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Twin Heteroclinic Connections of Reversible Systems

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Abstract—We examine smooth four-dimensional vector fields reversible under some smooth involution L that has a smooth two-dimensional submanifold of fixed points. Our main interest here is in the orbit structure of such a system near two types of heteroclinic connections involving saddle-foci and heteroclinic orbits connecting them. In both cases we found families of symmetric periodic orbits, multi-round heteroclinic connections and countable families of homoclinic orbits of saddle-foci. All this suggests that the orbit structure near such connections is very complicated. A non-variational version of the stationary Swift−Hohenberg equation is considered, as an example,where such structure has been found numerically.

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To jubileers — our old friends, with best wishes

1. INTRODUCTION

Reversible dynamical systems (both vector fields and diffeomorphisms) appear in different branches of science as representative models. As examples one can mention models in hydrodynamics [6], nonlinear optics [49], and engineering [42]. More references can be found in the reviews [12, 33]. So, their study is of great interest both from mathematical and applied points of view.

In this paper we study the orbit behavior near two types of heteroclinic connections in reversible systems. To be precise, we recall some needed notions. Let M be a smooth (C^∞) manifold of even dimension and $L : M \to M$ be a smooth mapping that is an involution, $L^2 = L \circ L = id_M$. We shall assume below that the set of fixed points of the involution L, Fix $(L) \equiv \{x \in M : L(x) = x\}$, is a smooth submanifold of the dimension equal to half the phase space dimension. In particular, for a four-dimensional case we study here, $\dim \mathrm{Fix}\ (L) = 2$. As an example of such a system we mention the system (1.2) in $\mathbb{R}^4$ presented below with two-dimensional plane Fix (L).

A smooth vector field v on M is reversible w.r.t. L if the identity $DL(v) \equiv -v \circ L$ holds. This implies that, if Φ^t is the flow generated by the vector field v, then the reversed flow, Φ^{-t}, is conjugate to the forward flow:

$$L\big(\Phi^t(x)\big) = \Phi^{-t}\big(L(x)\big). \tag{1.1}$$

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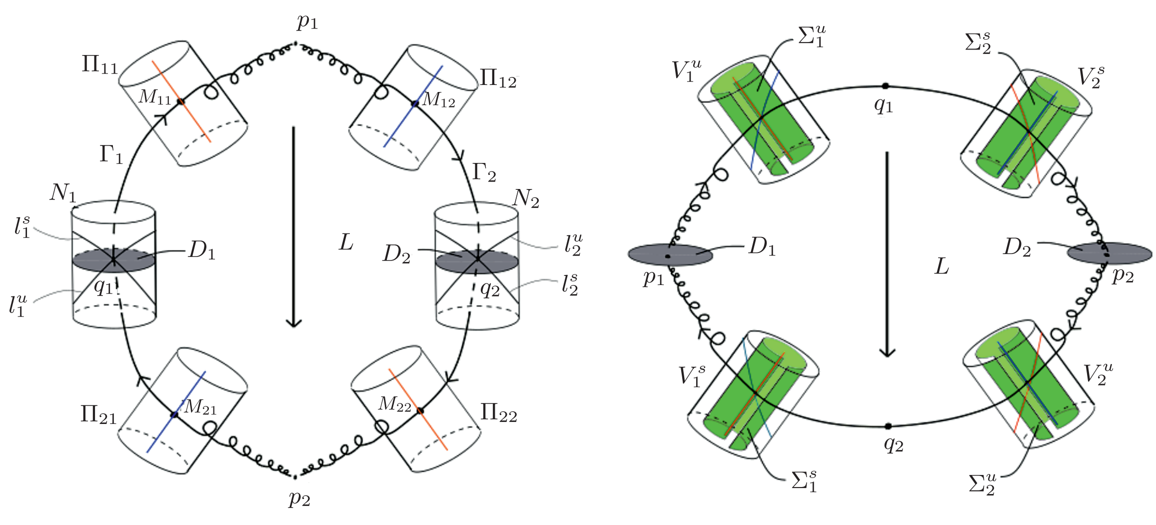
Henceforth, we assume the flow Φ^t to be complete, i. e., any its orbit is defined on all $\mathbb{R}$. Recall that an orbit γ of a reversible vector field is symmetric if it is invariant w.r.t. L: $L(\gamma) = \gamma$. Asymmetric orbits meet in pairs $\{\gamma, L(\gamma)\}$. Symmetric equilibria are those which belong to Fix (L), symmetric orbits are those which intersect Fix (L) once, and an orbit γ intersecting Fix (L) twice is periodic, its period is the double time of the passage from one intersection point with Fix (L) to another one [15].

We are concerned with the orbit structure of a smooth reversible vector field in neighborhoods of two types of twin heteroclinic connections. The first of them is made up of an asymmetric pair of saddle-foci p_1, p_2, $p_2 = L(p_1)$, and two symmetric nondegenerate heteroclinic orbits Γ_i, $i = 1, 2$, connecting these two saddle-foci (see Fig. 1, left panel). For a smooth four-dimensional vector field a saddle-focus is an equilibrium p such that the linearization operator at p, acting on tangent space T_pM, has the quadruple of eigenvalues $\alpha_1 \pm i\beta_1$, $\alpha_2 \pm i\beta_2$, $\alpha_i\beta_i \neq 0$, $\alpha_1\alpha_2 < 0$. Such an equilibrium is of saddle type, it possesses locally two smooth 2-dimensional invariant manifolds, stable $W^s(p)$ and unstable $W^u(p)$, transversally intersecting at p. A saddle value of a saddle-focus is the number $\sigma = \alpha_1 + \alpha_2$. For the asymmetric pair of saddle-foci p_1, p_2, $p_2 = L(p_1)$, their saddle values σ_1, σ_2 have opposite signs: $\sigma_2 = -\sigma_1$. This follows from the relation for linearization operators A_1, A_2 for p_1, p_2 $A_2 = -DL \circ A_1 \circ DL^{-1}$ implies that the eigenvalues of A_2 are minus eigenvalues of A_1. To be definite, we assume that $\sigma_1 < 0$, hence $\sigma_2 > 0$.

By a heteroclinic orbit of a vector field we mean here an orbit which tends to different equilibria as $t \to -\infty$ and $t \to \infty$. Other types of heteroclinic orbits which connect equilibria and periodic orbits, invariant tori, are also studied, but we restrict ourselves to equilibria as limit sets. In our case it will be an orbit which connects p_1 and p_2. We assume, in addition, that both orbits Γ_1, Γ_2 are symmetric ($L(\Gamma_i) = \Gamma_i$) and nondegenerate. The heteroclinic orbit belongs to the intersection of the stable manifold of one equilibrium and the unstable manifold of another equilibrium. Take a point q on this orbit and choose some cross-section N to the flow through this point. Intersection of N with the stable manifold of one saddle-focus and the unstable manifold of another saddle-focus for the four-dimensional case gives two smooth curves through q. Nondegeneracy of the heteroclinic orbit means that these two curves are noncollinear at q.



(a) Type 1 (b) Type 2

Fig. 1. Two types of twin heteroclinic connections.

Another type of twin heteroclinic connections in a reversible system to be considered is a connection which contains two *symmetric* saddle-foci $p_1, p_2 \in$ Fix (L) and two asymmetric heteroclinic orbits Γ_1, Γ_2 which join p_1, p_2 and are permuted by the involution, $\Gamma_2 = L(\Gamma_1)$ (see Fig. 1, right panel). In contrast to the first type of the twin connection, which is structurally stable in the class of reversible vector fields w.r.t. L, the second type is not, since the heteroclinic orbit Γ_1 (and Γ_2) is not symmetric and can be destroyed by a reversible perturbation. This follows from the fact that homo- and heteroclinic orbits to equilibria are not structurally stable objects in the class of generic vector fields. This latter type of a heteroclinic connection was studied earlier [29], where the existence of 1-round homoclinic orbits to any of p_i was proved.

In the case of the first type twin connection, the existence of two symmetric heteroclinic orbits Γ_i, $i = 1, 2$, means that the local unstable manifold $W^u(p_2)$, being extended by the flow, intersects the stable manifold $W^s(p_1)$ along Γ_1, moreover, the symmetry means $L(\Gamma_1) = \Gamma_1$. The same holds true for Γ_2, but $\Gamma_2 \subset W^u(p_1) \cap W^s(p_2)$. Recall [15] that the symmetry of an orbit implies that this orbit intersects the fixed point set Fix (L). Denote $q_i = \Gamma_i \cap \mathrm{Fix}\, L$, $i = 1, 2$. We denote the solutions $\varphi_i(t)$ of v which start at the points q_i at $t = 0$, $\varphi_i(0) = q_i$. Such a solution possesses the symmetry property $\varphi_i(t) = L\varphi_i(-t)$ and $\lim \varphi_1(t) = p_1$ as $t \to \infty$, $\lim \varphi_1(t) = p_2$ as $t \to -\infty$, and $\lim \varphi_2(t) = p_2$ as $t \to \infty$, $\lim \varphi_2(t) = p_1$ as $t \to -\infty$.

L. P. Shilnikov was the first who discovered the complicated orbit behavior near a homoclinic orbit to a saddle-focus with a positive saddle value in a 3-dimensional system [44, 45]. Later these results were extended to systems of greater dimension Shilnikov' results cannot be carried over directly to the reversible and Hamiltonian systems, since their saddle values are always zero due to symmetry of the spectrum at equilibrium (for a reversible system at a symmetric equilibrium). Devaney [14] found a hyperbolic subset (a suspension over Bernoulli's scheme) in a neighborhood of a transversal homoclinic orbit for a Hamiltonian system and found a one-parameter family of symmetric periodic orbits (SPOs) in a four-dimensional reversible system near a nondegenerate homoclinic orbit to a symmetric saddle-focus [15]. The complete orbit behavior on the degenerate level of a Hamiltonian and bifurcations in varying the level set of a Hamiltonian near a transverse homoclinic loop of a saddle-focus were described in [35, 37]. Heteroclinic connections were also much studied both for general systems [50] and for special systems like Hamiltonian [36, 53] and reversible ones [29, 51]. Much information can be found in the review [26]. Many details of the orbit behavior near a homoclinic orbit to saddle-focus, including the reversible case, can be found in [2, 27]. The methods which allow one to discover homoclinic/heteroclinic orbits in Hamiltonian systems were developed in many papers, for example, [13, 16, 17, 34, 40], to mention but a few. Such orbits either appear in Hamiltonian systems close to integrable ones or arise via local bifurcations of equilibria.

As an example, where a reversible system possesses a heteroclinic connection ofthe first type, consider a PDE whose stationary (not depending on time) solutionsare described by an ODE that is transferred to a reversible system of ODEs. This isa variant of the Swift – Hohenberg equation [48]. Some versions of this equation are obtained from variational principles and their reductions are Hamiltonian [4, 9, 18]; however, there are also non-Hamiltonian versions [31]. One such case has stationary solutions $u(x)$ that obey the ODE

$$(1 + \partial_x^2)^2 u - \alpha u - \beta u \partial_x u + u^3 = 0,$$

where parameter α can be arbitrary, but β will be assumed positive, since the change of variable $u \to -u$ makes it positive if $\beta < 0$. Upon defining the variables $q_1 = u$, $q_2 = u'$, $p_1 = -u' - u'''$, and $p_2 = u + u''$, the equation transforms to the four-dimensional first order system

$$\begin{aligned} q_1' &= q_2, & q_2' &= p_2 - q_1\ , \\ p_1' &= p_2 - \alpha q_1 - \beta q_1 q_2 + q_1^3, & p_2' &= -p_1. \end{aligned} \tag{1.2}$$

This system is reversible with respect to the linear involution $L : (q_1, q_2, p_1, p_2) \to (-q_1, q_2, p_1, -p_2)$ and volume-preserving. It has up to three equilibria: the origin, which is symmetric, and (when $\alpha > 1$) the asymmetric pair $(\pm\sqrt{\alpha - 1}, 0, 0, \pm\sqrt{\alpha - 1})$ arising at $\alpha = 1$ from the symmetric one. This system is not Hamiltonian, as can be verified by computing the eigenvalues of the asymmetric pair. Indeed,the characteristic polynomial at these equilibria,

$$P(\lambda) = \lambda^4 + 2\lambda^2 \mp \beta\sqrt{\alpha - 1}\lambda + 2(\alpha - 1),$$

is not even, as it would have to be if the system were Hamiltonian, the zeroth coefficient at λ^3 is due to volume preservation. By contrast, the characteristic polynomial for the symmetric equilibrium at the origin is

$$P(\lambda) = \lambda^4 + 2\lambda^2 + 1 - \alpha,$$

so that when $\alpha < 0$ it is a saddle-focus. It can be shown this equilibrium does have symmetric homoclinic orbits. Indeed, here we get the reversible Hopf bifurcation when α crosses zero, since at $\alpha = 0$ the equilibrium has two double pure imaginary eigenvalues with two-dimensional Jordan boxes for each of them. For $\alpha < 0$ the equilibrium is a saddle-focus and it is an elliptic point for positive $0 < \alpha < 1$. Generically there are two types of this bifurcation depending on the sign of some coefficient in the normal form of the third order in r.h.s. calculated through terms of the second and third order at $\alpha = 0$ (if the linear part has already been transformed to the standard Jordan form). For the equation above this coefficient is $27 - \beta^2$ as in the Hamiltonian case for the case of the usual Swift – Hohenberg equation [10, 18]. This means that for $|\beta| < 3\sqrt{3}$ the bifurcation is subcritical and two symmetric one-round homoclinic orbits exist in this case [28].

