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A Complete Complexity Dichotomy of the Edge-Coloring Problem for All Sets of 8-Edge Forbidden Subgraphs

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Abstract—For a given graph, the edge-coloring problem is to minimize the number of colors sufficient to color all the graph edges so that any adjacent edges receive different colors. For all classes defined by sets of forbidden subgraphs, each with 7 edges, the complexity status of this problem is known. In this paper, we obtain a similar result for all sets of 8-edge prohibitions.

Keywords: *edge-coloring problem, computational complexity, monotone class*

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INTRODUCTION

The present paper considers *ordinary* graphs, i.e., undirected graphs without loops and multiple edges. A *hereditary* class of graphs is a set of graphs closed under isomorphism and removal of vertices. Each hereditary class $\mathcal{X}$ (and only hereditary class) can be specified by the set of its forbidden generated subgraphs $\mathcal{Y}$ and is denoted as $\mathcal{X} = \mathrm{Free}(\mathcal{Y})$. A *monotone* class of graphs is a hereditary class that is also closed with respect to the removal of edges. Each monotone class (and only monotone class) $\mathcal{X}$ can be defined by the set of its forbidden subgraphs $\mathcal{Y}$ and is denoted by $\mathcal{X} = \mathrm{Free}_s(\mathcal{Y})$.

Let $G = (V, E)$ be a graph. Any mapping $c\colon E \to \{1, 2, \ldots, k\}$ such that $c(e_1) \neq c(e_2)$ for all adjacent edges e_1 and e_2 is called an *edge k-coloring* of the graph G. The *chromatic index* of G is the smallest number k for which there exists an edge k-coloring of G. It is denoted by $\chi'(G)$.

The *edge k-coloring problem* (k-PP *problem*) for a given graph G is to recognize whether the inequality $\chi'(G) \leq k$ holds. The *edge coloring problem* (PP *problem*) for a given graph G and a number k is to recognize whether the inequality $\chi'(G) \leq k$ holds. It is well known that 3-PP problem (and therefore PP problem) is NP-complete [1].

According to the well-known result of V. G. Vizing [2], the following inequality holds: $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices of G. Thus, PP problem for a graph G is equivalent to recognizing whether the equality $\chi'(G) = \Delta(G)$ is true or not.

The paper [3] presents a complete classification of the complexity of k-PP problem for any k for all hereditary classes defined by one forbidden generated subgraph. A complete dichotomy of the complexity of 3-PP problem for pairs of 6-vertex forbidden generated fragments is obtained in [4], and a similar result for PP problem and families of monotone classes defined by the prohibition of subgraphs with at most 7 vertices or 7 edges each, in [5, 6].

Some results for vertex analogues of k-PP and PP problems are presented in [8–36].

By $G_1 + G_2$ we denote the disjoint union of the graphs G_1 and G_2 with disjoint sets of vertices, and by P_n and O_n, a simple path and an empty graph with n vertices. The paper [7] considered trees B_1^*, B_{1+}^*, ${}^+B_1^*$, and B_1^{+*} (Fig. 1) and proved the following assertion.

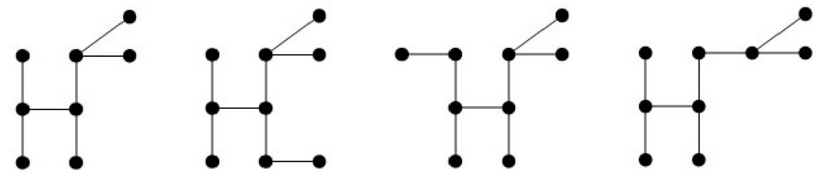

Fig. 1. Graphs B_1^*, B_{1+}^*, $^+B_1^*$, and B_1^{+*}.

Theorem 1. *Let F be an arbitrary 8-edge forest not belonging to the set*

$$\{B_1^* + P_2 + O_n,\ ^+B_1^* + O_n,\ B_1^{+*} + O_n,\ B_{1+}^* + O_n \mid n \geq 0\}.$$

Then PP *problem is polynomially solvable in the class* $\mathrm{Free}_s(\{F\})$. *If F belongs to this set, then* PP *problem is polynomially solvable in the class* $\{G \in \mathrm{Free}_s(\{F\}) \mid \Delta(G) \geq 4\}$.

The present paper improves the results in [6, 7]. Namely, a complete classification of the complexity of PP problem is produced for all sets of 8-edge prohibitions.

1. SOME DEFINITIONS, NOTATION, AND FACTS

The *girth* of a graph is the length of the shortest cycle contained in the given graph. If the graph is acyclic, then its girth is assumed to be equal to infinity. For a graph $G = (V, E)$, the operation of contracting its (connected) subgraph $H = (V', E')$ to a vertex consists of removing all vertices of the subgraph H from G and adding a new vertex v and all edges of the form vu such that $u \in V \setminus V'$ and there exists an edge $wu \in E$, where $w \in V'$.

