ISSN 1560-3547, Regular and Chaotic Dynamics, 2024, Vol. 29, No. 1, pp. 156–173. © Pleiades Publishing, Ltd., 2024.

# On Homeomorphisms of Three-Dimensional Manifolds with Pseudo-Anosov Attractors and Repellers

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Received September 17, 2023; revised December 25, 2023; accepted January 10, 2024

**Abstract**—The present paper is devoted to a study of orientation-preserving homeomorphisms on three-dimensional manifolds with a non-wandering set consisting of a finite number of surface attractors and repellers. The main results of the paper relate to a class of homeomorphisms for which the restriction of the map to a connected component of the non-wandering set is topologically conjugate to an orientation-preserving pseudo-Anosov homeomorphism. The ambient Ω-conjugacy of a homeomorphism from the class with a locally direct product of a pseudo-Anosov homeomorphism and a rough transformation of the circle is proved. In addition, we prove that the centralizer of a pseudo-Anosov homeomorphisms consists of only pseudo-Anosov and periodic maps.

MSC2010 numbers: 37B99, 37E30

**DOI:** 10.1134/S1560354724010106

Keywords: pseudo-Anosov homeomorphism, two-dimensional attractor

*On the occasion of the 70th anniversary of S. Gonchenko,*
*the glorious successor of the traditions*
*of the Nizhny Novgorod school of nonlinear oscillations,*
*founded by academician A. Andronov*

## 1. INTRODUCTION

In [3, 6] the dynamics of three-dimensional $A$-diffeomorphisms was studied under the assumption that their nonwandering set consists of surface two-dimensional basic sets. It is proved that diffeomorphisms of this class are ambiently Ω-conjugate to locally direct products of an Anosov diffeomorphism of a two-dimensional torus and a rough transformation of a circle. This work is a generalization of these results to a wider class $\mathcal{G}$ of maps, which we define as follows.

This work is a generalization of the results of [3, 6] to a wider class of maps $\mathcal{G}$, which we define as follows. The set $\mathcal{G}$ consists of orientation-preserving homeomorphisms $f$ of a closed orientable topological 3-manifold $M^3$ with the nonwandering set $NW(f)$ consisting of a finite number of connected components $B_0, \ldots, B_{m-1}$ satisfying for any $i \in \{0, \ldots, m-1\}$ the following conditions:

1) $B_i$ is a cylindrical[1)] embedding of a closed orientable surface of genus greater than 1;

2) there is a natural number $k_i$ such that $f^{k_i}(B_i) = B_i$, $f^{\tilde{k}_i}(B_i) \neq B_i$ for any natural number $\tilde{k}_i < k_i$ and the restriction of the map $f^{k_i}|_{B_i}$ is topologically conjugate to an orientation-preserving pseudo-Anosov homeomorphism (see the definition in Section 2.1);

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1) A subspace $X$ of a topological space $Y$ is called *a cylindrical embedding into $Y$ of a topological space $\bar{X}$* if there is a homeomorphism onto the image $h : \bar{X} \times [-1, 1] \to Y$ such that $X = h(\bar{X} \times \{0\})$.

3) $B_i$ is either an attractor[2)] or a repeller for the homeomorphism $f^{k_i}$.

The simplest representatives of the class $\mathcal{G}$ are homeomorphisms of the set $\Phi$ which are constructed as follows.

Represent the circle as a subset of the complex plane $\mathbb{S}^1 = \{e^{i2\pi\theta} | 0 \leqslant \theta < 1\}$ and define a covering $p\colon \mathbb{R} \to \mathbb{S}^1$ so that $p(r) = s$, where $s = e^{i2\pi r}$.

Consider sets of numbers $n, k, l$ such that $n, k \in \mathbb{N}$, $l \in \mathbb{Z}$, where $l = 0$ if $k = 1$, and $l \in \{1, \ldots, k-1\}$ is coprime to $k$ if $k > 1$. For each set $n, k, l$ we define a diffeomorphism $\bar{\varphi}_{n,k,l} : \mathbb{R} \to \mathbb{R}$ by the formula

$$\bar{\varphi}_{n,k,l}(r) = r + \frac{1}{4\pi nk} \sin(2\pi nkr) + \frac{l}{k}.$$

Since $\bar{\varphi}_{n,k,l}(r) + 1 = \bar{\varphi}_{n,k,l}(r+1)$, it follows that the diffeomorphism $\bar{\varphi}_{n,k,l}$ is the lift of the circle map $\varphi_{n,k,l}(s) = p\Big(\bar{\varphi}_{n,k,l}\big(p^{-1}(s)\big)\Big)$, where $p^{-1}(s)$ is the preimage of the point $s \in \mathbb{S}^1$ (see Statement 8).

**Theorem 1.** *A homeomorphism of a closed orientable surface that commutes with a pseudo-Anosov one is either pseudo-Anosov, or periodic*[3)].

Denote by $S_g$ a closed orientable surface of genus $g > 1$, by $Z(P)$ the centralizer $Z(P) = \{J\colon S_g \to S_g | \, PJ = JP\}$ of a homeomorphism $P\colon S_g \to S_g$ and by $\mathcal{P}$ the set of all pseudo-Anosov homeomorphisms on the surface $S_g$.

Consider orientation-preserving homeomorphisms $P \in \mathcal{P}$ and $J \in Z(P)$ such that the map $J^l P^k$ is a pseudo-Anosov homeomorphism. Let us represent the manifold $M_J$ as the quotient space of the manifold $S_g \times \mathbb{R}$ by the action of the group $\Gamma = \{\gamma^i, i \in \mathbb{Z}\}$ of degrees of homeomorphism $\gamma\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$, given by the formula $\gamma(z, r) = \big(J(z), r - 1\big)$, with natural projection $p_{_J}\colon S_g \times \mathbb{R} \to M_J$.

Define the map $\bar{\varphi}_{P,J,n,k,l}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ by the formula

$$\bar{\varphi}_{P,J,n,k,l}(z, r) = \big(P(z), \bar{\varphi}_{n,k,l}(r)\big).$$

It is readily verified that $\bar{\varphi}_{P,J,n,k,l}\gamma = \gamma\bar{\varphi}_{P,J,n,k,l}$. Then the orientation-preserving homeomorphism $\varphi_{P,J,n,k,l} : M_J \to M_J$ is correctly defined (see Statement 8) and given by the formula

$$\varphi_{P,J,n,k,l}(w) = p_{_J}\Big(\bar{\varphi}_{P,J,n,k,l}\big(p_{_J}^{-1}(w)\big)\Big),$$

where $w \in M_J$ and $p_J^{-1}(w)$ is the preimage of the point $w \in M_J$. We call homeomorphisms of the form $\varphi_{P,J,n,k,l}$ *model maps*. Denote by $\Phi$ the set of all model maps.

**Theorem 2.** *Any homeomorphism from the class $\Phi$ belongs to the class $\mathcal{G}$.*

**Theorem 3.** *Any homeomorphism from the class $\mathcal{G}$ is ambiently $\Omega$-conjugate*[4)] *to a homeomorphism from the class $\Phi$.*

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2) An invariant set $B$ of a homeomorphism $f$ is called an *attractor* if there is a closed neighborhood $U$ of the set $B$ such that $f(U) \subset \operatorname{int} U$, $\bigcap\limits_{j \geqslant 0} f^j(U) = B$. The attractor for the homeomorphism $f^{-1}$ is called the *repeller* of the homeomorphism $f$.

3) A homeomorphism $f$ is called periodic if there exists $m \in \mathbb{N}$ such that $f^m = id$.

4) Recall that homeomorphisms $f_1\colon X \to X$ and $f_2\colon Y \to Y$ of topological manifolds $X$ and $Y$ are called *ambiently $\Omega$-conjugated* if there is a homeomorphism $h\colon X \to Y$ such that $h\big(NW(f_1)\big) = NW(f_2)$ and $hf_1|_{NW(f_1)} = f_2 h|_{NW(f_1)}$.

## 2. MAIN DEFINITIONS AND AUXILIARY STATEMENTS

### *2.1. Pseudo-Anosov Homeomorphisms*

Let $M^n$ be a topological manifold of dimension $n$.

A family $\mathcal{F} = \{L_\alpha; \alpha \in A\}$ of path-connected subsets in $M^n$ is called a *k-dimensional foliation* if it satisfies the following three conditions:

- $L_\alpha \cap L_\beta = \emptyset$ for any $\alpha, \beta \in A$ such that $\alpha \neq \beta$;
- $\bigcup\limits_{\alpha \in A} L_\alpha = M^n$;
- for any point $p \in M^n$ there is a local map $(U, \varphi)$, $p \in U$, so that if $U \cap L_\alpha \neq \emptyset$, $\alpha \in A$, then the path-connected components of the set $\varphi(U \cap L_\alpha)$ have the form $\{(x_1, x_2, \ldots, x_n) \in \varphi(U);$ $x_{k+1} = c_{k+1}, x_{k+2} = c_{k+2}, \ldots, x_n = c_n\}$, where the numbers $c_{k+1}, c_{k+2}, \ldots, c_n$ are constant on the path-connected components.

*A foliation $\mathcal{F}$ with a set of singularities $S$ of $M^n$* is a family of path-connected subsets of $M^n$ such that the family of sets $\mathcal{F} \setminus S$ is a foliation of $M^n \setminus F$.

Let $q \in \mathbb{N}$. *The foliation $W_q$ on $\mathbb{C}$ with the standard saddle singularity at the point $O$ and $q$ separatrices* is a family of path-connected subsets in $\mathbb{C}$ such that $W_q \setminus O$ is a foliation on $\mathbb{C} \setminus O$ and $Im\ z^{\frac{q}{2}} =$ const on leaves of $W_q \setminus O$. Rays $l_1, \ldots, l_q \in W_q$ satisfying equality $Im\ z^{\frac{q}{2}} = 0$ are called separatrices of the point $O$.

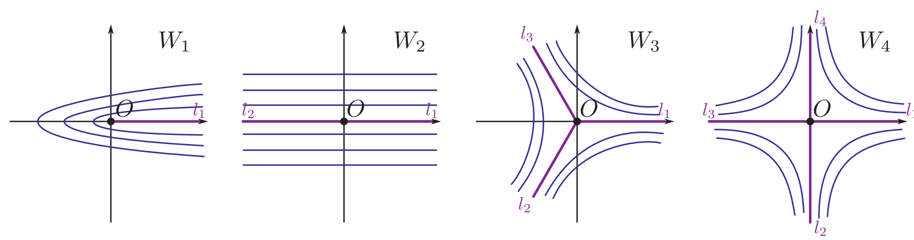


**Fig. 1.** The foliation $W_q$ on $\mathbb{C}$ with the standard saddle singularity at the point $O$ and $q$ separatrices for $q = 1, 2, 3, 4$.

A one-dimensional foliation $\mathcal{F}$ on $M^2$ is called a *foliation with saddle singularities* if the set $S$ of singularities of the foliation $\mathcal{F}$ consists of a finite number of points $s_1, \ldots, s_c$ and for any point $s_i$ $(i \in \{1, \ldots, c\})$ there is a neighborhood $U_i \subset M^2$, a homeomorphism $\psi_i\colon U_i \to \mathbb{C}$ and a number $q_i \in \mathbb{N}$ such that $\psi_i(s_i) = O$ and $\psi_i(\mathcal{F} \cap U_i) = W_{q_i} \setminus \{O\}$. The leaf containing the curve $\psi_i^{-1}(l_j)$, $j \in \{1, \ldots, q_i\}$, is called the separatrix of the point $s_i$. The point $s_i$ is called *a saddle singularity with $q_i$ separatrices.*

*The transversal measure* $\mu$ for a foliation $\mathcal{F}$ with saddle singularities on $M^2$ associates with each arc $\alpha$ transversal to $\mathcal{F}$ a nonnegative Borel measure $\mu|_\alpha$ with the following properties:

1) if $\beta$ is a subarc of the arc $\alpha$, then $\mu|_\beta$ is a restriction of the measure $\mu|_\alpha$;
2) if $\alpha_0$ and $\alpha_1$ are two arcs transversal to $\mathcal{F}$ and connected by a homotopy $\alpha\colon [0,1] \times [0,1] \to M^2$ such that $\alpha([0,1] \times \{0\}) = \alpha_0$, $\alpha([0,1] \times \{1\}) = \alpha_1$ and $\alpha(\{t\} \times [0,1])$ for any $t \in [0,1]$ is contained in a leaf of $\mathcal{F}$ (see Fig. 2), then $\mu|_{\alpha_0}(\alpha_0) = \mu|_{\alpha_1}(\alpha_1)$.

