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Creating New Mathematics by Schoolchildren

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Abstract—The paper discusses an example of an educational project in modern mathematics in which school students create mathematics that is new to them. The mathematical results produced by the students in the theory of definability also have an "absolute" novelty, i.e., are the basis for professional publications. The described course was based on recent results of this article's authors in the theory of definability. The new results were obtained by a group of 10 schoolchildren from different regions of Russia.

Keywords: Russian mathematical school, Konstantinov school, active learning, learning by doing, mathematical education, research activities of schoolchildren, definability theory, highly motivated students

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1. INTRODUCTION

Among the educational principles (rules) of the great John Amos Comenius (see [1], Chap. XXI) we find learning by doing (Fabricando fabricamur: in making, we make ourselves): to learn to forge, we need to do forging; to learn to reason, we need to use reasoning, etc. School should be a place where work is in full swing. In the 20th century, Paul Halmos wrote that the only way of studying mathematics is to create mathematics ([2], p. 7). The approach he discussed goes back to Robert Moore [3].

In the Soviet Union, this approach was used in university education, specifically, in Nikolai Nikolaevich Luzin's school (Lusitania) and in school mathematical circles, starting from the mid-1930s. In the mid-1960s, it became an element of regular school education for highly motivated students. Its ideologist and proponent, the head of the whole direction in pedagogy for several decades was Nikolai Nikolaevich Konstantinov [4].

An analysis of the continuing tradition of Russian mathematical education [5], [6] suggests that an effective way of mastering mathematics can rely on

—autonomous creation of mathematics: experiment, discovery of formulations of theorems and definitions, and the construction of proofs;

—a high level of novelty of problems for learners, i.e., solving problems that are "not-known-how-to-solve."

We believe that such an approach becomes necessary in the 21st century, since the most demanded qualities of humans in the workplace and everyday life today become the ability to independently search for solutions and the ability to solve new, unexpected tasks.

Importantly, these qualities are needed not only by creative people, researchers, inventors, etc. They are also useful for all people. Hence, it is desirable to form them in all students of mass school. We believe that the most effective way of solving this task is through mathematical education, although we do not deny this possibility in any school subject.

The task of such formation is greatly facilitated by digital technologies. Unloading people (including students) from noncreative tasks, memorization of routine operations, and their automatic execution, they make it possible to focus on creative tasks.

2. PROJECT

In this paper, we describe the implementation of the above-mentioned principles in work not with mass schoolchildren, but rather with highly motivated high-caliber students. This is the first stage of research aimed to study the effectiveness of teaching methods and the possibility of their further application in mass school.

In Russia, a center of work with highly motivated students is the Sirius Educational Center, located near

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the city of Sochi by the sea. This paper's authors were invited to carry out the project "The theory of definability" at the Sirius Center within the May project program in mathematics and theoretical computer science on May 1–24, 2022 [7] (in what follows for brevity we will write simply the Project).

3. MATHEMATICAL CONTENT

The mathematical subject of the Project was the theory of definability. The foundations of this theory were laid as early as the 19th century. Several leading mathematicians gave talks concerning this theory at the famous International Congresses of Mathematics and Philosophy held in Paris in 1900. Later, these issues were studied by Polish (Alfred Tarski) and American (Edward Huntington) logicians. There was a certain renaissance of the topic in the 1950s, when Lars Svenonius proved his remarkable theorem, which plays the role of a completeness theorem in definability [8]. In that period, the methods of finite automata were intensively developed. In this direction, Rabin obtained his outstanding definability result on infinite trees, which was reported at the congress in Nice [9]. The results that attracted much interest and were widely quoted included the Cobham–Semenov theorem [10], the results of Semenov's student, Andrey Muchnik, concerning the solution of Rabin's problem [11], and a new proof of the Cobham–Semenov theorem [12]. Muchnik's proof of the last theorem was based on the development of Tarski's idea of self-definability. Despite dozens of papers on definability theory having appeared in recent decades, compared to other fields of mathematical logic, this topic remains little developed, and there are great chances to get new nice results there. In recent years, this paper's authors obtained a number of results in this domain and set up problems concerned with definability on numerical and graph structures that are not homogeneous (for homogeneous structures, results were obtained earlier by other authors).

Our choice of the subject for the Project mini-course was based on the following circumstances:

—the possibility of quickly getting students into the topic—a small amount of theory to learn before starting work;

—the possibility of carrying out an experiment and discussing results, the "olympiad" style of arising problems; the possibility of using numerical and graph intuition;

—high probability of obtaining a "totally" new result (i.e., known to neither the Project leaders nor, presumably, the international mathematical community) based on available results and approaches.

Let us describe the system of mathematical concepts used in our Project (see [13]).

Suppose that we are given a set of objects—a *universe*. In the Project, we began with the universes of integers and rational numbers. Additionally, positive integers and graph generalizations of integers and positive integers—tree structures—were considered.

On the universe, we define *relations*. The main cases considered in the Project were the successor relation ($y = x + 1$) and the usual relation of linear order.

