [43]. In this regard, it is worth mentioning one of the most complex and undoubtedly important achievements of mathematics in the 20th century—classification of finite simple groups. Here, the computer is already considered as a tool for increasing the reliability and availability of evidence, for example, for the Coq proof of the Feit–Thomson theorem in the work of Georges Gontier [44].

Note that an interesting effect has also arisen in the field of computer simulation of human intuitive activity. Machine-learning specialists have recently tried to treat datasets of mathematical experiments and datasets of mathematical proofs and other texts as raw material for machine learning. It is argued that, in this case, the machine finds patterns that are significant for a person, offers correct texts for solving problems, etc. [45, 46].

Computer identification of the coincidence of the values of two differently specified numerical constants calculated with high accuracy leads to the hypothesis that this coincidence is not accidental, but the exact values of the constants are equal [46]. As a final example, we point to the experimental discovery in 1995 by the Bailey–Borwein–Pluff formulas to calculate binary expansion digit π according to its numbering [47].

5. VISUALIZATION IN A MATHEMATICAL EXPERIMENT AND MATHEMATICAL PROOF

Another quote from Atiyah is that "One of the advantages of current computers, which mathematicians are just beginning to appreciate, is their ability to display information graphically (and even in color). For many complex mathematical problems involving geometrical properties, this provides a new, extremely effective tool for studying phenomena" ([35], p. 10).

Atiyah's last consideration can be attributed, in particular, to the language in which we formulate mathematical statements. The greatest example of the construction of a mathematical theory in the history of mankind was ancient Greek geometry, which is reflected in the *Elements* of Euclid. This theory consisted in a combination of precise reasoning with visual representations. As we now understand, visualization played an essential role, which made it possible to use and not prove certain "obvious" premises that were not formulated explicitly in proofs. The Cartesian algebraization of geometry—analytic geometry—in a certain sense completed the question of what is a true geometric statement. Thereafter, however, it turned out that matters were not so simple. School constructions of geometry done in the style of Euclid still suffer from gaps.

Returning to the language of formulating mathematical statements, today we see the following possibility. A mathematical statement is formulated in the form of a picture, for example, in the form of a partition of a part of the plane into polygons [48]. The polygons of the picture correspond to some partition of the mathematical plane, each of which is given by a system of inequalities. The picture captures a huge number of statements about abstract objects: certain polygons do not intersect, they border each other, etc. Each of these statements can be verified by a computer. The assertion that the mathematical reality is exactly as it appears in the picture receives an exact computer proof. A computer proof can also receive a statement expressed by a picture. Finally, the assertion that computer constructions correspond to mathematical reality can also be proved, perhaps, but not necessarily, with some help from a computer. The picture becomes a no less accurate representation of an abstract mathematical statement than a formula or text.

A simpler example is the representation of a representation of a finite number of finite graphs justified by a computer calculation, and the assertion that these graphs exhaust all possibilities for the realization of certain conditions (cf. [49]).

For the possibilities of formulating mathematical theorems in natural language, see [50].

6. DEBUGGING AS A MATH EXPERIMENT

Let us pay attention to the fact that a huge array of mathematical activities are going on in the field of IT. As a rule, designers in this field deal with mathematical objects and mathematical methods of working with these objects. At the same time, they often have to deal with mathematically new, unexpected situations. As soon as the situation becomes repetitive, an appropriate software tool is invented that replaces a person's repetitive, routine actions.

In the same area of IT, a special kind of computer mathematical experiment has been created—debugging. In the process of debugging, the constructed mathematical object is experimentally compared with some condition and requirement. The identified discrepancy leads to a change in the object and the previously constructed formal or intuitive "proof" of the "correct" operation of the object. Sometimes this also leads to a change in the formal requirement.

Briefly touching on the problem area "computer in mathematical proofs," we deliberately did not mention the obvious: a computer is used in the implementation of mathematical algorithms in numerical modeling of objects and processes, accounting calculations, writing texts, image processing, etc. The list is huge; people use digital technologies in almost every field of activity.

7. ROLE IN THE FORMATION OF MATHEMATICAL INVENTION AND DISCOVERY THROUGH EXPERIMENT

So, as Jonathan Borwein stated [51, 52], "... the power of modern computers, combined with modern mathematical software and powerful mathematical methods, is changing the way we approach mathematical activity."

We believe that even more significance is presented by the regular change in our understanding of the mathematical activity of all people who study mathematics, starting from the very first stages of such study in elementary school and, perhaps, even earlier.

In this section, we will try to explain how the perspective of the professional activity of a modern mathematician helps to solve the aforementioned problems of mathematical education, focusing on one key aspect in this perspective—the role of an experiment. Let us repeat after Halmos and Atiyah that the mathematical experiment is a key element of mathematical activity.