An orientation-preserving homeomorphism $P\colon S_g \to S_g$ of a closed orientable surface of genus $g > 1$ is called a *pseudo-Anosov map* (*pA-homeomorphism*) with *dilatation* $\lambda > 1$ if on surface $S_g$ there is a pair of $P$-invariant transversal foliations $\mathcal{F}^s_P$, $\mathcal{F}^u_P$ with a set of saddle singularities $S$ and transversal measures $\mu_s$, $\mu_u$ such that:

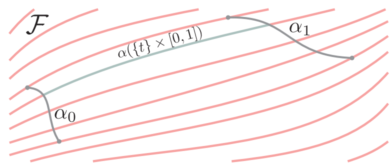


**Fig. 2.** Curves $\alpha_0$ and $\alpha_1$ are connected by homotopy $\alpha$.

- each saddle singularity from $S$ has at least three separatrices;
- $\mu_s\big(P(\alpha)\big)=\lambda\mu_s(\alpha)$ $(\mu_u\big(P(\alpha)\big)=\lambda^{-1}\mu_u(\alpha))$ for any arc $\alpha$ transversal to $\mathcal{F}^s_P$ $(\mathcal{F}^u_P)$.

Let $P\colon S_g\to S_g$ be a pseudo-Anosov homeomorphism. Define the stable (unstable) manifold $W^s(x)=\{y\in M^3: d(P^n(x),P^n(y))\to 0, n\to+\infty\}$ $(W^u(x)=\{y\in M^3: d(P^n(x),P^n(y))\to 0, n\to-\infty\})$ of $x\in S_g$, where $d$ is a metric on $S_g$. Note that the stable (unstable) manifold of the point $x\notin S$ is a leaf of the foliation $\mathcal{F}^s_P$ $(\mathcal{F}^u_P)$ and a stable (unstable) manifold of the point $x\in S$ is the union of a finite number of separatrices belonging to the foliation $\mathcal{F}^s_P$ $(\mathcal{F}^u_P)$ and the point $x$.

*A rectangle* is a subset $\Pi\subset S_g$ that is the image of a continuous map $\upsilon$ of the square $[0,1]\times[0,1]$ into $S_g$ with the following properties: $\upsilon$ is one-to-one on the interior of the square and maps segments of its horizontal partition into arcs of leaves $\mathcal{F}^s_P$, and segments of its vertical partition into arcs of leaves $\mathcal{F}^u_P$. Denote by $\dot\Pi$ the image of the interior of the square. We will call the images of the horizontal and vertical sides contracting and stretching sides of the rectangle $\Pi$.

*A Markov partition for a pseudo-Anosov homeomorphism* $P$ is a finite family of rectangles $\tilde\Pi=\{\Pi_1,\dots,\Pi_n\}$ for which the following conditions are satisfied:

- $\bigcup\limits_i\Pi_i=S_g$; $\dot\Pi_i\cap\dot\Pi_j=\emptyset$ for $i\neq j$;
- let $\partial^s\tilde\Pi$ $(\partial^u\tilde\Pi)$ be the union of all contracting (stretching) sides of rectangles $\Pi_1,\dots,\Pi_n$, then $P(\partial^s\tilde\Pi)\subset\partial^s\tilde\Pi$; $P(\partial^u\tilde\Pi)\supset\partial^u\tilde\Pi$.

**Statement 1 ([1, Proposition 10.17]).** *A pseudo-Anosov homeomorphism has a Markov partition.*

A foliation $\mathcal{F}$ is called *uniquely ergodic* if it admits a single transversal measure (up to multiplication by a scalar).

**Statement 2 ([1, Theorem 12.1]).** *The foliations $\mathcal{F}^s_P$ and $\mathcal{F}^u_P$ of the pseudo-Anosov homeomorphism $P$ are uniquely ergodic.*

**Statement 3 ([1, Theorem 12.5]).** *Two homotopic pseudo-Anosov diffeomorphisms are conjugate by a diffeomorphism isotopic to the identity.*

**Statement 4 ([9, Lemma 3.1]).** *A homeomorphism that is topologically conjugate to a pseudo-Anosov homeomorphism is also pseudo-Anosov.*

**Statement 5 ([9, Theorem 3.2]).** *The set of periodic points of a pseudo-Anosov homeomorphism is dense everywhere on the surface.*

**Statement 6 ([9, Note 3.6]).** *Every leaf of foliations $\mathcal{F}^s_P$ and $\mathcal{F}^u_P$ of the pseudo-Anosov homeomorphism $P$ is everywhere dense on $S_g$.*

### 2.2. Group Action on a Topological Space

Let us recall some facts related to the action of a group on a topological space (for more details, see [4]).

For a continuous mapping $h\colon X \to Y$ of a topological space $X$ into a topological space $Y$, denote by $h^{-1}(V)$ the preimage of the set $V \subset Y$, that is, $h^{-1}(V) = \{x \in X | h(x) \in V\}$.

Let the action of a group $G$ be free and discontinuous on a Hausdorff space $X$ and let the orbits space $X/G$ be connected. The definition of the projection $p_{X/G}\colon X \to X/G$ implies that $p^{-1}_{X/G}(x)$ is an orbit of some point $\bar{x} \in p^{-1}_{X/G}(x)$. Let $c$ be a path in $X/G$ for which $c(0) = c(1) = x$. The monodromy theorem implies that there is a unique path $\bar{c}$ in $X$ starting from $\bar{x}$ ($\bar{c}(0) = \bar{x}$) which is a lift of the path $c$. Therefore, there is an element $g \in G$ for which $\bar{c}(1) = g(\bar{x})$. Hence, the map $\eta_{X/G,\bar{x}}\colon \pi_1(X/G, x) \to G$ defined by $\eta_{X/G,\bar{x}}([c]) = g$ is well defined, i. e., it is independent of the choice of the path in the class $[c]$.

**Statement 7 ([4, Statement 10.32]).** *The map $\eta_{X/G,\bar{x}}\colon \pi_1(X/G, x) \to G$ is a nontrivial homomorphism. It is called the homomorphism induced by the cover $p_{X/G}\colon X \to X/G$.*

Let $G$ be an abelian group and let $\bar{c}'$ be the lift of a path $c \in \pi_1(X/G, x)$ starting from a point $\bar{x}' = \bar{c}'(0)$ distinct from the point $\bar{x}$ and let $g'(\bar{x}') = \bar{c}'(1)$. Since there is the unique element $g'' \in G$ for which $g''(\bar{x}) = \bar{x}'$ the monodromy theorem implies $g''(\bar{c}) = \bar{c}'$. Then $g''g = g'g''$ and, therefore, $g' = g$. Thus, $\eta_{X/G,\bar{x}} = \eta_{X/G,\bar{x}'}$ and from now on we omit the index $\bar{x}$ in the notation of the epimorphism $\eta_{X/G,\bar{x}}$ and we write $\eta_{X/G}$ if $G$ is an abelian group.

**Statement 8 ([4, Statement 10.35]).** *Let cyclic groups $G$, $G'$ act freely and discontinuously on $G$, $G'$-space $X$ and let $g$, $g'$ be their respective generators. Then*

1) *if $\bar{h}\colon X \to X$ is a homeomorphism for which $\bar{h}(g(\bar{x})) = g'(\bar{h}(\bar{x}))$ for every $\bar{x} \in X$, then the map $h\colon X/G \to X/G'$ defined by $h = p_{X/G'}\Big(\bar{h}\big(p^{-1}_{X/G}(x)\big)\Big)$ is a homeomorphism and $\eta_{X/G} = \eta_{X/G'}h_*$;*

2) *if $h\colon X/G \to X/G'$ is a homeomorphism for which $\eta_{X/G} = \eta_{X/G'}h_*$, then there is a unique homeomorphism $\bar{h}\colon X \to X$ which is a lift of $h$ and such that $\bar{h}\big(g(\bar{x})\big) = g'\big(\bar{h}(\bar{x})\big)$, $\bar{h}(\bar{x}) = \bar{x}'$ for $\bar{x} \in X$ and $\bar{x}' \in p^{-1}_{X/G'}(x')$, where $x' = h\big(p_{X/G}(\bar{x})\big)$.*

## 3. ON THE CENTRALIZER OF A PSEUDO-ANOSOV MAP

In this section we prove that a homeomorphism $J \in Z(P)$, where $P \in \mathcal{P}$, is either a pseudo-Anosov homeomorphism or a periodic homeomorphism.

Let $P \in \mathcal{P}$ and $J \in Z(P)$. Since $P = JPJ^{-1}$, it follows that $J$ maps stable manifolds of $P$ into stable ones, and unstable ones into unstable ones. Therefore, $J(\mathcal{F}^s_P) = \mathcal{F}^s_P$ and $J(\mathcal{F}^u_P) = \mathcal{F}^u_P$. The foliations $\mathcal{F}^s_P$, $\mathcal{F}^u_P$ have transversal measures $\mu_s$, $\mu_u$. Let us define for the foliation $\mathcal{F}^s_P$ ($\mathcal{F}^u_P$) a transversal measure $\tilde{\mu}_s(\alpha_s) = \mu_s\big(J(\alpha_s)\big)$ $\Big(\tilde{\mu}_u(\alpha_u) = \mu_u\big(J(\alpha_u)\big)\Big)$, where $\alpha_s$ ($\alpha_u$) is the arc transversal to the foliation $\mathcal{F}^s_P$ ($\mathcal{F}^u_P$). Since foliations $\mathcal{F}^s_P$, $\mathcal{F}^u_P$ are uniquely ergodic (Proposition 3), there exist numbers $\nu_s, \nu_u \in \mathbb{R}_+$ such that $\tilde{\mu}_s = \nu_s\mu_s$ and $\tilde{\mu}_u = \nu_u\mu_u$. Thus, $\mu_s\big(J(\alpha_s)\big) = \nu_s\mu_s(\alpha_s)$, $\mu_u\big(J(\alpha_u)\big) = \nu_u\mu_u(\alpha_u)$ for arc $\alpha_s$ transversal to $\mathcal{F}^s_P$ and the arc $\alpha_u$ transversal to $\mathcal{F}^u_P$.