Suppose that an arbitrary system S of relations is defined on the universe. A relation R is *definable* in terms of S if there exists a logical formula defining R with values of relation names taken from S. For example, the between relation on the rational numbers can be defined in terms of the less-than order relation. This simple example is a starting point for understanding the whole situation. Indeed, it can be seen that the less-than relation, on the contrary, cannot be defined in terms of the between relation. Both professional mathematicians and capable students relatively quickly can find a proof of this fact: there is a transformation (permutation) of rational numbers that preserves the between relation (and, hence, preserves everything that can be defined in terms of the between relation) but fails to preserve the less-than relation. Such a permutation—a between automorphism—is the reversion of the rational line, for example, the sign change in a rational number. The following naturally defined concepts appear in the general case:

—definability *space* i.e., a set of relations closed under definability;

—*automorphism groups* of a space;

—*Galois correspondence* between definability spaces and their automorphism groups.

The *main problem* is to describe the *definability lattice* of all subspaces of a given space.

It is natural to try to describe the definability lattice by considering closed (in the natural topology) *supergroups of the automorphism group* of the original space. However, simple examples quickly found by students show that this approach is productive, but insufficient: the automorphism groups are too poor.

Lars Svenonius' achievement—the completeness theorem for definability—states that automorphisms are sufficient if, along with the basic structure, we consider its *elementary extensions*. The idea of this natural concept is that certain "ideal elements" are added to the universe, and the truth in a smaller structure of any statement with parameters from it is equivalent to its truth in a larger structure. For example, in the case of the successor of integers, an elementary extension can be obtained by adding to them several unconnected copies of this set. In the case of the order of rational numbers, any of its extensions is isomorphic to the ordered rational numbers themselves.

We have briefly described all the requisites needed to launch student research. As you can see, they represent a relatively small amount of information and rely on well-known school concepts.

4. STRUCTURE OF THE TRAINING PROGRAM OF THE PROJECT

Our work with students consisted of the following stages.

1. The formation of the content of the Project as one of four fields of work during the May session at the Sirius Center.

2. Introduction of preparatory cycle problems to student candidates with explanations.

3. Candidates chose their own set of problems for the first stage of the preparatory cycle and solved them.

4. Personal online interview concerning problems of the second stage of the preparatory cycle.

5. Face-to-face work with selected students at the Sirius Center.

The format of face-to-face work in the Sirius school includes mini-courses consisting of four stages, each four days long. Between the stages, there is a day of rest with excursions and other types of recreation. The entire school day, with lunch breaks and brief recesses, consists of working together in a small audience and little managed independent work of students outside the classroom. The general daily schedule is regulated by the breakfast, lunch, and dinner times; meals for students are free. In terms of the number of hours, such a mini-course is equivalent to a semester university course or research seminar.

In addition to the authors of this article, A.Ya. Kanel-Belov also participated in the implementation of the Project. The Project program was preliminarily tested in 2021 within the Summer conference of the International Mathematical Tournament of Towns at Berendeev Glades in the Kostroma region [14], [15], where several school students worked for eight days.

After the entrance and personal selection of candidates, the Project involved 10 students of 10th and 11th grades from the following cities: Zhukovskii (Moscow region), Kurgan, Novouralsk (a closed town in the Sverdlovsk region), Moscow (two students, one of them was from the Yaroslavl region, but studied in a Moscow boarding school), Tomsk, Kemerovo, Samara, and St. Petersburg.

During the first of the four stages of the Project, the students studied the basic concepts, except for the concept of elementary extension, and constructed examples of relations defined in terms of other relations. This allowed them to master the basic system of objects, its building blocks used to create constructions and arguments. They built formulas that corresponded to an intuitive idea of how one thing could be defined in terms of something else. An important point here was the understanding of what CANNOT be used in a definition. Specifically, in definitions, we cannot use names of objects that were not previously declared as part of definitions; additionally, we cannot use variables for sets, sequences, and functions: variables can be only objects, i.e., elements of the universe. In this discussion, we did not immediately give a formal definition of a formula, rules for using brackets, etc., which is usually done in courses of mathematical logic. At some point of the general discussion, the concept of a formula and the meaning of brackets became clear to everyone.

The first stage of the Project ended with setting up the problem of how to prove that something CANNOT be defined. In our context, as in many other cases in mathematics, the proof of undefinability required going beyond the constructed system of concepts. Using the particular example of the above-mentioned less-than and between relations, students came up with the idea of an automorphism. A formal definition was easy to give, and the understanding of the method of automorphisms was an important achievement. It is well known that the understanding of the key role of automorphisms was the basis for the famous Erlangen program of Felix Klein [16].

The second stage of the Project included the construction of automorphisms of structures where possible. Specifically, the students constructed supergroups of automorphisms corresponding to known, previously discussed examples of Huntington relations and established undefinability in the necessary cases. An isomorphism of definability lattices for elementary equivalent structures, in particular, for elementary extensions was discussed. For a certain pair of subspaces in the space of the successor of integral numbers, the students proved their distinction via the transition to elementary equivalent spaces. Pairwise and group discussion became the main model of problem solving and solution checking.

The last two stages of the Project involved mainly research. They will be described in the next section.