It is generally accepted that a *story* told to children about mathematical experiments and inventions can be useful: it inspires and motivates them. We believe that this story will motivate them even more if the invention is made by *them*! It is possible, but not necessary, that they will repeat the path of some great mathematician of antiquity. It is possible, and very likely, that the solution of some new task or system of tasks will lead the student to universal (for him) discoveries and inventions, which form the understanding of "big ideas." The optimal situation is when the task is initially comprehended and interesting to the student, and this interest increases in the course of his search for a way to solve, set up, and conduct an experiment and unexpected discoveries.

A few examples related mainly to elementary school are as follows.

• Even before school, children know (albeit perhaps tentatively) the names of numbers up to 9 and how to write them in numbers. The task that the teacher sets for the students is to invent a way to write down larger numbers and name large numbers (quantities). The process in which students can be assisted by a teacher leads to their invention of the decimal number system. The experiment here is to look at different collections of the same items, group them together, and try to invent a way to write down the answer to the question "How many items are there?"; i.e., by counting them. One will need many identical objects, such as matches or beans, and some way of physically grouping them together, such as tying them with an elastic band or folding them into a separate container. In the process of work, for naming groups of objects, of course, one needs the appropriate words, with these words being provided (or recalled) by the teacher, without making a secret of the fact that children are now inventing what humanity invented thousands of years ago—“ten,” “hundred,” and so on. This grouping is then transferred to the grouping of objects drawn on paper (for example, small beads), with the binding here being replaced by a line. Objects on a sheet can be placed exactly in rows or randomly. The result of the grouping is reflected in the table, where there are columns for “units,” “tens,” “hundreds,” etc. One can see how much less space the record of the quantity takes up than the page with drawn beads itself. Now, one can erase the names of the columns of the table and invite the children to find out what number is written down by laying out the required number of matches on the table: units or connected tens and hundreds. One can discuss what to do if columns are added to the table and numbers are written in them, but the names of the columns are not set. This is how the discovery/invention of the positional number system occurs.

• Students create addition and multiplication tables by counting areas (the number of unit squares) in lines and rectangles. The experiment consists in drawing various rectangles in cells or on a grid, counting the number of unit squares in them, and writing the result in the correct cell of the table. Important effects arise in the interaction of two students who have obtained different products for the same pair of numbers, as well as those who have obtained the same products for two different pairs.

• Experience with the area of a rectangle leads to important ideas about names (notations). The symbol × has a fixed “multiply” value, and the L and W in the expression L × W can have different values for the length and width of the rectangle. The gradual invention of a correspondence between name and value begins. There are names the meaning of which we try to determine and always consider the same, such as the symbols of addition and multiplication, and names the meanings of which can change, for example, “length and width.” A general formula is invented for the area of the sum of two rectangles with the same width—this being humanity’s most important mathematical discovery, algebra—and it is great if the student makes this discovery on his own.

• Students invent algorithms for addition and multiplication with detailed (sometimes, graphic) recording on paper; a method is invented and used for writing two-digit numbers in one cell, divided by a diagonal from the upper right corner to the lower left: tens are written above the diagonal, and units are written under the diagonal. The multiplication algorithm according to the Indian method brought to Europe by Leonardo of Pisa (Fibonacci) is invented: two-digit products of single-digit numbers are written in cells with a diagonal, a method is invented for organizing the record of multiplying a single-digit number by an integer number of tens, etc. [53].

• Students invent a way to find the area of an arbitrary polygon with vertices at integer points and discover the additivity property of the area.

• Students invent ways to organize an exhaustive search to find an object that satisfies a condition, for example, searching for the right one among the objects on the page, or finding one of the solutions to an equation.

• Students invent general formulas for solving linear and quadratic equations that work for any values of the names (coefficients) included in them.

• Students develop winning strategies in games with pebbles, discover the general concept of strategy, and define it. They open a method for inductively proving the correctness of the program and strategy.

• Students invent rational numbers by experimenting with areas.

• Students invent an algorithm for finding a common measure of segments (Euclid’s algorithm), finding the greatest common divisor of two numbers and discovering a geometric situation where the algorithm works indefinitely - with decreasing similar figures, thereby discovering irrational numbers and, possibly, their expansion into continued fractions.

• Students invent a way to decompose a number into prime factors, discover the main theorem of arithmetic.

• And so on.

To show the range in which a school mathematical experiment can unfold, we will give one example of a purely mathematical problem, the search for a solution to which can be programmed by the students themselves by enumeration. In 1953, Mordell proposed [54–57] to find solutions to the equation:

$$x^3 + y^3 + z^3 = 3 \text{ in integers except}$$
$$1^3 + 1^3 + 1^3 = 3,\ 4^3 + 4^3 - 5^3 = 3.$$

The next largest set of numbers yield the equality

$$569\,936\,821\,221\,962\,380\,720^3$$
$$-\ 569\,936\,821\,113\,563\,493\,509^3$$
$$-\ 472\,715\,493\,453\,327\,032^3 = 3.$$

Above, we have already mentioned the search for large prime numbers, such a search is also available to a school student.

