# LASCOUX POLYNOMIALS AND SUBDIVISIONS OF GELFAND–ZETLIN POLYTOPES

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ABSTRACT. We give a new combinatorial description for stable Grothendieck polynomials in terms of subdivisions of Gelfand–Zetlin polytopes. Moreover, these subdivisions also provide a description of Lascoux polynomials. This generalizes a similar result on key polynomials by Kiritchenko, Smirnov, and Timorin.

## 1. Introduction

In this paper, we provide a new combinatorial description of Lascoux polynomials in terms of subdivisions of Gelfand–Zetlin polytopes and certain collections of their faces. Lascoux polynomials, denoted by  $\mathcal{L}_{\alpha}$ , form a basis for  $\mathbb{Z}[\beta][x_1, x_2, \ldots]$ , where  $\alpha$  runs over the set of weak compositions (i.e., infinite sequences of nonnegative integers with finitely many positive entries). They can be viewed as simultaneous generalizations of key polynomials, i.e. the characters of Demazure modules, and stable Grothendieck polynomials; the latter family represents classes of structure sheaves of Schubert varieties in the connective K-theory of a Grassmannian, as shown by A. Buch [Buc02]. Both of these families are superfamilies of Schur polynomials.

Lascoux polynomials were defined by A. Lascoux [Las04] in terms of homogeneous divided difference operators; just as many other remarkable families of polynomials, they have nonnegative coefficients. Originally Lascoux defined these polynomials were defined for  $\beta = -1$ ; the idea of considering  $\beta$  as a parameter is motivated by considering the connective K-theory and goes to works of S. Fomin and An. Kirillov [FK96], [FK94]. Although Lascoux polynomials do not have a description in geometric or representation-theoretic terms and still remain a purely combinatorial object, they admit several combinatorial descriptions: for example, V. Buciucas, T. Scrimshaw, and K. Weber [BSW20] establish their connection to the five-vertex model, while T. Yu [Yu21] provides their description in terms of set-valued tableaux, generalizing simultaneously Buch's description of stable Grothendieck polynomials in terms of set-valued Young tableaux and the description of key polynomials using skyline fillings.

In [LS89] A. Lascoux and M.-P. Schützenberger conjectured an expansion of Schubert polynomials into key polynomials, proven by V. Reiner and M. Shimozono in [RS95]. This expansion was generalized by V. Reiner and A. Yong in [RY21], who found a K-theoretic counterpart of this expansion, conjecturing an expansion of Grothendieck polynomials into Lascoux polynomials. Their conjecture was proven by Shimozono and Yu in [SY23] (see also [SY22]).

The top-degree components of Lascoux polynomials have also recently attracted attention of researchers. In [PY23] by J. Pan and T. Yu describes the top-degree components of Lascoux polynomials, establishing their connection to the Castelnuovo–Mumford polynomials, aka the top-degree components of Grothendieck polynomials, defined by O. Pechenik, D. Speyer and A. Weigandt in [PSW21]. A recent preprint [Yu23] describes the connection between the top Lascoux polynomials and Schubert polynomials.

Lascoux polynomials  $\mathcal{L}_{\alpha}$  specialized at  $\beta = 0$  are equal to key polynomials. Suppose  $w \in S_n$  is a permutation such that  $\alpha = (\alpha_1, \dots, \alpha_n) = w(\lambda)$  for a suitable partition  $\lambda = (\lambda_1, \dots, \lambda_n)$ . The key polynomials  $\kappa_{\alpha} = \kappa_{w,\lambda}$  are defined as the characters of Demazure modules  $D_{w,\lambda}$ , i.e. B-submodules in the irreducible GL(n)-representation  $V_{\lambda}$  with the highest weight  $\lambda$ . The module  $D_{w,\lambda}$  is defined as the smallest B-submodule containing the extremal vector  $wv_{\lambda} \in V_{\lambda}$ , where  $B \subset GL(n)$  is a fixed Borel subgroup. Demazure modules appear in many problems in representation theory; their characters admit a number of explicit combinatorial descriptions.

In [KST12], V. Kiritchenko, E. Smirnov, and V. Timorin provide a formula for key polynomials in terms of integer points in Gelfand–Zetlin polytopes. Let  $\lambda$  be a strictly dominant weight for GL(n); then it defines an integer convex polytope  $GZ(\lambda) \subset \mathbb{R}^{n(n-1)/2}$ , called the Gelfand–Zetlin polytope (precise definitions follow in § 2.2). This polytope admits a projection  $\pi \colon GZ(\lambda) \to \operatorname{wt}(\lambda)$  into the weight polytope of  $V_{\lambda}$ . For each permutation  $w \in S_n$ , one can construct a collection of faces  $F_w$ ,  $\lambda$  of  $GZ(\lambda)$ , such that  $k_{w,\lambda}$  equals the sum of formal exponentials  $\sum \exp(\pi(z))$ , where z ranges over the set of integer points in  $F_{w,\lambda}$  (see [KST12, Corollary 5.2]).

The main purpose of this paper is to generalize this result, constructing a combinatorial description of Grothendieck and Lascoux polynomials in terms of subdivisions of Gelfand–Zetlin polytopes. For this we construct a cellular decomposition  $\mathscr{C}$  of  $GZ(\lambda)$  such that the set of its 0-cells coincides with the set of integer points in  $GZ(\lambda)$ . Now, to each *i*-dimensional cell  $C_i$  we assign a monomial  $m(C_i)$  in  $x_1, \ldots, x_n$ ; for a 0-cell  $z \in GZ(\lambda)$  this monomial is just  $\exp(\pi(z))$ . Some cells correspond to the zero monomial. Our main result is as follows:

$$\mathscr{L}_{w,\lambda} = \sum_{C_i \in \mathscr{C} \cap F_{w,\lambda}} \beta^i m(C_i),$$

where the sum is taken over all cells situated inside the collection of faces  $F_{w,\lambda}$ .

Informally, the Lascoux polynomial  $\mathcal{L}_{w,\lambda}$  can be viewed as a "weighted Euler characteristic" of the subdivision  $\mathcal{C} \cap F_{w,\lambda}$  for the collection of faces  $F_{w,\lambda}$ . Namely, *i*-dimensional cells of this subdivision correspond to monomials of degree  $i + \ell(w)$  with coefficient  $\beta^i$  in front of them.

This paper is organized as follows. We recall the definitions of Lascoux polynomials and Gelfand–Zetlin polytopes in Section 2. In Section 3 we consider the first interesting example, describing the construction for n = 3. The main result is stated in Section 4. Its proof is given in Section 5 for the case of the longest permutation  $w = w_0$  and in Section 6 for an arbitrary permutation w, respectively.

#### 2. Preliminaries

In the first part of this section we begin with definitions of divided difference operators and Demazure–Lascoux operators and their properties. Next, we define the Lascoux polynomials and state how they are related to key polynomials, stable Grothendieck polynomials, and Schur polynomials. In the second part of this section we will describe the Gelfand–Zetlin polytopes and their faces.

2.1. **Lascoux polynomials.** To define Lascoux polynomials, we need two families of operators: divided difference operators  $\partial_i$ , with  $1 \leq i \leq n-1$ , acting on the polynomial ring  $\mathbb{Z}[x_1, \ldots, x_n]$  and Demazure-Lascoux operators  $\pi_i^{(\beta)}$ , again with  $1 \leq i \leq n-1$ , acting on the ring  $\mathbb{Z}[\beta, x_1, \ldots, x_n]$  equipped with a formal parameter  $\beta$ .

The parameter  $\beta$  appears in the connective K-theory of a Grassmannian; taking  $\beta = -1$  and  $\beta = 0$ , we recover the usual K-theory and the cohomology ring of a Grassmannian respectively.

**Definition 2.1.** The  $i^{th}$  divided difference operator  $\partial_i$  acts on polynomial  $f = f(x_1, x_2, ...)$  in the following way:

$$\partial_i(f) = \frac{f - s_i f}{x_i - x_{i+1}},$$

where  $s_i f$  is obtained from f by permuting variables  $x_i$  and  $x_{i+1}$ .

**Definition 2.2.** The *i*<sup>th</sup> Demazure–Lascoux operator  $\pi_i^{(\beta)}$  acts on polynomial  $f \in \mathbb{Z}[\beta][x_1, x_2, \ldots]$  in the following way:

$$\pi_i^{(\beta)}(f) = \partial_i(x_i f + \beta x_i x_{i+1} f).$$

The following properties of Demazure–Lascoux operators are immediate.

**Proposition 2.3.** Demazure–Lascoux operators  $\pi_i^{(\beta)}$  are idempotent linear operators satisfying the braid relations. Namely:

• If 
$$f = s_i f$$
, then  $\pi_i^{(\beta)}(f) = f$ ;

$$\begin{split} \bullet \ & (\pi_i^{(\beta)})^2 = \pi_i^{(\beta)}; \\ \bullet \ & \pi_i^{(\beta)} \pi_j^{(\beta)} = \pi_j^{(\beta)} \pi_i^{(\beta)} \ if \ |i-j| > 1; \\ \bullet \ & \pi_i^{(\beta)} \pi_{i+1}^{(\beta)} \pi_i^{(\beta)} = \pi_{i+1}^{(\beta)} \pi_i^{(\beta)} \pi_{i+1}^{(\beta)}. \end{split}$$

*Proof.* Straightforward computation.

Let  $\alpha = (\alpha_1, \alpha_2, ...)$  be an infinite sequence of nonnegative integers with finitely many positive entries.