As a digression from the main line of the course, classes were offered where two remarkable results of definability theory were given also in the form of a chain of problems:

—Tarski's theorem on quantifier elimination and, as a consequence, on the solvability of elementary algebra and geometry;

—the Gödel—Tarski indefinability theorem and, as a consequence, Gödel's incompleteness theorem.

5. RESULTS

Four research teams were created at the second stage of the Project. With participation of teachers, each team formulated an open problem to be solved. Each of the problems was concerned with the construction of a definability lattice for the corresponding structure. The following structures were considered.

1. Order on the nonnegative rational numbers (Irina Shatova, Novouralsk).

2. Successor for positive integers (Aleksei Rutkovskii, Zhukovskii; Fedor Kolotilin, Samara;

Anatolii Slavnov, Moscow; and Leonid Mikhailov, St. Petersburg).

3. Addition of rational numbers (Kirill Dik, Moscow; Konstantin Zyubin, Tomsk).

4. Infinite cycle-free graph with all vertices of degree 3 (Artemii Denisov, Kemerovo; Mikhail Sibiryaev, Kurgan; Andrei Dmitrienko, St. Petersburg; and Leonid Mikhailov, St. Petersburg).

Consideration of automorphism supergroups in elementary extensions was used as a basic tool for solving the problems. The students proved Svenonius' theorem in the form of a sequence of problems.

At the subsequent two stages of the Project, all students obtained new results unknown to the teachers.

—They described some series of spaces of relations.

—They proved some inclusions for relations in certain cases and the absence of inclusions in other cases.

The participants wrote down their constructions, rather quickly moving from paper notes to typesetting in the TeX editor. The resulting texts in the form of draft papers were posted on the session website. At two final stages of the Project, each of the teams of students reported their advances to the other participants.

One of the participants—Leonid Mikhailov, St. Petersburg—on his own initiative independently constructed a proof of a well-known complicated theorem describing a definability lattice for the order of rational numbers. The first proof of this theorem was obtained in 1965, and it contained more than a hundred of pages [17]. The subsequent proofs involved subtle group-theoretic constructions. Mikhailov's proof was of the same algebraic nature and, in many respects, was similar in style to the proof of the well-known mathematician Peter Cameron [18] (of course, Mikhailov was not familiar with Cameron's proof).

The results obtained by each team were initially written for its internal use. At this stage, most of these notes occupied tens of pages. Further collaboration between members of each of the teams and teachers continued online. Now results with verified proofs deserve publication in professional mathematical journals. Presumably, the results of all teams can be expanded and supplemented. The collaboration of the teams is being continued.

In their mathematical activities, the students mastered and used a system of concepts, including

—structure (universe with named relations);

—logical language;

—definability;

—automorphisms;

—elementary extensions;

—lattices;

—Galois correspondences;

as well as they gained experience of constructing and presenting proofs that involve these concepts. All students practiced teamwork skills. A question concerning the authorship of certain constructions was raised and settled in one of the teams.

6. UNIVERSITY EXAMPLE

In a more massive variant, this approach has been applied in recent years to about 200 third-semester students of the MSU Faculty of Mechanics and Mathematics, Lomonosov studying the obligatory course of mathematical logic and the theory of algorithms [19]. Of course, actual research problems are no longer assigned to all students. In this case, we mean that many problems in the course are unexpected for most students.

This course was found especially effective during the COVID-19 pandemic. The use of remote technologies ensured a good quality of direct dialogue between the lecturer and each student who took initiative. For other students, this educational situation brings them closer to the mathematical kitchen of their peers, not just professional mathematicians. In the post-pandemic period, this quality of meaningful communication would likely require additional technological enhancement. It is not replaced by written communications (notes to the lecturer). The ability of each student to use voice communications through a mobile phone would require additional technical and organizational-pedagogical measures.

7. CONCLUSIONS

The implementation of the Project within the May project program at the Sirius Center has confirmed the possibility of effective mastering of new mathematics by schoolchildren in the regime of intensive solution of new problems within 20 days. In the case of highly motivated students of the level of prizewinners of all-Russia olympiads, with a certain choice of problems, this can lead to totally new mathematical results.

It is usually difficult for young mathematicians to prepare the first publication. The process is hindered by a large amount of literature to be studied, by the technical nature of results, or by difficulties in presenting research in the necessary format. The described project makes it possible to overcome these difficulties to a significant degree and, at the same time, to obtain high-quality results.

Today, all 11th-grade participants of the project have enrolled at universities of their own choice (Mathematical Faculty of the Higher School of Economics, St. Petersburg State University, ITMO University, and the Moscow Institute of Physics and Technology) and successfully studied, while 10-graders generally have ensured their enrollment in desirable higher education institutions. The Project leaders hope that the work on the topics will continue and will be completed with professional publications of participants.

The use of modern digital technologies in the Project was not absolutely necessary, but we believe it is critical because:

—the initial enrollment of students and their choice of subjects took place online;

—students and teachers used a graphical editor for creating mathematical texts and screen presentations;

—during the session and after it, professional communication was maintained in an online environment.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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