The pair of nonsymmetric equilibria arising for $\alpha > 1$ are saddles for $\mu = \sqrt{\alpha - 1}$ small enough, since at $\alpha = 0$ the degenerate symmetric equilibrium has two simple eigenvalues $\pm i\sqrt{2}$ and double zero eigenvalue. Simple eigenvalues continue as follows: $\lambda_{1,2} = \pm i\sqrt{2} - \beta\mu/4 + O(\mu^2)$, they give a stable focus on the stable manifold. Zeroth eigenvalues become two real positive, when $\alpha > 1$ with small $\mu > 0$ and $\beta^2 > 8$, their expansion in μ looks as follows:

$$\lambda_{3,4} = \frac{1}{4}(\beta \pm \sqrt{\beta^2 - 8})\mu + O(\mu^2).$$

For $0 < \beta^2 < 8$ we have complex conjugate eigenvalues, their expansion in μ looks as follows:

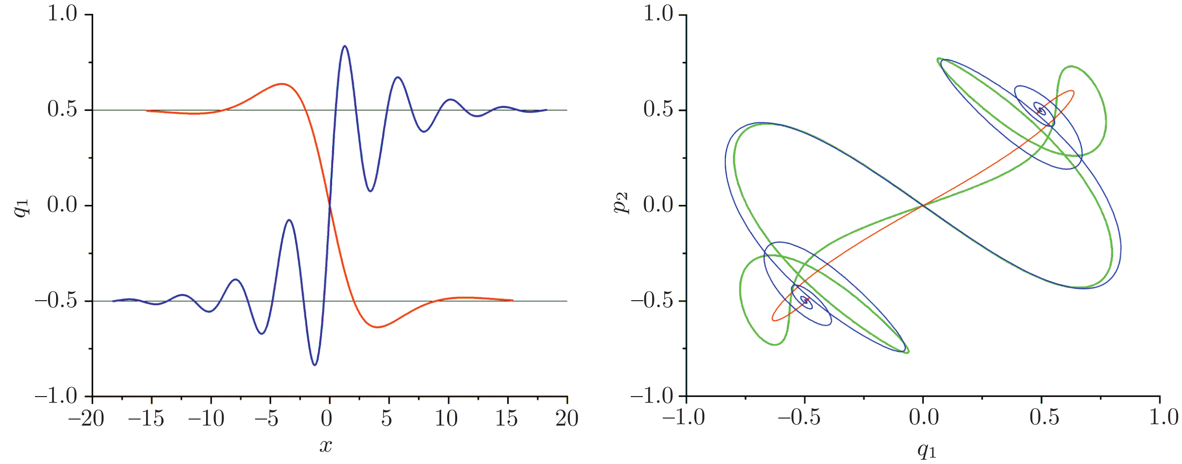
$$\lambda_{3,4} = \frac{1}{4}(\beta \pm i\sqrt{8 - \beta^2})\mu + O(\mu^2).$$

For instance, at $\alpha = 1.25$ and $\beta = 2$ eigenvalues for the upper equilibrium are $\lambda_{1,2} \approx -0.2751 \pm 1.4087i$, $\lambda_{3,4} \approx 0.2751 \pm 0.4087i$, that is, we have a saddle-foci of the type $(2, 2)$ at two symmetrically connected points. But if we take $\alpha = 1.25$ and $\beta = 5$, then these points are saddles of the types $(2, 2)$ with one saddle having a stable manifold with a focus on it ($\lambda_{1,2} \approx -0.4849 \pm 1.5887i$) and an unstable node on a two-dimensional unstable manifold ($\lambda_{3,4} \approx 0.7170, 0.2527$), respectively, with opposite signs for eigenvalues of the symmetric equilibrium. The figures presenting the graphs of $u(x) = q_1(x)$ for both heteroclinic orbits and projections of these orbits onto the plane (q_1, p_2) for parameters $\mu = \sqrt{\alpha - 1} = 0.5$, $\beta = 2$ are shown in Fig. 2, and for $\mu = 0.5$, $\beta = 5$, in Fig. 3.

For completeness, we recall the behavior of multipliers for a family of symmetric periodic orbits. Let γ be some symmetric periodic orbit of the family. Such an orbit intersects the submanifold Fix (L) twice. Let q be a point of intersection $\gamma \cap \mathrm{Fix}\ (L)$. Recall that there exists a cross-section N near q which is invariant w.r.t. L and contains a disk $D \subset \mathrm{Fix}\ (L)$. Thus the Poincaré map$P$ on N near the symmetric periodic orbit is reversible and its linearizationsatisfies the equality $DL \circ DP = DP^{-1} \circ DL$.The tangent vector to the trace of SPO at any its point is invariant w.r.t. DP, so the unity is the root of the characteristic equation. Two other roots make up the pair μ, μ^{-1}, where we denote by μ the root for which $|\mu| \leqslant 1$. Thus, the characteristic equation for this orbit is of the form $-(\mu - 1)(\mu^2 - \tau\mu + 1) = 0$. There are the following types of symmetric periodic orbits

1) *quasi-hyperbolic orientable*, when $\tau > 2$, so that $0 < \mu < 1$, ;
2) *quasi-hyperbolic nonorientable*, when $\tau < -2$, so that $-1 < \mu < 0$;
3) *quasi-elliptic*, when $|\tau| < 2$, so that $\mu = e^{\pm 2\pi i\omega}$, $0 < \omega < \frac{1}{2}$;
4) *parabolic*, when $\tau = \pm 2$, then $\mu = \pm 1$.

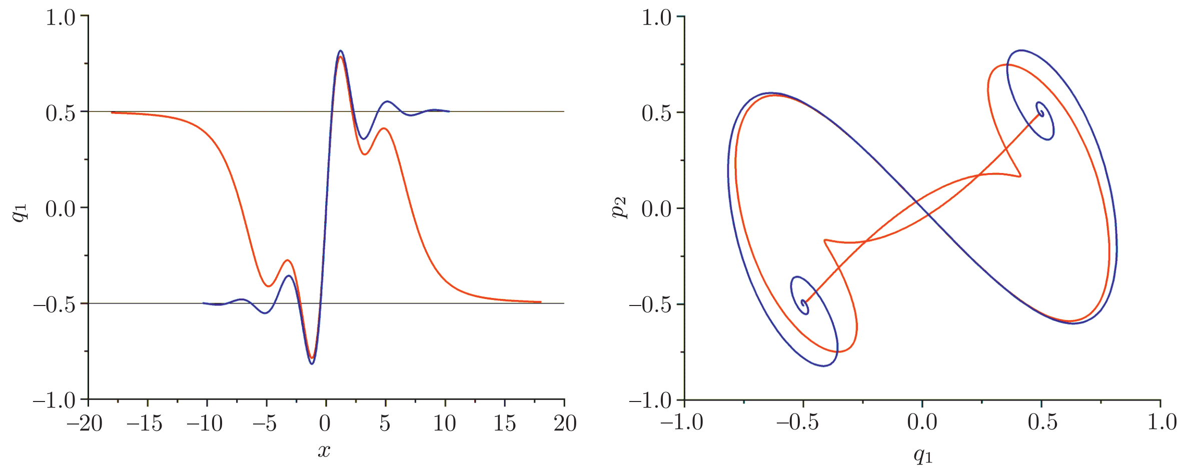
Thus, we expect, as in the case of a nondegenerate homoclinic orbit to a symmetric saddle-focus [15], that, when moving along the spiral which is the trace of SPOs on the disk D (see Theorem 3), the value τ will pass infinitely many times through ± 2 providing the transition from orientable



(a) The graphs $q_1(x)$ for Γ_1, Γ_2.

(b) Projection of the connection and a symmetric PO on the plane (q_1, p_2).

Fig. 2. Parameters $\mu = 0.5$, $\beta = 2$.



(a) The graphs $q_1(x)$ for Γ_1, Γ_2.

(b) Projection of the connection on the plane (q_1, p_2).

Fig. 3. Parameters $\mu = 0.5$, $\beta = 5$.

quasi-hyperbolic SPOs to quasi-elliptic SPOs, then to nonorientable quasi-hyperbolic SPOs and again to quasi-elliptic SPOs. But for the case of heteroclinic connections, the situation is much more complicated and results in this direction remain unsolved so far. We hope to fill this gap elsewhere.

The structure of the paper is as follows. In the next section we provide precise formulations of the problems and formulate the results obtained. In Section 3 we present necessary technical theorems concerning the local normal forms near saddle-foci used in the proofs and forms of the global maps. Section 4 contains the proofs for theorems for the first type twin heteroclinic connection. Section 5 does the same for the second type heteroclinic connection. In the Conclusion we discuss the results obtained and further avenues of research.

2. SET-UP AND MAIN RESULTS

In this section we formulate the conditions on the twin heteroclinic connection under which we study the orbit behavior nearby. The symmetric heteroclinic orbit $\Gamma_1 \subset W^s(p_1) \cap W^u(p_2)$ intersects Fix (L) at the point q_1, hence q_1 belongs to the intersection $W^s(p_1)$ and Fix (L). Both these sets near $q_1 \in M$ are smooth two-dimensional submanifolds and we assume their intersection to be transverse, such a symmetric heteroclinic orbit will be called *elementary*, similar to [15]. The same is assumed for $\Gamma_2 \subset W^u(p_1) \cap W^s(p_2)$. Below we assume a more strong property for the intersection of $W^s(p_1)$ and $W^u(p_2)$ (and for Γ_2 as well). As was noted above, there is a cross-section N_1 through the point q_1 such that N_1 contains a piece D_1 of Fix (L) near q_1 and N_1 is invariant w.r.t. the action of L. The intersection of $W^s(p_1)$ with N_1 is a smooth curve l_1^s which can be transverse to D_1 at q_1, but can be tangent to D_1 at q_1. The same holds true for the intersection of $W^u(p_2)$ and N_1, it is a smooth curve l_1^u. Due to symmetry of Γ_1 and invariance of N_1 under the action of L, the relation $L(l_1^s) = l_1^u$ holds. So, if l_1^s is transverse to D_1 in N_1, then l_1^u is also transverse to D_1 at q_1 and curves l_1^s, l_1^u are noncollinear at q_1 in N_1. In this case we call Γ_1 nondegenerate. We assume this to hold true later on.

For a twin heteroclinic connection of the second type the orbit Γ_1 connects two symmetric saddle-foci $p_1, p_2 \in$ Fix (L), it belongs to the set $W^u(p_1) \cap W^s(p_2)$. Since this intersection contains the curve Γ_1, the intersection cannot be transverse in the four-dimensional M, so we assume this intersection to be simplest degenerate. This means the following: fix some point $q_1 \in \Gamma_1$, then in the tangent space $T_{q_1}M$ 2-planes $T_{q_1}W^u(p_1), T_{q_1}W^s(p_2)$ intersect each other along a straight line tangent to Γ_1 at q_1. Let us choose a smooth 3-dimensional disk $N_1 \ni q_1$ being a cross-section to Γ_1. Then the intersections of $W^u(p_1)$ and $W^s(p_2)$ with N_1 are two smooth curves containing q_1, their tangent vectors at q_1 are assumed to be noncollinear. By symmetry, the same holds true for $\Gamma_2 \subset W^s(p_1) \cap W^u(p_2)$.