**Definition 2.4.** The Lascoux polynomial  $\mathcal{L}_{\alpha} \in \mathbb{Z}[\beta][x_1, x_2, \ldots]$  associated with  $\alpha$  is defined by:

$$\mathcal{L}_{\alpha} = \begin{cases} x^{\alpha} & \text{if } \alpha \text{ is a partition: } \alpha_{1} \geq \alpha_{2} \geq \dots \\ \pi_{i}^{(\beta)}(\mathcal{L}_{s_{i}\alpha}) & \text{otherwise, where } \alpha_{i} < \alpha_{i+1} \end{cases}$$

Since the Demazure–Lascoux operators satisfy the braid relations, we can associate a Lascoux polynomial to partition  $\lambda$  and permutation  $w \in S_n$  in the following way:

$$\mathscr{L}_{w,\lambda} = \pi_{i_k}^{(\beta)} \dots \pi_{i_2}^{(\beta)} \pi_{i_1}^{(\beta)}(x^{\lambda}),$$

where  $(s_{i_k}, \ldots, s_{i_1})$  is a reduced word for permutation  $w = s_{i_1} \ldots s_{i_k}$ .

It is well-known (cf., for instance, [Yu21]) that specializations of Lascoux polynomials provide other nice families of polynomials.

**Theorem 2.5.** (1) Key polynomials are obtained by specializing the Lascoux polynomials at  $\beta = 0$ :

$$\kappa_{w,\lambda} = \mathscr{L}_{w,\lambda} \mid_{\beta=0};$$

(2) Stable Grothendieck polynomials are equal to Lascoux polynomials with permutation  $w_0$ :

$$G_{\lambda}^{(\beta)} = \mathscr{L}_{w_0,\lambda} = \pi_{w_0}^{(\beta)}(x^{\lambda});$$

(3) Schur polynomials are equal to key polynomials for permutation  $w_0$ , or, equivalently, to stable Grothendieck polynomials for  $\beta = 0$ :

$$S_{\lambda} = \kappa_{w_0,\lambda}^{(\beta)} = \pi_{w_0}^{(\beta)}(x^{\lambda})|_{\beta=0}.$$

2.2. Gelfand–Zetlin polytopes and Gelfand–Zetlin patterns. Let  $\lambda$  be a partition, i.e. a sequence of nonnegative integers  $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ . Consider the space  $\mathbb{R}^d$ , where  $d = \frac{n(n-1)}{2}$ , with coordinates  $y_{ij}$  indexed by pairs (i,j) of positive integers satisfying  $i+j \leq n$ . Consider the system of inequalities defined by the following tableau:

where every triple of variables a, b, c in each small triangle a > b satisfies the inequalities  $a \le c \le b$ .

**Definition 2.6.** A Gelfand–Zetlin polytope  $GZ(\lambda)$  is the set of points in  $\mathbb{R}^{n(n-1)/2}$  satisfying the set of inequalities defined by (1). A Gelfand–Zetlin pattern is a tableau of integer coordinates  $y_{ij}$  satisfying the same inequalities. In other words, a Gelfand–Zetlin pattern is the set of coordinates of an integer point in  $GZ(\lambda)$ .

The following theorem is classical.

**Theorem 2.7** ([GC50]). The number of Gelfand–Zetlin patterns is equal to the dimension of GL(n)-module  $V(\lambda)$  with the highest weight  $\lambda$ . It can be computed using Weyl's dimension formula:

$$\#(GZ(\lambda) \cap \mathbb{Z}^d) = \dim V(\lambda) = \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j - i + j}{j - i}.$$

Gelfand–Zetlin patterns parametrize elements of a certain basis in  $V(\lambda)$ , constructed as follows. Consider the upper-left corner subgroup  $\operatorname{GL}(n-1) \subset \operatorname{GL}(n)$  and restrict  $V(\lambda)$  to this subgroup. As a  $\operatorname{GL}(n-1)$ -module, it will be reducible, but multiplicity free, meaning that every irreducible component occurs at most once:  $V(\lambda) = \bigoplus_{\mu} V(\mu)$ . The weights  $\mu = (\mu_1, \dots, \mu_{n-1})$  are highest weights of  $\operatorname{GL}(n-1)$ -modules; they satisfy the intermittence condition:  $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n$ . Write the components  $\mu_i$  from right to left as the second row of a Gelfand–Zetlin patterns. Now restrict each of  $\operatorname{GL}(n-1)$ -modules  $V(\mu_i)$  to the subgroup  $\operatorname{GL}(n-2)$ , etc.; at the end we obtain a set of one-dimensional subspaces of  $V(\lambda)$  (representations of  $\operatorname{GL}(1)$ ). Picking a nonzero vector in each of these subspaces, we obtain a  $\operatorname{Gelfand-Zetlin} \operatorname{basis}$  of  $V(\lambda)$  indexed by the sets of intermitting highest weights, i.e., by  $\operatorname{Gelfand-Zetlin} \operatorname{patterns}$ .

2.3. Faces of Gelfand–Zetlin polytopes. Faces of a Gelfand–Zetlin polytope are obtained by replacing some of the defining inequalities by equalities. Following [KST12], we will represent them by face diagrams. An example of face diagram is given in Fig. 2. Dots in this diagram correspond to coordinates  $y_{ij}$ , while an edge between  $y_{ij}$  and  $y_{i-1j}$  means that in the system of inequalities,  $y_{i-1j} \leq y_{ij}$  is replaced by  $y_{i-1j} = y_{ij}$ . The same happens with edges between  $y_{ij}$  and  $y_{i-1j+1}$ . Here we formally set  $y_{0i} = \lambda_{n+1-i}$ , so some variables may be equal to entries  $\lambda_i$  from the top row.

**Example 2.8.** Let  $\lambda = (2,1,0)$ . The polytope  $GZ(\lambda)$  is shown in Fig. 1, the diagram corresponding to the colored face is shown in Fig. 2.

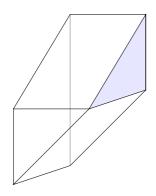


FIGURE 1. Gelfand–Zetlin polytope

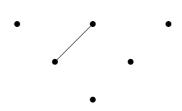


FIGURE 2. Face diagram of the colored face

Note that these equalities are in general not independent: for each four vertices forming a "diamond" in three consecutive rows, the top two equalities imply the two bottom ones, and vice versa. Indeed, for a "diamond"  $y_{i,j-1}$   $y_{i,j-1}$   $y_{i,j}$   $y_{i,j}$ , the equalities  $y_{i,j-1} = y_{i-1,j} = y_{i,j}$  imply that  $y_{i+1,j-1}$  is also equal to these three values, since both  $y_{i+1,j-1}$  and  $y_{i-1,j}$  are "squeezed" between  $y_{i,j-1}$  and  $y_{i,j}$ ; the same holds for equalities in the bottom row.

This means that Gelfand–Zetlin polytopes are not simple (in fact, they are "highly non-simple"): the intersection of certain facets can have codimension strictly less than the number of facets.

2.4. Dual Kogan faces. These are faces of Gelfand–Zetlin polytopes of some special form.

**Definition 2.9.** A face of Gelfand–Zetlin polytope is called a *dual Kogan face* if it is defined only by equations of the form  $y_{ij} = y_{i+1,j-1}$  for  $i \ge 0$ .

Equivalently, dual Kogan faces are exactly those containing the "maximal" vertex defined by the equations  $\lambda_1 = y_{1,n-1} = y_{2,n-2} = \cdots = y_{n-1,1}$ ,  $\lambda_2 = y_{1,n-2} = \cdots = y_{n-2,1}$ , and so on. Note that the "maximal" dual Kogan vertex is simple. We also formally consider the whole polytope  $GZ(\lambda)$  as a dual Kogan face, defined by the empty set of equations.

Now we will assign a permutation to each dual Kogan face. Consider the following reduced word for the longest permutation  $w_0 \in S_n$ :

$$\mathbf{w}_0 = (s_{n-1}, \dots, s_1, s_{n-2}, \dots, s_1, \dots, s_1, s_2, s_1)$$

To each dual Kogan face F we assign a subword  $\mathbf{w}^-(F)$  of  $\mathbf{w}_0$  as follows. Consider the face diagram and write simple transposition  $s_{n-j}$  on all edges corresponding to the equalities  $y_{i,j+1} = y_{i+1,j}$  (see Fig. 3).

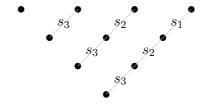


FIGURE 3. Assigning word to a dual Kogan face

Now, for a face F, the corresponding word  $\mathbf{w}^-(F)$  is obtained by reading the face diagram from bottom to top, from right to left, and taking the simple transpositions corresponding to the equations. Face F is said to be reduced, if the word  $\mathbf{w}^-(F)$  is reduced, and non-reduced in the opposite case. For a reduced face, we denote  $w(F) = w_0 w^-(F)$ , where  $w^-(F)$  is the permutation obtained by taking the product of simple transpositions in  $\mathbf{w}^-(F)$ . Fig. 4 below shows all the face diagrams for reduced dual Kogan faces in a three-dimensional Gelfand–Zetlin polytope and the corresponding permutations w(F). Note that the dimension of F is equal to the length of w(F).

