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MATHEMATICAL PROBLEMS OF NONLINEARITY

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# On a Classification of Chaotic Laminations which are Nontrivial Basic Sets of Axiom A Flows

**V. S. Medvedev, E. V. Zhuzhoma**

We prove that, given any $n \geqslant 3$ and $2 \leqslant q \leqslant n-1$, there is a closed $n$-manifold $M^n$ admitting a chaotic lamination of codimension $q$ whose support has the topological dimension $n-q+1$. For $n=3$ and $q=2$, such chaotic laminations can be represented as nontrivial 2-dimensional basic sets of axiom A flows on 3-manifolds. We show that there are two types of compactification (called casings) for a basin of a nonmixing 2-dimensional basic set by a finite family of isolated periodic trajectories. It is proved that an axiom A flow on every casing has repeller-attractor dynamics. For the first type of casing, the isolated periodic trajectories form a fibered link. The second type of casing is a locally trivial fiber bundle over a circle. In the latter case, we classify (up to neighborhood equivalence) such nonmixing basic sets on their casings with solvable fundamental groups. To be precise, we reduce the classification of basic sets to the classification (up to neighborhood conjugacy) of surface diffeomorphisms with one-dimensional basic sets obtained previously by V. Grines, R. Plykin and Yu. Zhirov [16, 28, 31].

Keywords: chaotic lamination, basic set, axiom A flow

## Introduction

Roughly speaking, a lamination is a foliation with no singularities on a closed subset of manifold (for exact definitions, see Section 1). Following the definition of Devaney's chaos [10], Churchill [9] introduced the notation of chaotic foliation. A good example of chaotic foliation is a transitive Anosov flow [1]. A natural generalization of chaotic foliation is the notation of chaotic lamination introduced in [33]. By definition, a chaotic lamination is a transitive

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lamination containing a dense subset of compact leaves. Due to Proposition 2 [33], there are no transversally oriented chaotic laminations of codimension one on closed 3-manifolds. However, many examples of chaotic laminations of codimension $q \geqslant 2$ can be constructed starting with hyperbolic dynamical systems with nontrivial basic sets (see Corollary 1 [33]). The reason for the existence of such examples is the Spectral Decomposition Theorem by Smale [30] saying that each basic set is a transitive set containing a dense subset of periodic orbits.

Our first result is the following generalization of Theorem 2 [33]. Below, $\mathbb{R}^k$ is a $k$-dimensional Euclidean space, $\mathbb{S}^l$ is an $l$-dimensional sphere, and $\mathbb{T}^s$ is an $s$-dimensional torus.

**Theorem 1.** *Given any $n \geqslant 3$ and $2 \leqslant q \leqslant n-1$, there is a closed orientable $n$-manifold $M^n$ admitting a chaotic lamination $\Lambda$ of codimension $q$ such that the support of $\Lambda$ has the topological dimension $n-q+1$. Moreover, the compact leaves of $\Lambda$ are homeomorphic to $\mathbb{T}^{n-q}$, while the noncompact leaves are homeomorphic to $\mathbb{R}^1 \times \mathbb{T}^{n-q-1}$.*

For the particular case $n = 3$ and $q = 2$, Theorem 1 was proved in [33]. Nontrivial 2-dimensional basic sets of axiom A flows on 3-manifolds are a good example of chaotic laminations (for basic definitions and notation of the theory of dynamical systems, see the books [3, 19, 29] and surveys [17, 30]). Such basic sets give codimension two chaotic laminations whose supports have the topological dimension 2.

Later on, we consider carefully chaotic laminations that are nontrivial 2-dimensional basic sets of axiom A flows (in short, A-flows) on closed orientable 3-manifolds. It was proved in [24] that a nontrivial 2-dimensional basic set of A-flow on a closed orientable 3-manifold is either an expanding attractor or a contracting repeller. For definiteness, we will consider two-dimensional basic sets that are attractors.

Let $f^t$ be an A-flow on a closed 3-manifold $M^3$ such that $f^t$ has a 2-dimensional attractor $\Lambda_a$. Since $\Lambda_a$ is a codimension one attractor, the stable manifold $W^s(\Lambda_a)$ called a basin of $\Lambda_a$ is an open 3-dimensional submanifold of $M^3$. We know that $\Lambda_a$ has a local fractal structure [28]. Therefore, the complement $M^3 \setminus W^s(\Lambda_a)$ of $W^s(\Lambda_a)$ can have a complicated topological structure (in the example by Franks and Williams [12], such a complement is locally homeomorphic to the product of a Cantor set and a 2-dimensional plane). It is natural to try to embed the basin $W^s(\Lambda_a)$ into a topological or smooth manifold $L$ with the simplest complement $L \setminus W^s(\Lambda_a)$ and the simplest dynamics. By (topological or smooth) compactification of $W^s(\Lambda_a)$ we mean a (topological or smooth) closed manifold $L$ containing an embedding of $W^s(\Lambda_a)$ such that the complement $L \setminus W^s(\Lambda_a)$ consists of finitely many compact components with a topological structure which is simple in a sense (we will call such a compactification a *casing*). To get such compactifications (and corresponding casings), we consider for simplicity nonmixing attractors $\Lambda_a$. The next result says that there are two kinds of smooth compactification of the basin $W^s(\Lambda_a)$, and both compactifications lead to simple dynamics of attractor-repeller type on the corresponding casings.