In the traditional school, there is no stage of invention, but considerable time is allotted for the student to learn the algorithm, sometimes even in some of its verbal formulation, and then train to quickly and accurately apply it. It takes more (sometimes much more) time for the students and the teacher to invent something than for the teacher to say something at the blackboard and the students write it down in a notebook. However:

• To write down is not to understand.

• Repeatedly mechanically applying what has been learned does not mean understanding; such application may act against understanding.

• The concept can be transferred to a computer that will do it instead of a person. We have already mentioned this general situation above as a model of the activity of a professional in the field of programming and, in general, in the field of IT.

8. PHYSICALITY AND VISUALIZATION IN A SCHOOL COMPUTER MATHEMATICAL EXPERIMENT

One mathematician who clearly realized the enormous power of a computer as a device for mathematical experiment and discovery in the hands of a child was Simor Papert [58, 59]. The most important childhood impression for him, in precomputer reality, was an active bodily acquaintance with gears, a differential in an old auto repair shop. Papert came up with the idea of visualization and materialization of the numerical mathematical world in the digital age [60]. Taking a version of Lisp designed by his friends as a learning environment for children, he proposed to attach a robot to the computer, first a turtle on the floor, and then on the screen [61]. Numerical entities—a given distance to move, a given angle of rotation—were represented (materialized) as actions of moving and turning the turtle.

Mathematical education, according to Papert, begins with programming [62]. Let us explain why programming at school can be an essential element in the development of mathematical thinking.

• The tasks of creating programs with an expected result can be more varied and meaningful than solving an equation or a word problem ("resulting in a quadratic equation") in most cases.

• The work of the program and its result can obtain a visual, meaningful representation in the form of an image, action in the real world, text, melody, or animation. Numerical results can also be graphically presented visually.

• The process of creating (inventing) an algorithm, transferring it to a computer, and executing this algorithm by the computer with various initial data, as well as the ability for the student to independently detect and eliminate computational errors, creates a positive emotional context, as well as debugging a program, leading to the desired result and, sometimes, to a new, unexpected, and interesting one.

• The construction and proof of the correctness of the program forms an area parallel to the geometric construction and proof. In this area, important general concepts and constructions appear (have been invented): invariants, induction, division of a problem into subtasks, analysis of cases, etc. The strategies of reasoning and action developed at the same time are transferred beyond the limits of programming and mathematics.

• Debugging, including stepping, is a rich experimental environment that enhances natural motivation. In the 21st century, debugging, correcting one's actions as a result of receiving feedback, and self-criticism are becoming much more relevant personality traits than memorization and deterministic execution of a given order or a given algorithm.

The Logo environment, which has been used for decades in the mathematical education of children in dozens of countries around the world, is today complemented by Scratch, developed in the same constructionist circle and in the same constructionist philosophy of Papert as Logo [63]. The programming of LEGO devices (by Seymour Papert, LEGO Professor at the Massachusetts Institute of Technology) were supplemented by the Arduino electronic designer [64].

Another powerful "microworld" for the development of mathematical thinking is a "Robot in a Maze." The environment for the existence of the Robot in the classical version is a rectangle on checkered paper bounded by a wall, inside which there are also walls. The size of the rectangle and the location of the walls are a priori unknown. The Robot executes commands to move one cell in one of four directions; in addition, it determines when it has hit a wall. The task is to write a program that will ensure the movement of the Robot, for example, from the upper left corner to the lower right margin. A large spectrum of concretizations arises from one general statement. For example, one can consider restrictions on the location of walls. For example, inside a rectangle, there can be exactly two walls, with both of them going north to south, etc. The cells of the labyrinth can be prepainted, and the Robot can determine their shading and can repaint them, etc. The nontriviality of the task lies in planning in advance the moves of the Robot that achieves the desired result in the specified class of mazes [65].

The effective implementation of the function of developing mathematical thinking can be facilitated by reducing the "purely linguistic" complexity (vocabulary and syntax) of the programming system due to

• allocation of the minimum core of algorithmic constructions (in the format of structured programming operators or flowcharts), fixing this core even in primary classes;

• the use of native language or pictograms (see, for example, Pervologo, a programming language without words and, at first, even numbers [66] and PictoMir [67];

• a block structure editor that allows one to construct programs with arbitrary functions and reduce the possibility of syntactic errors (similar to a spell checker when editing texts).

We indicate another basic example of the effective use of digital technology to form a mathematical representation.

• The student moves in a straight line in the classroom (for example, along the blackboard). At the end of his path, a sensor (ultrasonic, infrared) is installed that measures the distance to the student's body.