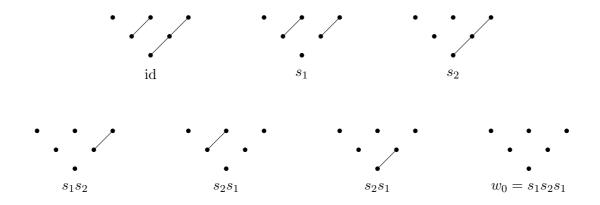


FIGURE 4. Reduced dual Kogan faces and permutations for n = 3.

Remark 2.10. In [KM05] and [KM04], A. Knutson and E. Miller define subword complexes and, more specifically, pipe dream complexes. It readily follows from the definitions that as a CW-complex the link of the "maximal" dual Kogan vertex is nothing but the subword complex corresponding to the word  $\mathbf{w}_0$  of the longest permutation  $w_0$ . Each dual Kogan face F of  $GZ(\lambda)$  corresponds to a pipe dream with permutation  $w^-(F)$ . To construct the pipe dream from a face diagram, for each equality  $y_{ij} = y_{i+1,j-1}$ , put a cross in the box (n-j+1,i+1), and elbows in all the remaining boxes.

2.5. **Key polynomials.** The following theorem is due to V. Kiritchenko, E. Smirnov, and V. Timorin [KST12]. It provides a description of key polynomials in terms of integer points in dual Kogan faces. Here we state it in a different (but equivalent) way: the authors of the original paper used Kogan faces, as opposed to dual Kogan faces that we are using, and lowest-weight modules instead of highest-weight ones.

Before we proceed, define the following projection map  $\pi: \mathbb{R}^{\frac{n(n-1)}{2}} \to \mathbb{R}^n$  from  $GZ(\lambda)$  to the weight polytope wt( $\lambda$ ) of the GL(n)-module  $V(\lambda)$ :

$$\pi \colon \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1,n-1} \\ & y_{21} & \cdots & & y_{2,n-2} \\ & & \ddots & \vdots & \ddots & \\ & & y_{n-1,1} & & \end{pmatrix} \mapsto \begin{pmatrix} y_{n-1,1} \\ & (y_{n-2,1} + y_{n-2,2}) - y_{n-1,1} \\ & \vdots \\ & (y_{11} + \cdots + y_{1,n-1}) - (y_{21} + \cdots + y_{2,n-2}) \end{pmatrix}.$$

This map takes a Gelfand–Zetlin pattern into the vector with the *i*-th component equal to the difference of sums of entries in *i*-th and (i + 1)-th rows of the pattern, counted from below.

**Definition 2.11.** For an integer point  $z \in GZ(\lambda) \subset \mathbb{R}^{\frac{n(n-1)}{2}}$ , define its *character* ch  $z \in \mathbb{Z}[x_1,\ldots,x_n]$  as follows: ch  $z=x_1^{a_1}\ldots x_n^{a_n}$ , where  $(a_1,\ldots,a_n)=\pi(z)$ . More generally, for an arbitrary subset  $S \subset \mathbb{R}^{\frac{n(n-1)}{2}}$ , define its *character* ch  $S \in \mathbb{Z}[x_1,\ldots,x_n]$  as the sum of all monomials ch z for all integer points  $z \in S \cap \mathbb{Z}^{\frac{n(n-1)}{2}}$ .

**Theorem 2.12** ([KST12, Theorem 5.1]). The key polynomial  $\kappa_{w,\lambda}$  is equal to the character of the union of the dual Kogan faces in  $GZ(\lambda)$  corresponding to the permutation  $w \in S_n$ :

$$\kappa_{w,\lambda} = \operatorname{ch}\left(\bigcup_{w(F)=w} F\right).$$

**Example 2.13.** Let  $S = GZ(\lambda)$  be the whole Gelfand–Zetlin polytope. Then, according to Theorem 2.7, its character ch S is nothing but the character of representation  $V(\lambda)$ , i.e., the Schur polynomial  $S_{\lambda}(x_1, \ldots, x_n)$ .

## 3. The three-dimensional case

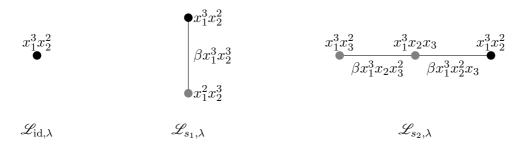
In this section we study the three-dimensional case. We start with describing a cellular decomposition for  $GZ(\lambda)$ , where  $\lambda = (3,2,0)$ , and show how monomials of Lascoux polynomials  $\mathcal{L}_{w,\lambda}$  correspond to cells of this decomposition. Next, we will describe the construction in general for arbitrary  $GZ(\lambda_1, \lambda_2, \lambda_3)$ . The construction presented in this section is *ad hoc*; we provide the cell decomposition in the general case in the subsequent section.

3.1. **Example.** Let  $\lambda = (3, 2, 0)$ . The Gelfand–Zetlin polytope GZ(3, 2, 0) is given by the following tableau:

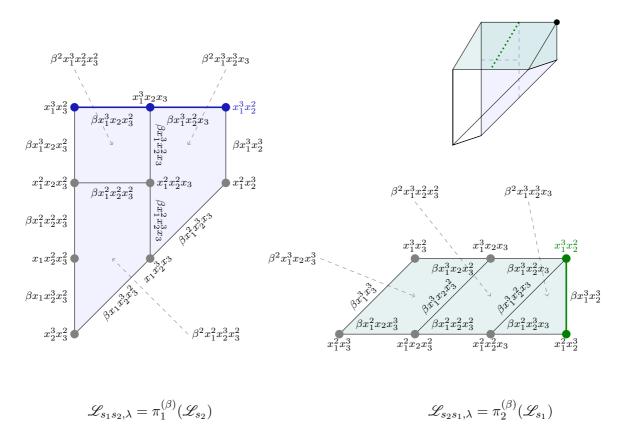
$$\begin{array}{cccc}
0 & 2 & 3 \\
x & y & 3
\end{array}$$

Our goal is to construct a cellular decomposition of  $GZ(\lambda)$  and assign to each cell a monomial in  $x_1, \ldots, x_n, \beta$  with the following properties: first, each face of  $GZ(\lambda)$  should be the union of some cells. And second, for the set of dual Kogan faces corresponding to a permutation  $w \in S_n$ , the sum of monomials corresponding to the cells forming these faces should be equal to the Lascoux polynomial  $\mathcal{L}_{w,\lambda}$ 

We will proceed by induction on the length of w. First put the monomial  $x_1^3x_2^2$  into the vertex corresponding to the identity permutation. Then take the integer points on one-dimensional dual Kogan faces, thus getting their subdivision. Then we assign monomials occurring in  $\pi_1^{(\beta)}(x_1^3x_2^2)$  and  $\pi_2^{(\beta)}(x_1^3x_2^2)$  to vertices and segments of corresponding one-dimensional faces.



Acting again with  $\pi_1^{(\beta)}$  and  $\pi_2^{(\beta)}$  respectively, we obtain the following subdivisions for two-dimensional faces.



Finally, acting by  $\pi_1^{(\beta)}$  on  $\mathcal{L}_{s_2s_1,\lambda}$ , we get  $\mathcal{L}_{s_1s_2s_1,\lambda}$ , with monomials corresponding to the cells in the cellular decomposition of the polytope shown below. Moreover, only the colored one-dimensional and two-dimensional faces correspond to zero, while exactly one nonzero monomial corresponds to each of the remaining cells. The proof is by direct computation.

3.2. **General case.** Note that if a polynomial f is symmetric over  $x_i$  and  $x_{i+1}$ , then for every polynomial g we have  $\pi_i^{(\beta)}(f \cdot g) = f \cdot \pi_i^{(\beta)}(g)$ . This implies that it is enough to describe the construction for GZ(a,b,0), where  $a \geq b \geq 0$ .

Next, we will describe the Lascoux polynomials for permutation  $u \in S_3$  and partition  $\lambda = (a, b, 0)$ . As before, first put the monomial  $x_1^a x_2^b$  in the vertex corresponding to the identity permutation. Then put monomials of the polynomial  $\pi_1^{(\beta)}(x_1^a x_2^b)$  along the corresponding one-dimensional face. Under the action of  $\pi_2^{(\beta)}$ , each monomial of  $\mathcal{L}_{s_1,\lambda}$  "expands" into a row in a subdivision of a two-dimensional face. For instance, the highest row corresponds to  $\mathcal{L}_{s_2,\lambda}$ .

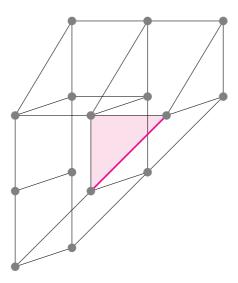
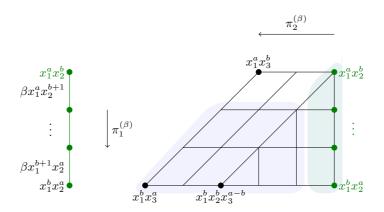


Figure 5. Cellular decomposition of GZ(3,2,0)



Next we split the monomials of  $\mathcal{L}_{s_2,\lambda}$  into several groups. The first one (colored blue) can be represented as  $x_3\pi_2^{(\beta)}\pi_1^{(\beta)}(x_1^{a-1}x_2^b)$ . Since  $x_3$  is symmetric with respect to  $x_1$  and  $x_2$ , it follows that  $\pi_1^{(\beta)}(x_3\pi_2^{(\beta)}\pi_1^{(\beta)}(x_1^{a-1}x_2^b)) = x_3\pi_1^{(\beta)}(\pi_2^{(\beta)}\pi_1^{(\beta)}(x_1^{a-1}x_2^b))$ , hence it is the case for smaller a and b. The monomials in the second group (colored green) form a polynomial that is symmetric over  $x_1$  and  $x_2$ ; denote it by f. It is easily shown that  $\pi_1^{(\beta)}(f) = f$ . To complete the construction in is enough to check that the remaining monomials expand into columns of the "correct" size, consistent with the size of the polytope GZ(a,b,0). The proof is by direct computation.