The study of orbits in a neighborhood U of the heteroclinic connection in both cases will be carried out by investigating the related Poincaré map on some cross-sections for Γ_i, $i = 1, 2$. Usually the most technically burdened part of this study is related to the investigation of the orbit behavior near equilibria. For a saddle equilibrium of general type the boundary value method due to Shilnikov is used here (see [47] for details). For the saddle-focus point we shall use two normal form theorems. For a twin connection of the first type, where both saddle-foci are nonsymmetric, we apply Belitskii's linearization theorem used in a similar problem in [11, 22].

Theorem 1. *Let $f : U \to \mathbb{R}^n$ be a C^2-smooth diffeomorphism of a neighborhood U of the origin, $f(0) = 0$, with the spectrum of eigenvalues $\lambda_1, \ldots, \lambda_n$. If the inequalities*

$$|\lambda_i| \neq |\lambda_j||\lambda_k|, \ (\forall |\lambda_j| \leqslant 1 \leqslant |\lambda_k|)$$

hold for all $\{i, j, k\}$, then there is a C^1-smooth diffeomorphism $h : U \to U$ such that $h^{-1} \circ f \circ h$ is the linear map in $\mathbb{R}^n$ defined by $Df(0)$.

Now standard arguments show that, if a vector field v has an equilibrium of saddle-focus type at the origin with the local flow φ_t, then the linearization of the map φ_1 at the equilibrium has numbers $\exp[\alpha_1 \pm i\beta_1]$ and $\exp[\alpha_2 \pm i\beta_2]$ as eigenvalues, whose absolute values are $\exp[\alpha_1] < 1$ and $\exp[\alpha_2] > 1$, therefore, φ_1 is linearizable. Hence, the linearization of the flow in a neighborhood of the saddle-focus follows (see, for instance, [23]). Due to symmetry, the linearization near p_1 implies the linearization near $p_2 = L(p_1)$, see Section 3 for details.

For the case of two symmetric equilibria we apply the theorem on the normal form that follows from the results [8] in the analytic case, from [39] for the C^∞-smooth case and from [1, 7] for a finitely smooth case $C^r, r \geqslant 12$.

Theorem 2. *There is a neighborhood of a symmetric equilibrium in M and coordinates (x_1, x_2, y_1, y_2) such that in these coordinates the involution L acts as $(x_1, x_2, y_1, y_2) \to (-y_2, -y_1, -x_2, -x_1)$ and the system casts as*

$$\begin{aligned} \dot{x}_1 &= -H_1(\xi,\eta)x_1 + H_2(\xi,\eta)x_2, & \dot{y}_1 &= H_1(\xi,\eta)y_1 + H_2(\xi,\eta)y_2, \\ \dot{x}_2 &= -H_2(\xi,\eta)x_1 - H_1(\xi,\eta)x_2, & \dot{y}_2 &= -H_2(\xi,\eta)y_1 + H_1(\xi,\eta)y_2, \end{aligned} \tag{2.1}$$

where H_i, $i = 1, 2$, are two functions in variables $\xi = x_1y_1 + x_2y_2$ and $\eta = x_1y_2 - x_2y_1$, defined in U, $H_1(0, 0) = \alpha$ and $H_2(0, 0) = \beta$. Functions H_i are real analytic if M and v are analytic, and they are C^∞-smooth if M and v are such. For a finitely differentiable case functions H_i are polynomials.

The great advantage of this normal form is its integrability that allows one to construct the local map near a symmetric saddle-focus.

Now we are ready to formulate our results. The first result deals with the first-type heteroclinic connection where the theorem was formulated without a proof by Devaney [15]

Theorem 3. *There is a neighborhood U of the heteroclinic connection $C = \overline{\Gamma_1 \cup \Gamma_2}$ such that U contains a smooth one-parameter family of symmetric periodic orbits γ_τ that accumulate at C. The parametrization of the family can be taken as the period of γ_τ and γ_τ tend topologically to C as $\tau \to \infty$.*

The second result here is the existence of countably many two-round symmetric heteroclinic connections involving p_1, p_2. The roundness of the orientable closed curve γ lying in a neighborhood of a given orientable closed curve C is called the integer which expresses the class of loose homotopy for γ w.r.t. C: $[\gamma] = n[C]$.

Theorem 4. *There is a neighborhood U of the heteroclinic connection $C = \overline{\Gamma_1 \cup \Gamma_2}$ such that U contains a countable set of symmetric two-round nondegenerate heteroclinic orbits going from p_2 to p_1 as time increases. Similarly, there is a countable set of symmetric two-round nondegenerate heteroclinic orbits going from p_1 to p_2 as time increases. Thus, taking by one heteroclinic orbit from these two families, we get countably many twin heteroclinic connections of the first type.*

This theorem allows one to prove the existence of 2^n-round symmetric nondegenerate heteroclinic connections existing in a neighborhood of the primary twin connection C. But in order to prove the existence connections of any roundness, one needs to prove symmetric heteroclinic orbits on an odd roundness. To that goal, we prove the following

Theorem 5. *For any neighborhood of the connection C there is a finite number of symmetric 3-round heteroclinic orbits going, as time increases, from p_2 to p_1. There exists another similar, finite family of symmetric 3-round heteroclinic orbits going, as time increases, from p_1 to p_2.*

Our next theorem is concerned with unfoldings of reversible vector fields depending smoothly on a parameter. Let v_μ be such a family, and let each vector field v_μ be reversible w.r.t. the smooth involution L of the same type as above. We assume a critical vector field v_0 to have a heteroclinic connection of the first type.

Theorem 6. *Suppose the family v_μ satisfies the genericity condition, namely, $[\beta_1(\mu)/\beta_2(\mu)]' \neq 0$ at $\mu = 0$. Then for any fixed neighborhood V of the heteroclinic connection C at $\mu = 0$ there is a sequence of μ_n accumulating at the critical value $\mu = 0$ such that the vector field v_{μ_n} has a symmetric pair of nonsymmetric homoclinic orbits, one to p_1 and the other to p_2. Both homoclinic orbits belong to the neighborhood V.*

Theorem 6 has a corollary that if, in addition, the saddle values at p_i do not vanish, i. e., the inequality $\alpha_1 + \alpha_2 \neq 0$ holds, then for $\mu = \mu_n$ the vector field v_{μ_n} satisfies the conditions of theorem by Shilnikov [46] on the existence of a countable set of saddle periodic orbits (nonsymmetric here) in a neighborhood of the homoclinic orbit for p_1. By symmetry, there is a similar family of nonsymmetric periodic orbits in a neighborhood of the pairing homoclinic orbit for p_2. Another result, proved in [41], says that, if the saddle value is negative at p_1, then in a generic two-parameter unfolding there are systems which have stable periodic orbits near a homoclinic orbit of p_1. Definitely, such two-parameter unfolding can be constructed to be reversiblewith two nonsymmetric saddle-foci and a heteroclinic connection. Due to reversibility, such a system has also completely unstable periodic orbits near a pairing homoclinic orbit of p_2 where the saddle value is positive. Such a situation says that in this case the system has mixed dynamical behavior [19], when the phase space contains periodic orbits of stable, saddle, unstable types, as well as elliptic symmetric periodic orbits. For instance, this type of the heteroclinic connection is encountered in one model of a celtic stone [20]. So, one can indeed assert that in that model of the celtic stone stable periodic orbit exist.

Further results concern the existence of periodic and homoclinic orbits for the second-type connection. The first of them is the following.

Theorem 7. *For any neighborhood U of the second-type twin heteroclinic connection C and any $n \in \mathbb{N}$ there are countable families of n-round nondegenerate symmetric homoclinic orbits for p_i, $i = 1, 2$, and countably many of one-parameter families of symmetric periodic orbits.*

This theorem was proved in fact in [29], here we present another geometric proof.

Another result concerns the existence of 2-round connections near the primary C for generic reversible one-parameter unfoldings of a reversible system with the connection $C$of the second type. These connections involve two symmetric saddle-foci p_1, p_2 that are continuations of the initial ones and each such connection contains two nondegenerate nonsymmetric 2-round heteroclinic orbits permuted by involution L.

Theorem 8. *Suppose the family v_μ satisfies some genericity condition at $\mu = 0$ to be formulatedin the Section* 5. *Then for any fixed neighborhood V of the heteroclinic connection C at $\mu = 0$ there is a sequence of μ_n accumulating to the critical value $\mu = 0$ such that the vector field v_{μ_n} has a heteroclinic connection of the second type involving a pair of symmetric saddle-foci and two nonsymmetric nondegenerate two-round heteroclinic orbits connecting saddle-foci and permuted by the involution. Both heteroclinic orbits belong to the neighborhood V.*

3. LOCAL AND GLOBAL MAPS

In this section we utilize the approach of [32] (Section 2.1). We should make first more precise the choice of linearizing coordinates in symmetrically defined neighborhoods U, U' of the equilibria p_1, p_2, $U' = L(U)$. Denote by (U, φ) the chart near the point p_1 in which the vector field v is linear (4.1), thus $\varphi : (x, y) \to U$, (x, y) are Belitskii's coordinates in $\mathbb{R}^4$. Integration of v in these coordinates gives the representation $T(t)$ of the flow:

$$\begin{aligned} x_1(t) &= e^{t\alpha_1}[x_1^0 \cos(\beta_1 t) - x_2^0 \sin(\beta_1 t)],\ y_1(t) = e^{t\alpha_2}[y_1^0 \cos(\beta_2 t) - y_2^0 \sin(\beta_2 t)], \\ x_2(t) &= e^{t\alpha_1}[x_1^0 \sin(\beta_1 t) + x_2^0 \cos(\beta_1 t)],\ y_2(t) = e^{t\alpha_2}[y_1^0 \sin(\beta_2 t) + y_2^0 \cos(\beta_2 t)]. \end{aligned} \tag{3.1}$$

If we denote by $\Phi^t : M \to M$ the flow generated by v on M, then we have $T(t) = \varphi^{-1} \circ \Phi^t \circ \varphi$ for the flow in U in Belitskii's coordinates. By reversibility, we have $L \circ \Phi^t = \Phi^{-t} \circ L$.

Let now $\varphi_1 : (u, v) \to U'$ be a coordinate frame in the symmetrically chosen neighborhood $U' = L(U)$ of the point p_2. We search for φ_1 in the form $\varphi_1 = L \circ \varphi \circ R^{-1}$, where R is some diffeomorphism $R : (x, y) \to (u, v)$. Thus, we have the following representation:

$$T(t) = \varphi^{-1} \circ \Phi^t \circ \varphi = \varphi^{-1} \circ L^{-1} \circ \Phi^{-t} \circ L \circ \varphi,$$

or, using the representation for φ_1, we come to

$$T(t) = R^{-1} \circ \varphi_1^{-1} \circ \Phi^{-t} \circ \varphi_1 \circ R.$$

Denote $T_1(t) = \varphi_1^{-1} \circ \Phi^t \circ \varphi_1$, i. e., the representation of Φ^t in coordinates (u, v) in U'. But since R is the diffeomorphism, $R^{-1} \circ \Phi^{-t} \circ R$ is nothing than the representation of $T_1(-t)$ in (u, v)-coordinates. Then we have the connection between $T(t)$ and $T_1(t)$:

$$T(t) = R^{-1} \circ T_1(-t) \circ R. \tag{3.2}$$

Until now, the choice of R has been arbitrary, but now we take as R the linear mapping $R(x_1, x_2, y_1, y_2) = (v_1, v_2, u_1, u_2)$. Differentiating both sides of equality (3.2) and setting $t = 0$ gives the relation for the related vector fields in coordinates (x, y) and (u, v), respectively

$$-\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & E_2 \\ E_2 & 0 \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} \begin{pmatrix} 0 & E_2 \\ E_2 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives the matrix of the linear vector field in (u,v) variables

$$\begin{pmatrix} -\alpha_2 & \beta_2 & 0 & 0 \\ -\beta_2 & -\alpha_2 & 0 & 0 \\ 0 & 0 & -\alpha_1 & \beta_1 \\ 0 & 0 & -\beta_1 & -\alpha_1 \end{pmatrix}. \tag{3.3}$$

Recall that we have assumed $\alpha_1 < 0, \alpha_2 > 0$, so R transforms the stable plane of p_1 to the unstable plane of p_2 and vice versa.