• A graph of movement on the screen is shown to the whole class: positions (coordinates on a straight line), a graph of speed, acceleration, and distance traveled can be added to it.

• The student may have a tablet in his hands, which also displays graphs.

For more than 40 years, the research module based on the described experimental environment has been an extremely effective way for students to understand the motion schedule and other neighboring concepts of physics and mathematics. The task for the student may be to match the given schedule as much as possible or, after completing the movement and not seeing the screen during the movement, drawing a schedule, etc.

One environment that, in recent decades, has significantly influenced the study of geometry in many schools around the world, including in Russia, is *dynamic geometry*. Exact constructions are possible in it, using straight lines and circles, equality of segments, parallelism, etc. These constructions are displayed on the screen. Of course, the accuracy of construction on the screen is limited, but the computer can use internal symbolic representations, using calculations with radicals, and "rounding correctly" on the screen, "understanding" what the student wanted to construct. On the screen, one can specify (with the cursor, that is, with one's hand) a point on a segment or circle and give it a name. In common school implementations of dynamic geometry, such as GeoGebra [68], Live Mathematics (the Russian version of Geometer's Sketchpad) [69, 70], and 1C Mathematical Constructor [71], as well as already in the earliest one, along with Geometer's Sketchpad, Cabri Geometry [72], the key idea of dynamic geometry was implemented: the student changes the configuration on the screen (it "takes" a point and moves it), the computer changes the drawing, keeping the necessary ratios (for example, an inscribed triangle remains inscribed, although the radius of the circle, angles, etc., change). The student sees that some properties are preserved, can formulate a hypothesis that these properties will always be true, and try to prove his hypothesis.

Note that, in many of the above examples of data entry for an experiment, we are dealing with a whole series of experiments, parameterized by a numerical parameter (vector). In particular, the following methods of setting and generating data for the experiment are possible.

• Generation of a numerical parameter (including a vector of numbers) as a reflection of the position or movement of a hand moving a mouse, a finger on a tactile screen, an object or a student's body in relation to an ultrasonic (or other remote, for example, infrared, radio) sensor, or a head wearing virtual glasses. In the future, a more complex response to the position of the hands, the body, the direction of the gaze, the electrical activity of the brain, the reaction to the speed of movement, etc., is possible. In most cases, the use of feedback is fundamental: the student sees the result of the movement in the form of the movement of an object (cursor, point, slider); changes in the situation, for example, transformations of a geometric figure; and changes in the "position and point of view of the observer" and, in fact, the result of the experiment: in the form of numerical parameters, as a rule, reflected in the new configuration, animation, etc.

• Automatic generation of a random numerical parameter as an analogue of spatial input, with visualization similar to the previous case.

• The student's choice of a combinatorial, discrete object, for example, an initial state in the game Life (or in another cellular automaton), a move in a game with a discrete set of states: a game of pebbles or a card game.

• Random selection of a combinatorial object by a computer.

• Organization of enumeration by the student in a suitable (including visualized) environment.

9. COMPUTER PROCESSING OF RESULTS OF AN ACTUAL EXPERIMENT

The use of a computer to process data from a real physical experiment is a separate topic. We started with this use in the previous section. Further examples are obvious: almost all school physics experiments become much more efficient, "operational," if one uses the digitization of the quantities with which this experiment works. There are also new features, for example, digitizing the position and speed of a point in a video recording. In any case, data visualization helps to hypothesize about a mathematical pattern that links physical quantities.

Computer simulation of a real process, obtaining and checking predictions, is a related topic. In this case, the model can be built by the student, for example, in the form of a system of equations, not necessarily "school" ones. A computer can find an explicit solution to the system or "calculate" it given some initial data. The program for this calculation can be written by the student.

When speaking about the transfer of mathematical activity to the school context, as in other cases, we must "scale" the situation. At both the school and university levels, we can note a number of key points when experimentation is important both *for understanding* and for finding a nontrivial proof. Some examples can be found in Shabat [73, 74]. It is important that understanding is not necessarily accompanied by a strict definition and the same proof. An obvious and important example of this is represented by the concepts of mathematical analysis and probability theory in school. It would be superfluous to note the difficulty and not obvious usefulness of the formalization of these concepts in school. One can construct derivatives and antiderivatives "by eye," determine the area under the curve, carry out an experiment with throwing a die, etc.

10. MAIN PROBLEMS IN THE IMPLEMENTATION OF THE PROPOSED APPROACH AND WAYS TO SOLVE THEM

Teacher inertia is a natural obstacle to any change in school. Such inertia, which is characteristic of a person in general, has always been a part of professionalism in the case of a school employee: the task of the school is to transmit the knowledge accumulated by humanity in the past so that the graduate can use it tomorrow. Today, however, the task of education should be different: to prepare a person for an unpredictable tomorrow, to acquire the necessary knowledge himself and learn how to apply it. If the education system is not reconstructed, it will become less and less needed, and people, including children, will switch to education outside of school. The prediction of Illich in *Deschooling Society* [75] will pass from a warning and a constructive metaphor to something obvious and necessary.