**Theorem 2.** *Let $f^t$ be an A-flow on an orientable closed 3-manifold $M^3$ such that the nonwandering set $NW(f^t)$ contains a 2-dimensional nonmixing attractor $\Lambda_a$. Then there are two compactifications $M(\Lambda_a)$ and $N(\Lambda_a)$ of the basin $W^s(\Lambda_a)$ by the family of circles $l_1, \ldots, l_k$ such that*

- *both $M(\Lambda_a)$ and $N(\Lambda_a)$ are closed smooth orientable 3-manifolds;*
- *the restriction $f^t|_{W^s(\Lambda_a)}$ is extended continuously to the structurally stable nonsingular flows $\widetilde{f}^t_M$ and $\widetilde{f}^t_N$ on $M(\Lambda_a)$ and $N(\Lambda_a)$, respectively, with the nonwandering set consisting of the attractor $\Lambda_a$ and the repelling isolated periodic trajectories $l_1, \ldots, l_k$;*

- *the family* $\{l_1, \ldots, l_k\} \subset M(\Lambda_a)$ *is a fibered link in* $M(\Lambda_a)$;
- *the manifold* $N(\Lambda_a)$ *is the total space of a locally trivial fiber bundle over a circle*;
- *the flow* $\widetilde{f}^t_N$ *is topologically equivalent to the dynamical suspension* $sus(\vartheta)$ *over some structurally stable diffeomorphism* $\vartheta\colon M^2 \to M^2$ *of a closed surface* $M^2 \subset N(\Lambda_a)$ *with the nonwandering set* $NW(\vartheta)$ *consisting of the expanding one-dimensional attractor* $\lambda_a = \Lambda_a \cap M^2$ *and the isolated repelling periodic orbits* $\{l_1, \ldots, l_k\} \cap M^2$.

After Theorem 2, it is natural to consider the problem of classification for structurally stable flows with two-dimensional basic sets which are dynamical suspensions over structurally stable surface diffeomorphisms with one-dimensional basic sets.

Let $f_i\colon M_i \to M_i$ be a diffeomorphism and $\Omega_i$ an invariant set of $f_i$, $i = 1, 2$. Recall that the restrictions $f_1|_{\Omega_1}$, $f_1|_{\Omega_1}$ are *neighborhood conjugate* if there is a homeomorphism $h\colon M_1 \to M_2$ such that $h(\Omega_1) = \Omega_2$ and $f_2 \circ h|_{\Omega_1} = h \circ f_1|_{\Omega_1}$ [15, 19, 34]. Let $f^t_i$ be a flow on a manifold $N_i$ and $\Lambda_i$ an invariant set of $f^t_i$, $i = 1, 2$. We say that the restrictions $f^t_1|_{\Lambda_1}$, $f^t_2|_{\Lambda_1}$ of $f^t_1$, $f^t_2$ on the invariant sets $\Lambda_1$, $\Lambda_2$, respectively, are *neighborhood equivalent* if there is a homeomorphism $\varphi\colon N_1 \to N_2$ taking any trajectory from $\Lambda_1$ onto a trajectory from $\Lambda_2$ preserving a time direction. The first step to solve the problem of classification for structurally stable flows with two-dimensional basic sets which are dynamical suspensions over structurally stable surface diffeomorphisms with one-dimensional basic sets is to find necessary and sufficient conditions for neighborhood equivalence. The following result says that the neighborhood equivalence of such flows reduces to the neighborhood conjugacy of corresponding structurally stable surface diffeomorphisms provided that support manifolds have a solvable fundamental group. We restrict ourselves to orientation-preserving diffeomorphisms and orientable basic sets.

**Theorem 3.** *Let* $f^t_i$ *be a structurally stable flow that is a dynamical suspension* $sus(\vartheta_i)$ *over an orientation-preserving structurally stable diffeomorphism* $\vartheta_i\colon M^2 \to M^2$ *of a closed orientable surface* $M^2$ *such that* $\vartheta_i$ *has a one-dimensional orientable basic set* $\lambda_i$ *(thus,* $f^t_i$ *has a two-dimensional orientable basic set* $\Lambda_i$*),* $i = 1, 2$*. Suppose that the fundamental groups* $\pi_1\left(M^3_{\vartheta_1}\right)$, $\pi_1\left(M^3_{\vartheta_2}\right)$ *of the supporting manifolds* $M^3_{\vartheta_1}$, $M^3_{\vartheta_2}$ *for* $f^t_1$ *and* $f^t_2$*, respectively, are solvable. Then the flows* $f^t_1|_{\Lambda_1}$, $f^t_2|_{\Lambda_1}$ *are neighborhood equivalent if and only if the diffeomorphisms* $\vartheta_1|_{\lambda_1}$, $\vartheta_2|_{\lambda_2}$ *are neighborhood conjugate.*

Let $f^t$ be a flow satisfying the conditions of Theorem 3. To be precise, $f^t$ is a structurally stable flow that is a dynamical suspension $sus(\vartheta)$ over an orientation-preserving structurally stable diffeomorphism $\vartheta\colon M^2 \to M^2$ of a closed orientable surface $M^2$ such that $\vartheta$ has a one-dimensional orientable basic set $\lambda$. This implies that $f^t$ has a two-dimensional orientable basic set $\Lambda$. Grines and Plykin [15, 16, 28] introduced the complete invariant of neighborhood conjugacy for the diffeomorphism $\vartheta|_\lambda$. This invariant has in a sense an algebraic nature concerning an action in the fundamental group of casing for $\lambda$. Zhirov [31, 32] introduced a more convenient complete invariant of neighborhood conjugacy for the diffeomorphism $\vartheta|_\lambda$[1]. This invariant is combinatorial, and there is a finite algorithm deciding either two invariants coincide or not. It is natural to assign Zhirov's invariant denoted by $Z\left(f^t|_\lambda\right)$ to the flow $f^t$ (it would be interesting to construct Zhirov's invariant $Z\left(f^t|_\lambda\right)$ directly for the flow $f^t$). The following statement follows immediately from Theorem 3.