Since the pseudo-Anosov homeomorphism $P$ has a Markov partition (see Statement 1) consisting of $n$ rectangles $\Pi_1, \ldots, \Pi_n$, it follows that on each rectangle $\Pi_i$ ($i\ in\{1, \ldots, n\}$) the measure $\mu_s \otimes \mu_u$ is defined by the formula $\mu_s \otimes \mu_u(\Pi_i) = \mu_s(\alpha_{s,i})\mu_u(\alpha_{u,i}) = \mu_i$, where $\alpha_{s,i}$ is the stretching side of the rectangle $\Pi_i$ and $\alpha_{u,i}$ is the contracting side. Since the foliations $\mathcal{F}^s_P$, $\mathcal{F}^u_P$ are invariant under $J$, it follows that the set $J(\Pi_i)$ ($i \in \{1, \ldots, n\}$) is also a rectangle with measure $\mu_s \otimes \mu_u\big(J(\Pi_i)\big) = \mu_s\big(J(\alpha_{s,i})\big)\mu_u\big(J(\alpha_{u,i})\big) = \ nu_s\nu_u\mu_i$. Thus, $\mu_s \otimes \mu_u(S_g) = \mu_s \otimes \mu_u(\bigcup\limits_i \Pi_i) = \bigcup\limits_i \mu_i$

and $\mu_s \otimes \mu_u\big(J(S_g)\big) = \mu_s \otimes \mu_u\Big(\bigcup_i\big(J(\Pi_i)\big)\Big) = \nu_s\nu_u(\bigcup_i \mu_i)$. Since $J(S_g) = S_g$, it follows that $\nu_s\nu_u = 1$. Let $\nu = \nu_s$.

Consider the case $\nu \neq 1$. The homeomorphism $J$ has a pair of invariant transversal foliations $\mathcal{F}^s_P$, $\mathcal{F}^u_P$ with a common set of saddle singularities having at least three separatrices, and transversal measures $\mu_s$, $\mu_u$ such that $\mu_s\big(J(\alpha)\big) = \nu\mu_s(\alpha)$ $\big(\mu_u\big(J(\alpha)\big) = \nu^{-1}\mu_u(\alpha)\big)$ for any arc $\alpha$ transversal to $\mathcal{F}^s_P$ ($\mathcal{F}^u_P$). Consequently, for $\nu > 1$ ($\nu < 1$) the homeomorphism $J$ is a pseudo-Anosov map with dilatation $\nu > 1$ $\big(\frac{1}{\nu} > 1\big)$.

Consider the case $\nu = 1$. Since the foliation $\mathcal{F}^s_P$ is invariant under $J$, it follows that separatrices of saddle singularities under the action of $J$ are mapped into separatrices of saddle singularities. Since the set of separatrices is finite, there exists $m \in \mathbb{N}$ such that $J^m(s_i) = s_i$ and $J^m(l) = l$ for some separatrix $l$ of the saddle singularity $s_i$ of the foliation $\mathcal{F}^s_P$.

Let us prove that $J^m(x) = x$ for any point $x \in l$. Let $[s_i, x]$ be the arc of the curve $l$ bounded by points $s_i$ and $x$. Since $\mu_u(J^m[s_i, x]) = \mu_u([s_i, x])$, it follows that $J^m([s_i, x]) = [s_i, x]$. Therefore, $J^m(x) = x$.

Since the leaf $l$ is dense everywhere on $S_g$ (see Statement 6) and $J^m|_l = id$, it follows that $J^m(z) = z$ for any $z \in S_g$.

Consequently, the map $J$ is a periodic homeomorphism for $\nu = 1$ and is pseudo-Anosov for $\nu \neq 1$.

## 4. ON THE MODEL MAPS

In this section we prove Theorem 2 and auxiliary lemmas.

Recall that a map $f_2 \colon Y \to Y$ of a topological space $Y$ is called a *factor* of a map $f_1 \colon X \to X$ of a topological space $X$ if there is a surjective continuous map $h \colon X \to Y$ such that $hf_1 = f_2h$. The map $h$ is called *semiconjugacy*.

**Lemma 1.** *Let $f_1 \colon X \to X$, $f_2 \colon Y \to Y$ be homeomorphisms of topological spaces $X$ and $Y$ such that $f_2$ is a factor of $f_1$ with semiconjugacy $h \colon X \to Y$. Then:*

1) *$h\big(NW(f_1)\big) \subset NW(f_2)$;*

2) *if $f_2^k(V_y) = V_y$ for some $k \in \mathbb{N}$, $V_y \subset Y$, then $f_1^k(V_x) \subset V_x$ for $V_x = h^{-1}(V_y)$;*

3) *if $f_1^k(V_x) = V_x$ for some $k \in \mathbb{N}$, $V_x \subset X$, then $f_2^k(V_y) = V_y$ for $V_y = h(V_x)$.*

*Proof.* Let $f_1 \colon X \to X$, $f_2 \colon Y \to Y$ be homeomorphisms of topological spaces $X$ and $Y$ such that $f_2$ is a factor of $f_1$ with semiconjugacy $h \colon X \to Y$, that is, $hf_1 = f_2h$. Let us prove each point of the lemma separately.

1) Consider the point $x \in NW(f_1)$ and the point $y = h(x)$ with an arbitrary open neighborhood $U_y$. Let $U_x = h^{-1}(U_y)$. Since $h$ is a continuous map, the inverse image $U_x$ of the open set $U_y$ is also open. Then, by the definition of a nonwandering point $x$, there exists $n \in \mathbb{N}$ such that $f_1^n(U_x) \cap U_x \neq \emptyset$. Let $f_1^n(U_x) \cap U_x = \hat{U}_x$ and $\hat{U}_y = h(\hat{U}_x)$. Since $\hat{U}_x \subset U_x$, then $h(\hat{U}_x) \subset h(U_x)$, that is, $\hat{U}_y \subset U_y$. Note that $hf_1^n = f_2^nh$. Since $\hat{U}_y \subset h\big(f_1^n(U_x)\big)$, then $\hat{U}_y \subset f_2^n(h(U_x)) = f_2^n(U_y)$. Therefore, $f_2^n(U_y) \cap U_y \neq \emptyset$. Thus, $y = h(x) \in NW(f_2)$.

2) Let $f_2^k(V_y) = V_y$, where $k \in \mathbb{N}$, $V_y \subset Y$, $V_x = h^{-1}(V_y)$ and $f_1^k(V_x) = V'_x$. Then $f_2^k\big(h(V_x)\big) = f_2^k(V_y) = V_y$ and $h\big(f_1^k(V_x)\big) = h(V'_x)$. Since $hf_1^k = f_2^kh$, it follows that $h(V'_x) = V_y$. Therefore, $V'_x \subset V_x$, that is, $f_1^k(V_x) \subset V_x$.

3) Let $f_1^k(V_x) = V_x$, where $k \in \mathbb{N}$, $V_x \subset X$ and $V_y = h(V_x)$. Then $h\big(f_1^k(V_x)\big) = h(V_x) = V_y$. Since $hf_1^k = f_2^kh$, then $f_2^k\big(h(V_x)\big) = f_2^k(V_y) = V_y$. Therefore, $f^k(V_y) = V_y$. □

We will call a set of numbers $n, k, l$ *correct* if $n, k \in \mathbb{N}$, $l \in \mathbb{Z}$, where $l = 0$ for $k = 1$ and $l \in \{1, \dots, k-1\}$ is coprime to $k$ for $k > 1$. Everywhere else in this section the set of numbers $n, k, l$ is correct. Let us recall the main notation and formulas.

- The manifold $M_J$ is the quotient space of $S_g \times \mathbb{R}$ under the action of the group $\Gamma = \{\gamma^i, i \in \mathbb{Z}\}$ of degrees of homeomorphism $\gamma\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ given by the formula $\gamma(z, r) = (J(z), r - 1)$, where $J\colon S_g \to S_g$ is an orientation-preserving homeomorphism;
- $p_J\colon S_g \times \mathbb{R} \to M_J$ is a natural projection inducing the homomorhisms $\eta_{M_J}\colon M_J \to \mathbb{Z}$;
- $\bar{\varphi}_{n,k,l}\colon \mathbb{R} \to \mathbb{R}$ is a diffeomorphism given by the formula

$$\bar{\varphi}_{n,k,l}(r) = r + \frac{1}{4\pi nk}\sin(2\pi nkr) + \frac{l}{k}; \tag{4.1}$$

- $\mathbb{S}^1 = \{e^{i2\pi\theta} | 0 \leqslant \theta < 1\}$, $p\colon \mathbb{R} \to \mathbb{S}^1$ is a covering given by the formula $p(r) = s$, where $s = e^{i2\pi r}$;
- $\varphi_{n,k,l}\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a diffeomorphism given by the formula

$$\varphi_{n,k,l}(s) = p\Big(\bar{\varphi}_{n,k,l}\big(p^{-1}(s)\big)\Big); \tag{4.2}$$

- $\bar{\varphi} = \bar{\varphi}_{P,J,n,k,l}(z, r)\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ is a homeomorphism given by the formula

$$\bar{\varphi}(z, r) = \big(P(z), \bar{\varphi}_{n,k,l}(r)\big), \tag{4.3}$$

  where $P\colon S_g \to S_g$ is an orientation-preserving pseudo-Anosov homeomorphism such that $J \in Z(P)$;
- model homeomorphism $\varphi = \varphi_{P,J,n,k,l}\colon M_J \to M_J$ is given by the formula

$$\varphi(w) = p_{_J}\Big(\bar{\varphi}\big(p_{_J}^{-1}(w)\big)\Big); \tag{4.4}$$

- $\Phi$ is a set of model homeomorphisms.

Let us introduce the following notation:

- $\mathcal{B}_i = p_{_J}(S_g \times \{\frac{i}{2nk}\}) \in M_J$ $(i \in \{0, \dots, 2nk-1\})$;
- $b_i = p(\frac{i}{2nk}) \in \mathbb{S}^1$ $(i \in \{0, \dots, 2nk-1\})$;
- $p_{J,r}\colon S_g \times \{r\} \to p_{_J}(S_g \times \{r\})$ is a homeomorphism given by the formula

$$p_{J,r} = p_J|_{S_g \times \{r\}},\ r \in \mathbb{R}; \tag{4.5}$$

- $\rho\colon S_g \times \mathbb{R} \to S_g$ is a canonical projection given by the formula

$$\rho(z, r) = z; \tag{4.6}$$

- $\rho_r\colon S_g \times \{r\} \to S_g$ is a homeomorphism given by the formula

$$\rho_r = \rho|_{S_g \times \{r\}},\ r \in \mathbb{R}. \tag{4.7}$$

Note that the Eq. (4.4) is obtained from the relation

$$p_{_J}\bar{\varphi} = \varphi p_{_J}, \tag{4.8}$$

and Eq. (4.2) is obtained from the relation

$$p\bar{\varphi}_{n,k,l} = \varphi_{n,k,l}p. \tag{4.9}$$

Since $p_J\colon S_g \times \mathbb{R} \to M_J$ is a natural projection, it follows that

$$p_J\gamma = p_J. \tag{4.10}$$

Denote by $h_J\colon M_J \to \mathbb{S}^1$ the continuous surjective map given by the formula

$$h_J(w) = p(r), \text{ where } w = p_J(z,r) \in M_J. \tag{4.11}$$

It is readily verified that $h_J\varphi = \varphi_{n,k,l}p_J$. Thus, the following lemma is true.

**Lemma 2.** *The homeomorphism $\varphi_{n,k,l}\colon \mathbb{S}^1 \to \mathbb{S}^1$ is the factor of the homeomorphism $\varphi\colon M_J \to M_J$ with semiconjugacy $h_J\colon M_J \to \mathbb{S}^1$.*

It is directly verified (see Eqs. (4.1) and (4.2)) that the nonwandering set of the diffeomorphism $\varphi_{n,k,l}$ consists of $2nk$ points $b_0, \ldots, b_{2nk-1}$ of period $k$ such that points with odd indices $i$ are sinks and points with even indices are sources.

Let us prove Theorem 2, that is, prove the inclusion $\Phi \subset \mathcal{G}$.

*Proof.* Consider the model homeomorphism $\varphi = \varphi_{P,J,n,k,l}\colon M_J \to M_J$. Since the homeomorphism $J$ preserves orientation, it follows that the manifold $M_J$ is orientable. Preserving the orientation of homeomorphisms $P$ and $\varphi_{n,k,l}$ implies preserving orientation by homeomorphism $\varphi$ inducing by map $\bar{\varphi}(z,r) = \big(P(z), \bar{\varphi}_{n,k,l}(r)\big)$.