The next task is to clarify the form of the global maps. We choose cross-sections Σ_i^s, Σ_i^u, $i = 1, 2$, near the points p_1, p_2 as some solid tori. The intersections of heteroclinic orbits Γ_1, Γ_2 with these cross-sections are points M_{ij}, where j means the number of a heteroclinic orbit and i enumerates the equilibria. In coordinates (x, y) near p_1 the coordinates of the point M_{11} (for entering the p_1 heteroclinic orbit) is $x = x^*, y = 0$, similar for the point M_{12} (for leaving the p_1 heteroclinic orbit) $x = 0, y = y^*$. In polar coordinates near p_1 we have $x_1^* = \rho_s \cos\theta_*$, $x_2^* = \rho_s \sin\theta_*$, $r = 0$ and $y_1^* = r_u \cos\varphi_*$, $y_2^* = r_u \sin\varphi_*$, $\rho = 0$. Applying the involution, whose action in coordinates is R, we have the points $M_{21} = R(M_{11}), M_{22} = R(M_{12})$ with coordinates $u = 0, v = x^*$, $v = 0, u = y^*$. The corresponding polar coordinates near p_2 are $v_1^* = \rho_s \cos\theta_*$, $v_2^* = \rho_s \sin\theta_*$, $u_1^* = r_u \cos\varphi_*$, $u_2^* = r_u \sin\varphi_*$.

To understand the properties of the global maps and afterwards the Poincaré map, we present global maps in a convenient form. First, we choose some local cross-sections near the points M_{ij} in such a way that a cross-section containing the point M_{11} is L-symmetric to the cross-section containing M_{21} and, similarly, a cross-section containing the point M_{12} is L-symmetric to the cross-section containing M_{22}. This has been done earlier, we need only to choose sufficiently small neighborhoods Π_{ij} of the points in the related solid tori. Let us emphasize that cross-sections Σ_1^s and Σ_2^u are permuted by the involution L and cross-sections Σ_1^u, Σ_2^s are permuted by L. Similarly, their pieces Π_{ij} are also symmetrically connected: Π_{11} with Π_{21} and Π_{12} with Π_{22}.

Now recall that cross-sections N_1, N_2 near points $q_1 = \Gamma_1 \cap \mathrm{Fix}(L)$ and $q_2 = \Gamma_2 \cap \mathrm{Fix}(L)$ have also been chosen, both of them are invariant w.r.t. the action of L and each contains the related disk from the submanifold $\mathrm{Fix}(L)$. Denote by F_1 the transition map $F_1 : N_1 \to \Pi_{11}$ generated by the flow. F_1 is a diffeomorphism that is defined in a small enough neighborhood of the point q_1. We wish to express the transition map $h_1 : \Pi_{21} \to \Pi_{11}$ via F_1 and L. Take a point $b \in \Pi_{21}$ close enough to M_{21} and consider the point $\Phi^{t_1}(b) \in N_1$ where t_1, t_2 are the times of passage by the flow orbit $\Phi^t(b)$ from point b to N_1 and from N_1 to Π_{11}. If the flow orbit through b is not symmetric w.r.t. L, then points $\Phi^{t_1}(b)$ and $L \circ \Phi^{t_1}(b) \in N_1$ generate a symmetric pair of orbits through them[1]. Thus, one has $L^{-1}b \in \Pi_{11}$ and, due to reversibility of the flow and invariance N_1 w.r.t. L, we get (a picture is needed here)

$$h_1(b) = F_1 \circ L \circ F_1^{-1} \circ L^{-1}(b), \tag{3.4}$$

with its inverse map $h_1^{-1} : \Pi_{11} \to N_1$

$$h_1^{-1} = L \circ F_1 \circ L^{-1} \circ F_1^{-1}. \tag{3.5}$$

Let us express this in coordinates. As stated above, the coordinates (ξ_1, η_1, ζ_1) in N_1 can be chosen in such a way that the action of L casts as $(\xi_1, \eta_1, \zeta_1) \to (\xi_1, \eta_1, -\zeta_1)$ and the trace of the stable manifold $W^s(p_1)$ in N_1 is a smooth curve through q_1 being transverse to $\mathrm{Fix}(L) = \{\zeta_1 = 0\}$. So, in coordinates (ξ_1, η_1, ζ_1) in N_1 and (θ_1, y_1, y_2) on Σ_1 near the point M_{11} we have the representation

[1] In fact, points $\Phi^{t_1}(b)$ and $L \circ \Phi^{t_1}(b)$ will be different even if the orbit through b is symmetric, but its intersection point with $\mathrm{Fix}(L)$ does not belong to N_1.

for F_1

$$\theta_1 - \theta_1^* = g(\xi_1, \eta_1, \zeta_1),$$
$$y_1 = f_1(\xi_1, \eta_1, \zeta_1),$$
$$y_2 = f_2(\xi_1, \eta_1, \zeta_1),$$

where functions f_i, g are smooth, $f_1(0,0,0) = f_2(0,0,0) = g(0,0,0) = 0$, the Jacobian does not vanish at $(0,0,0)$ and the transversality means that at the point $(0,0,0)$, the following inequality holds:

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial \xi_1} & \frac{\partial f_1}{\partial \xi_2} \\ \frac{\partial f_2}{\partial \xi_1} & \frac{\partial f_2}{\partial \eta_1} \end{pmatrix} \neq 0,$$

which geometrically means that the F_1-image in Π_{11} of the disk $\mathrm{Fix}(L) \subset N_1$ is transverse to the trace of the stable manifold $W^s(p_1)$ in Σ_1. The involution L restricted on Σ_1 acts in coordinates as $R(\theta_1, y_1, y_2) = (u_1, u_2, \varphi_2) = (u_1, u_2, \theta_1)$. In particular, one has $\varphi_2^* = \theta_1^*$.

In a similar way, the mapping $h_2 : \Pi_{12} \to \Pi_{22}$ is constructed. Denote by $F_2 : \Pi_{12} \to N_2$ the transition map generated by the flow, which is a diffeomorphism as well. Then the map h_2 is expressed via F_2, L as follows

$$h_2 = L \circ F_2^{-1} \circ L \circ F_2.$$

The map F_2 in coordinates (φ_1, x_1, x_2) in Π_{12} and (ξ_2, η_2, ζ_2) in N_2 is expressed as follows

$$\begin{aligned} \xi_2 &= A_1(\varphi_1 - \varphi_1^*, x_1, x_2), \\ \eta_2 &= A_2(\varphi_1 - \varphi_1^*, x_1, x_2), \\ \zeta_2 &= B(\varphi_1 - \varphi_1^*, x_1, x_2), \end{aligned} \tag{3.6}$$

with smooth functions A_i, B, $A_1(0,0,0) = A_2(0,0,0) = B(0,0,0) = 0$, the Jacobian does not vanish at $(0,0,0)$ and the transversality means that at the point $(0,0,0)$, the following inequality holds:

$$\frac{\partial B}{\partial \varphi_1} \neq 0 \text{ at } (\varphi_1^*, 0, 0).$$

4. PROOFS

We start with the proof of Theorem 3. To find a symmetric periodic orbit (briefly, SPO), we need to prove that there is an orbit that intersects the set Fix (L) at two different points. To that end, we apply first the theorem from [5] which says

Theorem 9. *For any point $m \in$ Fix (L) there is a neighborhood V of m and smooth coordinates (a_1, a_2, b_1, b_2) in V such that V is invariant w.r.t. L and acts as $L(a_1, a_2, b_1, b_2) = (a_1, a_2, -b_1, -b_2)$. In particular, the set* Fix $(L) \cap V$ *is given as $b_1 = b_2 = 0$.*

Suppose $m \in$ Fix (L) is a point such that the vector $v(m)$ does not belong to the tangent plane T_mFix (L). For instance, so are the points q_1 and q_2. The following assertion is well known

Lemma 1. *There is a cross-section $N \ni m$ to the flow such that N contains disk $D \subset$* Fix (L) *and N is invariant w.r.t. the action of L: $L(N) = N$.*

According to Lemma 1, we choose two L-invariant cross-sections N_1, N_2 to the flow such that $q_i \in N_i$, $i = 1, 2$, and these cross-sections contain disks $D_i \subset$ Fix $(L) \cap N_i$ containing the points q_i. It will be shown that the transition map $G_1 : N_1 \to N_2$, $G_1 = F_2 \circ T_1 \circ F_1$, generated by the flow near $\Gamma_1 \cup \Gamma_2$, transforms the disk D_1 transversely to the disk D_2 and their intersection is a spiral $\sigma \subset N_2$ winding up at the point q_2, hence, symmetric periodic orbits pass through points of σ.

Now recall that $W^s(p_1)$ is transverse to Fix (L) at the point q_1 (the same holds true for $W^u(p_1)$ at q_2). Along with the assumption on the nondegeneracy of Γ_1 this implies that the intersection $W^s(p_1) \cap N_1 = l_1^s$ is a smooth segment which in N_1 is transverse to D_1 at q_1. By the symmetry, the curve $L(l_1^s) = l_1^u$ is the trace of $W^u(p_2)$ in N_1 with the same property.

To construct the transition map $G_1 : N_1 \to N_2$, we choose in a neighborhood U of p_1, where the Belitskii linearization theorem works, two more cross-sections Σ_1^s, Σ_1^u to the orbits on $W^s(p_1), W^u(p_1)$, respectively. In Belitskii's coordinates (x_1, x_2, y_2, y_2) the system near p_1 is written as

$$\begin{aligned} \dot{x}_1 &= \alpha_1 x_1 - \beta_1 x_2, \qquad \dot{y}_1 = \alpha_2 y_1 - \beta_2 y_2, \\ \dot{x}_2 &= \beta_1 x_1 + \alpha_1 x_2, \qquad \dot{y}_2 = \beta_2 y_1 + \alpha_2 y_2, \end{aligned} \tag{4.1}$$

recall that we suppose $\alpha_1 < 0, \alpha_2 > 0$, and both $\beta_i > 0$.

It is more convenient to work in polar coordinates in U: $x_1 = \rho_1 \cos\theta_1$, $x_2 = \rho_1 \sin\theta_1$, $y_1 = r_1 \cos\varphi_1$, $y_2 = r_1 \sin\varphi_1$. As cross-sections near p_1 we take the solid tori $\Sigma_1^s : x_1^2 + x_2^2 = \rho_s^2$, $y_1^2 + y_2^2 \leqslant \delta_s^2$, and $\Sigma_1^u : y_1^2 + y_2^2 = r_u^2$, $x_1^2 + x_2^2 \leqslant \delta_u^2$. The heteroclinic orbit Γ_1 hits Σ_1^s at the point $M_{11} = (\rho_s \cos\theta_1^*, \rho_s \sin\theta_1^*, 0, 0)$, and the heteroclinic orbit Γ_2 hits Σ_1^u at the point $M_{12} = (0, 0, r_u \cos\varphi_1^*, r_u \sin\varphi_1^*)$. Then we choose the neighborhoods of these points on the related circles by inequalities $|\theta_1 - \theta_1^*| \leqslant \varepsilon$ and $|\varphi_1 - \varphi_1^*| \leqslant \varepsilon$ for positive ε small enough.

For the first twin connection we have two nonsymmetric saddle-foci p_1, p_2 permuted by the involution L. We have already introduced linearizing coordinates in neighborhoods U, U' of p_1, p_2 consistent with the action of L. Related cross-sections Σ_i^s, Σ_i^u near p_1, p_2 will be denoted by the same letters with indices $1, 2$. Because of the symmetry of orbits Γ_1, Γ_2, it will be convenient to represent transition maps $h_1 : \Sigma_2^u \to \Sigma_1^s$ and $h_2 : \Sigma_1^u \to \Sigma_2^s$, generated by the flow, via L and transition mappings F_1, F_2 from cross-sections N_1 to Σ_1^s (F_1) and from Σ_1^u to N_2 (F_2). Both F_1, F_2 are local diffeomorphisms defined near points $q_1 \in N_1$ and $M_{12} \in \Sigma_1^u$. This implies that the F_1-image of disk D_1 is the disk $D_1^s \subset \Sigma_1^s$ transversal at the point M_{11} to the curve — the trace of $W^s(p_1)$. Similarly, the F_2-preimage of the disk $D_2 \subset \mathrm{Fix}(L) \cap N_2$ is the disk $D_1^u \subset \Sigma_1^u$ transversal at M_{12} to the curve — the trace of $W^u(p_1)$. This means that the disk D_1^s can be written as a graph of the smooth function $\theta_1 = h_s(r_1 \cos\varphi_1, r_1 \sin\varphi_1)$, $h_s(0,0) = \theta_1^*$. Analogously, we have the representation for $D_1^u : \varphi_1 = h_u(\rho_1 \cos\theta_1, \rho_1 \sin\theta_1)$, $h_u(0,0) = \varphi_1^*$.