Let G be some graph, and let x be a vertex of G. The neighborhood of x is denoted by $N(x)$. $\deg(x)$ denotes the degree of x, and $\Delta(G)$ is the maximum degree of the vertices of G. If $\Delta(G) \leq 3$, then G is called *subcubic*. If the degrees of all vertices of the graph are equal to 3, then it is called *cubic*.

The following assertion was proved in Sec. 28.1 in the monograph [37] (see the proof of Theorem 28.1).

Lemma 1. *For any graph G containing a vertex x such that $|\{y \in N(x) \mid \deg(y) = \Delta(G)\}| \leq 1$, one has the relation*

$$\chi'(G) = \Delta(G) \Leftrightarrow \chi'(G \setminus \{x\}) \leq \Delta(G).$$

A *cutpoint* is a vertex of a graph the removal of which increases the number of its connected components. A connected graph G without cutpoint is said to be *incompressible* if any vertex G has at least two neighbors of degree $\Delta(G)$. In [7, Sec. 2] it is noted that the following assertion holds.

Lemma 2. PP *problem for graphs from an arbitrary monotone class reduces polynomially to the same problem for incompressible graphs from this monotone class.*

Let G be a graph, and let $V' \subseteq V(G)$. Then $G[V']$ is a subgraph of the graph G generated by V', and $G \setminus V'$ is the result of removing all elements of V' from G.

Let G_1 and G_2 be graphs. The notation $G_1 \cong G_2$ means that the graphs G_1 and G_2 are isomorphic. If $V(G_1) \cap V(G_2) = \emptyset$, then we denote the graph $(V(G_1) \cup V(G_2),\ E(G_1) \cup E(G_2))$ by $G_1 + G_2$. For the graph G and number k we set $kG = \underbrace{G + G + \cdots + G}_{k \text{ times}}$.

Let G, $H_1, H_2, \ldots, H_k$ be graphs. Then the notation $\langle G; H_1, H_2, \ldots, H_k \rangle$ means that G contains each of the graphs $H_1, H_2, \ldots, H_k$ as a subgraph.

As usual, O_n, K_n, P_n, and C_n denote the empty graph, a complete graph, a simple path, and a simple cycle on n vertices. A complete bipartite graph with p vertices in one part and q vertices

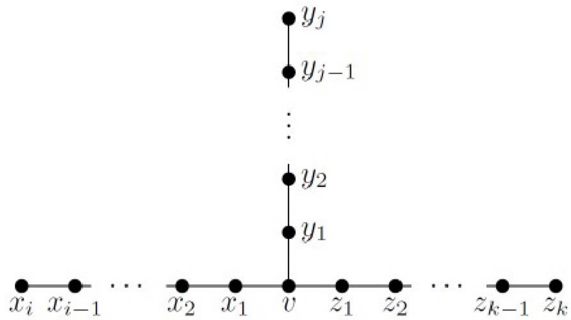

Fig. 2. Graph $T_{i,j,k}$.

in the other is denoted by $K_{p,q}$. By $K_4 - e$ and $K_{3,3} - e$ we denote the results of removing an edge from K_4 and $K_{3,3}$, respectively.

By $T_{i,j,k}$, $i, j, k \geq 0$, we denote a tree, called a *triode*, obtained by identifying the ends of three simple paths $(v = x_0, x_1, \ldots, x_i)$, $(v = y_0, y_1, \ldots, y_j)$, and $(v = z_0, z_1, \ldots, z_k)$ by a vertex v (Fig. 2).

Further in the proofs, for the vertices of the graph $T_{i,j,k}$ we will use the notation introduced when defining it. The class of all forests, each connected component of which is a triode, is denoted by $\mathcal{T}$. The following assertion is connected with this class.

Lemma 3 [7, Lemma 3]. *Let $H' \in \mathcal{T}$ and let $\mathcal{X}$ be a class of graphs, and assume that for some graph H we have $\mathcal{X} \subseteq \mathrm{Free}_s(\{H + H'\})$. Then* PP *problem in the class $\mathcal{X}$ can be polynomially reduced to the same problem in the class $\mathcal{X} \cap \mathrm{Free}_s(\{H\})$.*

A *monotone closure* of a class of graphs $\mathcal{X}$ is the set of all graphs that are subgraphs of graphs from $\mathcal{X}$. It is denoted by $[\mathcal{X}]_s$. The set of pairwise nonadjacent vertices of a graph is called *independent*.

2. NP-COMPLETENESS OF PP PROBLEM FOR SOME CLASSES OF SUBCUBIC GRAPHS

The transformations called *vertex replacement by a triangle* and *vertex replacement by a* (2,3)-*biclique* are well known. They are applied to a vertex x of a graph whose neighborhood consists exactly of the vertices x_1, x_2, x_3 and are defined as follows. In the first one, we remove x and add vertices x_1', x_2', x_3' and edges $x_1'x_1$, $x_2'x_2$, $x_3'x_3$, $x_1'x_2'$, $x_2'x_3'$, $x_1'x_3'$. In the second, we remove x and add vertices y_1, y_2, z_1, z_2, z_3 and edges y_1z_1, y_1z_2, y_1z_3, y_2z_1, y_2z_2, y_2z_3, x_1z_1, x_2z_2, x_3z_3. It is easy to see that a 3-edge coloring of the original graph exists if and only if it exists for the resulting graph.