Changing things, on the other hand, begins with a change in the role of the teacher: from an authority who knows the answers to all questions to a master of teaching who really does not know these answers, but searches with the children and, at the same time, teaches them to learn, to look for these answers. Such a change in the role of the teacher obviously also provides an (incomplete, of course) solution to the problem of the teacher's work with the constantly updated content of education. as he is not at all obliged to know everything in advance.

It is most natural to try to change the attitude of the public-school teacher by starting work with teachers of pedagogical universities, and their position should be "learning throughout life," including mastering the constantly updated digital technologies in their field of knowledge. In our case, we are talking about mathematics, but the possession of a microphone and a video camera and searching the Internet should obviously characterize a professor.

Another significant group consists of teachers of engineering, economics, and similar universities. We hear from them the fair statement that a significant proportion of the students entering them do not know the "formula for the sine of the sum" and other things that they themselves knew when they entered the institute. The proposed changes are unlikely to improve matters. Applicants will be even worse than today at solving equations and inequalities. However, graduates will be able to prove the simplest formulas on their own, understand their meaning, and remember the experiment in which they themselves reached this formula. And that is all—they will do it more successfully than today's students, and at the same time, they will ask university teachers why they are required to find the 20th integral in parts, although they understood the idea, and the computer finds all these integrals perfectly by itself.

Naturally, the question arises about the goals and content of university mathematical education for different areas of training and the role of digital technologies in this training. This issue requires serious professional discussion.

Parents are the next hurdle. The most successful and most influential parents, although few in number, turn out to be the greatest adherents of the old school: and the role of the school in obtaining a good education for them could be positive and significant. The ides of the "learning teacher" may be rejected by parents, and the school will have to defend it as a pedagogical device.

A certain role in shaping the position of parents can be played by the dialogue that the school will have with them, as well as the demonstration by the school and the student of his success and interest in learning. This can be facilitated by the implementation of the concept of effective education, where the goal of parents is not "an excellent student in all subjects" and "a gold medalist," but a young person who achieves real goals constructed by him together with his parents and the school [15]. These goals may include the result of the final assessment and the possibility of continuing education, joint prediction of this result and opportunities, based on the progress of the child's learning. However, the preservation of physical and mental health, interest in life, and harmonious relationships in the family are goals that the school should also have in mind.

The proposed approach to the involvement of teachers and parents in the proposed process is based on the voluntary use of digital technologies in educational work for all. On an exam, one will have the option to use a computer or not when completing assignments. In the same way, regardless of computers, one can use or not use a drawing in a geometric problem, solve an algebraic problem in one way or another, use limited enumeration or logical reasoning, etc.

Let us pay attention to the problem of the state final certification in the form of the USE or another certification. In our opinion, the main negative element of the existing USE system is the very high degree of predictability of the tasks received by the graduate in the exam. This narrows the range of preparation for the exam (on which a public school focuses) in comparison with the content of textbooks and leads to coaching and tutoring for speed. As the reader may note, one of the main features of the proposed changes is the increase in the variety of tasks to be solved, as well as the surprise that they may hold for the student, including in the exam.

The use of digital technologies is not prohibited by federal standards and curricula. It is just not assumed by default, and so it is not on the exam. As a result, teachers generally prohibit the use of digital technologies based on the fact that, earlier, in particular, when they themselves studied, students did not use digital technologies, and the use of technologies is not expected in exams, as well as in problems in textbooks.

The following may be significant factors in the implementation of the proposed approach:

• gradualness and predictability; advance planning of changes; at the first stages, the time spent on the experiment will be small and the share of nonstandard tasks will increase gradually; the computer will be used as a tool for educational work will be allowed only in individual tasks and used in exams only, for example, in revisions, etc.;

• voluntariness of changes; and

• more explicit highlighting of the possibility of changes in federal regulations (Federal State Educational Standards, etc.) and the inclusion of a requirement that the use, as well as the nonuse, of digital technologies in certain types of activities and topics, be clearly indicated in the main educational program of the school and curriculum on the school website.