Remark 3.1. Note that for general a and b, "most" cells look like (open) standard unit cubes; in particular, this is true for the "interior" cells, i.e. those with the closure not meeting the boundary of GZ(a,b,0). Moreover, the cells corresponding to the zero monomial also lie in the boundary of the polytope; these are two cells of dimension 1 or 2 adjacent to the nonsimple vertex, shown in purple in Fig 5. This is the case in general: such cells always belong to the boundary of the polytope and cannot have dimension 0 or n(n-1)/2. However, for Gelfand–Zetlin polytopes of nondominant weights dim  $GZ(\lambda) < n(n-1)/2$ , and cells corresponding to zero monomials can have maximal dimension, i.e. dimension equal to the dimension of the polytope itself.

## 4. Enhanced Gelfand–Zetlin patterns, cellular decomposition and formula for Lascoux polynomials

In this section we give the main results of this paper. We start with constructing a cellular decomposition for  $GZ(\lambda)$  and assigning a monomial to each cell. A cell is called *efficient* if this monomial is nonzero and *inefficient* otherwise. Then we state the main theorem: a Lascoux

polynomial  $\mathcal{L}_{w,\lambda}$  is equal to the sum of monomials for the cells located in the set of dual Kogan faces corresponding to w. The proof of this result is given in the two subsequent sections.

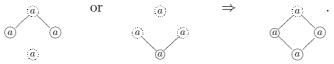
4.1. Enhanced Gelfand–Zetlin patterns. Here we present a construction for cells that provide a cellular decomposition of  $GZ(\lambda)$ . These cells are convex polytopes without boundary; by construction, the set of zero-dimensional cells will coincide with the set of integer points of  $GZ(\lambda)$ . This decomposition is regular in the following sense: if the closure of a given cell does not meet the boundary of  $GZ(\lambda)$ , this cell is just a face of the standard unit cube.

The cells are indexed by the so-called *enhanced Gelfand–Zetlin patterns*, i.e. Gelfand–Zetlin patterns with some additional data, which we will call *enhancement*. These data are of two kinds: first, some elements in a pattern may be encircled, and second, some pairs of neighbor elements in consecutive rows can be joined by an edge.

Informally, the pattern without enhancement stands for the "maximal" point of the closure of the corresponding cell, i.e. the point with the largest sum of coordinates.

**Definition 4.1.** A Gelfand–Zetlin pattern with the top row  $(\lambda_n, \ldots, \lambda_1)$  with some entries marked by circles and with edges between certain neighboring entries is said to be an *enhanced Gelfand–Zetlin patterns*, if these elements satisfy the following conditions:

- (1) The numbers in the first row are encircled.
- (2) The two entries joined by an edge must be equal, and the bottom entry should be encircled. The converse does not have to be true: two equal neighboring entries are not necessarily joined by an edge.
- (3) Two neighboring entries in a row are joined by edges with an entry above them if and only if they are joined with an entry below them. Pictorially this can be presented as follows:



(a dotted circle around an entry means that it may be either encircled or not).

- (4) If two entries in the topmost row are equal, then the entry below them (which is equal to both of them) is encircled and connected to both of them by edges.
- (5) If a < b and the pattern contains the following triangle: a a b, then the lowest entry in the triangle is encircled, and there is an edge between the two a's. The entry a in the upper row can be either encircled or not. Pictorially:



(6) If a < b and the pattern contains the following triangle:  $a_b^b$  with the bottom entry encircled, then there is an edge between the two b's:



- (7) For a triangle  $a_a$ : if the two top entries can be connected by a path of edges, the bottom entry should be encircled and connected with them.
- (8) If in a triangle  $a_a$  the bottom entry is encircled, then it should be connected with at least one of them by an edge.

We denote the set of all enhanced patterns with the first row  $\lambda$  by  $\mathscr{P}(\lambda)$ .

**Example 4.2.** The pattern  ${}^{0}$   ${}^{1}$   ${}^{2}$   ${}^{2}$  has eight enhancements.





**Example 4.3.** The pattern  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 \end{bmatrix}$  has four enhancements.



Note that according to Definition 4.1 (4), the last entry in the second row must be encircled and connected to the middle entry in the first row.

An enhanced pattern can be viewed as a graph (with marked vertices). Consider the connected components of this graph.

**Lemma 4.4.** The connected components of an enhanced Gelfand–Zetlin pattern satisfy the following:

- (1) the entries in the topmost row belong to the same connected component if and only if they have the same value;
- (2) each connected component either has a unique highest vertex or contains one or more entries from the topmost row;
- (3) all vertices in a connected component, possibly except the highest one, are encircled. In particular, the number of connected components is not less than the number of distinct  $\lambda_i$ 's plus the number of entries without circles.

*Proof.* This follows immediately from Definition 4.1.

We conclude with two definitions involving enhanced patterns.

**Definition 4.5.** The rank rk P of an enhanced pattern P is the number of entries without circles.

**Definition 4.6.** An enhanced pattern P is said to be *inefficient* if it contains a triangle of the form  $a_a$  such that its bottom entry is not connected with the right one by an edge, and *efficient* otherwise. The set of all efficient enhanced patterns with the first row  $\lambda$  is denoted by  $\mathscr{P}^+(\lambda)$ .

For instance, in Example 4.3 the first two enhanced patterns are inefficient, while the last two are efficient, and all patterns in Example 4.2 are efficient.

**Proposition 4.7.** Every enhanced pattern of rank zero is efficient.

*Proof.* Take an inefficient enhanced pattern P. This means that it contains a triangle of the form  $a_a$  such that there is no edge between the bottom and the right entries. Definition 4.1 implies that these two entries are contained in different connected components, both marked with the same number a. This means that at least one of these components contains a vertex without circle, so the rank of P cannot be zero.

Moreover, it turns out that for an efficient enhanced pattern, the edges provide redundant data. Namely, we have the following lemma.

**Lemma 4.8.** The edges in an efficient pattern are uniquely determined by positions of encircled vertices.

*Proof.* The conditions listed in Definition 4.1 imply that positions of edges are defined by positions of encircled vertices in all cases except for case (8). In the latter case there are two possibilities of joining the bottom vertex in the triangle  ${}^a{}_a{}^a$  with one of its neighbors in the upper row, and only one of them defines an efficient pattern.

For an efficient enhanced GZ-pattern P, we assign to it a monomial  $x^P$  in the following way. Let  $S_i(P)$  be the sum of numbers in the *i*-th row of the pattern P, with  $S_0(P) = \lambda_1 + \cdots + \lambda_n$ , and let  $D_i(P)$  stand for the number of entries without circles in the i-th row of P. Denote  $d_{n+1-i} = d_{n+1-i}(P) = S_{i-1}(P) - S_i(P) + D_i(P)$ . Then

$$x^P = \beta^{\operatorname{rk} P} x_1^{d_1} \dots x_n^{d_n}.$$

For an inefficient enhanced GZ-pattern P we formally set  $x^P = 0$ .

In the next section we construct a cellular decomposition of  $GZ(\lambda)$ , with cells corresponding to enhanced patterns, and with the dimension of a cell being equal to the rank of its enhanced pattern. Some of these cells will correspond to monomials in Lascoux polynomials; as we will see, these will be exactly the cells constructed from the efficient enhanced patterns. This is the motivation behind Definition 4.6.

4.2. Cellular decomposition of Gelfand–Zetlin polytope. In this subsection, we construct a cellular decomposition of  $GZ(\lambda)$ . The cells of this decomposition are indexed by enhanced patterns; moreover, the dimension of a cell is equal to the rank of the corresponding pattern.

Consider an enhanced Gelfand–Zetlin pattern P. For each such pattern we write a set of equalities and inequalities that, together with the inequalities defining  $GZ(\lambda)$ , defines a subset  $C_P \subset GZ(\lambda)$ . As we will show further, these sets are pairwise disjoint, open in their affine spans and homeomorphic to open balls; they define a cellular decomposition of  $GZ(\lambda)$  compatible with the polytope structure (i.e., every face of  $GZ(\lambda)$  is also a union of cells).

Recall that we denote the coordinates in  $\mathbb{R}^{\frac{n(n-1)}{2}} \supset GZ(\lambda)$  by  $y_{ij}$ , with  $1 \leq i \leq n-1$ and  $1 \le j \le n+1-i$ . We also fix the topmost row of a Gelfand-Zetlin tableau by setting  $y_{0,j} = \lambda_{n+1-j}$ . The inequalities that define the polytope are given by the tableau (1) on p. 3: these are

$$y_{i-1,j} \le y_{ij} \le y_{i-1,j+1},$$

for each (i, j) in the aforementioned range.

Now we define the cellular decomposition of  $GZ(\lambda)$ .