Let $\mathcal{Z}_k$ denote the set of cubic graphs of girth at least $k+1$, i.e., not containing cycles of length up to k inclusive. It is clear that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ coincide with the set of all cubic graphs. Let us denote by $\mathcal{Z}_k^*$ the set of graphs that are obtained from graphs of class $\mathcal{Z}_k$ by sequentially replacing all their vertices with triangles, with repeated replacement of vertices in newly added triangles not allowed. Let $\mathcal{Z}_k^{**}$ denote the set of graphs that are obtained from graphs of class $\mathcal{Z}_k$ by sequentially replacing all their vertices with (2,3)-bicliques, with repeated replacement of vertices in newly added (2,3)-bicliques not allowed.

The following statement in part of the class $\mathcal{Z}_k^*$ is Lemma 9 in the paper [6], and in the class $\mathcal{Z}_k^{**}$ it can be proven by analogy with it.

Lemma 4. *For each k* PP *problem is* NP-*complete for graphs from the classes $\mathcal{Z}_k^*$ and $\mathcal{Z}_k^{**}$.*

The proof is based on the NP-completeness of 3-PP problem for any k in the set of subcubic graphs of girth at least k (see [38]), and also the equivalence of 3-PP problems for a graph before and after replacing a vertex with a (2,3)-biclique.

Note that every 8-edge graph in $[\mathcal{Z}_4^*]_s$ does not contain cycles of length other than from 3. Note also that every 8-edge graph in $[\mathcal{Z}_4^{**}]_s$ does not contain cycles of length other than 4. It is clear that $[\mathcal{Z}_4^*]_s \subseteq \mathrm{Free}_s(\{B_1^*\})$ and $[\mathcal{Z}_4^{**}]_s \subseteq \mathrm{Free}_s(\{B_{1+}^*\})$.

3. POLYNOMIAL SOLVABILITY OF PP PROBLEM FOR SOME CLASSES OF SUBCUBIC GRAPHS WITHOUT SUBGRAPH B^*_{1+}

By Belt_k, $k \geq 2$, we denote a graph such that

$$V(\mathrm{Belt}_k) = \{v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_k\},$$
$$E(\mathrm{Belt}_k) = \{v_i u_i \mid 1 \leq i \leq k\} \cup \{v_i v_{i+1},\, u_i u_{i+1} \mid 1 \leq i \leq k-1\}.$$

In other words, Belt_k is obtained from C_{2k} by adding $k-2$ parallel chords.

Lemma 5. *Let $G = (V, E) \in \mathrm{Free}_s(\{B^*_{1+}\})$ be an incompressible subcubic graph, and let* Belt_k *be an inclusion-maximal subgraph contained in G as a (not necessarily generated) subgraph but not contained in each of the graphs $K_4 - e$, $K_{2,3}$, and $K_{3,3} - e$, with each of the edges v_1u_1 and v_ku_k not included in any triangle of the graph G. Then*

$$\chi'(G) \leq 3 \Leftrightarrow \chi'\big(G \setminus V(\mathrm{Belt}_k)\big) \leq 3$$

if none of the following conditions is satisfied:

1. $k \in \{3, 4\}$, $v_1v_k \in E \vee u_1u_k \in E$, *there exists an $w \in V$ such that*

$$wu_1, wu_k \in E \vee wv_1, wv_k \in E.$$

2. $k = 3$, *there exist vertices $w_1, w_2, w_3 \in V$ such that*

$$w_1v_1,\ w_1v_3,\ w_2u_1,\ w_2u_3,\ w_1w_3,\ w_2w_3 \in E.$$

In these cases one has $\chi'(G) = 4$.

Proof. Since the graph G is incompressible, it does not contain dangling vertices. Since the subgraph Belt_k under consideration is not contained in any of the graphs $K_4 - e, K_{3,3} - e$, then v_1u_2, v_2u_1, v_1u_3 , $v_3u_1 \notin E$.

Assume that $v_1u_k \in E$. Then $k \geq 4$. If u_1 or v_k has a neighbor outside $V(\mathrm{Belt}_k)$, then $\langle G; B^*_{1+}\rangle$. If neither u_1 nor v_k has a neighbor outside $V(\mathrm{Belt}_k)$, then $\chi'(G) = 3$. Throughout what follows we will assume that v_1u_k, $u_1v_k \notin E$.