[1]Actually, Grines, Plykin and Zhirov constructed complete invariants for the more wide classes of diffeomorphisms and homeomorphisms.

**Corollary 1.** *Let $f_i^t$ be a structurally stable flow that is a dynamical suspension $sus(\vartheta_i)$ over an orientation-preserving structurally stable diffeomorphism $\vartheta_i \colon M^2 \to M^2$ of a closed orientable surface $M^2$ such that $\vartheta_i$ has a one-dimensional orientable basic set $\lambda_i$ (thus, $f_i^t$ has a two-dimensional orientable basic set $\Lambda_i$), $i = 1, 2$. Suppose that the fundamental groups $\pi_1\left(M^3_{\vartheta_1}\right)$, $\pi_1\left(M^3_{\vartheta_2}\right)$ of the supporting manifolds $M^3_{\vartheta_1}$, $M^3_{\vartheta_2}$ for $f_1^t$ and $f_2^t$, respectively, are solvable. Then the flows $f_1^t|_{\Lambda_1}$, $f_2^t|_{\Lambda_1}$ are neighborhood equivalent if and only if Zhirov's invariants $Z\left(f_1^t|_{\Lambda_1}\right)$, $Z\left(f_2^t|_{\Lambda_2}\right)$ coincide.*

We mention some results concerning the subject. Chaotic foliations were considered in [6]. High-dimensional flows with nonmixing codimension one basic sets were classified in [4]. Laminations and its applications in dynamical systems were considered in the book [19].

The structure of the article is as follows. Section 1 provides preliminary information and results. The main theorems are proved in Section 2.

## 1. Preliminaries

Here, we introduce the main definitions and notions. In addition, we prove Lemma 1, which we need in the proof of Theorem 3.

*Chaotic laminations.* Following [2], we introduce the definition of local lamination, see also [19]. For simplicity, we actually give the definition of local lamination without singularities. Let $M^n$ be an $n$-manifold, $n \geqslant 2$, and $\mathcal{M} \subset M^n$ a subset which can coincide with $M^n$. Suppose $\mathcal{M}$ is a union $\bigcup\limits_{\alpha} L_\alpha$ of pairwise disjoint connected injectively embedded $d$-manifolds, $1 \leqslant d \leqslant n-1$, where $\alpha$ runs some set of indexes. The family $\{L_\alpha\}$ denoted by $\mathcal{D}$ is called a *local lamination* if, given any point $x \in \mathcal{M}$, there is a neighborhood $U(x) \subset M^n$ of $x$ and a $C^r$-diffeomorphism $\psi \colon U(x) \to \mathbb{R}^n$, $r \geqslant 0$, such that any component of the intersection $U(x) \cap L_\alpha$ (if nonempty) is mapped under $\psi$ onto a hyperplane $x_{d+1} = c_1, \ldots, x_n = c_{n-d}$. The set $\mathcal{M}$ is called a *support* of the local lamination $\mathcal{D}$. Each $L_\alpha$ is called a *leaf.* A component of the intersection $L_\alpha \cap U(x)$ is called a *local leaf.* This local leaf induces an interior topology on every leaf $L_\alpha$. Keeping such a topology in mind, we will say that a leaf is closed, compact or homeomorphic to some manifold.

The notation of local lamination is a generalization of notation of foliation and lamination. If the support $\mathcal{M}$ coincides with the manifold $M^n$, then a local lamination $\mathcal{D}$ is a *foliation* (sometimes, it is said to be a foliation without singularities). A good example of foliation is provided by trajectories of nonsingular flow. For other examples of foliations, see [8, 26]. If the support $\mathcal{M}$ is a closed subset of $M^n$, then a local lamination is called a *lamination.* A good example of lamination is a geodesic lamination. Many examples of laminations have dynamical origins [19].

Let $\mathcal{L}$ be a lamination and $|\mathcal{L}|$ the support of $\mathcal{L}$. The lamination $\mathcal{L}$ is called *transitive* if $\mathcal{L}$ has a leaf that is dense in the support $|\mathcal{L}|$. A lamination $\mathcal{L}$ is called *chaotic* if $\mathcal{L}$ is transitive and the set of closed leaves of $\mathcal{L}$ is dense in $|\mathcal{L}|$.

The notation of chaotic foliation was induced by the notation of a chaotic dynamical system introduced by Devaney [10]. Recall that a dynamical system is chaotic in the Devaney sense provided that the following conditions hold:

1) the dynamical system is transitive (there is a dense orbit) and the set of periodic orbits is dense in the phase space of the dynamical system;

2) there is a sensitive dependence on initial conditions.

In the paper [5], it was proved that condition (1) implies condition (2). This allowed Churchill [9] to introduce the notation of chaotic foliation as follows. A foliation $\mathcal{F}$ on a manifold $M$ is called *chaotic* if $\mathcal{F}$ is transitive (there is a leaf that is dense in $M$) and the set of closed leaves of $\mathcal{F}$ is dense in $M$. We see that the notation of chaotic lamination generalizes the notation of chaotic foliation (a chaotic lamination becomes a chaotic foliation when a support of the lamination becomes a supporting manifold).