11. CONCLUSIONS

It is beyond the scope of this article to experiment with the use of modern technologies for working with big data ("intuitive artificial intelligence"), and we also did not consider the use of artificial-intelligence technologies in school to check evidence and the use of virtual and augmented-reality technologies. We tried to limit ourselves to the most natural, important, and reliable applications of digital technologies, but this does not mean that we do not believe that these or other areas will appear promising in the future. On the other hand, we did not consider what are called "computer simulators," for example, arithmetic ones. They can be quite effective; however, in very many cases, we believe, as can be seen from the preceding text, that their use is consistent with the goal of "training skills," and we suggest that such goals be treated with great caution and proceed primarily from the interests of the student.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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REFERENCES

1. P. R. Halmos, *A Hilbert Space Problem Book* (Springer, New York, 1982). https://doi.org/10.1007/978-1-4684-9330-6
2. M. Guseltseva and A. Asmolov, "Education as a space of opportunities: From human capital to human potential," in *Psychology of Subculture: Phenomenology and Contemporary Tendencies of Development*, Ed. by T. Martsinkovskaya and V. R. Orestova, European Proceedings of Social and Behavioural Sciences, Future Academy (2019), Vol. 64, pp. 40–45. https://doi.org/10.15405/epsbs.2019.07.6
3. A. G. Asmolov, "Race for the future," Russ. Soc. Sci. Rev. **59** (6), 484–492 (2018). https://doi.org/10.1080/10611428.2018.1547054
4. A. Ya. Khinchin, "On the educational effect of math lessons," Mat. Prosveshchenie, Ser. 2, No. 2, 7–28 (1961). http://mi.mathnet.ru/mp676
5. V. V. Firsov, "Methods for teaching mathematics as a scientific discipline," in *Teaching via Mathematics* (Prosveshchenie, Moscow, 2012), pp. 160–172. https://www.mathedu.ru/text/firsov_uchim_matematikoy_2012/p160/
6. V. Firsov and A. Semenov, "School mathematics in Russia," National Presentations: Russia, *10th International Congress on Mathematical Education, Copenhagen, 2004.*
7. A. L. Semenov, *Today's Two Cultures: Mathematics and Literature: Literature Lessons in Humanitarian and Mathematical Classes: Compositions, Games, and Travel* (Mosk. Inst. Otkryt. Obrazovan., Inst. Nov. Tekhnol., Moscow, 2013) [in Russian].
8. E. I. Ignat'ev, *In the Realm of Ingenuity,* Book 1 (Novoe Vremya, St. Petersburg, 1914) [in Russian].