Integration of equations (4.1) in polar coordinates

$$\begin{aligned} &\rho_1(t) = \rho_s \exp[-\alpha_1 t],\ \theta_1(t) = \theta_1^0 + \beta_1 t, \\ &r_1(t) = r_1^0 \exp[\alpha_2 t],\ \varphi_1(t) = \varphi_1^0 + \beta_2 t. \end{aligned} \tag{4.2}$$

gives the representation for the map $T_1 : \Sigma_1^s \to \Sigma_1^u$ generated by the flow. To find the passage time t_p of an orbit from Σ_1^s to Σ_1^u, we solve the equation $r_u = r_1^0 \exp[\alpha_2 t_p]$. After inserting the passage time into (4.2) we have

$$\begin{aligned} &\rho_1 = \rho_s \Big(\frac{r_0}{r_u}\Big)^{-\alpha_1/\alpha_2} = C r_0^{\nu_1},\ \theta_1 = \theta_0 + \gamma_1 \ln(r_u/r_0))\ \ (\mathrm{mod}\ 2\pi),\ \nu_1 = -\alpha_1/\alpha_2,\ \gamma_1 = \beta_1/\alpha_2, \\ &\varphi_1 = \varphi_0 + \gamma_2 \ln(r_u/r_0))\ \ (\mathrm{mod}\ 2\pi),\ C = \rho_s r_u^{-\alpha_1/\alpha_2},\ \gamma_2 = \beta_2/\alpha_2, \end{aligned} \tag{4.3}$$

where $(\theta_0, r_0, \varphi_0)$ is an initial point in Σ_1^s, and $(\varphi_1, \rho_1, \theta_1)$ is the hit point in Σ_1^u for the flow orbit through the initial point, and one has to select only those initial points where $|\theta_0 - \theta_0^*| \leqslant \varepsilon$.

Later on we will need some lemma that is used for proving the existence of multi-round heteroclinic orbits. Notice that the noncollinearity at the point M_{11} of some smooth curve and a piece of $W^s(p_1) \cap \Pi_{11}$ allows one to represent this smooth curve, more exactly, both halves of it without M_{11}, as a smooth function in r_0.

Lemma 2. *Let $\theta_0 = a(r_0), \varphi_0 = b(r_0)$, $0 \leqslant r_0 < r_0^*$, be a smooth curve in Σ_1^s with $a(0) = \theta_0^*, b(0) = \varphi_0^*$, and let its tangent vector $\big(a'(0), 1, b'(0)\big)$ to the curve at $r_0 = 0$ be noncollinear with the vector* $(1, 0, 0)$ (*the tangent vector to the trace of $W^s(p_1)$*)*. Then the T_1-image of this curve in Σ_1^u is an* *infinite spiral such that the intersection of this spiral with the neighborhood of the point M_{12} defined by inequality $|\varphi_1 - \varphi_1^*| \leqslant \varepsilon$ is a countable set of segments J_n which accumulate in C^1-topology, as $n \to \infty$, to the segment $\rho_1 = 0$ – the trace of $W^u(p_1)$ in Σ_1^u.*

Proof. The T_1-image of the curve is given in the parametric form as follows:

$$\begin{aligned} \theta_1 &= a(r_0) + \gamma_1 \ln(r_u/r_0) \pmod{2\pi}, \\ \rho_1 &= C r_0^{\nu_1}, \\ \varphi_1 &= b(r_0) + \gamma_2 \ln(r_u/r_0) \pmod{2\pi}. \end{aligned} \tag{4.4}$$

We consider now the coordinate φ_1 to be infinite (in the covering of the solid tori Σ_1^u which is then an infinite solid cylinder) and express r_0 as a function of φ_1 from the last equation. Since function b is smooth with bounded derivative, the derivative $d\varphi_1/dr_0 = b'(r_0) - \gamma_2/r_0$ is large enough in modulus for r_0^* small enough, therefore it does not vanish and there is an inverse function $r_0 = \Phi(\varphi_1)$ defined for $\varphi_1 \geqslant \varphi_1^0$ or $\varphi_1 \leqslant \varphi_1^0$ depending on the sign of γ_2. This function tends to zero as $\varphi_1 \to \infty$ or $\varphi_1 \to -\infty$ with the exponential estimate $\Phi(\varphi_1) \leqslant \kappa \exp[-\varphi_1/\gamma_2]$, $\kappa > 0$. We assume below that $\varphi_1 \geqslant \varphi_1^0$ is definite. Differentiating the identity $\varphi_1 = b(\Phi) + \gamma_2 \ln(r_u/\Phi)$, we come to the equality

$$\Phi'(\varphi_1) = \frac{-r_u}{\gamma_2 - b'(\Phi)\Phi} \exp[(b(\Phi) - \varphi_1)/\gamma_2].$$

This gives the exponential estimate for Φ' as well. So, packing the curve $(\rho_1(\varphi_1), \theta_1(\varphi_1))$ into the solid torus and intersecting the spiral obtained with the neighborhood Π_{12}, where $|\varphi_1 - \varphi_1^*| \leqslant \varepsilon$, we come to the conclusion of the lemma. Indeed, the expression for θ_1 is as follows: $\theta_1 = a(\Phi) - \frac{\beta_1}{\beta_2} b(\Phi) + \frac{\beta_1}{\beta_2}\varphi_1$ with bounded $c(\Phi) = a(\Phi) - \frac{\beta_1}{\beta_2} b(\Phi)$. Returning to the Cartesian coordinates $x_1 = \rho_1 \cos\theta_1$, $x_2 = \rho_1 \sin\theta_1$, this gives

$$x_1 = C\Phi^{\nu_1} \cos\Big[c(\Phi) + \frac{\beta_1}{\beta_2}\varphi_1\Big],\ x_2 = C\Phi^{\nu_1} \sin\Big[c(\Phi) + \frac{\beta_1}{\beta_2}\varphi_1\Big],$$

with exponentially small estimates for $dx_1/d\varphi_1$ and $dx_2/d\varphi_1$. □

To find a symmetric periodic orbit, we need to prove that the T_1-image of the disk D_1^s intersects the disk D_2^u, SPOs pass through any intersection point. The T_1-image of disk D_1^s is expressed in a parametric form with parameters (r, φ) as follows:

$$\begin{aligned} \theta_1 &= h_s(r\cos\varphi, r\sin\varphi) + \gamma_1 \ln(r_u/r)) \pmod{2\pi}, \\ \rho_1 &= C r^{\nu_1},\ \nu_1 > 0, \\ \varphi_1 &= \varphi + \gamma_2 \ln(r_u/r)) \pmod{2\pi}. \end{aligned} \tag{4.5}$$

To understand the shape of this set and its position w.r.t. the disk D_1^u, let us fix the value $r = r_0$ assuming r_0 small enough. In Σ_1^s this equality singles out a thin cylinder $r = r_0, 0 \leqslant \varphi_0 \leqslant 2\pi, |\theta_0 - \theta_1^*| \leqslant \varepsilon$, whose points are close to the segment $r_0 = 0$. Because of the transversality of D_1^s and the curve $r_0 = 0$ — the trace of $W^s(p_1)$, — the intersection of the thin cylinder with D_1^s gives a smooth closed curve $\theta_0 = h_s(r_0 \cos\varphi_0, r_0 \sin\varphi_0)$, for small r_0 this curve is close to the point $r_0 = 0, \theta_0 = \theta_1^*$. The T_1-image of this small closed curve in the whole cross-section Σ_1^u is a closed curve on the torus $\rho_1 = C r_0^{\nu_1}$ that is very close to the closed curve $\rho_1 = 0$. The resulting closed curve on this torus makes the complete go-round in φ_1 and is almost constant in θ_1, since $h_s(r\cos\varphi, r\sin\varphi)$ is close to θ_1^* and the second term is constant in the expression for θ_1. This follows from the first and third relations in (4.5).

The restriction of the resulting closed curve to the cylinder $|\varphi_1 - \varphi_1^*| \leqslant \varepsilon$, gives a segment. Now we see that, as $r_0 \to 0$, the union of these segments in Σ_1^u makes up a smooth scroll-shaped two-dimensional surface which is wrapped around the central segment $\rho_1 = 0$ in Σ_1^u. Moreover, each segment of this surface, corresponding to the fixed r_0, is C^1-close to the segment $\rho_1 = 0$. This implies that for r_0 small enough each segment intersects the disk D_1^u transversely at a single point. The union of these intersection points makes up a smooth spiral σ on D_1^u winding up at the point M_{12}. The preimage w.r.t. T_1 of this smooth spiral is also a spiral in D_1^s winding up at the point M_{11}. So, Theorem 3 has been proved. □

In order to prove Theorem 4, we consider again the smooth curve l_1^s which is the trace of $W^s(p_1)$ in the cross-section N_1 and shall find the trace of its continuation by the flow (in backward direction in time) in N_2 after one round passing near lower halves of $\Gamma_1 \cup \{p_2\} \cup \Gamma_2$ (see Fig. 1, left panel). If we prove that some orbit through a point on $l_1^s \setminus q_1$ intersects at some point m the disk $D_2 \subset \text{Fix }(L)$, then the second half of this orbit, when time changes in backward direction, after passing the point m and $t \to -\infty$, forms by symmetry a 2-round heteroclinic orbit connecting p_2 and p_1. Along this orbit the time moves points from p_2 to p_1. In a similar way, heteroclinic orbits going from p_1 to p_2, as time increases, are sought for, starting from the l_2^s — the trace of $W^s(p_2)$ on N_2.

So, consider the curve l_1^s. Its point q_1 divides the curve into two pieces, each piece has a representation in coordinates (ξ_1, η_1, ζ_1) on N_1, in which $q_1 = (0, 0, 0)$, $L(\xi_1, \eta_1, \zeta_1) = (\xi_1, \eta_1, -\zeta_1)$: $\xi_1 = a(\zeta_1)$, $\eta_1 = b(\zeta_1)$, with smooth functions a, b, $a(0) = b(0) = 0$, due to the transversality of this curve to D_1. According to the action of L in these coordinates, the curve l_1^u (the trace of $W^u(p_2)$ in N_1) has the representation $\xi_1 = a(-\zeta_1)$, $\eta_1 = b(-\zeta_1)$. These two curves l_1^s, l_1^u are noncollinear to each other at q_1, since Γ_1 is nondegenerate, and both of them are transverse to D_1. This implies that for the transition map $\tilde{F}_1 : \Pi_{21} \to N_1$, generated by the flow, $\tilde{F}_1$-preimages of two curves l_1^s, l_1^u are two smooth curves in Π_{21}, one of which is a piece of the trace of $W^s(p_1) \cap \Sigma_2^u$ and the other is a piece of $W^u(p_2) \cap \Sigma_2^u$ near the point M_{21}. Since $\tilde{F}_1$ is a diffeomorphism, these two smooth curves are also noncollinear. So, each half of the smooth segment $W^s(p_1) \cap \Pi_{21}$ near the point M_{21} is written as $\varphi_2 = g(\rho_2)$, $\theta_2 = h(\rho_2)$, $g(0) = \varphi_2^*$, $h(0) = \theta_2^0$ with smooth functions g, h defined on some segment $[0, \rho_*]$, $\rho_* > 0$. Here we utilize the assertion of Lemma 2 for the map T_2^{-1} (similar to (4.3)) and we conclude that the T_2-preimages in Σ_2^s of two halves of the segment are two infinite spirals winding at the closed curve — the trace of $W^s(p_2) \cap \Sigma_2^s$.

Therefore, in the neighborhood Π_{22} we get two countable sets of smooth segments accumulating in C^1-topology to the segment $r_2 = 0$. Consequently, in N_2 we get similar sets of segments accumulating in C^1-topology to the curve l_2^s — the trace of $W^s(p_2)$. The curve l_2^s intersects transversely the disk D_2, so for ρ_* small enough all curves of both countable sets of segments intersect transversely D_2, giving two countable sets of points with the limit point at q_2 for both of them. Every such intersection point is the trace of a symmetric heteroclinic orbit going twice around the connection C. All these heteroclinic orbits go, as time increases, from p_2 to p_1.

In a similar way, starting with the curve l_2^s, which is a portion of $W^s(p_2) \cap N_2$, we shall find a countable set of symmetric heteroclinic orbits going twice around the connection C, when time increases from p_1 to p_2. Taking one symmetric heteroclinic 2-round orbit from each countable family, we shall get a network of heteroclinic connections. This proves the first part of Theorem 4.