Construction 4.9. Let P be an enhanced pattern with entries  $a_{ij}$ . To each coordinate  $y_{ij}$  we assign an equality or a double inequality as follows:

- (1) if there is an edge going up from  $a_{ij}$  to  $a_{i-1,j}$  or  $a_{i-1,j+1}$  (or both), then  $y_{ij} = y_{i-1,j}$  or  $y_{ij} = y_{i-1,j+1}$ , respectively;
- (2) if there are no edges going up from  $a_{ij}$ , and this entry is encircled, then  $y_{ij} = a_{ij}$ ;
- (3) if there are no edges going up from  $a_{ij}$  and this entry is not encircled, we impose a double inequality on  $y_{ij}$  as follows:
  - (a) If the entry  $a_{i-1,j}$  satisfies  $a_{ij} a_{i-1,j} \ge 2$ , then  $a_{ij} 1 < y_{ij}$ ; otherwise,  $y_{i-1,j} < y_{ij}$ ; (b) If  $a_{i-1,j+1}$  is equal to  $a_{ij}$ , we set  $y_{ij} < y_{i-1,j+1}$ ; otherwise,  $y_{ij} < a_{ij}$ .

Denote the set defined by these equalities and inequalities by  $\widehat{C}_P$ . This is "almost" the required cell corresponding to P; however, it does not necessarily lie in  $GZ(\lambda)$ . To get an actual cell, take the affine span L of  $\widehat{C_P}$  and intersect  $\widehat{C_P}$  with the relative interior of  $GZ(\lambda) \cap L$  in L:

$$C_P = \widehat{C_P} \cap (GZ(\lambda) \cap L)^0.$$

This set is convex and open in L (as the intersection of two open convex sets); we shall see in Lemma 5.2 that it is nonempty.

This means the following. For each connected component in P containing only encircled entries with the same numbers, all the corresponding coordinates are equal to this number. On the other hand, if a connected component has a non-encircled vertex, the corresponding coordinate can take values in an interval determined by conditions (3) or (4); note that the length of this interval does not exceed i-1, where i is the row number. All the remaining coordinates in the same connected component (corresponding to encircled entries) are equal to this coordinate.

4.3. Main results. The following theorems are the main results of this paper.

**Theorem 4.10.** The cells  $C_P$  for  $P \in \mathcal{P}(\lambda)$  form a cellular decomposition of  $GZ(\lambda)$ .

The proof of this theorem is given in § 5.1.

**Theorem 4.11.** Let  $\lambda$  be a partition. Then the stable Grothendieck polynomial  $G_{\lambda}^{(\beta)}$  can be computed as follows:

$$G_{\alpha}^{(\beta)} = \mathscr{L}_{w_0,\lambda} = \sum_{P \in \mathscr{P}^+(\lambda)} x^P.$$

The proof of this theorem is given in § 5.2.

We can generalize this result to get a description of Lascoux polynomials. Denote by  $\mathscr{P}^+(w,\lambda)$  the set of efficient patterns such that the corresponding cells are contained in the union of dual Kogan faces corresponding to  $GZ(\lambda)$  and permutation w.

**Theorem 4.12.** Let  $w \in S_n$  be a permutation and  $\lambda$  be a partition. Then the Lascoux polynomial  $\mathcal{L}_{w,\lambda}$  is equal to

$$\mathscr{L}_{w,\lambda} = \sum_{P \in \mathscr{P}^+(w,\lambda)} x^P.$$

We give the proof of this theorem in Section 6.

Corollary 4.13. Let  $\lambda$  be a partition and  $u, w \in S_n$  be permutations such that  $u \leq w$  in the Bruhat order on  $S_n$ . Then the polynomial  $\mathcal{L}_{w,\lambda} - \mathcal{L}_{u,\lambda}$  has nonnegative coefficients.

*Proof.* Denote the union of dual Kogan faces of  $GZ(\lambda)$  corresponding to permutation w by  $\Gamma_{w,\lambda}$ . The definition of Kogan faces in terms of subwords (cf. § 2.4) implies that  $\Gamma_{u,\lambda} \subseteq \Gamma_{w,\lambda}$  if  $u \leq w$  in the Bruhat order. Applying Theorem 4.12 completes the proof.

- 5. Proofs of the main results for the case of the longest permutation
- 5.1. **Proof of Theorem 4.10.** In this section we show that the cells described in § 4.2 form a cellular decomposition of  $GZ(\lambda)$ . We split the proof into several lemmas.

**Lemma 5.1.** For every point  $y \in GZ(\lambda)$  there exists a unique cell  $C_P$  such that  $y \in C_P$ .

*Proof.* To construct such a P, let us first draw the edges for all pairs of equal neighboring coordinates, no matter whether they are integer or not. Then we need to assign integer values  $a_{ij}$  to all the vertices of P. This is done for the top vertex in each connected component, row by row from top to bottom. Here we follow Construction 4.9.

We construct the pattern P row by row, going from top to bottom. The topmost row of the pattern is given by  $\lambda$ . Now suppose we have already filled the row number i-1; consider the i-th row. If an element  $y_{ij}$  is equal to  $y_{i-1,j}$  or  $y_{i-1,j+1}$  (or both), this means that there is an edge going up from this position; then the corresponding  $a_{ij}$  is already determined. We encircle this entry.

Otherwise, we distinguish between three cases. If  $a_{i-1,j}+1 \le y_{ij} < a_{i-1,j+1}$  and  $y_{ij}$  is integer, we set  $a_{ij} = y_{ij}$  and encircle this element. If  $y_{ij} < a_{i-1,j}+1$ , we set  $a_{ij} = a_{i-1,j}+1$  and do not encircle it. Finally, for  $y_{ij} \notin \mathbb{Z}$  and  $a_{i-1,j}+1 < y_{ij}$ , we set  $a_{ij} = \lceil y_{ij} \rceil$ , and the corresponding entry also has no circle. Construction 4.9 implies that for the enhanced pattern P obtained in such a way, the corresponding cell  $C_P$  contains the point y.

**Lemma 5.2.** Every cell  $C_P$  is nonempty and homeomorphic to an open ball of dimension  $\operatorname{rk} P$ .

*Proof.* First, the set  $\widehat{C^P}$  defined by inequalities in Construction 4.9 is a convex set open in its affine span L. Replacing these strict inequalities by non-strict ones, we obtain the closure of  $\widehat{C_P}$ . It contains the point with coordinates  $y_{ij} = a_{ij}$ , so this closure is nonempty, and  $C_P$  is nonempty as well. Similarly, the relative interior  $(L \cap GZ(\lambda))^0$  is convex and nonempty. So it remains to show that their intersection is nonempty.

Indeed, consider the point  $y = (y_{ij})$  defined by  $y_{ij} = a_{ij}$ ; it belongs to the closure of both  $\widehat{C}_P$  and  $(L \cap GZ(\lambda))^0$ . Moreover, in a neighborhood of y both these sets coincide. So  $C_P$  also has

dimension  $\operatorname{rk} P$ ; being the intersection of two nonempty open convex bodies, it is homeomorphic to an open ball.

**Lemma 5.3.** Let  $y \in GZ(\lambda)$ , and let  $C_P$  be the cell containing it. If for some cell C we have  $y \in \overline{C}$ , then  $C_P \subset \overline{C}$ .

*Proof.* Given  $y \in GZ(\lambda)$ , let us find all enhanced patterns Q such that  $y \in \overline{C_Q}$ . This will be done similarly to the proof of Lemma 5.1.

We construct all such patterns row by row from top to bottom. On each step, the procedure may not be unique. The topmost row is given by  $\lambda$ ; this is the induction base.

Now suppose that the first i-1 rows are filled, and consider the coordinates  $y_{ij}$  in the *i*-th row, starting from the first one. For a given coordinate  $y_{ij}$ , we proceed exactly as in Lemma 5.1. Namely, if  $y_{ij} < a_{i-1,j} + 1$ , we set  $a_{ij} = a_{i-1,j} + 1$ ; otherwise, if it is not an integer, we set  $a_{ij} = [y_{ij}]$ . In both of these cases, the corresponding entry has no circle.

If  $y_{ij}$  is an integer and  $a_{i-1,j}+1 \leq y_{ij}$ , we can construct the corresponding entry of the pattern in at most three different ways. In the first case, we set  $a_{ij} = y_{ij}$ , put a circle around this vertex and join it with entries in the previous row, just like in Lemma 5.1. In the second and the third case, we set  $a_{ij} = y_{ij}$  or  $a_{ij} = y_{ij} + 1$  and do not put a circle around this vertex, provided that the resulting diagram (or the constructed part of it) satisfies the conditions of Definition 4.1.

By construction, for all the patterns Q obtained by this procedure the corresponding cell closure  $\overline{C}_Q$  contains y, and this set includes  $C_P$ . Moreover, for every constructed Q we have  $\operatorname{rk} P \leq \operatorname{rk} Q$ , with the equality only in the case P = Q. It is also clear that for all points  $y \in C_P$  the set of such patterns Q will be the same.

This lemma immediately implies that the boundary  $\overline{C} \setminus C$  of each cell consists of cells of smaller dimension. So  $GZ(\lambda) = \bigsqcup_{P \in \mathscr{P}(\lambda)} C_P$ , is indeed a cellular decomposition. This concludes the proof of Theorem 4.10.

5.2. **Proof of Theorem 4.11.** In this section we establish a bijection between the set  $\mathscr{P}^+(\lambda)$  of efficient enhanced patterns and the set of monomials (with coefficient 1) in  $\pi_{w_0}^{(\beta)}(x^{\lambda})$ .