Let us prove that the connected component $\mathcal{B}_i$ ($i \in \{0, \ldots, 2nk-1\}$) is a cylindrical embedding of the surface $S_g$. For $i \in \{0, \ldots, 2nk-1\}$ we set $\bar{U}_i = S_g \times [\frac{i}{2nk} - \frac{i}{4nk}, \frac{i}{2nk} + \frac{i}{4nk}]$ and $U_i = p_J(\bar{U}_i)$. Since $p_J\colon S_g \times \mathbb{R} \to M_J$ is a covering, it follows that for any $i \in \{0, \ldots, 2nk-1\}$ its restriction $p_J|_{\bar{U}_i}\colon \bar{U}_i \to U_i$ is a homeomorphism. In addition, $p_J|_{\bar{U}_i}(S_g \times \{\frac{i}{2nk}\}) = \mathcal{B}_i$. Therefore, $\mathcal{B}_i$ $\big(i \in \{0, \ldots, 2nk-1\}\big)$ is a cylindrical embedding of $S_g$.

Let us prove that $\varphi^k(\mathcal{B}_i) = \mathcal{B}_i$, $\varphi^{\tilde{k}_i}(\mathcal{B}_i) \neq \mathcal{B}_i$ ($i \in \{0, \ldots, 2nk-1\}$) for any natural number $\tilde{k}_i < k$. In accordance with Lemma 2, the map $\varphi_{n,k,l}$ is the factor of a homeomorphism $\varphi$ with semiconjugacy $h_J$. Note that $h_J^{-1}(b_i) = \mathcal{B}_i$ ($i \in \{0, \ldots, 2nk-1\}$), where $b_i \in \mathbb{S}^1$ is a point of period $k$. It follows from Lemma 1 that $\varphi^k(\mathcal{B}_i) \subset \mathcal{B}_i$. Since the map $\varphi^k$ is a homeomorphism and the component $\mathcal{B}_i$ is homeomorphic to $S_g$, it follows that $\varphi^k(\mathcal{B}_i) = \mathcal{B}_i$. Suppose that $\varphi^{\tilde{k}}(\mathcal{B}_i) = \mathcal{B}_i$ for some natural number $\tilde{k} < k$. Then Lemma 1 implies that $\varphi^{\tilde{k}}_{n,k,l}(b_i) = b_i$. We come to a the contradiction that point $b_i$ has period $k$.

Let us prove that the map $\varphi^k|_{\mathcal{B}_i}$ ($i \in \{0, \ldots, 2nk-1\}$) is topologically conjugate to the orientation-preserving pseudo-Anosov homeomorphism. Since

$$\gamma^l\Big(\bar{\varphi}^k\Big(z, \frac{i}{2nk}\Big)\Big) = \Big(J^l\Big(P^k(z)\Big), \frac{i}{2nk}\Big), \tag{4.12}$$

it follows that

$$\rho_{\frac{i}{2nk}}\Big(\gamma^l\Big(\bar{\varphi}^k\Big(\rho^{-1}_{\frac{i}{2nk}}(z)\Big)\Big)\ Big) = J^l\Big(P^k(z)\Big). \tag{4.13}$$

For any point $w \in \mathcal{B}_i$ we get

$$\varphi^k(w) \overset{(4.4)}{=} p_J\Big(\bar{\varphi}^k(p_J^{-1}(w))\Big) \overset{(4.10)}{=} p_J(\gamma^l(\bar{\varphi}^k(p_J^{-1}(w))))$$

$$\overset{(4.12)}{=} p_{J,\frac{i}{2nk}}\Big(\gamma^l\Big(\bar{\varphi}^k\big(p^{-1}_{J,\frac{i}{2nk}}(w)\big)\Big)\Big) \overset{(4.13)}{=} p_{J,\frac{i}{2nk}}\Big(\rho^{-1}_{\frac{i}{2nk}}\big(J^l\big(P^k(\rho_{\frac{i}{2nk}}(p^{-1}_{J,\frac{i}{2nk}}(w)))\big)\big)\Big).$$

Consequently, the homeomorphism $\varphi^k|_{\mathcal{B}_i}$ is topologically conjugate to the orientation-preserving pseudo-Anosov homeomorphism $J^lP^k$ via the homeomorphism $p_{J,\frac{i}{2nk}}\rho^{-1}_{\frac{i}{2nk}}$.

Lemmas 1 and 2 imply that $NW(\varphi) \subset (\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{2nk-1})$.

Since the set of periodic points of a pseudo-Anosov homeomorphism is dense everywhere on the surface (Proposition 5) and $\varphi^k(\mathcal{B}_i) = \mathcal{B}_i$ $(i \in \{0, \ldots, 2nk-1\})$, it follows that $NW(\varphi) = \mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{2nk-1}$.

Let us prove that the connected components $\mathcal{B}_i$ with odd indices $i$ belong to the set of attractors of the homeomorphism $\varphi$. Points $b_i$ with odd indices $i$ are sink points of the diffeomorphism $\varphi^k_{n,k,l}$. Therefore, $\varphi^k(u_i) \subset int\, u_i$ and $\bigcap\limits_{j\geqslant 0} \varphi^{jk}_{n,k,l}(u_i) = b_i$ for the neighborhood $u_i = h_J(U_i) = p([\frac{i}{2nk} - \frac{i}{4nk}, \frac{i}{2nk} + \frac{i}{4nk}])$ of point $b_i$ with odd index $i$. Since $h_J^{-1}(p[a,b]) = p_J(S_g \times [a,b])$ for any $a, b \in \mathbb{R}$, $h_J\varphi^{jk} = \varphi^{jk}_{n,k,l}h_J$ and $h_J^{-1}(b_i) = \mathcal{B}_i$, it follows that $\varphi^k(U_i) \subset \text{int}\, U_i$, $\bigcap\limits_{j\geqslant 0}\varphi^{jk}(U_i) = \mathcal{B}_i$. Consequently, connected components $\mathcal{B}_i$ with odd indices $i$ are attractors of the map $\varphi^k$.

Analogously one proves that connected components $\mathcal{B}_i$ with even indices $i$ belong to the set of repellers.

Thus, $\varphi \in \mathcal{G}$. □

## 5. THE AMBIENT Ω-CONJUGACY OF A HOMEOMORPHISM $f \in \mathcal{G}$ TO A MODEL MAP

Recall that the set $\Phi$ consists of model homeomorphisms of the form $\varphi_{P,J,n,k,l}$. This section contains a proof of Ω-conjugacy of homeomorphisms of the class $\mathcal{G}$ with homeomorphisms of the set $\Phi$ and auxiliary lemmas. We will also use the notation introduced in Section 3 below.

Let us denote by $\mathcal{H}$ the set of all homeomorphisms $f$ satisfying the following conditions:

1) there exists an orientation-preserving homeomorphism $J\colon S_g \to S_g$ such that $f\colon M_J \to M_J$;
2) $f$ preserves the orientation of $M_J$;
3) there exists $m \in \mathbb{N}$ such that the nonwandering set $NW(f)$ of the homeomorphism $f$ consists of $2m$ connected components $\mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{2m-1}$;
4) for any $i \in \{0, \ldots, 2m-1\}$ there is a natural number $k_i$ such that $f^{k_i}(\mathcal{B}_i) = \mathcal{B}_i$, $f^{\tilde{k}_i}(\mathcal{B}_i) \neq \mathcal{B}_i$ for any natural $\tilde{k}_i < k_i$ and the map $f^{k_i}|_{\mathcal{B}_i}$ preserves the orientation of $\mathcal{B}_i$;
5) $f(\mathcal{B}_i) = \mathcal{B}_j$, where the numbers $i, j \in \{0, \ldots, 2m-1\}$ are either even or odd at the same time.

Note that the homeomorphisms of the set $\Phi$ belong to the class $\mathcal{H}$.

For $m \in \mathbb{N}$ we denote by $\mathcal{T}_m$ the set $\mathcal{T}_m = \{\frac{i}{2m}, i \in \mathbb{Z}\}$. Then $p_J^{-1}(NW(f)) = S_g \times \mathcal{T}_m$, where $f \in \mathcal{H}$.

**Lemma 3.** *For any homeomorphism $f \in \mathcal{H}$ with a nonwandering set consisting of $2m$ connected components, there exist a unique correct set of numbers $n, k, l$ and a lift $\bar{f}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ such that*

$$\bar{f}(z, r) = \left(f_r(z), r + \frac{l}{k}\right), \ \forall r \in \mathcal{T}_{nk},$$

*where $nk = m$ and $f_r\colon S_g \to S_g$ is an orientation-preserving homeomorphism given by*

$$f_r = \rho_{r+\frac{l}{k}}\bar{f}\rho_r^{-1}.$$

*Proof.* Let $f\colon M_J \to M_J$ be a homeomorphism from the class $\mathcal{H}$.

Let us prove that there is a lift $\bar{f}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ of the homeomorphism $f$. By Statement 8 it sufficies to show that $\eta_{M_J} = \eta_{M_J} f_*$.

Consider the loop $c \in M_J$ which is the projection of the curve $\bar{c} \in S_g \times \mathbb{R}$ $(p_J(\bar{c}) = c)$, bounded by points $\bar{c}(0) = (z, 1)$, $\bar{c}(1) = \gamma\big(\bar{c}(0)\big) = \big(J(z), 0\big)$ and intersecting each set $S_g \times \{\frac{i}{2m}\}$, $i \in \{0, \ldots, 2m-1\}$ at exactly one point. By construction, the curve $c$ intersects each connected component $\mathcal{B}_0, \ldots, \mathcal{B}_{2m-1}$ at exactly one point and $\eta_{M_J}([c]) = 1$. We set $C = f(c)$ and $C(0) =$

$f\big(c(0)\big)$. Since $f$ is a homeomorphism such that $f(\mathcal{B}_i) = \mathcal{B}_{i'}$, $i, i' \in \{0, \ldots, 2m-1\}$, it follows that the curve $C = f(c)$ also intersects each component of $\mathcal{B}_0, \ldots, \mathcal{B}_{2m-1}$ at exactly one point. We set $\mathcal{B}_j = f(\mathcal{B}_0)$. If we choose a point $\bar{C}(0) \in p_J^{-1}\big(C(0)\big)$ such that $\bar{C}(0) \in S_g \times \{\frac{j}{2m} + 1\}$, by the monodromy theorem there is a unique lift $\bar{C}$ of the path $C$ starting at the point $\bar{C}(0)$. Since the loop $C$ intersects each component $\mathcal{B}_0, \ldots, \mathcal{B}_{2m-1}$ at exactly one point, it follows that there are 2 cases: 1) $\bar{C}(1) = \gamma^{-1}\big(\bar{C}(0)\big)$, 2) $\bar{C}(1) = \gamma\big(\bar{C}(0)\big)$.

Let us show that case 1) does not take place.

Consider the case $m = 1$. Then $f(\mathcal{B}_0) = \mathcal{B}_0$. Since the homeomorphism $f$ preserves the orientation $M_J$ and the orientation $\mathcal{B}_0$, it follows that the curve $C(t)$ must be parameterized in one direction with the parameterization of the curve $c(t)$ with respect to the surface $\mathcal{B}_0$. Thus, $\bar{C}(1) = \gamma\big(\bar{C}(0)\big)$.