Remark 1. The method can be iterated, since at every step we have a heteroclinic connection involving nondegenerate symmetric heteroclinic orbits. So, we can find connections of roundness 2^n for any $n \in \mathbb{N}$.

Now we shall prove Theorem 5 on the existence of 3-round symmetric nondegenerate heteroclinic orbits. Here we are able to prove the existence of a finite number of such orbits, in contrast to the 2-round ones. We again start with the smooth curve l_1^s. As was proved above, the two halves of $l_1^s \setminus \{q_1\}$ are transformed under the map $G_2^{-1} \circ T_2^{-1} \circ G_1^{-1} : N_1 \to N_2$ into two countable families of smooth segments which C^1-smoothly tend to the segment l_2^s. Segments of both families which are C^1-close to l_2^s intersect transversely disk D_2 and 2-round heteroclinic orbits pass through intersection points. We are interested in such orbits on these segments which do not belong to the disk D_2 and go further in backward direction in time to hit D_1. Such an orbit will be 3-round symmetric heteroclinic, since, by symmetry, its second part composes such an orbit. To find such an orbit, we consider the image of the disk D_1 under the map $F_2 \circ T_1 \circ F_1$. As was proved above, this image is a scroll that infinitely wraps the curve l_2^u which is its topological limit. This scroll is composed from smooth curves, they tend to l_2^u in C^1-topology. The curve l_2^s is noncollinear to l_2^u and intersects it at the point q_2. This implies that l_2^s intersects the scroll transversely (except for q_2) at infinitely many points. So, each curve of a countable family, which is C^1-close to l_2^s, intersects the scroll transversely at only finitely many points. In principle, among curves of both countable families there may exist curves which have tangency with the scroll (or, one can find such a tangency one deals with the generic reversible unfolding of the vector field under consideration). Thus, we get a finite number of 3-round heteroclinic orbits going from p_2 to p_1. In a similar way, we prove the existence of finitely many symmetric 3-round nondegenerate heteroclinic orbits going from p_1 to p_2. This proves Theorem 5. □

Our last task for the first type connection is to prove Theorem 6. Consider an unfolding v_μ of reversible vector fields on M which at the critical value of the parameter $\mu = 0$ has the vector field v_0 that satisfies the above-mentioned conditions on the existence of a heteroclinic connection C of the first type. The Belitskii theorem works also for all small enough values of $|\mu|$, the only difference with the parameterless case is the smooth dependence $\alpha_i(\mu), \beta_i(\mu),\ i = 1, 2$. Without loss of generality, one may suppose equilibria p_i to be fixed, we shall assume this later on and therefore we omit their explicit dependence on μ. So, one can suppose that all cross-sections for C, constructed above, remain cross-sections for all vector fields of the unfolding for μ close enough to the critical one. Also, traces of stable and unstable manifolds of saddle-foci in these coordinates will be expressed in the same way. Henceforth, we assume this holds true.

Thus, for all $|\mu|$ small enough the vector fields v_μ have two saddle-foci p_1, p_2 permuted by the involution and their stable and unstable manifolds intersect each other along two symmetric nondegenerate heteroclinic orbits $\Gamma_1(\mu)$ and $\Gamma_2(\mu)$ for any $|\mu|$ small enough. These orbits intersect cross-sections N_1, N_2 at the points $q_1(\mu), q_2(\mu)$ which belong to the disks $D_1 \subset \mathrm{Fix}\ (L)$ and $D_1 \subset \mathrm{Fix}\ (L)$. We shall prove the existence of nonsymmetric homoclinic orbits for p_2, assuming that the saddle value for p_2 is positive ($0 < \nu_2(0) < 1$). By symmetry, pairing homoclinic orbits will exist for p_1, the only difference is that the saddle value for p_1 is negative ($\nu_1(\mu) > 1$).

To this end, for some small values of $|\mu|$ we need to find intersections of $W^u(p_2)$ with $W^s(p_2)$. For the vector field v_0 we know that the trace of the unstable manifold $W^u(p_2)$ in the cross-section N_1 is the smooth curve l_1^u passing through the point q_1, and the trace of the stable manifold $W^s(p_1) \cap N_1$ is the curve l_1^s, $L(l_1^u) = l_1^s$. Similarly, the curves $l_2^s = W^s(p_2) \cap N_2$ and $l_2^u = L(l_2^s)$ are defined in N_2, in both cases they intersect each other noncollinearly at the points q_1 and q_2, respectively.

Let us drag the curve l_2^s by the flow in backward direction in time into the neighborhood of the point p_1 up to the cross-section Π_{12} and get there a smooth curve $\tilde{w}_2^s$. This latter curve is noncollinear to the curve $w_1^u = W^u(p_1) \cap \Pi_{12}$. Similarly, we drag the curve l_1^u by the flow in forward direction up to the cross-section Π_{11} and get a smooth curve $\tilde{w}_2^u$ — the trace of $W^u(p_2)$. This latter curve intersects the curve $w_1^s = W^s(p_1) \cap \Pi_{11}$ at the point M_{11} noncollinearly. After continuing by the flow in forward direction in time through a neighborhood of p_1 of orbits passing through the points of $\tilde{w}_2^u$, we get in Π_{12} two countable families of smooth segments accumulating in C^1-topology to the curve w_1^u. For the unfolding v_μ the related curve $\tilde{w}_2^u$ will depend smoothly on μ (by smooth dependence of $W^u(p_2)$ on μ) and similarly, the curve $\tilde{w}_2^s$ also smoothly depends on μ. Now we need to prove that varying μ allows one to find an intersection of segments from the countable set with $\tilde{w}_2^s$. The transition from Π_{11} to Π_{12} is given by the map (4.4) all coefficients of which depend smoothly on μ. The following lemma holds.

Lemma 3. *Consider the map* (4.4) *and two smooth segments in the neighborhood of p_1, one $\tilde{w}_2^u$ intersecting noncollinearly w_1^s at the point M_{11} and the other $\tilde{w}_2^s$ intersecting noncollinearly w_1^u at the point M_{12}. Then there are two sequences $\mu_n^{(\sigma)} \to 0$, $\sigma = \pm 1$, such that at $\mu = \mu_n^{(\sigma)}$ the vector field $v_{\mu_n^{(\sigma)}}$ has two orbits which start on the curve $\tilde{w}_2^u(\mu_n^{(\sigma)})$ and pass through the curve $\tilde{w}_2^s(\mu_n^{(\sigma)})$.*

Proof. We outline the proof omitting some details of the calculation. The beginning of the proof resembles that of Lemma 2. In coordinates $(\theta_0, r_0, \varphi_0)$ in Π_{11} the curve $\tilde{w}_2^u$, due to its noncollinearity at the point M_{11} with the segment $w_1^s = \{r_0 \equiv 0\}$, has the representation $\theta_0 = a_\pm(r_0, \mu)$, $\varphi_0 = b_\pm(r_0, \mu)$, $0 \leqslant r_0 \leqslant \delta$. Here the smooth curve $\tilde{w}_2^u$ is represented as the union of its two halves with the common point M_{11}. These halves are given by the differentiable functions $a_\pm, b_\pm$ with the equalities $a_\pm(0,\mu) = \theta_1^*(\mu)$, $\big(a'_+(0,\mu), b'_+(0,\mu)\big) = -\big(a'_-(0,\mu), b'_-(0,\mu)\big)$. We shall prove the statements for the half of the curve, therefore we omit below the indices $\pm$. As in Lemma 2, we have the image of the curve $\tilde{w}_2^u$ in Π_{12} under the local map $T_1(\mu)$

$$\theta_1 = a(r_0,\mu) + \gamma_1(\mu)\ln(r_u/r_0) \pmod{2\pi},$$
$$\rho_1 = C r_0^{\nu_1(\mu)},$$
$$\varphi_1 = b(r_0,\mu) + \gamma_2(\mu)\ln(r_u/r_0) \pmod{2\pi}.$$

We again solve the third equation w.r.t. r_0, considering μ as a parameter and φ_1 as the infinite coordinate in the universal covering of the solid torus Σ_1^u. We get an inverse function $r_0 = \Phi(\varphi_1, \mu)$ defined on the region $\varphi_1 \geqslant \varphi_1^0$, $|\mu| \leqslant \kappa$. This function decays exponentially fast to zero as $\varphi_1 \to \infty$. Inserting this function into the first and second relations gives the representation of the curve in the solid cylinder being the half ($\varphi_1 \geqslant \varphi_1^0$) of the covering of the solid torus Σ_1^u

$$\rho_1 = C\Phi^{\nu_1(\mu)}(\varphi_1,\mu),\ \nu_1(0) > 1,$$
$$\theta_1 = \Theta(\varphi_1,\mu) = a(\Phi,\mu) + \gamma_1(\mu)\ln(r_u/\Phi)) = c(\Phi(\varphi_1,\mu),\mu) + \frac{\beta_1(\mu)}{\beta_2(\mu)}\varphi_1,$$

where $c(\Phi(\varphi_1,\mu),\mu) = a(\Phi,\mu) - \big(\beta_1(\mu)/\beta_2(\mu)\big)b(\Phi,\mu)$. The function $C\Phi^{\nu_1(\mu)}$ and its derivative in φ_1 satisfy the exponential estimates uniformly in μ. In particular, at any fixed μ this function decays to zero exponentially fast as φ_1 tends to infinity. Hence, if we restrict the graph of the vector-function (ρ_1, θ_1) on the set $|\varphi_1 - \varphi_1^*(\mu)| \leqslant \varepsilon$ in the solid torus Σ_1^u, we get in Π_{12}, as in Lemma 2, a countable set of smooth segments corresponding each to intervals $2\pi n + \varphi_1^* - \varepsilon \leqslant \varphi_1 \leqslant 2\pi n + \varphi_1^* + \varepsilon$. In Π_{12} these countable sets of curvilinear segments accumulate in C^1-topology, as $n \to \infty$, to the segment $\rho_1 = 0$. The dependence of any such segment in the angular variable θ_1 is described by the function $\Theta(\varphi_1, \mu)$ at a fixed n.

The second curve $\tilde{w}_2^s$ has a similar representation in Π_{12} $\theta_1 = A_\pm(\rho_1,\mu)$, $\varphi_1 = B_\pm(\rho_1,\mu)$ with bounded differentiable functions $A_\pm, B_\pm$, $A_+(0,\mu) = \theta_2^1(\mu)$, $A_-(0,\mu) = \theta_2^1(\mu) + \pi$, $B_\pm(0,\mu) = \varphi_1^*(\mu)$ and related equalities for their derivatives expressing the smoothness of the whole curve $\tilde{w}_2^s$ at the point M_{12}. We again work only with the half of this curve and therefore omit the indices $\pm$.

Now we want to understand how each segment from the countable set in Π_{12} rotates in angular direction θ_1 when μ varies. To this purpose, we calculate the derivative of Θ in μ using the notation $b_\mu, b_{r_0}, c_\mu, c_{r_0}$ for the partial derivatives in the related variables

$$\frac{\partial\Theta}{\partial\mu} = c_{r_0}\frac{\partial\Phi}{\partial\mu} + c_\mu + \frac{\beta_1'\beta_2 - \beta_1\beta_2'}{\beta_2^2}\varphi_1 = c_{r_0}\frac{\Phi b_\mu + \gamma_2'\Phi\ln(r_u/\Phi)}{\gamma_2 - \Phi b_{r_0}} + c_\mu + \frac{\beta_1'\beta_2 - \beta_1\beta_2'}{\beta_2^2}\varphi_1. \tag{4.6}$$

The functions $c_{r_0}, c_\mu, b_{r_0}, b_\mu$ are bounded and continuous, Φ as the function in φ_1 tends to zero exponentially fast as $\varphi_1 \to \infty$ uniformly in μ. Thus, if the quantity $(\beta_1/\beta_2)'(0)$ does not vanish (this is just the genericity condition on the unfolding), then for φ_1 large enough, i.e., for segments of the family with large numbers n, the derivative is very large in modulus (they rotate fast in θ_1-direction as μ changes) or the inverse functions $\mu = M_n(\theta_1, \varphi_1)$ exist and their derivatives $\partial M_n/\partial\theta_1$ tends to zero as $n \to \infty$ uniformly in φ_1 for any n.