Denote by  $c_k$  the following Coxeter element  $s_k \dots s_1 \in S_k$ ; here  $1 \le k \le n-1$ . The longest permutation  $w_0$  can be presented as the product of such elements:

(2) 
$$w_0 = s_1(s_2s_1) \dots (s_{n-1} \dots s_1) = c_1c_2 \dots c_{n-1}.$$

**Definition 5.4.** A polynomial  $p(\beta, x_1, ..., x_n)$  is said to be *multiplicity free*, if all its nonzero coefficients are equal to 1.

**Lemma 5.5.** Let  $\mu$  be a partition. For a monomial  $x^{\mu} = x_1^{\mu_1} \dots x_n^{\mu_n}$ , the polynomial  $c_k x^{\mu}$  is multiplicity free.

*Proof.* The proof is by induction on k. If k = 1, the statement is obvious:

$$c_1 x^{\mu} = \pi_1^{(\beta)} x^{\mu} = \left( (x_1^{\mu_1} x_2^{\mu_2} + \dots + x_1^{\mu_2} x_2^{\mu_1}) + \beta (x_1^{\mu_1} x_2^{\mu_2 + 1} + \dots + x_1^{\mu_2 + 1} x_2^{\mu_1}) \right) \cdot x_3^{\mu_3} \dots x_n^{\mu_n}.$$

Note that all monomials in the right-hand side have different bidegrees over  $(\beta, x_1)$ .

Applying  $\pi_2^{(\beta)}$  does not change the  $x_1$ -degree of a given monomial, only affecting  $x_2$ ,  $x_3$ , and  $\beta$ . This means that all monomials in  $c_2x^{\mu}$  will have different tridegrees over  $(\beta, x_1, x_2)$ , and so on.

We need one more definition concerning monomials.

**Definition 5.6.** Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  be a partition, i.e.  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . We shall say that monomial  $x^{\mu} = x_1^{\mu_1} \dots x_n^{\mu_n}$ , where  $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ , is  $\lambda$ -alternating, if  $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \dots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$ , and  $\lambda$ -nonalternating otherwise. Denote the sum of all nonalternating monomials of a polynomial  $p(x_1, \dots, x_n)$  by  $[p]_{\lambda}$ .

**Example 5.7.** Let n=3. Take a partition  $\lambda=(a,b,0)$  and apply  $\pi_{s_2s_1}^{(\beta)}$  to  $x^{\lambda}$ . The resulting monomials are shown in Figure 6, with all  $\lambda$ -alternating and nonalternating monomials located in the blue and green area, respectively. Also note that the alternating monomials such that the degree of  $x_2$  equals b are on the blue diagonal.

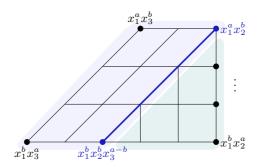


Figure 6. Alternating and nonalternating monomials

Moreover, the sum of the nonal ternating monomials in every column is equal to  $\pi_1^{(\beta)}(x_1^{a-c}x_2^bx_3^c)$  $x_1^{a-c}x_2^bx_3^c$  or  $\pi_1^{(\beta)}(\beta x_1^{a-c}x_2^bx_3^{c+1}) - \beta x_1^{a-c}x_2^bx_3^{c+1}$  respectively. Recall that the monomials  $x_1^{a-c}x_2^bx_3^c$  correspond to vertices on the diagonal, while the monomials  $\beta x_1^{a-c}x_2^bx_3^{c+1}$  correspond to edges between them (cf. Sec. 3.2). This means that  $\pi_1^{(\beta)}([\pi_{s_2s_1}^{(\beta)}x^{\lambda}]_{\lambda})=0$ : the sum of all nonalternating monomials belongs to the kernel of  $\pi_1^{(\beta)}$ .

**Lemma 5.8.** Let  $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$  be a partition. Then we have

$$[\pi_{c_{n-1}}^{(\beta)} x^{\lambda}]_{\lambda} \in \operatorname{Ker} \pi_{c_{1} \dots c_{n-2}}^{(\beta)}.$$

*Proof.* The proof is by induction on n. For n=1 or 2, there is nothing to prove. The case n=3is treated in Example 5.7.

The general case is treated similarly to the case n=3. Consider a monomial  $x^{\lambda}$  and apply to it  $\pi_2^{(\beta)}\pi_1^{(\beta)}$ . This is the sum of monomials  $x_1^{\nu_1}x_2^{\nu_2}x_3^{\nu_3}x_4^{\lambda_4}\dots x_n^{\lambda_n}$ , where  $\nu_3 \geq \lambda_4 \geq \dots$  is a partition. Denote by A the sum of all monomials satisfying the condition  $\lambda_1 \geq \nu_2 \geq \lambda_2$ . (They correspond to the blue area in Fig. 6). The remaining monomials (those in the green area) can be represented as

$$\sum_{i=0}^{\lambda_1-\lambda_2} \left( \pi_1(x_1^{\lambda_1-j} x_2^{\lambda_2} x_3^{\lambda_3+j} x_4^{\lambda_4} \dots) - x_1^{\lambda_1-j} x_2^{\lambda_2} x_3^{\lambda_3+j} x_4^{\lambda_4} \dots \right),$$

where monomials  $x_1^{\lambda_1-j}x_2^{\lambda_2}x_3^{\lambda_3+j}x_4^{\lambda_4}\dots$  are located on the blue diagonal (see Fig. 6). This implies that the sums  $\left(\pi_1(x_1^{\lambda_1-j}x_2^{\lambda_2}x_3^{\lambda_3+j}x_4^{\lambda_4}\dots)-x_1^{\lambda_1-j}x_2^{\lambda_2}x_3^{\lambda_3+j}x_4^{\lambda_4}\dots\right)$  are located in the corresponding columns in the green area (see Fig. 6). The sum of all such monomials will be denoted by B. Then  $\pi_2^{(\beta)} \pi_1^{(\beta)}(x_\lambda) = A + B$ . Now denote  $(\pi_{n-1}^{\beta} \dots \pi_3^{(\beta)})$  by  $\pi$ . We have

$$\pi_{c_{n-1}}^{(\beta)} x_{\lambda} = (\pi_{n-1}^{\beta} \dots \pi_3^{(\beta)}) (\pi_2^{(\beta)} \pi_1^{(\beta)} (x_{\lambda})) = \pi(A+B).$$

Since  $\pi$  is a linear operator, we have

$$\pi(A+B) = \pi(A) + \pi(B).$$

Monomials in  $x_3, x_4, \ldots$  are symmetric in  $x_1$  and  $x_2$ , hence

$$\begin{split} \pi_1(x_1^{\lambda_1-j}x_2^{\lambda_2}x_3^{\lambda_3+j}x_4^{\lambda_4}\dots) - x_1^{\lambda_1-j}x_2^{\lambda_2}x_3^{\lambda_3+j}x_4^{\lambda_4}\dots = \\ &= (x_3^{\lambda_3+j}x_4^{\lambda_4}\dots) \cdot \pi_1(x_1^{\lambda_1-j}x_2^{\lambda_2}) - x_1^{\lambda_1-j}x_2^{\lambda_2}x_3^{\lambda_3+j}x_4^{\lambda_4}\dots = \\ &= (x_3^{\lambda_3+j}x_4^{\lambda_4}\dots)(\pi_1^{(\beta)}(x_1^{\lambda_1-j}x_2^{\lambda_2}) - x_1^{\lambda_1-j}x_2^{\lambda_2}). \end{split}$$

Since  $(\pi_1^{(\beta)}(x_1^{\lambda_1-j}x_2^{\lambda_2}) - x_1^{\lambda_1-j}x_2^{\lambda_2})$  is symmetric in  $x_3, x_4, ...,$  we have:

$$\pi\left((x_3^{\lambda_3+j}x_4^{\lambda_4}\dots)(\pi_1^{(\beta)}(x_1^{\lambda_1-j}x_2^{\lambda_2})-x_1^{\lambda_1-j}x_2^{\lambda_2})\right)=\pi(x_3^{\lambda_3+j}x_4^{\lambda_4}\dots)\cdot(\pi_1^{(\beta)}(x_1^{\lambda_1-j}x_2^{\lambda_2})-x_1^{\lambda_1-j}x_2^{\lambda_2})$$

Let us introduce some extra notations. Denote by  $B_0^j$  the sum of all alternating monomials in  $\pi(x_3^{\lambda_3+j}x_4^{\lambda_4}\dots)$ , and by  $B_1^j$  the sum of all nonalternating monomials respectively. Then  $\pi(x_3^{\lambda_3+j}x_4^{\lambda_4}\dots)=B_0^j+B_1^j$ . Note that  $B_0^j$  and  $B_1^j$  are elements of  $\mathbb{Z}[x_3,x_4,\dots]$ . Also we denote by  $A_0$  and  $A_1$  the sums of all alternating and nonalternating monomials in  $\pi(A)$ , respectively. Then we have:

$$\pi_{c_{n-1}}^{(\beta)} x_{\lambda} = \pi(A) + \pi(B) = A_0 + A_1 + \sum_{j=0}^{\lambda_1 - \lambda_2} \left( (B_0^j + B_1^j) (\pi_1^{(\beta)} (x_1^{\lambda_1 - j} x_2^{\lambda_2}) - x_1^{\lambda_1 - j} x_2^{\lambda_2}) \right)$$