*Hyperbolic invariant sets.* Let $f^t$ be a smooth flow on a closed $n$-manifold $M^n$, $n \geqslant 3$. A subset $\Lambda \subset M^n = M$ is *invariant* provided that $\Lambda$ consists of trajectories of $f^t$. An invariant nonsingular set $\Lambda \subset M$ is called *hyperbolic* if the subbundle $T_\Lambda M$ of the tangent bundle $TM$ can be represented as a $Df^t$-invariant continuous splitting $E^{ss}_\Lambda \oplus E^t_\Lambda \oplus E^{uu}_\Lambda$ such that

1) $\dim E^{ss}_\Lambda + \dim E^t_\Lambda + \dim E^{uu}_\Lambda = n$;
2) $E^t_\Lambda$ is the line bundle tangent to the trajectories of the flow $f^t$;
3) there are $C_s > 0$, $C_u > 0$, $0 < \lambda < 1$ such that

$$\left\|df^t(v)\right\| \leqslant C_s \lambda^t \|v\|, \quad v \in E^{ss}_\Lambda; \quad \left\|df^{-t}(v)\right\| \leqslant C_u \lambda^t \|v\|, \quad v \in E^{uu}_\Lambda, \quad t > 0.$$

A singular point $x$ is hyperbolic if $x$ is an isolated hyperbolic equilibrium state. The topological structure of flow near $x$ is described by the Grobman–Hartman theorem, see, for example, [29]. In this case $E^t_x = 0$ and $\dim E^{ss}_\Lambda + \dim E^{uu}_\Lambda = n$.

If $\Lambda$ does not contain fixed points, then the bundles $E^{uu}_\Lambda \oplus E^t_\Lambda = E^u_\Lambda$, $E^{ss}_\Lambda \oplus E^t_\Lambda = E^s_\Lambda$, $E^{uu}_\Lambda$, $E^{ss}_\Lambda$ are uniquely integrable [22, 30]. The corresponding leaves $W^u(x)$, $W^s(x)$, $W^{uu}(x)$, $W^{ss}(x)$ through a point $x \in \Lambda$ are called *unstable*, *stable*, *strongly unstable*, and *strongly stable manifolds*, respectively.

*A-flows.* Given a set $U \subset M^n$, denote by $f_{t_0}(U)$ the shift of $U$ along the trajectories of $f^t$ at time $t_0$. Recall that a point $x$ is nonwandering if, given any neighborhood $U$ of $x$ and a number $T_0 > 0$, there is $t_0 \geqslant T_0$ such that $U \cap f_{t_0}(U) \neq \varnothing$. The *nonwandering set* $NW(f^t)$ of $f^t$ is the union of all nonwandering points.

Denote by $\mathrm{Fix}\left(f^t\right)$ the set of fixed points of $f^t$. Following Smale [30], we shall call $f^t$ an *A-flow* provided that its nonwandering set $NW\left(f^t\right)$ is hyperbolic and the periodic trajectories are dense in $NW\left(f^t\right) \setminus \mathrm{Fix}\left(f^t\right)$. A-flows are a wide class of dynamical systems containing structurally stable flows including Anosov flows and Morse–Smale flows [1, 20].

According to Smale's Spectral Decomposition Theorem [21, 30], the nonwandering set of A-flow is a disjoint union of closed, and invariant, and transitive sets called *basic sets*. A basic set is called *trivial* if it is either an isolated singularity or an isolated periodic trajectory. Otherwise, a basic set is *nontrivial*. Any nontrivial basic set of A-flow has the topological dimension no less than one, and a supporting manifold admitting a nontrivial basic set has the dimension no less than three.

*Fibered links.* Recall that a *link* in a 3-manifold $M^3$ is a collection of disjoint embedded circles $L = \{l_1, \ldots, l_k\} \subset M^3$. The link $L = \{l_1, \ldots, l_k\}$ is *fibered* if $M^3 \Big\backslash \left(\bigcup\limits_{i=1}^{k} l_i\right)$ is the total space of fiber bundle $p\colon \left(M^3 \setminus L\right) \to S^1$ and the boundary of the fibers $p^{-1}(\cdot)$ is $L$. In addition, the fibers $p^{-1}(\cdot)$ meet $L$ nicely. To be precise, consider the solid torus $\mathbb{P}_0 = \mathbb{S}^1 \times \mathbb{D}^2$ called a *canonical solid torus*. Here, $\mathbb{S}^1$ is a circle endowed with the cyclic coordinate $\vartheta$, and $\mathbb{D}^2$ a unit disk $\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$. Set $S^1 = \partial\mathbb{D}^2$. The mapping $p_0(\vartheta, z) = \frac{z}{|z|}$, $\vartheta \in \mathbb{S}^1$, $z \in \mathbb{D}^2 \setminus \{0\}$, is the fiber bundle

$$p_0\colon \mathbb{S}^1 \times \left(\mathbb{D}^2 \setminus \{0\}\right) \to S^1$$

over $S^1$ with the fiber an annulus denoted by $A_0$. There is a tubular neighborhood $T(l_i)$ of $l_i$ homeomorphic to $\mathbb{P}_0$ (so, we can assume $T(l_i) = \mathbb{P}_0$) such that $T(l_i) \setminus \{l_i\} = \mathbb{P}_0 \setminus \left(\mathbb{S}^1 \times \{0\}\right)$. By definition, $p|_{T(l_i)\setminus\{l_i\}}$ is isomorphic to $p_0$, $i = 1, \ldots, k$.