9. E. I. Ignat'ev, *In the Realm of Ingenuity,* Book 2 (Novoe Vremya, St. Petersburg, 1909) [in Russian].
10. E. I. Ignat'ev, *In the Realm of Ingenuity,* Book 3 (Novoe Vremya, St. Petersburg, 1915) [in Russian].
11. Ya. I. Perelman, *Fun Problems: 101 Puzzles for Young Mathematicians* (A.S. Suvorin's, Petrograd, 1916) [in Russian].
12. S. N. Olekhnik, Yu. V. Nesterenko, and M. K. Potapov, *Ancient Entertaining Mathematical Problems*, 2nd ed. revised (Nauka, Moscow, 1988) [in Russian].
13. Kangaroo: Mathematics for All. Contests for Schoolchildren. https://russian-kenguru.ru/konkursy/kenguru. Accessed December 1, 2022.
14. M. I. Bashmakov, *Mathematics in the Pocket of "Kangaroo": International Olympiads of Schoolchildren* (Drofa, Moscow, 2010) [in Russian]. https://www.mathedu.ru/text/bashmakov_matematika_v_karmane_kenguru_2010/p0/
15. N. N. Konstantinov and A. L. Semenov, "Productive education in mathematical schools," Dokl. Math. **106**, Suppl. 2, S1–S18 (2022). https://doi.org/10.22405/2226-8383-2021-22-1-413-446
16. Journal of Inquiry-Based Learning in Mathematics. https://jiblm.org/
17. S. Papert, *Mindstorms: Children, Computers, and Powerful Ideas* (Basic Books, New York, 1980).
18. A. P. Ershov, A. G. Kushnirenko, G. V. Lebedev, A. L. Semenov, and A. Kh. Shen, *Fundamentals of Informatics and Computer Science: A Trial Textbook for Secondary Educational Institutions*, Ed. by A. P. Ershov (Prosveshchenie, Moscow, 1988) [in Russian].
19. A. G. Kushnirenko, G. V. Lebedev, and R. A. Svoren', *Fundamentals of Informatics and Computer Science: Textbook for Grades 10 and 11 of General Educational Institutions* (Prosveshchenie, Moscow, 1990) [in Russian].
20. A. G. Kushnirenko, A. G. Leonov, Ya. N. Zaidelman, and V. V. Tarasova, *Informatics Grades 7–9* (Drofa, Moscow, 2017) [in Russian].
21. A. K. Zvonkin, S. K. Lando, A. L. Semenov, and N. M. Vyalyi, *Informatics and Algorithmics Grades 6–7* (Prosveshchenie, Moscow, 2006–2008) [in Russian].
22. A. L. Semenov and T. A. Rudchenko, *Informatics Grades* 5–6: *Educational and Methodological Set for General Educational Institutions* (Prosveshchenie, Moscow, 2019) [in Russian].
23. A. L. Semenov, M. A. Posicelskaya, S. E. Posicelsky, T. A. Rudchenko, et al., *Mathematics and Informatics Grades 1–4*: *Educational and Methodological Set* (*Textbooks, Workbooks*) *for General Educational Institutions* (Mosk. Tsentr Neprer. Mat. Obrazovan., Moscow, 2012–2019) [in Russian].
24. A. L. Semenov and T. A. Rudchenko, *Informatics Grades 3–4 in 3 Parts: Educational and Methodological Set for General Educational Institutions* (Prosveshchenie, Moscow, 2019) [in Russian].
25. T. A. Rudchenko and A. L. Semenov, *Informatics Grades 1–4: Educational and Methodological Set for General Educational Institutions* (Prosveshchenie, Moscow, 2020–2022) [in Russian].
26. "Mental calculation: In the popular school of S.A. Rachinskii." https://ru.wikipedia.org/?curid=1489692&oldid=119766377
27. L. S. Vygotsky, "The instrumental method in psychology," *Collected Works in Six Volumes* (Pedagogika, Moscow, 1982), Vol. 1 [in Russian]. http://elib.gnpbu.ru/text/vygotsky_ss-v-6tt_t1_1982/go,108;fs,1/
28. L. S. Vygotsky, *Mind in Society: The Development of Higher Psychological Processes* (Harvard Univ. Press, Cambridge, Mass., 1978).
29. V. I. Arnol'd, "On teaching mathematics," Russ. Math. Surv. **53** (1), 229–236 (1998). https://doi.org/https://doi.org/10.4213/rm5
30. P. R. Halmos, *I Want to Be a Mathematician: An Automathography* (Springer, New York, 1985). https://doi.org/10.1007/978-1-4612-1084-9
31. Yu. V. Matiyasevich, "Asymptotic structure of eigenvalues and eigenvectors of certain triangular Hankel matrixes," Dokl. Math. **106**, Suppl. 2, S250–S255 (2022).
32. A. Einstein, "Autobiographical notes," in *Albert Einstein, Philosopher–Scientist*, Ed. by P. A. Schilpp, 2nd ed. (Tudor, New York, 1951), pp. 2–95.
33. Einstein's Thought Experiments. https://en.wikipedia.org/w/index.php?title=Einstein%27s_thought_experiments&oldid=1120538785
34. G. Gonthier, "Formal proof—the four-color theorem," Not. Am. Math. Soc. **55** (11), 1382–1393 (2008). https://www.cs.cornell.edu/courses/JavaAndDS/files/Gonthier4ColorCoq.pdf
35. M. Atiyah, "Mathematics and the computer revolution," in *Collected Works*, Vol. 1: *Early Papers, General Papers* (Oxford Science, Clarendon, Oxford, 1988), pp. 327–347.
36. M. Atiyah, "Mathematics and the computer revolution," Izv. Math. **80** (4), 637–646 (2016). https://doi.org/10.1070/IM8512
37. H. A. Helfgott, "Major arcs for Goldbach's problem" (2014). https://doi.org/10.48550/arXiv.1305.2897
38. Great Internet Mersenne Prime Search. GIMPS. Finding World Record Primes Since 1996. https://www.mersenne.org/
39. List of Known Mersenne Prime Numbers. https://www.mersenne.org/primes/
40. N. A. Vavilov, "Computers as novel mathematical reality: III. Mersenne numbers and divisor sums," Komp'yut. Instrum. Obrazovan., No. 4, 5–58 (2020). https://doi.org/10.32603/2071-2340-2020-4-5-58
41. D. MacHale, *Comic Sections: The Book of Mathematical Jokes, Humour, Wit, and Wisdom* (Boole, Dublin, 1993).
42. V. Voevodsky, "Foundations of mathematics—their past, present, and future," *The 2014 Paul Bernays Lectures*, September 9–10, 2014, ETH Zurich.
43. V. Voevodsky, B. Ahrens, D. Grayson, et al., "UniMath: A computer-checked library of univalent mathematics." https://github.com/UniMath/UniMath. http://unimath.org

44. "Feit–Thompson theorem has been totally checked in Coq," *Mathematical Components*, September 20, 2012. https://web.archive.org/web/20161119094854/ http:/www.msr-inria.fr/news/feit-thomson-proved-in-coq/

45. K. V. Anokhin, K. S. Novoselov, S. K. Smirnov, A. R. Efimov, and F. M. Matveev, "AI for science and science for AI," Vopr. Filos., No. 3, 93–105 (2022). https://doi.org/10.21146/0042-8744-2022-3-93-105