Assume that $v_1v_k \in E$ for $k \geq 3$. If $\deg(u_1) = \deg(u_k) = 2$ or $u_1u_k \in E$, then $\chi'(G) = 3$. Since G is incompressible (consequently, does not contain bridges), we have

$$N(u_1) = \{z', v_1, u_2\},$$
$$N(u_k) = \{z'', v_k, u_{k-1}\}.$$

Since $\neg\langle G; B^*_{1+}\rangle$, we have $k \leq 4$. If $z' = z''$, then $\chi'(G) = 4$. If $z' \neq z''$, then $\deg(z') = \deg(z'') = 2$ because $\neg\langle G; B^*_{1+}\rangle$. It can readily be seen that

$$\chi'(G) = 3 \Leftrightarrow \chi'\big(G \setminus V(\mathrm{Belt}_k)\big) \leq 3;$$

this can be verified by taking the 3-coloring of the edges of the graph $G \setminus V(\mathrm{Belt}_k)$ and coloring $z'u_1$, $z''u_k$, v_1v_k, v_2u_2, $\ldots$, $v_{k-1}u_{k-1}$ in one color with further coloring of the remaining edges Belt_k in two colors. Throughout, we will assume that v_1v_k, $u_1u_k \notin E$ for $k \geq 3$.

Set

$$N' = \big(N(v_1) \cup N(u_1) \cup N(v_k) \cup N(u_k)\big) \setminus V(\mathrm{Belt}_k).$$

Due to the incompressibility of G, the set N' contains a vertex adjacent to some vertex of $\{v_1, u_1\}$, as well as a vertex adjacent to some vertex of $\{v_k, u_k\}$.

Suppose that for each pair of vertices $\{v_1, u_1\}$ and $\{v_k, u_k\}$ either at least one of these vertices has degree 2 or at least one of them is adjacent to a vertex of degree 2 from N'. Then

$$\chi'(G) = 3 \Leftrightarrow \chi'\big(G \setminus V(\mathrm{Belt}_k)\big) \leq 3.$$

Indeed, it suffices to consider the 3-coloring of edges $G \setminus V(\mathrm{Belt}_k)$ and in G, to color the edges between $(N(v_1) \cup N(u_1)) \setminus V(\mathrm{Belt}_k)$ and $\{v_1, u_1\}$ in one color, and to color the edges between $(N(v_k) \cup N(u_k)) \setminus V(\mathrm{Belt}_k)$ and $\{v_k, u_k\}$ also in one color. Thus, throughout what follows we assume that $\deg(u_1) = \deg(v_1) = 3$ and both of these vertices are adjacent to different vertices of degree 3 from N'.

Assume that

$$N(v_1) = \{z_1, v_2, u_1\},$$
$$N(u_1) = \{z_2, v_1, u_2\}.$$

Then each of the following statements is true:

- $\deg(z_1) = \deg(z_2) = 3$.
- $z_2 \notin N(z_1)$ in view of maximality of the subgraph Belt_k.
- For $k = 2$ one has $z_2 \notin N(v_2)$, since otherwise the subgraph Belt_2 under consideration is embedded in the subgraph $K_{3,3} - e$.
- For $k \geq 3$ one has $z_1 v_k$, $z_2 u_k \notin E$, since otherwise $\langle G; B^*_{1+}\rangle$.

If $k = 3$, then $z_2 \neq v_3$ and $z_1 \neq u_3$ due to the absence of bridges in G. Let us consider the situation when $z_2 u_3 \notin E$. Since $G \in \mathrm{Free}_s(\{B^*_{1+}\})$, we have $\deg(z_2) = 2$, and this case was analyzed earlier. The situation when $z_1 v_3 \notin E$ is considered similarly.

Consider the case where $N(z_1) = \{v_1, v_3, z_3\}$, $N(z_2) = \{u_1, u_3, z_4\}$. Since $G \in \mathrm{Free}_s(\{B^*_{1+}\})$, we have $\deg(z_3) = \deg(z_4) = 2$. If $z_3 \neq z_4$, then

$$\chi'(G) = 3 \Leftrightarrow \chi'\Big(G \setminus \big(V(\mathrm{Belt}_3) \cup \{z_1, z_2\}\big)\Big) \leq 3,$$

since it suffices to consider the 3-coloring of edges in $G \setminus (V(\mathrm{Belt}_3) \cup \{z_1, z_2\})$ and to color $z_1 z_3$, $z_2 z_4$, $v_1 u_1$, $v_2 u_2$, and $v_3 u_3$ in G in one color. If $z_3 = z_4$, then $\chi(G) = 4$.

If $k = 2$, then, due to the incompressibility of G and the conditions in the lemma, we have

$$\max\big(\deg(v_2), \deg(u_2)\big) = 3, \quad z_1 \notin N(u_2), \quad N(z_1) \cap N(v_2) = \{v_1\}.$$

The graph G contains the subgraph B^*_{1+} if $\deg(v_2) = 3$. If $\deg(v_2) = 2$, then $N(u_2) = \{v_2, u_1, z_5\}$, and $z_2 z_5 \notin E$ in view of the maximality of Belt_2. It can readily be seen that $\deg(z_2) = 2$, since $G \in \mathrm{Free}_s(\{B^*_{1+}\})$. This case was treated earlier. The proof of Lemma 5 is complete. $\square$

Lemma 6. *Let H^* and H^{**} be 8-edge graphs belonging to $[\mathcal{Z}^*_4]_s$ and $[\mathcal{Z}^{**}_4]_s$, respectively. Then* PP *problem is polynomially solvable for subcubic graphs of the class* $\mathrm{Free}_s(\{B^*_{1+}, H^*, H^{**}\})$.