Consider the case $m > 1$. Let us denote by $\xi_c \colon \mathbb{S}^1 \to c$, $\xi_C \colon \mathbb{S}^1 \to C$ homeomorphisms such that $\xi_c(b_i) = \mathcal{B}_i \cap c$, $\xi_C(b_i) = \mathcal{B}_i \cap C$, where $i \in \{0, \ldots, 2m-1\}$. Define the homeomorphism $\psi \colon \mathbb{S}^1 \to \mathbb{S}^1$ by the formula $\psi = \xi_C^{-1} f \xi_c$. Let us prove that the homeomorphism $\psi$ preserves orientation. Assume the converse. Let us prove that there exists $q \in \{0, \ldots, 2m-1\}$ such that $\psi(b_q) = b_q$. Let $\mathcal{B}_j = f(\mathcal{B}_0)$. Then $\psi(b_0) = b_j$. If $j = 0$, then $q = 0$. Let $j \neq 0$. By the condition of the class $\mathcal{H}$, the number $j$ is even. Since, by assumption, $\psi$ changes the orientation of $\mathbb{S}^1$ and the set $b_0 \cup \cdots \cup b_{2m-1}$ is invariant, it follows that the arc of the circle $(b_0, b_j)$ is mapped into itself and $\psi(b_i) = b_{j-i}$, $i \in \{0, \ldots, \frac{j}{2}\}$. Thus, $\psi(b_{\frac{j}{2}}) = b_{\frac{j}{2}}$ and $q = \frac{j}{2}$. Therefore, $f(\mathcal{B}_q) = \mathcal{B}_q$. Since $\psi$ changes orientation, it follows that the curve $C(t)$ is parameterized in the direction opposite to the parameterization of the curve $c(t)$ with respect to the surface $\mathcal{B}_q$ (see Fig. 3). Since the homeomorphism $f$ preserves the orientation $M_J$ and the orientation $\mathcal{B}_q$, then the parameterization of the curve $C(t)$ must be parameterized in one direction with the parameterization of the curve $c(t)$ with respect to the surface $\mathcal{B}_q$. We have got a contradiction. Consequently, the homeomorphism $\psi$ preserves the orientation of $\mathbb{S}^1$. Then $\bar{C}(1) = \gamma\big(\bar{C}(0)\big)$.

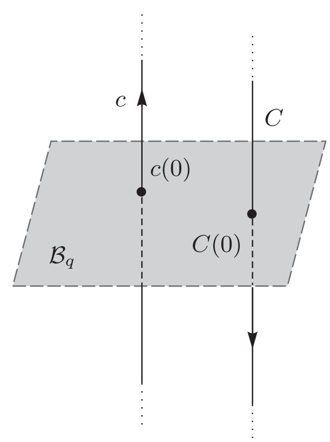


**Fig. 3.** Direction of increasing parameter $t \in [0, 1]$ on curves $c$ and $C$.

Thus, $\bar{C}(1) = \gamma\big(\bar{C}(0)\big)$ and $\eta_{M_J}\big(f_*([c])\big) = 1$. Consequently, $\eta_{M_J} = \eta_{M_J} f_*$ and there is a unique lift $\bar{f} \colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ of the homeomorphism $f$ such that $\bar{f}(\bar{c}(1)) = \bar{C}(1)$ and

$$\bar{f}\gamma = \gamma\bar{f}. \tag{5.1}$$

Let us find the correct set of numbers $n, k, l$ for the homeomorphism $f$. The case $m = 1$ corresponds to the correct set of numbers $n = 1$, $k = 1$ and $l = 0$. Consider the case $m > 1$. Since the homeomorphism $\psi$ is orientation-preserving, it follows that it has a rational rotation number $\frac{l}{k}$, where $k \in \mathbb{N}$, $l \in \{0, \ldots, k-1\}$ and $(l, k) = 1$ (see [7, Theorem 4.1]). From [7, Theorem 4.2] it follows that all periodic points of the homeomorphism $\psi$ have period $k$. Since point $b_i$ with even (odd) index $i$ is mapped to point $b_{i'}$ with even (odd) index $i'$, it follows that $2m$ points $b_0, \dots, b_{2m-1}$ are divided into 2 invariant sets of equal power, each of which consists of points of period $k$. Therefore, $m$ is divisible by $k$. We set $n = \frac{m}{k}$. Thus, $n, k, l$ is the required correct set of numbers.

Since the rotation number of $\psi$ is equal to $\frac{l}{k}$, it follows that $\psi(b_0) = b_{2nl}$, that is, $f(\mathcal{B}_0) = \mathcal{B}_{2nl}$.

Let us find a formula that defines the map $\bar{f}$ for the point $(z, r) \in S_g \times \mathcal{T}_{nk}$. Since $\bar{C}(1) = \gamma\big(\bar{C}(0)\big)$, it follows that $\bar{C}(1) \in S_g \times \{\frac{2nl}{2nk}\} = S_g \times \{\frac{l}{k}\}$. Invariance of the set $p_J^{-1}(NW(f)) = S_g \times \mathcal{T}_{nk}$ under $\bar{f}$ implies that $\bar{f}(S_g \times [0,1]) = S_g \times [\frac{l}{k}, 1 + \frac{l}{k}]$, where $\bar{f}(S_g \times \{0\}) = S_g \times \{\frac{l}{k}\}$. From this we find that $\bar{f}(S_g \times \{\frac{i}{2nk}\}) = S_g \times \{\frac{i}{2nk} + \frac{l}{k}\}$ for any $i \in \{0, \dots, 2nk-1\}$. Using Eq. (5.1), we find that $\bar{f} = \gamma^m \bar{f} \gamma^{-m}$ for any $m \in \mathbb{Z}$. Then $\bar{f}(S_g \times \{r\}) = \gamma^{[r]}\bigg(\bar{f}\Big(\gamma^{-[r]}(S_g \times \{r\})\Big)\bigg)$, where $[r]$ is the integer part of the number $r \in \mathbb{R}$. Thus, it is readily verified that $\bar{f}(S_g \times \{r\}) = S_g \times \{r + \frac{l}{k}\}$ for $r \in \mathcal{T}_{nk}$. Then for any $r \in \mathcal{T}_{nk}$ the homeomorphism $f_r \colon S_g \to S_g$ is correctly defined and given by the formula $f_r = \rho_{r+\frac{l}{k}} \bar{f} \rho_r^{-1}$. Thus, $\bar{f}(z, r) = \big(f_r(z), r + \frac{l}{k}\big)$ for any $r \in \mathcal{T}_{nk}$.

It remains to prove that $f_r$ preserves the orientation of $S_g$, where $r \in \mathcal{T}_{nk}$. Preserving the orientation of $M_J$ by $f$ implies preserving the orientation of $S_g \times \mathbb{R}$ by its lift $\bar{f}$. Since $\bar{f}(S_g \times \{r\} = f_r(S_g) \times \{r + \frac{l}{k}\}$ for any $r \in \mathcal{T}_{nk}$, it follows that the homeomorphism $\bar{f}$ preserves the orientation of $\mathbb{R}$. Therefore, $\bar{f}$ preserves the orientation of $S_g$, that is, $f_r$ preserves the orientation of $S_g$. □

Note that in the case $f = \varphi_{P,J,n,k,l}$ the equality $f_r(z) = P(z)$ holds for any $r \in \mathcal{T}_{nk}$ and $\bar{f} = \bar{\varphi}_{P,J,n,k,l}$.

**Lemma 4.** *Let $f \in \mathcal{H}$. Then $f_r$ is isotopic to $f_0$ for any $r \in \mathcal{T}_{nk}$.*

*Proof.* Let $f \in \mathcal{H}$. Let us prove that $f_r$ is isotopic to $f_0$ for any $r \in \mathcal{T}_{nk}$.

Define a family of continuous maps $F_{r,t} \colon S_g \to S_g$ by the formula $F_{r,t}(z) = \rho(\bar{f}(z, rt))$, where $t \in [0,1]$, $r \in \mathcal{T}_{nk}$. Then $F_{r,t}$ defines a homotopy connecting the maps $F_{r,0} = f_0$ and $F_{r,1} = f_r$. Thus, homeomorphisms $f_0$ and $f_r$ are homotopic. It follows from [10, p. 5.15] that they are isotopic for any $r \in \mathcal{T}_{nk}$. □

**Lemma 5.** *Let $f \colon M^3 \to M^3$ be a homeomorphism from the class $\mathcal{G}$. Then there exists a homeomorphism $f' \in \mathcal{H}$ topologically conjugate to $f$.*

*Proof.* Let $f \colon M^3 \to M^3$ be a homeomorphism from the class $\mathcal{G}$ with a nonwandering set consisting of $q$ connected components $B_0, \dots, B_{q-1}$.

In accordance with [2, Lemma 2.1], the set $M^3 \setminus (B_0 \cup \dots \cup B_{q-1})$ consists of $q$ connected components $V_0, \dots, V_{q-1}$, bounded by one connected component of an attractor and one connected component of a repeller. Therefore, $q = 2m$, where $m \in \mathbb{N}$. Without loss of generality, for $m > 1$ we can assume that $cl\ V_i \cap cl\ V_{i-1} = B_{i-1}$, where $i \in \{1, \dots, 2m-2\}$ and $cl\ V_0 \cap cl\ V_{2m-1} = B_{2m-1}$.

In accordance with [2, Lemma 2.2], each connected component $V_i$, $i \in \{0, \dots, 2m-1\}$ of the set $M^3 \setminus (B_0 \cup \dots \cup B_{2m-1})$ is homeomorphic to $S_g \times [0,1]$. It follows from [5, Lemma 2] that there exists a continuous surjective map $H \colon S_g \times [0,1] \to M^3$ (see Fig. 4) such that maps $H|_{S_g \times \{\frac{i}{m}\}} \colon S_g \times \{\frac{i}{m}\} \to B_i$ $(i \in \{0, \dots, 2m-1\})$, $H|_{S_g \times \{1\}} \colon S_g \times \{1\} \to B_0$ and $H|_{S_g \times (0,1)} \colon S_g \times (0,1) \to M^3 \setminus B_0$ are homeomorphisms.

Let $J(z) = \rho_0\Big((H|_{S_g \times \{0\}})^{-1}\big(H|_{S_g \times \{1\}}(\rho_1^{-1}(z))\big)\Big)$ (see Fig. 5).

Denote by $[r]$ the integer part of the number $r \in \mathbb{R}$. Define a continuous map $h \colon S_g \times \mathbb{R} \to M^3$ by the formula $h(z, r) = H\big(\gamma^{[r]}(z, r)\big)$.

Let the homeomorphism $\xi \colon M^3 \to M_J$ be given by the formula $\xi = p_J\big(h^{-1}(w)\big)$. Set $f' = \xi f \xi^{-1}$.