Next, we need the following statement.

**Lemma 1.** *Let $M^3_\varphi$ be a mapping torus that is the total space of the fiber bundle over a circle and a fiber a closed orientable surface $M^2$. Suppose that $M^3_\varphi$ is an orientable closed 3-manifold with a solvable fundamental group. Then $M^2$ is either $\mathbb{S}^2$ or $\mathbb{T}^2$.*

*Proof.* The combinatorial description of the fundamental group $\pi_1\left(M^3_\varphi\right)$ is as follows:

$$\left\langle \text{the generators of } \pi_1(M^2),\ c \mid \text{the relations in } \pi_1(M^2),\ c \cdot \gamma \cdot c^{-1} = \varphi_*(\gamma),\ \gamma \in \pi_1\left(M^2\right)\right\rangle.$$

Clearly, $\pi_1\left(M^2\right)$ is a subgroup of $\pi_1\left(M^3_\varphi\right)$. Since each subgroup of the solvable group is a solvable group, $\pi_1\left(M^2\right)$ is a solvable group. Hence, $M^2$ is either $\mathbb{S}^2$ or $\mathbb{T}^2$ because the fundamental groups of closed surfaces of Euler characteristic less than zero are not solvable. □

## 2. Proofs of main results

In this section we prove the main results.

*Proof of Theorem* 1. Choose and fix $n \geqslant 3$ and $2 \leqslant q \leqslant n-1$. Since a basic set of A-flow is transitive and periodic trajectories are dense in the basic set, any nontrivial basic set is a one-dimensional chaotic lamination. By Theorem 3 and Corollary 1 [24], any closed orientable $(q+1)$-manifold $M^{q+1}$ admits an A-flow $f^t$ such that the nonwandering set $NW\left(f^t\right)$ contains a two-dimensional attractor. As a consequence, $M^{q+1}$ admits a one-dimensional chaotic lamination, say $\Lambda$, of the topological dimension 2. Then the $n$-manifold $M^n = M^{q+1} \times \mathbb{T}^{n-q-1}$ admits an $(n-q)$-dimensional (i. e., codimension $q$) lamination $\Lambda \times \mathbb{T}^{n-q-1}$ with the topological dimension $n-q+1$. A product of periodic trajectory and $\mathbb{T}^{n-q-1}$ gives a compact leaf which is homeomorphic to $\mathbb{T}^{n-q}$. A product of nonperiodic trajectory and $\mathbb{T}^{n-q-1}$ gives a noncompact leaf homeomorphic $\mathbb{R}^1 \times \mathbb{T}^{n-q-1}$. Since periodic trajectories are dense in $\Lambda$, the compact leaves are dense in $\Lambda \times \mathbb{T}^{n-q-1}$. It follows from a transitivity of $\Lambda$ that $\Lambda \times \mathbb{T}^{n-q-1}$ contains a leaf which is dense in $\Lambda \times \mathbb{T}^{n-q-1}$. Hence, $\Lambda \times \mathbb{T}^{n-q-1}$ is also a chaotic lamination. This completes the proof. □

*Proof of Theorem* 2. According to [7], the restriction $f^t\big|_{\Lambda_a}$ of $f^t$ on $\Lambda_a$ is a dynamical $\tau$-time suspension over some homeomorphism $\varphi_*\colon \Pi_0 \to \Pi_0$ where $\Pi_0$ is the topological closure of $W^{uu}(x_0)$, $x_0 \in \Lambda_a$. Taking a circle $S^1$ as $[0;\,\tau]/0 \simeq \tau$, one gets the fiber bundle $p_a\colon \Lambda_a \to S^1 = [0;\,\tau]/0 \simeq \tau$ with the fiber $\Pi_0$, where $p_a(x) = t$ provided that $x \in f_t(\Pi_0)$. Due to Lemma 3 [24], this fiber bundle structure can be extended to some attracting neighborhood $U(\Lambda_a)$ of $\Lambda_a$ such that

- the boundary $\partial U(\Lambda_a)$ is transversal to $f^t$ and consists of finitely many components $T^2_i$, $i = 1, \ldots, k$, where each $T^2_i$ is homeomorphic to the 2-torus $\mathbb{T}^2$;
- the flow $f^t$ in $W^s(\Lambda_a)$ has a global section $S$ that is a locally flat surface transversal to $\partial U(\Lambda_a)$, and the intersection $S \cap \partial U(\Lambda_a)$ consists of pairwise disjoint closed simple curves;

- there is the fiber bundle $P_W: W^s(\Lambda_a) \to S^1 = [0;\tau]/0 \simeq \tau$ with the fiber $S$ where $P_W(x) = t$ provided that $x \in f_t(S)$; moreover, the fibers form a foliation denoted by $F_W$.

The idea of the next speculations is to glue solid tori to the boundary $\partial U(\Lambda_a)$ extending the fiber bundle $P_W$.