Proof. By Lemma 3, we will assume that each connected component of the graphs H^* and H^{**} does not belong to $\mathcal{T}$. By Theorem 1, we assume that either $H^* = B^{+*}_1$ or H^* is not a forest, and also that either $H^{**} \in \{{}^+B^*_1, B^{+*}_1\}$ or H^{**} is not a forest. By Lemma 2, we will consider only incompressible graphs of the class $\mathrm{Free}_s(\{B^*_{1+}, H^*, H^{**}\})$. Let $G = (V, E)$ be an arbitrary such graph.

If $N(x) = \{x_1, x_2, x_3\}$ for some vertex $x \in V$, then x belongs either to a triangle or to a generated C_4-cycle of the graph G. Indeed, suppose that $\{x_1, x_2, x_3\}$ is an independent set. Among its elements, at least two (say, x_1 and x_2) have degree 3 in G, and the vertex x_3 has degree at least 2. We can assume that

$$N(x_1) \cap N(x_2) = N(x_2) \cap N(x_3) = N(x_1) \cap N(x_3) = \{x\},$$

otherwise x belongs to the generated C_4-cycle of the graph G, but then G contains the subgraph B^*_{1+}.

In the graph G, the set $\mathfrak{B}$ of all its maximal subgraphs of the form Belt_k that are simultaneously not contained in the subgraphs $K_4 - e$, $K_{2,3}$ and $K_{3,3} - e$ can be found in polynomial time. If this set is not empty, then consider an arbitrary subgraph Belt_k. Based on the proof of Lemma 5, we can assume that v_1v_k, $u_1u_k \notin E$ and case (2) is not realized. By Lemma 5, we can assume that there exists a vertex $w \in V$ that forms a triangle with v_k and u_k. Let wv_1, $wu_1 \notin E$, otherwise $\chi'(G) = 4$. If $k = 2$, then

$$N(w) \cap \big(N(v_1) \cup N(u_1)\big) = \{v_2, u_2\}$$

in view of the maximality of the subgraph Belt_2. Since $\neg\langle G; B^*_{1+}\rangle$ and G incompressible, either v_1u_1 is included in a triangle or v_1 is adjacent with a vertex of degree 3 and $\deg(u_1) = 2$ or each of the vertices v_1 and u_1 is adjacent with its vertex of degree two, with these two vertices not being adjacent. In the last case, we contract $G[V(\mathrm{Belt}_k) \cup \{w\}]$ into the vertex w' to obtain a graph G' that will be ordinary (since G is incompressible) with

$$\chi'(G) = 3 \Leftrightarrow \chi'(G') \le 3.$$

By Lemma 1,

$$\chi'(G') \le 3 \Leftrightarrow \chi'\big(G' \setminus \{w'\}\big) \le 3,$$

with $G' \setminus \{w'\} \cong G \setminus (V(\mathrm{Belt}_k) \cup \{w\})$. Therewith we assume that either v_1u_1 is included in the triangle (v_1, u_1, w'') or v_1 is adjacent with a vertex of degree 3 and $\deg(u_1) = 2$.

Note that

$$\chi'(G) = 3 \Leftrightarrow \chi'\big(G \setminus V(\mathrm{Belt}_k)\big) \le 3,$$

if $\deg(w) = 2$, there exists a vertex $w_1 \notin \{v_k, u_k\}$, $\deg(w_1) = 2$ such that $ww_1 \in E$ and v_1u_1 does not lie in the triangle (v_1, u_1, w''), where w'' has a neighbor of degree 3 outside $V(\mathrm{Belt}_k)$; therefore in what follows, we assume that $\deg(w_1) = 3$.

In addition, suppose that (w_1, w_2, w_3) is a triangle in the graph G. Obviously,

$$\{v_1, u_1\} \cap \{w_2, w_3\} = \emptyset \vee \{v_1, u_1\} = \{w_2, w_3\},$$

and in the last case we have $\chi'(G) = 3$. In view of the incompressibility of the graph G, one of the vertices w_2 and w_3 has degree 3. Note that if $v_1w_2 \in E$, then $N(u_1) \subseteq \{v_1, u_2, w_3\}$, since $\langle G; B^*_{1+}\rangle$; therefore $|V(G)| = 2k + 4$, as otherwise G contains a cutpoint. It follows from our reasoning that $\langle G; H^*\rangle$, since G contains all 8-edge graphs in $[\mathcal{Z}^*_4]_s$ that are not forests such that each connected component does not belong to $\mathcal{T}$. If w_1 is a vertex of degree 2 of the subgraph $K_{2,3}$, then one of the other two vertices of degree 2 of this subgraph has degree 3 in G. It is easy to verify that $\langle G; H^{**}\rangle$.

Thus w_1 is a vertex of degree 2 of some element $\mathrm{Belt}_{k'} \in \mathfrak{B}$. By Lemma 5, we can assume that there exists a vertex that forms a triangle with two vertices $\mathrm{Belt}_{k'}$, but then

$$\chi'(G) = 3 \Leftrightarrow \chi'\big(G \setminus V(\mathrm{Belt}_k)\big) \le 3.$$

Hence, according to our reasoning, we can assume that each generated C_4-cycle of the graph G is included in a certain subgraph $K_{2,3}$, which can be considered generated by the incompressibility of G. By Lemma 3, let G contain $2T_{5,5,5}$ as a subgraph. Let us consider an arbitrary component of the connected components of this $2T_{5,5,5}$.