Let us prove that the homeomorphism $f'$ satisfies all 5 conditions of the class $\mathcal{H}$. Since $M^3$ is orientable and homeomorphic to $M_J$, it follows that $J$ preserves the orientation of $S_g$ and condition 1

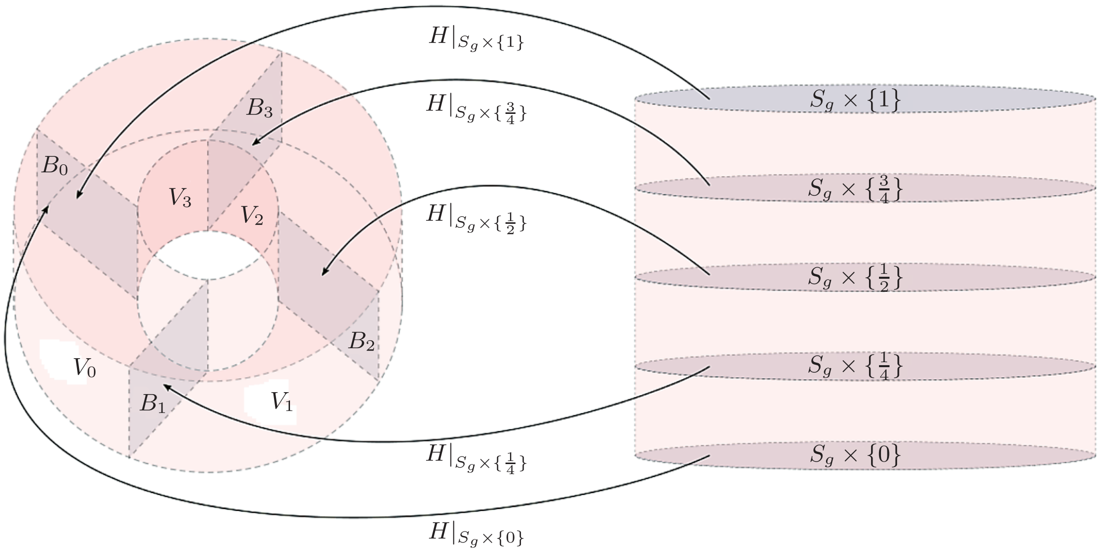


**Fig. 4.** Action of the homeomorphism $H$ in the case $m = 2$.

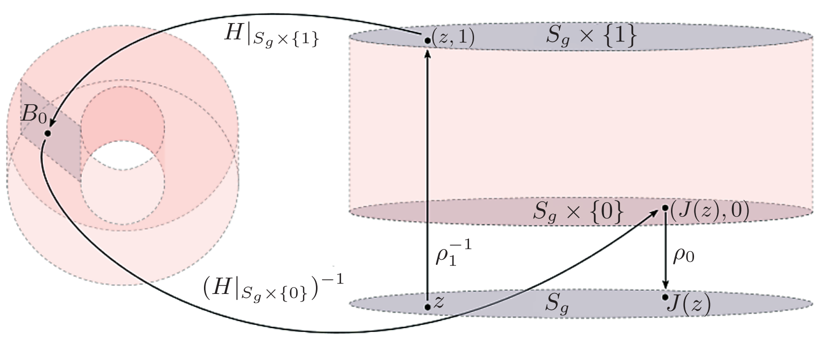


**Fig. 5.** Homeomorphism $J\colon S_g \to S_g$.

is satisfied. Since $f$ preserves the orientation of $M^3$, it follows that $f'$ preserves the orientation of $M_J$ and condition 2 is satisfied. Since $\xi(NW(f)) = NW(f')$ and $h^{-1}\big(NW(f)\big) = S_g \times \mathcal{T}_{nk}$, it follows that $NW(f') = p_J\Big(h^{-1}\big(NW(f)\big)\Big) = p_J(S_g \times \mathcal{T}_{nk}) = \mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{2m-1}$. Therefore, condition 3 is satisfied. Since for any $B_i$ ($i \in \{0, \ldots, 2m-1\}$) there is a natural number $k_i$ such that $f^{k_i}(B_i) = B_i$, $f^{\tilde{k}_i}(B_i) \neq B_i$ for any natural $\tilde{k}_i < k_i$ and the map $f^{k_i}|_{B_i}$ preserves the orientation of $B_i$, it follows that the same is true for the connected component $\mathcal{B}_i$ of the nonwandering set $NW(f')$, that is, condition 4 is satisfied. The connected components of the nonwandering set $NW(f)$ of the homeomorphism $f$ are numbered in such a way that, if $B_i$ is the connected component of an attractor of the homeomorphism $f$, then $B_{i+1 \pmod{2m}}$ is the connected component of a repeller of the homeomorphism $f$. Therefore, $f(B_i) = B_j$, where $i, j \in \{0, \ldots, 2m-1\}$ are either even or odd at the same time. Since $\xi(B_i) = \mathcal{B}_i$ ($i \in \{0, \ldots, 2m-1\}$), it follows that $f'(\mathcal{B}_i) = \mathcal{B}_j$, where $i, j \in \{0, \ldots, 2m-1\}$ are simultaneously either even or odd, that is, condition 5 is satisfied. Thus, $f' \in \mathcal{H}$. □

Everywhere below in this section we mean by $\bar{f}$, $f_r$ and $n, k, l$ the lift of the homeomorphism $f \in \mathcal{H}$, the homeomorphism $f_r\colon S_g \to S_g$, $r \in \mathcal{T}_{nk}$, and the correct set of numbers $n, k, l$ from Lemma 3.

**Lemma 6.** *Let $f \in \mathcal{H} \cap \mathcal{G}$. Then $f_0$ is isotopic either to some periodic homeomorphism or to some pseudo-Anosov homeomorphism.*

*Proof.* Let $f \in \mathcal{H} \cap \mathcal{G}$.

Let us prove that $f_0$ is isotopic either to some periodic homeomorphism or to some pseudo-Anosov homeomorphism.

Since $\bar{f}$ is a lift of a homeomorphism $f$, it follows that

$$p_J \bar{f} = f p_J. \tag{5.2}$$

Therefore,

$$f(w) = p_J\Big(\bar{f}\big(p_J^{-1}(w)\big)\Big). \tag{5.3}$$

For $r \in \mathcal{T}_{nk}$ denote by $\phi_r \colon S_g \to S_g$ the homeomorphism given by the formula

$$\phi_r = J^l f_{r+\frac{(k-1)l}{k}} \cdots f_{r+\frac{l}{k}} f_r. \tag{5.4}$$

Then it is readily verified that

$$\gamma^l\big(\bar{f}^k|_{S_g \times \mathcal{T}_{nk}}(z, r)\big) = \big(\phi_r(z), r\big), \quad \text{where } r \in \mathcal{T}_{nk}. \tag{5.5}$$

Therefore,

$$\phi_r = \rho_r \gamma^l \bar{f}^k \rho_r^{-1}. \tag{5.6}$$

Thus, $f^k|_{\mathcal{B}_0}(w) \overset{(5.3)}{=} p_J(\bar{f}^k(p_J^{-1}(w))) \overset{(4.10)}{=} p_J\bigg(\gamma^l\Big(\bar{f}^k\big(p_J^{-1}(w)\big)\Big)\bigg) \overset{(5.5)}{=} p_{J,0}\bigg(\gamma^l\Big(\bar{f}^k\big(p_{J,0}^{-1}(w)\big)\Big)\bigg) \overset{(5.6)}{=}$ $p_{J,0}\bigg(\rho_0^{-1}\Big(\phi_0\big(\rho_0(p_{J,0}^{-1}(w))\big)\Big)\bigg)$, that is,

$$f^k|_{\mathcal{B}_0} = p_{J,0}\rho_0^{-1}\phi_0\rho_0 p_{J,0}^{-1}. \tag{5.7}$$

Therefore, the homeomorphism $\phi_0$ is topologically conjugate to the homeomorphism $f^k|_{\mathcal{B}_0}$ via the map $p_{J,0}\rho_0^{-1}$. Since the homeomorphism $f^k|_{\mathcal{B}_0}$ is topologically conjugate to the pseudo-Anosov homeomorphism, it follows that the homeomorphism $\phi_0$ is also a pseudo-Anosov map (see Statement 4).

Equation (5.1) implies that $\Big(J\big(f_r(z)\big), r + \frac{l}{k} - 1\Big) = \Big(f_{r-1}\big(J(z)\big), r - 1 + \frac{l}{k}\Big)$ and

$$J f_r = f_{r-1} J \text{ for any } r \in \mathcal{T}_{nk}. \tag{5.8}$$

Therefore, $f_0 J^l = J^l f_l$. Then $f_0\bigg(J^l f_{\frac{(k-1)l}{k}} \cdots f_{\frac{l}{k}} f_0\bigg) = \bigg(J^l f_l f_{\frac{(k-1)l}{k}} \cdots f_{\frac{l}{k}}\bigg) f_0$, that is,

$$\phi_0 = f_0^{-1}\phi_{\frac{l}{k}} f_0. \tag{5.9}$$

It follows from Eq. (5.9) and Statement 4 that $\phi_{\frac{l}{k}}$ is also a pseudo-Anosov homeomorphism.

Since $f_r$ is isotopic to $f_0$ for any $r \in \mathcal{T}_{nk}$ by Lemma 4, it follows that $J^l f_{\frac{(k-1)l}{k}} \cdots f_{\frac{l}{k}} f_0$ is isotopic to $J^l f_l f_{\frac{(k-1)l}{k}} \cdots f_{\frac{l}{k}}$, that is, $\phi_0$ is isotopic to $\phi_{\frac{l}{k}}$. Then, according to Statement 3, there exists a homeomorphism $h \colon S_g \to S_g$, isotopic to the identity, such that

$$\phi_0 = h\phi_{\frac{l}{k}} h^{-1}. \tag{5.10}$$

Substituting Eq. (5.10) into Eq. (5.9), we find that $\phi_0 = f_0^{-1}(h^{-1}\phi_0 h) f_0$, that is, $(hf_0)\phi_0 = \phi_0(hf_0)$.

Since $\phi_0 \in \mathcal{P}$ and $hf_0 \in Z(\phi_0)$, it follows that the homeomorphism $hf_0$ is either periodic or pseudo-Anosov by Theorem 1. The isotopism of $h$ to the identity implies that $f_0$ is isotopic either to some periodic homeomorphism or to some pseudo-Anosov homeomorphism. □

**Lemma 7.** *Let $f \in \mathcal{H} \cap \mathcal{G}$ and $f_0$ be isotopic to some periodic homeomorphism. Then there exists a homeomorphism $f' \in \mathcal{H}$ such that $f'$ is topologically conjugate to $f$ and $f'_0$ is isotopic to some pseudo-Anosov homeomorphism.*

*Proof.* Let $f\colon M_J \to M_J$ be a homeomorphism from the class $\mathcal{H} \cap \mathcal{G}$ with a nonwandering set consisting of $2nk$ connected components of period $k$, and let $f_0$ be isotopic to some periodic homeomorphism.

Let us show that $k \neq 1$. Assume the converse. Then $l = 0$ and the homeomorphism $\phi_0$ has the form $\phi_0 = f_0$ (see Eq. (5.4)). According to Eq. (5.7), the homeomorphism $\phi_0$ is topologically conjugate to the pseudo-Anosov homeomorphism $f^k|_{\mathcal{B}_0}$. We come to a contradiction with the fact that $k = 1$. Therefore, $k > 1$.

Define the homeomorphisms $\bar{h}, \gamma'\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ by the formulas $\bar{h}(z, r) = (z, -r)$, $\gamma'(z, r) = (J^{-1}(z), r - 1)$. Recall that $\gamma(z, r) = (J(z), r - 1)$. Since $(J(z), -(r - 1)) = (J(z), (-r) + 1)$, it follows that $\bar{h}\gamma = (\gamma')^{-1}\bar{h}$. Therefore, the homeomorphism $\bar{h}$ projects into the homeomorphism $h\colon M_J \to M_{J^{-1}}$ (see Statement 8), given by the formula $h = p_{J^{-1}}\Big(\bar{h}\big(p_J^{-1}(w)\big)\Big)$, where $p_{J^{-1}}\colon S_g \times \mathbb{R} \to M_{J^{-1}}$ is a natural projection.

Set $f' = hfh^{-1}$. Recall that for a homeomorphism $f \in \mathcal{H}$ there is a unique lift $\bar{f}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ such that $\bar{f}_{S_g \times \mathcal{T}_{nk}}(z, r) = \big(f_r(z), r + \frac{l}{k}\big)$, where $n, k, l$ is the correct set of numbers. Consider the lift $\bar{f}'$ of the homeomorphism $f'$ given by the formula $\bar{f}' = \gamma^{-1}\bar{h}\bar{f}\bar{h}^{-1}$. Then for any $r \in \mathcal{T}_{nk}$ we have $\bar{f}'(z, r) = \Big(J\big(f_r(z)\big), r + \frac{k-l}{k}\Big)$. Since $k \neq 1$, it follows that $l \in \{1, \ldots, k-1\}$. Therefore, $(k - l) \in \{1, \ldots, k-1\}$ and is coprime to $k$. Thus, $n, k, (k - l)$ is the correct set of numbers and $f'_r = Jf_r$.