Let us now fix some $\kappa > 0$ small enough and consider the direct product $\Sigma_1^u \times (-\kappa, \kappa)$. For any $\mu \in (-\kappa, \kappa)$ we have a smooth segment in Σ_1^u given as $\theta_1 = A(\rho_1, \mu)$, $\varphi_1 = B(\rho_1, \mu)$ intersecting noncollinearly the trace $\rho_1 = 0$ of $W^u(p_1)$ at the point $\varphi_1 = \varphi_1^*(\mu)$, $\varphi_1^*(0) \in (\varphi_1^* - \varepsilon, \varphi_1^* + \varepsilon)$. We need to find solutions for the system of equations

$$\begin{cases} A(\rho_1,\mu) = \Theta(\varphi_1,\mu), \\ \varphi_1 - 2\pi n = B(\rho_1,\mu), \\ \rho_1 = C\Phi^{\nu_1(\mu)}(\varphi_1,\mu), \end{cases}$$

where $\mu \in (-\kappa, \kappa)$ and for fixed $n \in \mathbb{N}$ large enough, i. e., $\varphi_1 \in (2\pi n + \varphi_1^* - \varepsilon, 2\pi n + \varphi_1^* + \varepsilon)$. Inserting φ_1 from the second relation into the third relation gives the equation, depending of μ, with respect to ρ_1 from which we find ρ_1 as a function of μ: $\rho_1 = h_n(\mu)$. This is done using the exponential decay of the function Φ and large n. Finally, we substitute φ_1 into the first relation and after that we insert there the function h_n instead of ρ_1. This provides the equation w.r.t. to μ which is solved using the large derivative of Θ in the variable μ due to (4.6). Here the genericity assumption implies that, when varying μ, the curve $l_2^u(\mu)$ will move in such a way that it intersects at countably many values μ_n the curves of countable families noncollinearly, each intersection point gives the crossing stable and unstable manifolds of the point $p_2(\mu)$. This completes the proof. □

5. THE SECOND TWIN HETEROCLINIC CONNECTION

In this section we study homoclinic orbits and SPOs near a twin heteroclinic connection of the second type, i. e., consisting of two symmetric saddle-foci $p_1, p_2 \in \mathrm{Fix}\ (L)$ and two nondegenerate heteroclinic orbits Γ_1, Γ_2 permuted by involution: $\Gamma_2 = L(\Gamma_1)$. Also, we present the proof of the existence of two-round heteroclinic connections of the second type in a generic reversible unfolding of a system having a connection of the second type.

In neighborhoods of symmetric saddle-foci $p_i, i = 1, 2$, we use coordinates (2.1), where the functions ξ and η are local integrals of the vector field. We have worked so far in a neighborhood of the symmetric saddle-focus p_1, but the same holds true near p_2. To simplify notation, we omit here subindices $1, 2$.

Remark 2. In fact, one can also use the linearizing Belitskii coordinates as was done in [2, 22, 27, 29], this is sufficient for the results in this section, but this tool is not well suited for the purposes of the bifurcations which we intend to develop elsewhere. Therefore, we want to use another tool of the normal form which works for this case with any assumption of smoothness.

Since H_1 and H_2 depend only on invariants ξ, η, they are constant along the orbit, and (2.1) is effectively a linear system with constant coefficients and can be integrated

$$x(t) = e^{-tH_1} R_{tH_2} x(0),\ y(t) = e^{tH_1} R_{tH_2} y(0), \tag{5.1}$$

where H_1 and H_2 are evaluated at the constant values of the invariants ξ, η at the initial point $x(0), y(0)$, and R_θ is the rotation matrix at the angle θ. We assume further, without loss of generality, that $\alpha_1 > 0$, $\alpha_2 > 0$, this always can be achieved by the linear change of variables.

The stable and unstable manifolds of the point p are given in the form

$$W^s = \{y_1 = y_2 = 0\},\ W^u = \{x_1 = x_2 = 0\},$$

and the action of involution is defined as

$$L(x_1, x_2, y_1, y_2) = (-y_2, -y_1, -x_2, -x_1).$$

Thus, the plane of fixed points of the involution (in fact, it is a 2-disk) is determined by the equalities:

$$\mathrm{Fix}(L) = \{x_1 + y_2 = 0, x_2 + y_1 = 0\}.$$

The form (5.1) allows the local map to be constructed. To this end, we define two three-dimensional cross-sections, N^s and N^u, to the stable and unstable manifolds, respectively, as follows:

$$N^s = \{x_1^2 + x_2^2 = \rho^2, y_1^2 + y_2^2 \leqslant \delta^2\},$$

$$N^u = \{y_1^2 + y_2^2 = \rho^2, x_1^2 + x_2^2 \leqslant \delta^2\}.$$

Each of these sections is a solid torus. We have selected them to be symmetric to each other, so that $L(N^s) = N^u$, and vice versa.

Since the stable manifold of p_1 corresponds to the set $y = 0$, its intersection with N^s is the circle $x_1^2 + x_2^2 = \rho^2$, $y_1 = y_2 = 0$, and the intersection of the heteroclinic orbit Γ_2 (later on it will be p_1, p_2, Γ_1, Γ_2, N_1^s, N_1^u, etc.) with N^s is a point with coordinates $(x_1^*, x_2^*, 0, 0)$. The cross-sections are transposed by L, hence the trace of W^u on N^u is the circle $y_1^2 + y_2^2 = \rho^2$, $x_1 = 0$, $x_2 = 0$, and the trace of $\Gamma_1 \cap N_1^u$ corresponds to the point $(0, 0, -x_2^*, -x_1^*)$ in accordance with the action of L in coordinates.

It is convenient to use local integrals (ξ, η) for coordinates on N^s and N^u along with angular coordinates θ, φ. Let θ^* denote the angle on the circle, corresponding to the trace of Γ_1, defined by relations $x_1^* = \rho\cos\theta^*$, $x_2^* = \rho\sin\theta^*$. Combining these with the integrals implies that

$$\begin{aligned} x_1 &= \rho\cos(\theta+\theta^*), \quad y_1 = \rho^{-1}\big(\xi\cos(\theta+\theta^*) - \eta\sin(\theta+\theta^*)\big),\\ x_2 &= \rho\sin(\theta+\theta^*), \quad y_2 = \rho^{-1}\big(\xi\sin(\theta+\theta^*) + \eta\cos(\theta+\theta^*)\big), \end{aligned} \tag{5.2}$$

so that

$$N^s = \{(\xi, \eta, \theta) : \sqrt{\xi^2+\eta^2} \leqslant \rho\delta, \theta \in S^1\}\ .$$

Note that in these coordinates $W^s \cap N^s$ is the circle $\xi = \eta = 0$, and $\Gamma_1 \cap N^s = (0, 0, 0)$.

Similarly, we define an angle φ on N^u such that $\varphi = 0$ corresponds to $L(\Gamma_2) \cap N^u$. Symmetry implies that $\varphi^* = 3\pi/2 - \theta^*$, since then $-\rho\sin\theta^* = \rho\cos\varphi^*$, $-\rho\cos\theta^* = \rho\sin\varphi^*$. Thus, in N^u we have

$$\begin{aligned} x_1 &= \rho^{-1}\big(\xi\sin(\varphi-\theta^*) - \eta\cos(\varphi-\theta^*)\big), \qquad y_1 = \rho\sin(\varphi-\theta^*),\\ x_2 &= \rho^{-1}\big(-\xi\cos(\varphi-\theta^*) - \eta\sin(\varphi-\theta^*)\big), \quad y_2 = -\rho\cos(\varphi-\theta^*), \end{aligned} \tag{5.3}$$

where

$$N^u = \{(\xi, \eta, \varphi) : \sqrt{\xi^2+\eta^2} \leqslant \rho\delta, \varphi \in S^1\}.$$

As before, $W^u \cap N^u$ is the circle $\xi = \eta = 0$, the intersection $\Gamma_2 \cap N^u$ is the origin $(0, 0, 0)$.

Finally, in the new coordinate systems, the restriction of the involution $L : N^s \to N^u$ becomes

$$L(\xi, \eta, \theta) = (\xi, \eta, \varphi) = (\xi, \eta, -\theta)\ . \tag{5.4}$$

Now we are ready to construct the local map $T : N^s \to N^u$ generated by the local flow (5.1). The passage time t_p from N^s to N^u is derived from the equation $\|y(t_p)\|^2 = \rho^2$ and is equal to

$$t_p = \frac{1}{H_1(\xi,\eta)} \ln\frac{\rho}{\|y(0)\|}.$$

Since ξ, η are local integrals, the local map in these coordinates is given by

$$(\bar\xi, \bar\eta, \varphi) = T(\xi, \eta, \theta) = \big(\xi, \eta, s(\xi, \eta, \theta)\big), \tag{5.5}$$

where $s(\xi, \eta, \theta)$ is a circle map in the variable θ. To find the form of s, we insert t_p into the equations for $y(t)$ in (5.1). Using $\|y(0)\|^2 = (\xi^2+\eta^2)/\rho^2$ and (5.3), after easy calculations, we find

$$\varphi = s(\xi, \eta, \theta) = \theta + 2\theta^* + \pi/2 - \Delta(\xi, \eta) + \Phi(\xi, \eta) \ (\mathrm{mod}\ 2\pi), \tag{5.6}$$

where we have defined the polar angle $\Phi(\xi, \eta)$ in the (ξ, η) plane in such a way that

$$\xi = d\cos\Phi\ , \quad \eta = d\sin\Phi,$$

and the shift

$$\Delta(\xi, \eta) \equiv H_2 t_p = \frac{H_2}{H_1}\ln\frac{\rho^2}{d}.$$

We now prove the existence of nondegenerate symmetric homoclinic orbits. To this end we need to find the intersection of unstable manifold $W^u(p_2)$ with the disk $D_1 \subset \mathrm{Fix}\ (L)$ near p_1, and similarly, $W^u(p_1)$ with a disk $D_2 \subset \mathrm{Fix}\ (L)$ near p_2. By symmetry, we shall get the second halves of the related homoclinic orbits.

To do this, we prove first an auxiliary lemma.

Lemma 4. *The image in* N_1^u *under the local map* T_1 *of a local disk* $D_1 \subset \text{Fix }(L)$ *near* p_1 *is a scroll* Sc_1^u *that wraps infinitely many times onto the circle* $W_1^u \cap N_1^u$*. The scroll is transverse to any disk* $\varphi_1 = \text{const}$ *for any* φ_1 *in* $[0, 2\pi]$ *giving at the intersection a spiral with the limit point* $\xi = \eta = 0$ *of this disk.*

Proof. We choose a local disk $D_1 \subset \text{Fix }(L)$ near p_1 and find its image under the flow in N_1^u. In terms of (x, y) coordinates (2.1) we have $\text{Fix }(L) = \{x_1 = -y_2, x_2 = -y_1\}$. Hence, we may use y as the local coordinates in D_1. We will use polar coordinates for y, $(y_1, y_2) = (d_1 \cos\chi_1, d_1 \sin\chi_1)$, so that D_1 corresponds to the set $0 \leqslant d_1 \leqslant \rho_1/2$, $0 \leqslant \chi_1 \leqslant 2\pi$.

According to the local flow (5.1), the time for a point $y(0) \in D_1$ to reach N_1^u is

$$\tau_p = \frac{1}{H_1^1(\xi_1, \eta_1)} \ln\frac{\rho_1}{d_1} = \left(\frac{1}{\alpha_1} + O(d_1^2)\right) \ln\frac{\rho_1}{d_1},$$

since $\xi_1 = -d_1^2 \sin 2\chi_1$ and $\eta_1 = d_1^2 \cos 2\chi_1$. Using the coordinates $(\xi_1, \eta_1, \varphi_1)$ on N_1^u (see (5.3)), we obtain from the equations in (5.1) for y^1 the circle map $\chi_1 \to \varphi_1$:

$$\varphi_1 = \chi_1 + \pi/2 + \theta_1^* - \left(\beta_1/\alpha_1 + O(d_1^2)\right) \ln(d_1/\rho_1) \pmod{2\pi}. \tag{5.7}$$

This implies that each circle $||y^1(0)|| = d_1$ in D_1 is transformed to a closed curve in N_1^u that lies on the torus $\xi_1^2 + \eta_1^2 = d_1^4$ and has the $(1, 1)$ homology with respect to the standard generators φ_1 and $\varphi_1 = \text{const}$. Thus, only a segment $|\varphi_1| \leqslant \varepsilon_1$ of this curve belongs to the neighborhood $V_1^u \subset N_1^u$:

$$V_1^u = \{(\xi_1, \eta_1, \varphi_1) \in N_1^u : |\varphi_1| \leqslant \varepsilon_1\}.$$

The preimage of this segment in D_1 is an arc of the initial circle. From (5.6) the extreme points of the arc are $\chi_1^{\pm}(d_1) = \pm\varepsilon_1 - \pi/2 - \theta_1^* + (\beta_1/\alpha_1 + O(d_1^2)) \ln(\rho_1/d_1)$. Thus, as $d_1 \to 0$, we get two infinite rays through these extreme points which rotate spirally infinitely many times about the point $(0, 0)$ in D_1. These two spirals, along with a boundary arc on the circle $d_1 = \rho_1/2$, delineate a thick spiral that represents all points on D_1 that map to V_1^u by the local flow. The image of the thick spiral under the action of the map given by the flow orbits is a scroll $\Sigma_1^u \subset V_1^u$ that wraps infinitely many times onto the segment $W^u(p_1) \cap V_1^u$. □

Similarly, if one chooses a local disk $D_2 \subset \text{Fix }(L)$ near p_2 and finds its preimage under the flow in N_2^u, then, reasoning as above, we also get a scroll $\Sigma_2^s \subset V_2^s$ that wraps infinitely many times onto the trace of $W^s(p_2) \cap V_2^s$.