The transversality of each $T_i^2$ to the trajectories of $f^t$ and the inclusion $U(\Lambda_a) \subset W^s(\Lambda_a)$ imply that every positive semitrajectory starting at a point of $T_i^2$ belongs to $U(\Lambda_a)$ and never intersects again $\bigcup\limits_{j=1} T_j^2$. It follows that the union $T_a = \bigcup\limits_{j=1} T_j^2$ divides $W^s(\Lambda_a)$ into two domains $U(\Lambda_a)$ and $U_{out} = W^s(\Lambda_a) \setminus U(\Lambda_a)$. Clearly, every negative semitrajectory starting at a point of $T_a$ belongs to $U_{out}$ and never intersects $T_a$ again. Keeping in mind the continuous dependence of trajectories on initial conditions, one finds that $U_{out}$ is homeomorphic (in the interior topology) to $\left(\bigcup\limits_{i=1}^{k} T_i^2\right) \times (-\infty; 0)$ that is the disjoint union $\bigcup\limits_{i=1}^{k} \left(T_i^2 \times (-\infty; 0)\right) = U_{out}$. To construct compactifications of $W^s(\Lambda_a)$, it is enough to construct compactifications for every $T_i^2 \times (-\infty; 0)$ by a circle.

Let $\mathbb{P}_0 = \mathbb{S}^1 \times \mathbb{D}^2$ be the *canonical solid torus*. Denote by $\vec{v}$ a vector field on $\mathbb{P}_0$ such that $\vec{v}$ is transversal to the boundary $\partial\mathbb{P}_0$ directed outside of $\mathbb{P}_0$, and suppose that $\vec{v}$ has a unique periodic trajectory $l_0 = \mathbb{S}^1 \times \{0\}$ that is a repeller of $\vec{v}$. Using a homeomorphism $\vartheta_i: \partial\mathbb{P}_0 \to T_i^2$, one can construct a homeomorphism $\widetilde{\vartheta}_i: \mathbb{P}_0 \setminus \{l_0\} \to T_i^2 \times (-\infty; 0] = T_i^3$ for every $i = 1, \ldots, k$ as follows. Take a point $z \in \mathbb{P}_0 \setminus \{l_0\}$, and let $F_0^t$ be a flow induced by the vector field $\vec{v}$. Denote by $F_t$ a shift (the so-called $t$-shift) along the trajectories of $F^t$ at time $t$. It is well known that $F^t$ is a diffeomorphism for any fixed $t$. By the definition of $\vec{v}$, there is a unique $t_0 \geqslant 0$ such that $F_{t_0}(z) \in \partial\mathbb{P}_0$. Denote by $f_t$ the $t$-shift along the trajectories of $t^t$. Put by definition $\widetilde{\vartheta}_i(z) = f_{-t_0} \circ \vartheta_i \circ F_{t_0}(z)$.

The sets $T_1^3, \ldots, k$ are pairwise disjoint because each $T_i^3$ is generated by negative semitrajectories starting on $T_i^2$, an $T_i^2 \cap T_j^2 = \varnothing$ for $i \neq j$. We see that

$$W^s(\Lambda_a) = U(\Lambda_a) \bigcup_{i=1}^{k} T_i^3 = U(\Lambda_a) \bigcup_{i=1}^{k} \widetilde{\vartheta}_i^{-1}(\mathbb{P}_0 \setminus \{l_0\}).$$

To finish some compactification, we take copies $l_1, \ldots, l_k$ of $l_0$ and introduce a topological structure on every set $T_i^3 \cup l_i = \widetilde{\vartheta}_i^{-1}(\mathbb{P}_0 \setminus \{l_0\}) \cup l_i$ separately as follows. Take the set $T_i^3 \cup l_i$ where $l_i = l_0$. Remark that $T_i^3$ is endowed with the initial topology induced by $M^3$. Given a point $x \in l_i = l_0$, let $U(x)$ be a neighborhood of $x \in l_0 \subset \mathbb{P}_0$ in the solid torus $\mathbb{P}_0$. Clearly, $U(x) \setminus \{l_0\}$ is an open set in $\mathbb{P}_0$. Therefore, $\widetilde{\vartheta}_i(U(x) \setminus \{l_0\}) = \widetilde{U}(x)$ is an open set in $T_i^3$. We consider the union $\widetilde{U}(x) \cup (U(x) \cap l_0)$ as a neighborhood of $x$ in $T_i^3 \cup l_i$. It is easy to see that the set of such neighborhoods introduces the topological structure on $T_i^3 \cup l_i$. This gives the compactification of $T_i^3$ by the closed curve $l_i$ for every $i = 1, \ldots, k$. As a consequence, one gets the compactification $W^s(\Lambda_a) \bigcup\limits_{i=1}^{k} l_i$ denoted by $M(\Lambda_a)_{\vartheta_1,\ldots,\vartheta_k}$. One can easily check that $M(\Lambda_a)_{\vartheta_1,\ldots,\vartheta_k}$ is a closed topological manifold. Due to [25], every topological 3-dimensional manifold admits a unique structure of smooth manifold which is an extension of previous topological structure. Hence, $M(\Lambda_a, \vartheta_1, \ldots, \vartheta_k)$ is endowed with the structure of a smooth manifold which is the extension of the smooth structure on $W^s(\Lambda_a)$. Below, we describe the homeomorphisms $\vartheta_1, \ldots, \vartheta_k$ in detail to get the compactification $N(\Lambda_a)$.