To prove this lemma for an arbitrary n, we observe that  $c_1 \ldots c_{n-2} = w'_0$  is the longest permutation for the subgroup  $\langle s_1, \ldots, s_{n-2} \rangle \cong S_{n-1} \subset S_n$  and fix another word for  $w'_0$ . We denote by  $w''_0$  the longest permutation in the subgroup  $\langle s_2, \ldots, s_{n-2} \rangle \cong S_{n-2} \hookrightarrow S_{n-1} \hookrightarrow S_n$ . Then we have

(3) 
$$w_0' = (s_{n-2} \dots s_2 s_1) \cdot w_0'' = s_1(s_2 s_1) \dots (s_{n-2} \dots s_1).$$

Consider the polynomial  $\pi_{c_{n-1}}^{(\beta)}x^{\lambda}$ . According to Lemma 5.5, it is multiplicity free. Let  $x^{\mu}$  be a  $\lambda$ -nonalternating monomial occurring in it. By construction, we have  $\lambda_1 \geq \mu_1$  and  $\mu_i \geq \lambda_{i+1}$  for each  $1 \leq i \leq n-1$ . We distinguish between the two cases:

(1) There exists a k such that  $\mu_k > \lambda_k$ , with  $k \geq 3$ . Denote the sum of all such nonalternating monomials from  $[\pi_{c_{n-1}}^{(\beta)} x^{\lambda}]_{\lambda}$  by  $r_1(\beta, x_1, \ldots, x_n)$ . With the previous notation:

$$r_1(\beta, x_1, \dots, x_n) = A_1 + \sum_{j=0}^{\lambda_1 - \lambda_2} \left( B_1^j \cdot (\pi_1^{(\beta)}(x_1^{\lambda_1 - j} x_2^{\lambda_2}) - x_1^{\lambda_1 - j} x_2^{\lambda_2}) \right)$$

(2) We have  $\mu_2 > \lambda_2$  and  $\lambda_i \geq \mu_i$  for each  $i \geq 3$ . The sum of all such nonalternating monomials from  $[\pi_{c_{n-1}}^{(\beta)} x^{\lambda}]_{\lambda}$  will be denoted by  $r_2(\beta, x_1, \dots, x_n)$ . With the previous notation:

$$r_2(\beta, x_1, \dots, x_n) = \sum_{j=0}^{\lambda_1 - \lambda_2} \left( B_0^j \cdot (\pi_1^{(\beta)}(x_1^{\lambda_1 - j} x_2^{\lambda_2}) - x_1^{\lambda_1 - j} x_2^{\lambda_2}) \right)$$

Now we shall check that  $\pi_{w'_0}^{(\beta)}$  applied to each of  $r_1$  and  $r_2$  equals zero. For this, we take different words for  $w'_0$ . For the first one, take

$$w_0' = (s_{n-2} \dots s_2 s_1) \cdot w_0'',$$

where  $w_0'' = s_2(s_3s_2) \dots (s_{n-1}s_{n-2} \dots s_2)$  is the longest permutation in the subgroup  $\langle s_2, \dots, s_{n-2} \rangle \cong S_{n-2} \hookrightarrow S_{n-1}$ . Then  $\pi_{w_0''}^{(\beta)} r_1 = 0$  by the induction hypothesis, and so is  $\pi_{w_0'}^{(\beta)}$ .

In the second case, we claim that  $\pi_1^{(\beta)}$  annihilates  $r_2$ , hence so does  $\pi_{w_0'}^{(\beta)}$ , because the word (3) ends with  $s_1$ , we have:

$$\pi_1^{(\beta)}(r_2) = \pi_1^{(\beta)} \left( \sum_{j=0}^{\lambda_1 - \lambda_2} \left( B_0^j \cdot (\pi_1^{(\beta)}(x_1^{\lambda_1 - j} x_2^{\lambda_2}) - x_1^{\lambda_1 - j} x_2^{\lambda_2}) \right) \right) =$$

$$=\sum_{j=0}^{\lambda_1-\lambda_2}\pi_1^{(\beta)}\left(B_0^j\cdot(\pi_1^{(\beta)}(x_1^{\lambda_1-j}x_2^{\lambda_2})-x_1^{\lambda_1-j}x_2^{\lambda_2})\right)=\sum_{j=0}^{\lambda_1-\lambda_2}B_0^j\cdot\pi_1^{(\beta)}((\pi_1^{(\beta)}(x_1^{\lambda_1-j}x_2^{\lambda_2})-x_1^{\lambda_1-j}x_2^{\lambda_2}))$$

Recall, that  $\pi_i^{(\beta)}(\pi_i^{(\beta)}(f)-f)=0$ . It follows that:

$$\pi_1^{(\beta)}(r_2) = \sum_{j=0}^{\lambda_1 - \lambda_2} B_0^j \cdot \pi_1^{(\beta)}((\pi_1^{(\beta)}(x_1^{\lambda_1 - j}x_2^{\lambda_2}) - x_1^{\lambda_1 - j}x_2^{\lambda_2})) = \sum_{j=0}^{\lambda_1 - \lambda_2} B_0^j \cdot 0 = 0.$$

Now let us act on  $x^{\lambda}$  by  $\pi_{w_0}^{(\beta)}$ , where we take the word for  $w_0$  given by (2). Consider the following sequences of monomials:

$$x^{\lambda} = x^{\lambda^{(1,0)}} \xrightarrow{\pi_1^{(\beta)}} x^{\lambda^{(1,1)}} \xrightarrow{\pi_2^{(\beta)}} x^{\lambda^{(1,2)}} \xrightarrow{\pi_3^{(\beta)}} \dots \xrightarrow{\pi_{n-1}^{(\beta)}} x^{\lambda^{(1,n-1)}} = x^{\lambda^{(2,0)}};$$

$$x^{\lambda^{(2,0)}} \xrightarrow{\pi_1^{(\beta)}} x^{\lambda^{(2,1)}} \xrightarrow{\pi_2^{(\beta)}} x^{\lambda^{(2,2)}} \xrightarrow{\pi_3^{(\beta)}} \dots \xrightarrow{\pi_{n-2}^{(\beta)}} x^{\lambda^{(2,n-2)}} = x^{\lambda^{(3,0)}};$$

$$x^{\lambda^{(3,0)}} \xrightarrow{\pi_1^{(\beta)}} x^{\lambda^{(3,1)}} \xrightarrow{\pi_2^{(\beta)}} x^{\lambda^{(3,2)}} \xrightarrow{\pi_3^{(\beta)}} \dots \xrightarrow{\pi_{n-3}^{(\beta)}} x^{\lambda^{(3,n-3)}} \xrightarrow{\pi_1^{(\beta)}} x^{\lambda^{(4,0)}};$$

$$\vdots$$

$$x^{\lambda^{(n-1,0)}} \xrightarrow{\pi_1^{(\beta)}} x^{\lambda^{(n-1,1)}} \xrightarrow{\pi_2^{(\beta)}} x^{\lambda^{(n-1,2)}} = x^{(n,0)};$$

$$x^{\lambda^{(n,0)}} \xrightarrow{\pi_1^{(\beta)}} x^{\lambda^{(n,1)}} = x^{\nu}.$$

Here each  $x^{\lambda^{(i,j)}}$  occurs as a summand in  $\pi_j^{\beta} x^{\lambda^{(i,j-1)}}$ . Such a sequence is called a track of  $x^{\lambda}$ .

For each track with a nonzero  $x^{\nu}$ , we construct an efficient enhanced pattern P as follows. First, note that by Lemma 5.8, each  $\lambda^{(i,0)}$  is  $\lambda^{(i-1,0)}$ -alternating; in particular, it is a partition. Take a pattern P with entries  $p_{ij}$  such that its i-th row contains the first n-i parts of  $\lambda^{(i-1,0)}$ , written in the decreasing order (the initial row contains the parts of  $\lambda$  and has number 0), that is,  $p_{ij} = (\lambda^{(i-1,0)})_{n-i-j+1}$ .

is,  $p_{ij} = (\lambda^{(i-1,0)})_{n-i-j+1}$ . Next, draw a circle around a vertex if the degree of  $\beta$  in front of the corresponding monomials  $x^{\lambda^{(i,j-1)}}$  and  $x^{\lambda^{(i,j)}}$  is equal.

This gives us the location of encircled vertices. According to Lemma 4.8, this defines an efficient enhanced pattern. It is clear that all the efficient patterns can be obtained in such a way, and different patterns correspond to different tracks. Theorem 4.11 is proved.

**Example 5.9.** Let  $\lambda = (2, 1, 0)$ . Below is a track of monomial  $\beta^2 x_1^2 x_2 x_3^2$  and the corresponding enhanced Gelfand–Zetlin pattern, being filled row by row from the right to the left, from top to bottom.

$$x_1^2 x_2 \xrightarrow{\pi_1^{\beta}} \beta x_1^2 x_2^2 \xrightarrow{\pi_2^{\beta}} \beta^2 x_1^2 x_2 x_3^2 \xrightarrow{\pi_1^{\beta}} \beta^2 x_1^2 x_2 x_3^2.$$



#### 6. Proof for the general case

6.1. Face diagrams. For every permutation  $u \in S_n$  there exists a dual Kogan face F such that w(F) = u. Moreover, there can be more than one such Kogan face; they correspond to reduced subwords in

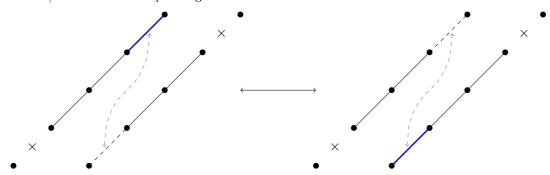
$$\mathbf{w}_0 = (s_{n-1}, \dots, s_1, s_{n-2}, \dots, s_1, \dots, s_1, s_2, s_1)$$

with the product equal to u.