46. G. Raayoni, S. Gottlieb, Y. Manor, et al., "Generating conjectures on fundamental constants with the Ramanujan machine," Nature **590**, 67–73 (2021). https://doi.org/10.1038/s41586-021-03229-4

47. D. H. Bailey, P. B. Borwein, and S. Plouffe, "On the rapid computation of various polylogarithmic constants," Math. Comput. **66** (218), 903–913 (1997). https://doi.org/10.1090/S0025-5718-97-00856-9

48. F. D. Rukhovich, "Outer billiards," Mat. Obrazovan. **1** (69), 42–57 (2014). https://www.mathnet.ru/rus/mo25

49. N. M. Adrianov and G. B. Shabat, "Calculating complete lists of Belyi pairs," Mathematics **10**, 258 (2022). https://doi.org/10.3390/math10020258

50. G. B. Shabat and G. E. Kreydlin, "Mathematical theorems in natural languages," Adv. Math. Res. **28**, 181–194 (2020).

51. P. Borwein, "The easier Waring problem," *Computational Excursions in Analysis and Number Theory* (Springer-Verlag, New York, 2002), Chap. 12, pp. 97–101.

52. N. A. Vavilov, "Computers as novel mathematical reality: V. The easier Waring problem," Komp'yut. Instrum. Obrazovan., No. 3, 1–53 (2022).

53. *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*, transl. by L. E. Sigler (Springer, New York, 2003).

54. L. J. Mordell, "On sums of three cubes," J. London Math. Soc. **17**, 139–144 (1942).

55. L. J. Mordell, "On Ryley's solution of $x^3 + y^3 + z^3 = n$," J. London Math. Soc. **17**, 194–196 (1942).

56. L. J. Mordell, "On the integer solutions of the equation $x^2 + y^2 + z^2 + 2xyz = n$," J. London Math. Soc. **28**, 500–510 (1953).

57. L. J. Mordell, "On an infinity of integer solutions of $ax^3 + ay^3 + bz^3 = bc^3$," J. London Math. Soc. **30**, 111–113 (1955).

58. G. S. Stager, "Seymour Papert (1928–2016)," Nature **537**, 308 (2016). https://doi.org/10.1038/537308a

59. A. L. Semenov, "Seymour Papert and we: Constructionism as educational philosophy of the 21st century," Vopr. Obrazovan. **1**, 269–294 (2017).

60. S. Papert, "Teaching children to be mathematicians vs. teaching about mathematics," Massachusetts Institute of Technology AI Laboratory, Artificial Intelligence Memo No. 249, Logo Memo, No. 4 (1971). https://archive.org/details/papert-teaching-children-mathematics/mode/2up

61. M. Resnick, S. Ocko, and S. Papert, "LEGO, Logo, and design," Children's Environ. Q. **5** (4), 14–18 (1988). https://dailypapert.com/lego-logo-and-design/

62. I. Harel and S. Papert, "Software design as a learning environment," Interactive Learn. Environ. **1** (1), 1–32 (1990). https://doi.org/10.1080/1049482900010102

63. Scratch. Create stories, games, and animations. Share with other around the word. https://scrat·sh.mit.edu. Accessed November 3, 2019.

64. Arduino. Empower Scientists and Artists of the Future. https://www.arduino.cc/education

65. V. B. Betelin, A. G. Kushnirenko, A. L. Semenov, and S. F. Soprunov, "On digital literacy and environments for its development," Systems and Means of Informatics **14** (4), 102–109 (2020). https://doi.org/10.14357/199222642004014

66. S. F. Soprunov, A. S. Ushakova, and E. I. Yakovleva, *Pervologo 4.0: Handbook* (Inst. Nov. Tekhnol. Moscow, 2012) [in Russian].

67. Homepage of the PiktoMir project. https://www.niisi.ru/piktomir. Accessed November 3, 2019.

68. Geogebra for Teaching and Learning Math. https://www.geogebra.org. Accessed December 1, 2022.

69. Live Mathematics. Virtual Mathematical Laboratory. https://www.int-edu.ru/content/rusticus-0. Accessed December 8, 2022.

70. The Geometer's Sketchpad. Resource Center. https://www.dynamicgeometry.com. Accessed December 8, 2022.

71. Mathematical Constructor: Best Russian Dynamic Mathematics Program. https://obr.1c.ru/mathkit. Accessed December 1, 2022.

72. Cabri Geometry. CabriLog. Making Math Success Simple. http://www.cabri.net. Accessed December 8, 2022.

73. G. B. Shabat, "On computer experiments in teaching mathematics," Monitor **6**, 122–125 (1995).

74. G. B. Shabat, "*Live Mathematics* and mathematical experiments," Vopr. Obrazovan. **4**, 156–165 (2005).

75. I. Illich, *Deschooling Society* (Harper & Row, 1971).