Let $P_i^3$ be a copy of $\mathbb{P}_0$, and $\vec{v}_i = \vec{v}$ the vector field with closed curve $l_i = l_0$ in $P_i^3$, $i = 1, \ldots, k$. By construction, $N(\Lambda_a, \vartheta_1, \ldots, \vartheta_k) = U(\Lambda_a) \underset{\vartheta_1}{\cup} P_1^3 \cup \cdots \underset{\vartheta_k}{\cup} P_k^3$. Slightly deforming the vector fields $\vec{v}_i$, $i = 1, \ldots, k$, one can assume that these fields and the restriction $f^t|_{U(\Lambda_a)}$ form the smooth flow $\widetilde{f}^t$ that is the extension of $f^t$ to $N(\Lambda_a, \vartheta_1, \ldots, \vartheta_k)$. Clearly, $NW\left(\widetilde{f}^t\right) = \Lambda_a \bigcup\limits_{i=1}^{k} l_i$. Since $l_i$ are repelling periodic trajectories of $\vec{v}_i$, $i = 1, \ldots, k$, the trajectories $l_1, \ldots, l_k$ are repelling isolated periodic trajectories of $\widetilde{f}^t$.

Take the foliation $F_W$ generated by the fibers of the bundle $P_W$. By construction, given any $T_i^2$, the intersections of the leaves with $T_i^2$ form a rational foliation $F\left(T_i^2\right)$ such that each leaf of $F\left(T_i^2\right)$ belongs to a leaf of $F_W$. Therefore, $P_W$ induces the fiber bundle $P_W|_{U(\Lambda_a)} : U(\Lambda_a) \to \to S^1 = [0; \tau]/0 \simeq \tau$ such that the restriction $F_W|_{U(\Lambda_a)}$ of $F_W$ on $U(\Lambda_a)$ is a foliation whose leaves are the fibers of the bundle $P_W|_{U(\Lambda_a)}$. This gives the continuation of the fiber bundle $P_W|_{U(\Lambda_a)}$ to $T_i^3$, $i = 1, \ldots, k$.

The intersections of these leaves with $\mathbb{T}^2$ produce the rational foliation $F_0$ generated by parallels on the torus $\mathbb{T}^2 = \partial\mathbb{P}_0$. We know that rational foliations are topologically equivalent [3, 27]. Hence, there are the mappings $\vartheta_i \colon \partial\mathbb{P}_0 \to T_i^2$ taking the leaves of the foliation $F_0$ to the leaves of $F\left(T_i^2\right)$. This gives the continuation of the fiber bundle $P_W|_{U(\Lambda_a)}$ to $T_i^3$, $i = 1, \ldots, k$.

Let $\mu_\alpha$, $\alpha \in \mathbb{S}^1$ be the family of meridians on the solid torus $\mathbb{P}_0 = \mathbb{S}^1 \times \mathbb{D}^2$, and these meridians form a rational foliation. Each meridian bounds a disk $\{\cdot\} \times \mathbb{D}^2$ that is transversal to the vector field $\vec{v}$. Recall that $S$ is the global section of the flow $f^t$ in $U(\Lambda_a)$, $S$ is transversal to $f^t$, and the intersection $S \cap T_i^2$, $i = 1, \ldots, k$, consists of closed simple curves $s_1, \ldots, s_{j_i}$. It follows that $s_1, \ldots, s_{j_i}$ belong to the same nonzero homotopy class. Therefore, there is $\vartheta_i$ taking some meridians $\mu_{\alpha,1}, \ldots, \mu_{\alpha,j_i}$ to the curves $s_1, \ldots, s_{j_i}$. The final flow $\widetilde{f}^t$ becomes $\widetilde{f}_N^t$. The locally flat surface $S$ and the disks bounded by the meridians $\mu_{\alpha,1}, \ldots, \mu_{\alpha,j_i}$ form the closed locally flat embedded surface which can be approximated by a smooth surface $M^2$ such that $M^2$ is a global section for the flow $\widetilde{f}_N^t$. Thus, by construction, $\widetilde{f}_N^t$ is a dynamical suspension $sus(\vartheta)$ over some A-diffeomorphism $\vartheta \colon M^2 \to M^2$ with the nonwandering set $NW(\vartheta)$ consisting of the expanding one-dimensional attractor $\lambda_a = \Lambda_a \cap M^2$ and the isolated repelling periodic orbits $\{l_1, \ldots, l_k\} \cap M^2$.

Obviously, the unstable manifolds of the repelling periodic trajectories $l_1, \ldots, l_k$ are three-dimensional open submanifolds of $M^3$. Hence, they intersect transversally the two-dimensional stable manifolds of the points of $\Lambda_a$. Since the nonwandering set $NW(\widetilde{f}^t) = \Lambda_a \bigcup\limits_{i=1}^{k} l_i$ has a hyperbolic structure, $\widetilde{f}_N^t$ is an A-flow satisfying a strong transversality condition. It follows from [20] that $\widetilde{f}_N^t$ is a structurally stable nonsingular flow.

The compactification $M(\Lambda_a)$ and the flow $\widetilde{f}_M^t$ were constructed in [24]. For the reader's convenience, we give a sketch of the proof. By construction above,

$$M(\Lambda_a, \vartheta_1, \ldots, \vartheta_k) = U(\Lambda_a) \underset{\vartheta_1}{\cup} P_1^3 \cup \cdots \underset{\vartheta_k}{\cup} P_k^3,$$