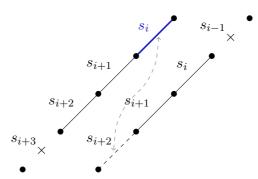
In this section we show how different diagrams of dual Kogan faces with the same permutation are related to each other. This is done in Lemma 6.1 and Lemma 6.3. As we mentioned before, (reduced) dual Kogan faces bijectively correspond to (reduced) pipe dreams; in terms of pipe dreams these lemmas are proved in [BB93], but we still provide their proofs for the sake of completeness of exposition. Then we use them to prove Theorem 4.12 in § 6.3.

## 6.2. Three lemmas about dual Kogan faces.

**Lemma 6.1.** Let F and G be two dual Kogan faces corresponding to permutations  $w_F$  and  $w_G$  respectively. If their diagrams are obtained one from another by moving one edge as shown in figure below, then we have  $w_F = w_G$ .



*Proof.* Recall that simple transpositions correspond to the edges of the diagram as shown below:



It follows that reading this diagram from bottom to top from right to left we get the word

$$w_F = \dots s_{i+1} s_{i+2} \dots s_i s_{i+1} \dots \underline{s_i} \dots$$

(here we underline the letter corresponding to the edge we are moving). Since simple transpositions satisfy braid relations, we get the word for face G:

$$\begin{split} w_F = \dots s_{i+1} s_{i+2} \dots s_i s_{i+1} \dots \underline{s_i} \dots &= \dots s_{i+1} s_{i+2} \dots s_i s_{i+1} \underline{s_i} \dots = \\ &= \dots s_{i+1} s_{i+2} \dots \underline{s_{i+1}} s_i s_{i+1} \dots = \dots \underline{s_{i+2}} s_{i+1} s_{i+2} \dots s_i s_{i+1} \dots = \\ &\underline{s_{i+2}} \dots s_{i+1} s_{i+2} \dots s_i s_{i+1} \dots = w_G. \end{split}$$

**Definition 6.2.** A diagram is called *right-adjusted* if all its edges are pushed towards the right side.

Note that every right-adjusted diagram corresponds to a word of the form

$$\mathbf{u} = (\ldots, s_3, s_4, \ldots, s_m, s_2, s_3, \ldots, s_k, s_1 s_2, \ldots, s_r).$$

Such a word is called *canonical*. It is easy to see that this canonical word is reduced. It follows that there exists a unique right-adjusted diagram for every permutation.

**Lemma 6.3.** Let F and G be two dual Kogan faces such that w(F) = w(G). Then the diagram for F is obtained from the diagram for G by applying transformations described in Lemma 6.1.

*Proof.* Note that it is enough to prove this lemma for a face G with the right-adjusted diagram. Suppose the diagram of F is not right-adjusted. It follows that there exists an edge such that there is no edge immediately to the right of it. Let  $e_0$  be the lowest rightmost edge with such a property. Since a word of simple transpositions is necessarily reduced, then we can move this edge several rows down using Lemma 6.1. Continuing this procedure, we will get a right-adjusted diagram.

**Lemma 6.4.** Suppose that the diagram of face F is right-adjusted. Then permutation w(F) is equal to the permutation u obtained by reading the empty places of diagram F from bottom to top from left to right in the following way:



*Proof.* First we recall that the word u for the face F described in § 2.2 obtained by reading a diagram from bottom to top from right to left. The permutation w(F) is equal to  $w_0u$ , there  $w_0 \in S_n$  is the longest permutation.

The proof is by induction on n. For n = 1 there is nothing to prove. We denote by  $w'_0 \in S_{n-1}$  the longest permutation, here  $S_{n-1}$  is generated by  $s_2, s_3, \ldots, s_{n-1}$ . Let the word for the face F be equal to  $u = u'(s_1s_2 \ldots s_k)$ , with exactly k edges in the first row, and let  $u' \in S_{n-1}$  be obtained by reading rows starting from the second. Then w(F) is equal to  $w(F) = w_0 u = (s_{n-1} \ldots s_2 s_1) w'_0 u'(s_1 s_2 \ldots s_k)$ . The permutation  $w'_0 u'$  corresponds to a smaller diagram obtained by restricting our diagram on rows starting from the second. Graphically, this is shown on Figure 7 (left).

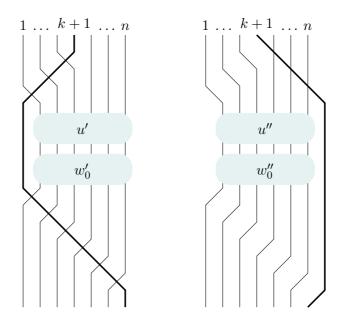


Figure 7. Wiring diagrams

We denote by u'' and  $w_0''$  permutations obtained by replacing every  $s_i$  by  $s_{i-1}$  in reduced words for u' and  $w_0'$ . Since  $u', w_0' \in S_{n-1}$ , where  $S_{n-1}$  is generated by  $s_2, \ldots, s_{n-1}$ , permutations u'' and  $w_0''$  are well defined. It follows that  $w(F) = w_0 u = (s_{n-1} \ldots s_2 s_1) w_0' u'(s_1 s_2 \ldots s_k) = w_0'' u''(s_{n-1} \ldots s_{k+2} s_{k+1})$  (See Figure 7, right).

By the induction hypothesis, the permutation  $w_0''u''(s_{n-1}...s_{k+2}s_{k+1})$  is obtained by reading empty places of diagram F from bottom to top from right to left.

This lemma will play a key role in the proof of the main result. Note that the empty places marked as shown in Lemma 6.4 correspond to actions of Demazure–Lascoux operators described in  $\S$  5.2.

Remark 6.5. Lemma 6.4 is a standard fact about permutations; see, for instance, [Man98, Rémarque 2.1.9].

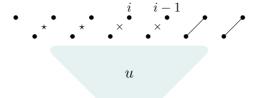
6.3. **Proof of Theorem 4.12.** In this section we show that the restriction of the construction described in Theorem 4.11 to the union of dual Kogan faces gives us an arbitrary Lascoux polynomial.

The main idea of this section is to fix the canonical word for the permutation  $u \in S_n$  described in Lemma 6.4. Then alternating monomials will be located in the face F with a right-adjusted diagram. But not all nonalternating monomials will necessarily cancel. Their offspring will be located in other diagrams corresponding to the permutation u.

Following Lemma 6.3 we recall that diagram of the face F is obtained from the right diagram by moving edges. Moreover, we firstly can move edges to the first row (from left to right), and at each step, the restriction of the diagram to rows starting from the second one will be right.

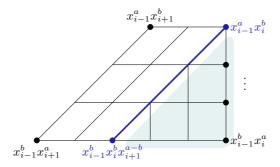
Let us enumerate the diagonals rotated to the right-up in a triangular tableau from left to right. Note, that all edges in diagrams of dual Kogan faces will be directed along such diagonals.

**Lemma 6.6.** Let us have a diagram F, where the rows starting from the second one form a right-adjusted diagram corresponding to the permutation u, and the first row filled as follows: places from 1 to k are filled by edges, places from k+1 to i>k+1 are empty, the remaining places can be filled in any way.



Then the nonalternating monomials appearing at the i-th step (that means, the power of  $x_i$  is bigger than necessary) do not cancel if and only if it is possible to move an edge on the place i in the first row.

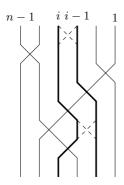
*Proof.* Following Example 5.7, recall that nonalternating monomials could be divided into groups  $\pi_{i-1}^{(\beta)}(m)-m$ , there m is an alternating monomial with the maximum allowed power of  $x_i$  (located on the diagonal in picture below).



Such monomials vanish under the action of operator  $\pi_{i-1}^{(\beta)}$ . Since  $\pi_{i-1}^{(\beta)}$  commutes with operators  $\pi_{i+1}^{(\beta)}, \ldots, \pi_{n-1}^{(\beta)}$  (that correspond to the last part of the first row), we should check whether a permutation u can start with a transposition  $s_{i-1}$ .

Recall that  $u = \dots s_{n-3} \dots s_{k_2} \cdot s_{n-2} \dots s_{k_1}$ . In braid terms this means that first the  $k_1$ -th line going to the end on the place n-1, then the  $k_2$ -th line going to the end on the place n-2, and so on. Then we can put transposition  $s_{i-1}$  on the first place if and only if the (i-1)-th and i-th lines intersect each other or, equivalently, the (i-1)-th line goes to the end earlier than i-th line or, equivalently, there is no edge in the corresponding place in the diagram. On the other hand, if (i-1)-th and i-th lines do not intersect, we can move the edge to the first row and intersect lines. Suppose the new diagram corresponds to permutation u'. It is easy to see that  $\pi_u^{(\beta)}(\pi_{i-1}^{(\beta)}(m))$  coincides with  $\pi_{u'}^{(\beta)}(m)$ . It follows that offsprings of  $\pi_{i-1}^{(\beta)}(m)$  will be located

in diagram u'. Since in this case we consider monomial m, we should add an edge to the i-th place in the first row.



Proof of Theorem 4.12 is obtained by filling the diagram row by row from top to bottom from left to right and applying Lemma 6.6 at each step.

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