Since G is incompressible and $\neg\langle G; B^*_{1+}\rangle$, from symmetry considerations we can assume that either $x_1y_1 \in E$ or $x_1y_2 \in E$ and $y_1x_2 \in E$ or there exists a vertex $t \notin V(T_{5,5,5})$ such that x_1t, y_1t, $z_1t \in E$. In each of these cases, we contract either $G[\{v, x_1, y_1\}]$ or $G[\{v, x_1, y_1, x_2, y_2\}]$ or $G[\{v, x_1, y_1, z_1, t\}]$ to the vertex v^* to obtain a graph G^* for which

$$\chi'(G) = 3 \Leftrightarrow \chi'(G^*) \le 3.$$

At the same time, if the vertex v^* has at most one neighbor of degree 3 in G^*, then by Lemma 1 we have

$$\chi'(G^*) = 3 \Leftrightarrow \chi'\big(G^* \setminus \{v^*\}\big) \le 3.$$

Since $G^* \setminus \{v^*\}$ is a generated subgraph of the graph G, we assume that this case is not realized.

If $x_1y_1 \in E$, then we can assume that the vertices x_2 and y_2 have degree 3 in G. Since $\neg\langle G; B^*_{1+}, H^*\rangle$, we conclude that x_2 and y_2 are contained in generated subgraphs each of which is isomorphic to $K_{2,3}$. Then $\langle G; H^{**}\rangle$, so we can assume that the first case is not realized for any of the components of $2T_{5,5,5}$. In the second case, we can assume that $\deg(x_3) = 3$. If x_3 is contained in the generated copy of $K_{2,3}$, then $\langle G; H^{**}\rangle$. If x_3 is contained in a triangle, then also $\langle G; H^{**}\rangle$ (note that the subgraph $2C_4$ is obtained from two different components of $2T_{5,5,5}$). The third case is similar to the second one.

Lemma 3 implies the assertion in the present lemma. The proof of Lemma 6 is complete. □

4. POLYNOMIAL SOLVABILITY OF PP PROBLEM FOR SOME CLASSES OF SUBCUBIC GRAPHS WITHOUT SUBGRAPHS $B_1^* + P_2$, ${}^+B_1^*$, B_1^{+*}

Lemma 7. *Let $H \in \{B_1^* + P_2, {}^+B_1^*, B_1^{+*}\}$, and let H^* be an 8-edge graph belonging to $[\mathcal{Z}_4^*]_s$. Then* PP *problem is polynomially solvable for subcubic graphs in* $\mathrm{Free}_s(\{H, H^*\})$.

Proof. Let us show that in the class $\mathrm{Free}_s(\{H, H^*\})$, PP problem is polynomially reducible to the same problem for graphs in $\mathrm{Free}_s(\{B_1^*, H,\ H^*, T_{5,5,5}\})$. Taking this into account, the validity of Lemma 7 will follow from Lemma 3. By Theorem 1 and Lemma 3, we will assume that $H \in \{{}^+B_1^*, B_1^{+*}\}$, H^* is not a forest for $H^* \neq B_1^{+*}$, and each connected component of H^* does not belong to $\mathcal{T}$.

By Lemma 2, it suffices to consider the incompressible graph $G = (V, E) \in \mathrm{Free}_s(\{H, H^*\})$. Suppose that G contains a subgraph B_1^* with the set of vertices $\{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2\}$ and the set of edges $\{a_1a_2,\ a_2a_3,\ b_1b_2,\ b_2b_3,\ a_2c_1,\ b_2c_1,\ c_1c_2\}$. For $H = {}^+B_1^*$, none of the vertices a_1, a_3, b_1, b_3 has a neighbor outside $V(B_1^*)$, so either $|V(G)| = 8$ or c_2 is a cutpoint in the graph G.

Consider the case of $H = B_1^{+*}$. It is easy to see that either $|V(G)| \leq 16$ or c_2 is a cutpoint in the graph G or there exists a path (u, u_1, u_2) in which $u \in \{a_1, a_3, b_1, b_3\}$ and $u_1, u_2 \notin V(B_1^*)$. We can assume that $u = b_1$. Since $\neg\langle G; B_1^{+*}\rangle$ and G is incompressible, we conclude that u_1c_2, u_1a_1, $u_1a_3 \notin E$, and if $\deg(b_1) = 2$, then $\deg(u_1) = 3$, $u_1b_3 \in E$, $\deg(b_3) = 2$, which is impossible due to the incompressibility of G. For the same reasons, if $\deg(b_1) = 3$, then

$$b_1a_1 \in E \vee b_1a_3 \in E, \quad \deg(u_1) = 2, \quad \deg(u_2) = 3;$$

this implies that $\langle G; B_1^{+*}\rangle$.