where $P_i^3$ is a copy of $\mathbb{P}_0$, $i = 1, \ldots, k$. Recall (see Section 1) that $p_0 \colon \mathbb{S}^1 \times \left(\mathbb{D}^2 \setminus \{0\}\right) \to S^1$ is the fiber bundle where $p_0(\vartheta, z) = \frac{z}{|z|}$, $\vartheta \in \mathbb{S}^1 = \partial\mathbb{D}^2$, $z \in \mathbb{D}^2 \setminus \{0\}$, $\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$. The fibers $p_0^{-1}(\cdot)$ form a foliation denoted by $\mathcal{F}$. The leaves of $\mathcal{F}$ are annuli transversal to the boundary $\mathbb{T}^2 = \partial\mathbb{P}_0$. The intersections of these leaves with $\mathbb{T}^2$ produce the rational foliation $F_0$ generated by parallels on the torus $\mathbb{T}^2 = \partial\mathbb{P}_0$. We know that rational foliations are topologically equivalent [3, 27]. Hence, there are the mappings $\vartheta_i\colon \partial\mathbb{P}_0 \to T_i^2$ taking the leaves of the foliation $F_0$ to the leaves of $F\left(T_i^2\right)$. It follows that the collection $\{l_1, \ldots, l_k\}$ is a fibered link in $M(\Lambda_a)$. Again, due to [20], $\widetilde{f}^t$ is a structurally stable flow. $\square$

*Proof of Theorem* 3. Obviously, if the diffeomorphisms $\vartheta_1|_{\lambda_1}$, $\vartheta_2|_{\lambda_2}$ are neighborhood conjugate, the flows $f_1^t|_{\Lambda_1}$, $f_2^t|_{\Lambda_2}$ are neighborhood equivalent. So, we have to prove the converse statement.

Suppose that the flows $f_1^t|_{\Lambda_1}$, $f_2^t|_{\Lambda_2}$ are neighborhood equivalent. Hence, there is a homeomorphism $h\colon M^3_{\vartheta_1} \to M^3_{\vartheta_2}$ taking each trajectory of the flow $f_1^t$ onto a trajectory of $f_2^t$ preserving the direction of time. According to Lemma 1, $M_1^2$ is either a sphere or a torus because the fundamental groups $\pi_1\left(M^3_{\vartheta_1}\right)$, $\pi_1\left(M^3_{\vartheta_2}\right)$ are solvable.

Since the fundamental groups $\pi_1\left(M^3_{\vartheta_1}\right)$, $\pi_1\left(M^3_{\vartheta_2}\right)$ are isomorphic, both $M_1^2$ and $M_2^2$ are either spheres or both $M_1^2$ and $M_2^2$ are tori. If each $M_1^2$ and $M_2^2$ is a sphere, then the result follows from [23], Theorem 4.1 (see also Corollary 4.3) where it was proved that dynamical suspensions over homeomorphisms over a sphere are orbitally topologically equivalent if and only if the corresponding homeomorphisms are conjugate.

Assume that $M_i^2$ is a torus denoted by $T_i^2$, $i = 1, 2$. Recall that the fundamental group $\pi_1\left(M^3_{\vartheta_i}\right)$ is generated by $\pi_1\left(T_i^2\right)$ and the element $l_i$ represented by the embedding of a circle $S^1$ in $M^3_{\vartheta_i}$, see Section 1. Clearly, the fundamental group $\pi_1\left(T_i^2\right)$ is a normal subgroup of $\pi_1\left(M^3_{\vartheta_i}\right)$, $i = 1, 2$. According to Grines [15], the isomorphism $(\vartheta_i)_*\colon \pi_1\left(T_i^2\right) \to \pi_1\left(T_i^2\right)$ induced by $\vartheta_i\colon T_i^2 \to T_i^2$ is hyperbolic. Since $\pi_1\left(T_i^2\right)$ isomorphic to the homology group $H_1\left(T_i^2\right)$, the isomorphism $(\vartheta_i)_*$ is represented by unimodular integer matrices which define the linear mapping of $T_i^2$ denoted by the same symbol $(\vartheta_i)_*\colon T_i^2 \to T_i^2$. It follows from [11] that there is a continuous map $h_i\colon T_i^2 \to T_i^2$ such that $(\vartheta_i)_* \circ h_i = h_i \circ \vartheta_i$. Since $(\vartheta_i)_*$ is hyperbolic, $(\vartheta_i)_*$ is an Anosov diffeomorphism. We see that $h_i$ is a semiconjugacy from $\vartheta_i$ to the Anosov diffeomorphism $(\vartheta_i)_*$. Clearly, the dynamical suspension $sus(\vartheta_i)_*$ is an Anosov flow on the mapping torus $M^3_{(\vartheta_i)_*}$. The semiconjugacy $h_i$ can be extended to the semiconjugacy $\overline{h}_i\colon M^3_{\vartheta_i} \to M^3_{(\vartheta_i)_*}$ from $f_i^t = sus(\vartheta_i)$ to $sus(\vartheta_i)_*$. Following [4] (see also [18]), put by definition

$$B_i = \left\{z \in M^3_{(\vartheta_i)_*} \mid \overline{h}_i^{-1}(z) \text{ consists of more than one point}\right\}.$$

It follows from [14] that $h_*\left(\pi_1\left(T_1^2\right)\right) = \pi_1\left(T_2^2\right)$ and $h_*(l_1) = l_2$. Applying [13], Aranson and Zhuzhoma [4], Theorem 1, proved that there is a mapping $\psi\colon M^3_{(\vartheta_1)_*} \to M^3_{(\vartheta_2)_*}$ taking $B_1$ to $B_2$ such that the restriction $\psi|_{T_1^2\times\{0\}}\colon T_1^2 \times \{0\} \to T_2^2 \times \{0\}$ is a linear map. It follows from [15] that $\vartheta_1|_{\lambda_1}$, $\vartheta_2|_{\lambda_2}$ are neighborhood conjugate. This completes the proof. $\square$

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## Conflict of interest

The authors declare that they have no conflicts of interest.

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