Let us prove that the homeomorphism $f'_0$ is isotopic to some pseudo-Anosov homeomorphism. By Lemma 6, the homeomorphism $f'_0$ is isotopic either to some periodic map or to some pseudo-Anosov map. Suppose that the homeomorphism $f'_0 = Jf_0$ is isotopic to a periodic homeomorphism. Then the homeomorphism $J = f'_0 f_0^{-1}$ is also isotopic to a periodic homeomorphism. Since $J$ and $f_0$ are isotopic to periodic homeomorphisms and, according to Lemma 4, $f_0$ is isotopic to $f_r$ for any $r \in \mathcal{T}_{nk}$, it follows that the homeomorphism $\phi_0 = J^l f_{\frac{(k-1)l}{k}} \cdots f_{\frac{l}{k}} f_0$ is also isotopic to a periodic homeomorphism. We come to a contradiction with the fact that $\phi_0$ is topologically conjugate to the pseudo-Anosov homeomorphism $f^k|_{\mathcal{B}_0}$ (see Eq. (5.7)). Consequently, the homeomorphism $f'_0$ is isotopic to the pseudo-Anosov homeomorphism. Thus, $f' \in \mathcal{H}$ is topologically conjugate to $f$ and $f'_0$ is isotopic to some pseudo-Anosov homeomorphism. □

**Lemma 8.** *Let $f \in \mathcal{H} \cap \mathcal{G}$ and $f_0$ be isotopic to some pseudo-Anosov homeomorphism $P$. Then there is a homeomorphism $f'\colon M_{J'} \to M_{J'}$ from the class $\mathcal{H}$ such that $f'$ is topologically conjugate to $f$, $J'P = PJ'$ and $f'_0$ is isotopic to $P$.*

*Proof.* Let $f\colon M_J \to M_J$ be a homeomorphism from the class $\mathcal{H} \cap \mathcal{G}$ and $P$ be a pseudo-Anosov homeomorphism of the surface $S_g$, isotopic to $f_0$.

Let us construct a homeomorphism $J'\colon S_g \to S_g$. Set

$$P' = J^{-1}PJ. \tag{5.11}$$

Denote by $F_t$ the isotopy connecting the homeomorphisms $F_0 = f_0$ and $F_1 = P$. Then the family of maps $J^{-1}F_tJ$ defines an isotopy connecting the maps $J^{-1}F_0J = J^{-1}f_0J = f_1$ and $J^{-1}F_1J = J^{-1}PJ = P'$. Since $f_0$ is isotopic to $f_1$ (see Lemma 4) and to $P$, $f_1$ is isotopic to $P'$, it follows that $P$ is isotopic to $P'$. Homeomorphism $P$ is topologically conjugate to the pseudo-Anosov homeomorphism $P'$, $P$ is isotopic to $P'$. Then by Statement 3 there exists a homeomorphism $\xi$, isotopic to the identity, such that

$$P' = \xi P \xi^{-1}. \tag{5.12}$$

Set

$$J' = J\xi,\ \gamma' = \big(J'(z), r-1\big). \tag{5.13}$$

Note that $J'P \overset{(5.13)}{=} J\xi P \overset{(5.12)}{=} JP'\xi \overset{(5.11)}{=} PJ\xi \overset{(5.13)}{=} PJ'$.

Let us construct a homeomorphism $Y\colon M_J \to M_{J'}$. Denote by $\xi_t$ the isotopy connecting the homeomorphism $\xi_0 = \xi$ and the identity map $\xi_1 = id$. Define the homeomorphism $y_r\colon S_g \to S_g$ by the formula

$$y_r = \begin{cases} \xi_{6nk(1-r)} & \text{for } r \in [1-\frac{1}{6nk}, 1]; \\ id & \text{for } r \in [0.1-\frac{1}{6nk}]. \end{cases}$$

Define the homeomorphism $y\colon S_g \times [0,1] \to S_g \times [0,1]$ by the formula $y(z,r) = (y_r(z), r)$. Note that

$$y(z,0) = (z,0) \text{ and } y\Big(z, \frac{l}{k}\Big) = \Big(z, \frac{l}{k}\Big). \tag{5.14}$$

Denote by $[r]$ the integer part of the number $r \in \mathbb{R}$. Define the homeomorphism $\bar{Y}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ by the formula

$$\bar{Y}(z,r) = (\gamma')^{-[r]}\Big(y\big(\gamma^{[r]}(z,r)\big)\Big). \tag{5.15}$$

Since $\gamma'\bar{Y} = \bar{Y}\gamma$, it follows that the homeomorphism $\bar{Y}$ projects into the homeomorphism $Y\colon M_J \to M_{J'}$ (see Statement 8), given by the formula $Y = p_{J'}\Big(\bar{Y}\big(p_J^{-1}(w)\big)\Big)$, where $p_J\colon S_g \times \mathbb{R} \to M_J$, $p_{J'}\colon S_g \times \mathbb{R} \to M_{J'}$ are natural projections.

Set $f' = YfY^{-1}\colon M_{J'} \to M_{J'}$. By construction, $f' \in \mathcal{H}$. Let us prove that $f'_0$ is isotopic to $P$. Consider the lift

$$\bar{f}' = \bar{Y}\bar{f}\bar{Y}^{-1} \tag{5.16}$$

of the homeomorphism $f$. It is readily verified that $\bar{f}'(z,r) = (f'_r(z), r+\frac{l}{k})$, where $r \in \mathcal{T}_{nk}$ and $f'_r$ is a homeomorphism of $S_g$. Let us show that $f'_0 = f_0$. Indeed, $\bar{f}'(z,0) \overset{(5.16)}{=} \bar{Y}\Big(\bar{f}\big(\bar{Y}^{-1}(z,0)\big)\Big) \overset{(5.15)}{=} \bar{Y}\Big(\bar{f}\big(y^{-1}(z,0)\big)\Big) \overset{(5.14)}{=} \bar{Y}\big(\bar{f}(z,0)\big) = \bar{Y}\left(f_0(z), \frac{l}{k}\right) \overset{(5.15)}{=} y_{\frac{l}{k}}\left(f_0(z), \frac{l}{k}\right) \overset{(5.14)}{=} \left(f_0(z), \frac{l}{k}\right)$. Thus, $f'_0$ is also isotopic to $P$. □

Let us prove that any homeomorphism from the class $\mathcal{G}$ is ambiently $\Omega$-conjugate to a homeomorphism from the class $\Phi$.

*Proof.* Let $f \in \mathcal{G}$.

According to Lemma 5, without loss of generality, we may assume that $f$ is defined on $M_J = S_g \times \mathbb{R}/\Gamma$ with natural projection $p_J\colon S_g \times \mathbb{R} \to M_J$, where $J$ is an orientation-preserving homeomorphism of the surface $S_g$ and $\Gamma = \{\gamma^i | i \in \mathbb{Z}\}$ is a group of degrees of the homeomorphism $\gamma\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ given by the formula $\gamma(z,r) = (J(z), r-1)$. It follows from Lemma 3 that the nonwandering set of the homeomorphism $f$ consists of $2nk$ connected components $\mathcal{B}_0, \ldots, \mathcal{B}_{2nk-1}$ and there is a lift $\bar{f}$ of the homeomorphism $f$ such that $\bar{f}(z,r) = (f_r(z), r+\frac{l}{k})$ for any $r \in \mathcal{T}_{nk}$, where $f_r\colon S_g \to S_g$ is an orientation-preserving homeomorphism of the surface and $n, k, l$ is the correct set of numbers.

According to Lemmas 4, 6, 7, 8, without loss of generality we may assume that $f_r$ is isotopic to some orientation-preserving pseudo-Anosov homeomorphism $P$ for any $r \in \mathcal{T}_{nk}$ and $J \in Z(P)$. Since $J$ preserves the orientation of $S_g$, it follows that the homeomorphism $J^lP^k$ also preserves the orientation of $S_g$.

Let us prove that the homeomorphism $J^lP^k$ is a pseudo-Anosov homeomorphism. Using Eqs. (5.5) and (5.6), we obtain

$$f^k|_{p_J(S_g\times\{r\})} = p_{J,r}\rho_r^{-1}\phi_r\rho_r p_{J,r}^{-1},\ r \in \mathcal{T}_{nk}, \tag{5.17}$$

that is, the homeomorphism $\phi_r$ ($r \in \mathcal{T}_{nk}$) is topologically conjugate to the pseudo-Anosov homeomorphism $f^k|_{p_J(S_g\times\{r\})}$. Since, by Lemma 4, the homeomorphism $f_r$ for any $r \in \mathcal{T}_{nk}$ is isotopic to $P$, it follows that the homeomorphism $\phi_r = J^l f_{r+\frac{(k-1)l}{k}} \cdots f_{r+\frac{l}{k}} f_r$ is isotopic to $J^l P^k$, that is, the homeomorphism $J^l P^k$ is isotopic to the pseudo-Anosov homeomorphism. According to Theorem 1, we find that the homeomorphism $J^l P^k$ is a pseudo-Anosov map.

Note that homeomorphisms $J^l P^k$ and $\phi_r$ are isotopic for any $r \in \mathcal{T}_{nk}$ and are pseudo-Anosov homeomorphisms. Then, according to Statement 3, maps $\phi_r$ and $J^l P^k$ are topologically conjugate for any $r \in T$ via some homeomorphism isotopic to the identity. Denote such a homeomorphism by $h_r$. Then for any $r \in \mathcal{T}_{nk}$ we find that

$$J^l P^k = h_r(\phi_r)h_r^{-1}. \tag{5.18}$$

Thus, each homeomorphism $f \in \mathcal{G}$ corresponds to the correct set of numbers $n, k, l$ and orientation-preserving homeomorphisms $P\colon S_g \to S_g$, $J\colon S_g \to S_g$ such that the homeomorphisms $P$, $J^l P^k$ are pseudo-Anosov and $J \in Z(P)$. Therefore, there is a correctly defined model map $\varphi_{P,J,n,k,l} \in \Phi$.

Let us prove that the homeomorphism $f$ is ambiently $\Omega$-conjugate to $\varphi_{P,J,n,k,l}$. We construct a homeomorphism $f'\colon M_J \to M_J$, topologically conjugate to $f$ and coinciding with the homeomorphism $\varphi_{P,J,n,k,l}$ on the nonwandering set ($f'|_{NW(f')} = \varphi_{P,J,n,k,l}|_{NW(\varphi_{P,J,n,k,l})}$).

We divide the construction into steps.

**Step 1.** Construct a homeomorphism $x\colon S_g \times U \to S_g \times U$, where $U = \bigcup\limits_{j\in\{0,\ldots,k-1\}} U_j$, $U_j = [-\frac{1}{4nk} - j\frac{l}{k}, \frac{1}{k} - \frac{1}{4nk} - j\frac{l}{k})$.