Thus, we can assume that G lies in $\mathrm{Free}_s(\{B_1^*\})$ and contains the subgraph $T_{5,5,5}$ by Lemma 3. Let us consider two options: (1) the set $\{x_1, y_1, z_1\}$ is independent, (2) the set $\{x_1, y_1, z_1\}$ is not independent.

1. In view of the incompressibility of the graph G we can assume that $\deg(x_1) = \deg(y_1) = 3$. Let $N(x_1) = \{x_1', v, x_2\}$ and $N(y_1) = \{y_1', v, y_2\}$. It is obvious that either $x_1' = y_1'$ or $x_1' = y_2$ or $y_1' = x_2$, otherwise $\langle G; B_1^*\rangle$.

   Assume that $x_1' = y_1'$. Then since $\neg\langle G; B_1^*\rangle$, for each vertex $u \in \{x_2, y_2, z_1\}$ either $\deg(u) = 2$ or $ux_1' \in E$, and if $x_1'z_1 \in E$, then $\deg(z_2) = 2$, while if $x_1'x_2,\ x_1'y_2,\ x_1'z_1 \notin E$, then either $\deg(x_1') = 2$ or x_1' is adjacent to the vertex x_1'' of degree 2. It can readily be seen that if $x_1'x_2$, $x_1'y_2 \notin E$, then

   $$\chi'(G) = 3 \Leftrightarrow \chi'\big(G \setminus \{v, x_1, y_1, x_1'\}\big) \leq 3,$$

   since it is sufficient to color y_1y_2 and z_1v in one color and x_1x_2 and $x_1'x_1''$ (if such a vertex x_1'' exists) also in one color.

   Suppose that $x_1'x_2 \in E$, $x_1'y_2 \notin E$ (the case of $x_1'x_2 \notin E$ and $x_1'y_2 \in E$ is treated similarly). Then $\deg(y_2) = \deg(z_1) = 2$. Let us contract the subgraph $G[\{v, x_1, y_1, x_2, x_1'\}]$ into the vertex u_1 to obtain a graph G_1 for which the following holds by Lemma 1:

   $$\chi'(G) = 3 \Leftrightarrow \chi'(G_1) \Leftrightarrow \chi'\big(G_1 \setminus \{u_1\}\big) \leq 3,$$
   $$G_1 \setminus \{u_1\} \cong G \setminus \{v, x_1, y_1, x_2, x_1'\}.$$

   Suppose that $x_1'z_1 \in E$. Then $\deg(x_2) = \deg(y_2) = \deg(z_2) = 2$. Let us contract $G[\{v, x_1, y_1, z_1, x_1'\}]$ into the vertex u_2 to obtain a graph G_2 for which the following holds by Lemma 1:

   $$\chi'(G) = 3 \Leftrightarrow \chi'(G_2) \leq 3 \Leftrightarrow \chi'\big(G_2 \setminus \{u_2\}\big) \leq 3,$$

$$G_2 \setminus \{u_2\} \cong G \setminus \{v, x_1, y_1, z_1, x_1'\}.$$

The cases where $x_1' = y_2$ or $y_1' = x_2$ are treated similarly. To do this, it is sufficient to consider a triode centered at the vertex x_1 or at the vertex y_1 and repeat the presented proof for this triode.

2. From symmetry considerations we can assume that $x_1 y_1 \in E$. Let us contract the triangle (v, x_1, y_1) to the vertex u_3 to obtain the graph G_3. It is clear that $\chi'(G) = 3 \Leftrightarrow \chi'(G_3) \leq 3$. If at most one vertex has degree 3 among x_2, y_2, z_1, then, by Lemma 1 we have

$$\chi'(G_3) \leq 3 \Leftrightarrow \chi'\big(G_3 \setminus \{u_3\}\big) \leq 3,$$
$$G_3 \setminus \{u_3\} \cong G \setminus \{v, x_1, y_1\}.$$

Taking this into account, due to symmetry, we can assume that $N(x_2) = \{x_2', x_1, x_3\}$ and $N(y_2) = \{y_2', y_1, y_3\}$.

Let us assume that $x_2' = y_2$. Let us contract the subgraph $G[\{v, x_1, y_1, x_2, y_2\}]$ into the vertex u_4 to obtain a graph G_4 such that $\chi'(G) = 3 \Leftrightarrow \chi'(G_4) \leq 3$. We can assume that $\deg(x_3) = 3$ or $\deg(y_3) = 3$, otherwise by Lemma 1,

$$\chi'(G_4) \leq 3 \Leftrightarrow \chi'\big(G_4 \setminus \{u_4\}\big) \leq 3,$$
$$G_4 \setminus \{u_4\} \cong G \setminus \{v, x_1, y_1, x_2, y_2\};$$

therefore $\langle G; B_1^* \rangle$, and further we assume that $x_2' \neq y_2$.