Let $T = \{0, \frac{1}{2nk}, \ldots, \frac{2n-1}{2nk}\}$. Note that $T = \mathcal{T}_{nk} \cap U_0$ and $r \in \mathcal{T}_{nk} \cap U_j$ has the form $r = i - j\frac{l}{k}$, where $j \in \{0, \ldots, k-1\}$ and the number $i \in T$ is uniquely determined. For $i \in T$ and $j \in \{0, \ldots, k-1\}$ we define the homeomorphism $\xi_{i,j}\colon S_g \to S_g$ by the formula

$$\xi_{i,j} = P^{-j} h_i \underbrace{f_{i-j\frac{l}{k}+(j-1)\frac{l}{k}} \cdots f_{i-j\frac{l}{k}}}_{j \text{ maps}}. \tag{5.19}$$

Since the homeomorphism $f_{i-j\frac{l}{k}+(j-1)\frac{l}{k}} \cdots f_{i-j\frac{l}{k}+\frac{l}{k}} f_{i-j\frac{l}{k}}$ is isotopic to $P^j$ for $j \in \{1, \ldots, k-1\}$ and the homeomorphism $h_i$ is isotopic to the identity, it follows that the homeomorphism $\xi_{i,j}$ is isotopic to the identity for any $j \in \{0, \ldots, k-1\}$. Let $\xi_{i,j,t}$ denote the isotopy connecting the homeomorphism $\xi_{i,j,0} = \xi_{i,j}$ and the identity map $\xi_{i,j,1} = id$.

For $r \in U$ we define the homeomorphism $x_r\colon S_g \to S_g$ by the formula

$$x_r = \begin{cases} \xi_{i,j,6nk|r-(i-j\frac{l}{k})|} & \text{for } |r-(i-j\frac{l}{k})| \leqslant \frac{1}{6nk}; \\ id & \text{for other } r \in U. \end{cases}$$

Define the homeomorphism $x\colon S_g \times U \to S_g \times U$ by the formula

$$x(z, r) = (x_r(z), r).$$

Note that

$$x\Big(z, i - j\frac{l}{k}\Big) = \Big(\xi_{i,j}(z), i - j\frac{l}{k}\Big). \tag{5.20}$$

**Step 2.** Let us extend the homeomorphism $x\colon S_g \times U \to S_g \times U$ to the homeomorphism $\bar{X}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$.

Let us prove that for any point $r \in \mathbb{R}$ there is a unique integer $m \in \mathbb{Z}$ such that $(r - m) \in U$.

Divide the half-interval $\Big[-\frac{1}{4nk}, 1 - \frac{1}{4nk}\Big)$ into $k$ half-intervals: $\Big[-\frac{1}{4nk}, 1 - \frac{1}{4nk}\Big) = \Big[-\frac{1}{4nk}, \frac{1}{k} - \frac{1}{4nk}\Big) \cup \Big[-\frac{1}{4nk} + \frac{1}{k}, \frac{2}{k} - \frac{1}{4nk}\Big) \cup \cdots \cup \Big[-\frac{1}{4nk} + \frac{k-1}{k}, 1 - \frac{1}{4nk}\Big)$. Obviously, for any $r \in \mathbb{R}$ there is a unique number $a \in \mathbb{Z}$ such that $r - a \in \Big[-\frac{1}{4nk}, 1 - \frac{1}{4nk}\Big)$. Let $r - a \in \Big[-\frac{1}{4nk} + \frac{j}{k}, \frac{j+1}{k} - \frac{1}{4nk}\Big)$, where $j \in \{0, \ldots, k-1\}$. Since $j$ runs through the complete system of residues $\{0, 1, \ldots, k-1\}$ modulo $k$ and $l$ is coprime to $k$, it follows that $(-jl)$ also runs through a complete system of residues $\{0, -l, \ldots, -l(k-1)\}$ modulo $k$ [8, p. 46]. Consequently, there are integers $i \in \{0, -l, \ldots, -l(k-1)\}$ and $b$ such that $j + bk = i$. Then $(r - a + b) \in \Big[-\frac{1}{4nk} + \frac{j+bk}{k}, \frac{j+1+bk}{k} - \frac{1}{4nk}\Big) = \Big[-\frac{1}{4nk} + \frac{i}{k}, \frac{1}{k} + \frac{i}{k} - \frac{1}{4nk}\Big) \subset U$. Thus, $m = a - b$ is the required integer such that $(r - m) \in U$.

Let $\varrho(r)$ denote an integer $\varrho(r) \in \mathbb{Z}$ such that $\big(r - \varrho(r)\big) \in U$. Define the map $\bar{X}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ by the formula $\bar{X}(z, r) = \gamma^{-\varrho(r)}\Big(x\big(\gamma^{\varrho(r)}(z, r)\big)\Big)$ for $(z, r) \in S_g \times \mathbb{R}$. Then $\bar{X}\gamma = \gamma\bar{X}$.

**Step 3.** Construct a homeomorphism $f'\colon M_J \to M_J$.

Let us set $\bar{f}' = \bar{X}\bar{f}\bar{X}^{-1}$. Since $\bar{X}\gamma = \gamma\bar{X}$ and $\bar{f}\gamma = \gamma\bar{f}$, it follows that $\bar{f}'\gamma = \gamma\bar{f}'$ and homeomorphisms $\bar{X}$ and $\bar{f}'$ project into homeomorphisms $f'\colon M_J \to M_J$, $X\colon M_J \to M_J$ (see Statement 8), given by the formulas $f' = p_J\Big(\bar{f}'\big(p_J^{-1}(w)\big)\Big)$, $X = p_J(\bar{X}\Big(p_J^{-1}\big((w)\big)\Big)$ and $f' = XfX^{-1}$.

Let us prove that $\bar{f}'|_{S_g \times \mathcal{T}_{nk}} = \bar{\varphi}_{P,J,n,k,l}|_{S_g \times \mathcal{T}_{nk}}$. Since $\bar{X}(S_g \times \{r\}) = S_g \times \{r\}$ and $\bar{f}(S_g \times \{r\}) = S_g \times \{r + \frac{l}{k}\}$ for any $r \in \mathcal{T}_{nk}$, it follows that $\bar{f}'(S_g \times \{r\}) = \bar{X}\Big(\bar{f}\big(\bar{X}^{-1}(S_g \times \{r\})\big)\Big) = S_g \times \{r + \frac{l}{k}\}$. Then for any $r \in \mathcal{T}_{nk}$ the homeomorphisms ${f'}_r\colon S_g \to S_g$, $X_r\colon S_g \to S_g$ are correctly defined by ${f'}_r = \rho_{r+\frac{l}{k}}\bar{f}'\rho_r^{-1}$, $X_r = \rho_{r+\frac{l}{k}}\bar{X}\rho_r^{-1}$ and

$$ {f'}_r = X_{r+\frac{l}{k}} f_r X_r^{-1}. \tag{5.21} $$

Then

$$ X_r = J^{-m(r)} x_r J^{m(r)}. \tag{5.22} $$

By construction, $\bar{\varphi}_{P,J,n,k,l}(z, r) = \big(P(z), r + \frac{l}{k}\big)$ and $\bar{f}'(z, r) = \big({f'}_r(z), r + \frac{l}{k}\big)$ for any $r \in \mathcal{T}_{nk}$.

Let us prove that ${f'}_r = P$ for any $r \in \mathcal{T}_{nk}$. Let us represent $r \in \mathcal{T}_{nk}$ in the form $r = i - j\frac{l}{k} + m$, where $i \in T$, $j \in \{0, \ldots, k-1\}$ and $m \in \mathbb{Z}$.

Let $k = 1$. Then

$$ \begin{aligned} {f'}_r = {f'}_{i+m} &\overset{(5.21)}{=} X_{i+m} f_{i+m} X_{i+m}^{-1} \overset{(5.22)}{=} J^{-m} x_i J^m f_{i+m} J^{-m} x_i^{-1} J^m \overset{(5.20)}{=} J^{-m} \xi_{i,0} J^m f_{i+m} J^{-m} \xi_{i,0}^{-1} J^m \\ &\overset{(5.8)}{=} J^{-m} \xi_{i,0} f_i \xi_{i,0}^{-1} J^m \overset{(5.19)}{=} J^{-m} h_i f_i h_i^{-1} J^m \overset{(5.4)}{=} J^{-m} h_i \phi_i h_i^{-1} J^m \overset{(5.18)}{=} J^{-m} P J^m = P. \end{aligned} $$

Let $k > 1$. We consider the cases 1) $j \geqslant 1$ and 2) $j = 0$ separately.

1) If $j \geqslant 1$, then $j - 1 \in \{0, \ldots, k-2\}$ and the homeomorphism $\xi_{i,j-1}$ is correctly defined. We find that

$$ \begin{aligned} {f'}_r = {f'}_{i-j\frac{l}{k}+m} &\overset{(5.21)}{=} X_{i-(j-1)\frac{l}{k}+m} f_{i-j\frac{l}{k}+m} X_{i-j\frac{l}{k}+m}^{-1} \overset{(5.22)}{=} J^{-m} x_{i-(j-1)\frac{l}{k}} J^m f_{i-j\frac{l}{k}+m} J^{-m} x_{i-j\frac{l}{k}}^{-1} J^m \\ &\overset{(5.20)}{=} J^{-m} \xi_{i,j-1} J^m f_{i-j\frac{l}{k}+m} J^{-m} \xi_{i,j}^{-1} J^m \overset{(5.8)}{=} J^{-m} \xi_{i,j-1} f_{i-j\frac{l}{k}} \xi_{i,j}^{-1} J^m \\ &\overset{(5.19)}{=} J^{-m} P^{-j+1} h_i f_{i-(j-1)\frac{l}{k}+(j-2)\frac{l}{k}} \ldots f_{i-(j-1)\frac{l}{k}} f_{i-j\frac{l}{k}} f_{i-j\frac{l}{k}}^{-1} \ldots f_{i-j\frac{l}{k}+(j-1)\frac{l}{k}}^{-1} h_i^{-1} P^j J^m \\ &= J^{-m} P^{-j+1} h_i h_i^{-1} P^j J^m = P. \end{aligned} $$

2) If $j=0$, then $r+\frac{l}{k}=i+\frac{l}{k}+m=i-(k-1)\frac{l}{k}+(m+l)$. We find that

$$f'_r=f'_{i+m}\overset{(5.21)}{=}X_{i-(k-1)\frac{l}{k}+(m+l)}f_{i+m}X^{-1}_{i+m}\overset{(5.22)}{=}J^{-m-l}\xi_{i,k-1}J^{m+l}f_{i+m}J^{-m}\xi^{-1}_{i,0}J^m$$
$$\overset{(5.8)}{=}J^{-m-l}\xi_{i,k-1}J^lf_i\xi^{-1}_{i,0}J^m\overset{(5.19)}{=}J^{-m-l}P^{-k+1}h_if_{i-(k-1)\frac{l}{k}+(k-2)\frac{l}{k}}\dots f_{i-(k-1)\frac{l}{k}}J^lf_ih_i^{-1}J^m$$
$$\overset{(5.8)}{=}J^{-m-l}P^{-k+1}h_iJ^lf_{i+(k-1)\frac{l}{k}}\dots f_{i-\frac{l}{k}}f_ih_i^{-1}J^m\overset{(5.4)}{=}J^{-m-l}P^{-k+1}h_i\phi_ih_i^{-1}J^m$$
$$\overset{(5.18)}{=}J^{-m-l}P^{-k+1}J^lP^kJ^m=P.$$

We find that $\bar{f}'(p_J^{-1}(NW(f'))=\bar{\varphi}_{P,J,n,k,l}(p_J^{-1}(\varphi_{P,J,n,k,l})$.

Consequently, $f'|_{NW(f')}=\varphi_{P,J,n,k,l}|_{NW(\varphi_{P,J,n,k,l})}$ and the homeomorphism $f$ is ambiently $\Omega$-conjugate to the homeomorphism $\varphi_{P,J,n,k,l}$ via the map $X$. □

## FUNDING

The work is supported by the Russian Science Foundation under grant 22-11-00027 except for the results of Section 3 which was supported by the Laboratory of Dynamical Systems and Applications NRU HSE, grant of the Ministry of Science and Higher education of the RF, ag. No. 075-15-2022-1101.

## CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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