If $x_2' x_3 \notin E$, then, due to the incompressibility of G, either $\deg(x_3) = 3$ or $\deg(x_2') = 3$, so $\langle G; B_1^* \rangle$. If $x_2' x_3$ and $x_2' y_2 \in E$, then $\langle G; B_1^* \rangle$. If $x_2' x_3 \in E$ and $x_2' y_2 \notin E$, then G contains all 8-edge graphs in $[\mathcal{Z}_4^*]_s$ that are not forests each connected component of which does not belong to $\mathcal{T}$, so $\langle G; H^* \rangle$.

Lemma 3 implies the assertion in the present lemma. The proof of Lemma 7 is complete. □

5. MAIN RESULT

The main result of the present paper is the following assertion.

Theorem 2. *Let $\mathcal{Y}$ be a set of graphs each of which has at most 8 edges. Then* PP *problem is polynomially solvable for graphs in $\mathcal{X} = \mathrm{Free}_s(\mathcal{Y})$ if*

1. *Either $\mathcal{Y}$ contains a subcubic forest not belonging to the set*

$$\Big[\big\{B_1^* + P_2 + O_n,\ {}^{+}B_1^* + O_n,\ B_{1+}^* + O_n \mid n \geq 0\big\}\Big]_s.$$

2. *Either $\mathcal{Y}$ simultaneously contains a graph in $[\mathcal{Z}_4^*]_s$ and a graph in*

$$\Big[\big\{B_1^* + P_2 + O_n,\ {}^{+}B_1^* + O_n \mid n \geq 0\big\}\Big]_s.$$

3. *Or $\mathcal{Y}$ simultaneously contains a graph in $[\{B_{1+}^* + O_n \mid n \geq 0\}]_s$ and graphs in $[\mathcal{Z}_4^*]_s$ and $[\mathcal{Z}_4^{**}]_s$.*

In all other cases it is NP*-complete for graphs in $\mathcal{X}$.*

Proof. Recall that PP problem is NP-complete in the class $\mathcal{Z}_k$ for any k (see [38]). Therefore, we can assume that $\mathcal{Z}_k \nsubseteq \mathcal{X}$ for any k. Note that $\mathcal{Y}$ is finite (up to the addition of isolated vertices) and for any graph G^* that is not a subcubic forest, there exists a k^* (which can be set equal to the girth of the graph G^*) such that $\mathcal{Z}_{k^*+1} \subseteq \mathrm{Free}_s(\{G^*\})$. Therefore, $\mathcal{Y}$ contains a subcubic forest.

By Lemma 3, we can assume that each connected component of each graph in $\mathcal{Y}$ does not belong to $\mathcal{T}$. Let $F \in \mathcal{Y}$ be a subcubic forest. Note that $[\{B_1^{+*} + O_n \mid n \geq 0\}]_s \subseteq [\mathcal{Z}_4^*]_s$. By Theorem 1 and Lemma 7, PP problem is polynomially solvable in the class $\mathrm{Free}_s(\{F\})$ if F does not belong to the set

$$\Big[\big\{B_1^* + P_2 + O_n,\ {}^+B_1^* + O_n,\ B_{1+}^* + O_n \mid n \geq 0\big\}\Big]_s.$$

If F is a graph in

$$\Big[\big\{B_1^* + P_2 + O_n,\ {}^+B_1^* + O_n \mid n \geq 0\big\}\Big]_s$$

and $\mathcal{Y} \cap [\mathcal{Z}_4^*]_s = \emptyset$, then $\mathcal{Z}_4^* \subseteq \mathcal{X}$. PP problem is NP-complete for graphs in $\mathcal{X}$ by Lemma 4. If however $\mathcal{Y} \cap [\mathcal{Z}_4^*]_s \neq \emptyset$, then PP problem is polynomially solvable for graphs in $\mathcal{X}$ by Lemma 7.

Assume that $F \in [\{B_{1+}^* + O_n \mid n \geq 0\}]_s$. If

$$\mathcal{Y} \cap [\mathcal{Z}_4^*]_s = \emptyset \vee \mathcal{Y} \cap [\mathcal{Z}_4^{**}]_s = \emptyset,$$

then $\mathcal{Z}_4^* \subseteq \mathcal{X} \vee \mathcal{Z}_4^{**} \subseteq \mathcal{X}$ and PP problem is NP-complete for graphs in $\mathcal{X}$ by Lemma 4. If however $\mathcal{Y} \cap [\mathcal{Z}_4^*]_s \neq \emptyset$ and $\mathcal{Y} \cap [\mathcal{Z}_4^{**}]_s \neq \emptyset$, then PP problem is polynomially solvable for graphs in $\mathcal{X}$ by Lemma 6. The proof of Theorem 2 is complete. □

CONCLUSIONS

The present paper gives a complete classification of the computational complexity of the edge coloring problem for all classes of graphs defined by sets of forbidden subgraphs each of which has no more than 8 edges. Namely, for each such set of prohibitions it is established that for the class of graphs it defines, the edge coloring problem is either polynomially solvable or NP-complete. This classification completes the development of the results in the paper [7].

The next natural step is to consider 9-edge prohibitions. Obtaining a complete complexity dichotomy for them and the edge coloring problem is an interesting goal for future research. Apparently, this problem is much more difficult than for the 8-edge case.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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