Consider now the global map $S_1 : N_1^u \to N_2^s$ defined near the point $q_1^u = \Gamma_1 \cap N_1^u$. This map is a diffeomorphism, it takes values near the point $q_1^s = \Gamma_1 \cap N_2^s$. The equality $S_1(q_1^u) = q_1^s$ holds and S_1 transforms the trace of $W^u(p_1)$ – smooth curve through the point q_1^u – to the smooth curve through the point q_1^s which is noncollinear at q_1^s to the curve $W^s(p_2) \cap N_2^s$. We shall show that two surfaces Σ_1^u and $S_1^{-1}(\Sigma_2^s)$ intersect along a countable set of spiral-shaped curves whose sizes decrease when approaching to the intersection point q_1^u of two noncollinear smooth curves — traces of unstable $W^u(p_1)$ and stable $W^s(p_2)$ manifolds. That is, the intersection of two scrolls gives a sequence of curves contracting to the intersection point of the traces $W^u(p_1)$ and $W^s(p_2)$. Each such spiral-shaped curve will correspond to a one-parameter family of symmetric periodic orbits which lie entirely in the vicinity of the heteroclinic connection C.

As was noted above, the coordinates on N_1^u are $(\xi_1, \eta_1, \varphi_1)$. The trace of the stable manifold $W^s(p_2)$ in N_1^u is a smooth curve l_1^s given parametrically as $(\xi_1(\gamma), \eta_1(\gamma), \varphi_1(\gamma))$ with smooth functions ξ_1, η_1, φ_1, where γ is a parameter on the curve. For instance, using the map S_1 one can take θ_2 varying near θ_2^* as γ. We assume that $\gamma = 0$ corresponds to the intersection point $q_1^s = (0, 0, \varphi_1^*)$, hence, $\lim \varphi_1(\gamma) = \varphi_1^*$, as $\gamma \to 0$. The assumption of nondegeneracy for Γ_1 means that the tangent vector $\left(\xi_1'(0), \eta_1'(0), \varphi_1'(0)\right)$ is not collinear to the tangent vector $(0, 0, 1)$ to the trace of $W^u(p_1)$, i. e., $[\xi_1'(0)]^2 + [\eta_1'(0)]^2 \neq 0$.

Now we search first for intersection points of Σ_1^u with l_1^s. They correspond to the traces of symmetric homoclinic orbits of p_2. As is known from Devaney's theorem [15], for each symmetric

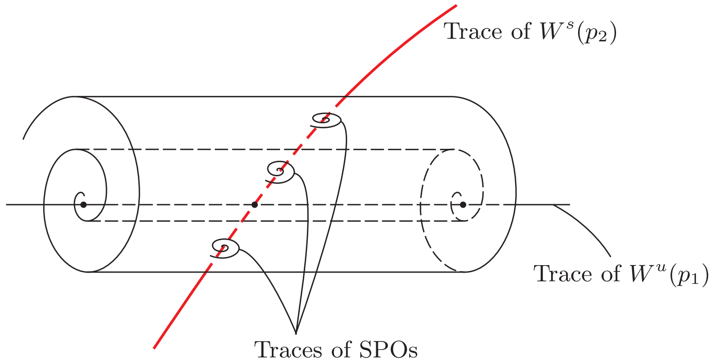


Fig. 4. Intersection of Σ_1^u and l_1^s with traces of SPOs.

homoclinic orbit to a symmetric saddle-focus (here it is p_2) there is a one-parameter family of SPOs accumulating to this homoclinic orbit. Here we prove the existence of a countable set of symmetric homoclinic orbits for p_2 and related sets of SPOs.

The scroll Σ_1^u in a parametric form with parameters (d_1, χ_1) varying on D_1 is given as follows, similarly to (5.7):

$$\begin{aligned} \xi_1 &= -d_1^2 \sin(2\chi_1), \\ \eta_1 &= d_1^2 \cos(2\chi_1), \\ \varphi_1 &= \chi_1 + \frac{\pi}{2} + \theta_1^* - \Big(\frac{\beta_1}{\alpha_1} + O(d_1^2)\Big) \ln \frac{d_1}{\rho_1} \pmod{2\pi}. \end{aligned}$$

Thus, the intersection points of the scroll and l_1^s are given by the solutions of the system

$$\begin{cases} \xi_1(\gamma) = -d_1^2 \sin(2\chi_1), \\ \eta_1(\gamma) = d_1^2 \cos(2\chi_1), \\ \varphi_1(\gamma) = \chi_1 + \dfrac{\pi}{2} + \theta_1^* - \Big(\dfrac{\beta_1}{\alpha_1} + O(d_1^2)\Big) \ln \dfrac{d_1}{\rho_1} \pmod{2\pi}. \end{cases} \tag{5.8}$$

Lemma 5. *There is d_1^0 small enough such that for $0 < d_1 \leqslant d_1^0$ the system (5.8) has a countable set of solutions $(\gamma_n, d_1^{(n)}, \chi_1^{(n)})$, where the following limits take place as $n \to \infty$:*

$$\lim \gamma_n = 0, \ \lim d_1^{(n)} = 0.$$

At the points of intersection Σ_1^u with l_1^s the intersection is transverse.

Proof. Because $[\xi_1'(0)]^2 + [\eta_1'(0)]^2 \neq 0$, at least one of derivatives $\xi_1'(0)$ or $\eta_1'(0)$ does not vanish. Suppose, for definiteness, that $\xi_1'(0)$ does not vanish. From the first two equations in (5.8) we can express $\xi_1(\gamma)\sqrt{1 + [\eta_1(\gamma)/\xi(\gamma)]^2} = d_1^2$ Applying l'Hopital's rule for the ratio under the square root, we conclude the existence of the limit $\tau_0 = \lim \eta_1(\gamma)/\xi_1(\gamma)$, as $\gamma \to 0$. Denote $\tau(\gamma) = \eta_1(\gamma)/\xi_1(\gamma)$. Then from the equation $\xi_1(\gamma)\sqrt{1 + \tau^2(\gamma)} = R$, $R = d_1^2$, we have by the implicit function theorem the unique solution $\gamma = b(R)$, $b(0) = 0$, $b'(0) = 1/\xi_1'(0)\sqrt{1 + \tau_0^2}$.

Now from the first equation in (5.8) we get $\sin(2\chi_1) = -\xi_1\big(b(R)\big)/R$. The function on the r.h.s. has the limit as $R \to 0$ equal to $-1/\sqrt{1 + \tau_0^2}$. So, the equation has two solutions $c(R) = \pm\frac{1}{2} \arcsin(\xi_1(b(R))/R)$ on each segment $[-\pi/2 + n\pi, -\pi/2 + n\pi]$.

At last, we consider the last equation of the system, where instead of γ and χ_1 the related functions $b(d_1^2)$ and $c(d_1^2) + n\pi/2$ are inserted. It can be rewritten in the form

$$\varphi_1(b(d_1^2)) - c(d_1^2) - \theta_1^* - (n+1)\frac{\pi}{2} = -\left(\frac{\beta_1}{\alpha_1} + O(d_1^2)\right) \ln \frac{d_1}{\rho_1} \pmod{2\pi}.$$

Taking into account that the function on the r.h.s. is monotonically increasing and tends to $+\infty$ as $d_1 \to +0$ and the function on the l.h.s. is smooth finite, we conclude that for positive d_1 small enough there are two solutions $d_1^{(k)}, \tilde{d}_1^{(k)}$ of the system on every 2π-period.

The resulting countable set of points on l_1^s are those through which symmetric homoclinic orbits of p_2 pass, since these orbits of $W^s(p_2)$ intersect Fix (L) and, by symmetry, they return in backward direction in time to the equilibrium p_2.

Now we shall show that at each intersection point the curve l_2^s is transverse to the scroll Σ_1^u. Let us change the parameter $d_1^2 = R$ and calculate at the intersection point the determinant composed from three tangent vectors: $(\xi_1', \eta_1', \varphi_1')$ to the curve l_1^s, and $(\partial\xi_1/\partial R, \partial\eta_1/\partial R, \partial\varphi_1/\partial R)$ and $(\partial\xi_1/\partial\chi, \partial\eta_1/\partial\chi, \partial\varphi_1/\partial\chi)$ — to the scroll Σ_1^u. Then we get

$$\begin{vmatrix} \xi_1' & -\sin(2\chi_1) & -2R\cos(2\chi_1) \\ \eta_1' & \cos(2\chi_1) & -2R\sin(2\chi_1) \\ \varphi_1' & -C\ln\frac{\sqrt{R}}{\rho_1} - \frac{1}{2}(\frac{\beta_1}{\alpha_1} + O(R))\frac{1}{R} & 1 \end{vmatrix}$$

To evaluate this we take into account that at the intersection point we have the equalities: $\sin(2\chi_1) = -\xi_1/R, \cos(2\chi_1) = \eta_1/R$. Substituting them into the determinant, we obtain the equality

$$-\frac{\xi_1^2}{R}\Big(\frac{\eta_1}{\xi_1}\Big)' + 2R^2\varphi_1' + (\xi_1^2 + \eta_1^2)'\left(C\ln\frac{\sqrt{R}}{\rho_1} + \frac{1}{2}\Big(\frac{\beta_1}{\alpha_1} + O(R)\Big)\frac{1}{R}\right).$$

Because $\eta_1/\xi_1 = -\cot(2\chi_1)$ from (5.8), the first term in the equality above equals $-2R$. So, the main term in the equality is $(\xi_1^2 + \eta_1^2)'((\frac{\beta_1}{\alpha_1} + O(R))\frac{1}{R})$, since $R_k \to 0$. Thus, we conclude that the determinant does not vanish for all k large enough. So, all found symmetric homoclinic orbits of p_2 are elementary and nondegenerate.

Because the scroll $\Sigma_{p_2}^s$ wraps and tends to the trace of $W^s(p_2)$ in N_2^s, two surfaces - the scrolls $\Sigma_{p_1}^u$ and $\Sigma_{p_2}^s$ - intersect along curves that wind up to the intersection points found above of the trace of the stable manifold $W_{p_2}^s$ and the scroll $\Sigma_{p_1}^u$. Their intersection provides all SPOs existing near the connection, but we may clearly separate only its part consisting of the countable set of spirals near the points corresponding to traces of symmetric homoclinic orbits of p_1 and p_2.

Similarly, considering the intersection of the trace of the unstable manifold $W^u(p_1)$ and the scroll Σ_2^s, we obtain a countable set of points through which symmetric homoclinic points of p_1 pass. □

Each found symmetric homoclinic orbit for a related saddle-focus is nondegenerate by construction. Therefore, use can be made of the result by Devaney [15] which suggests that there exists a one-parameter family of SPOs for any such homoclinic orbit Γ. The traces of SPOs of the family on the disk near the point $q = \Gamma \cap \mathrm{Fix}\ (L)$ form a spiral winding at the point q. Moreover, if one goes along this spiral and calculates multipliers for each SPO, then the types of these SPOs change from quasi-hyperbolic orientable to quasi-elliptic, then through double multiplier -1 to nonorientable quasi-hyperbolic and again through quadruple multiplier $+1$. Also, for each nondegenerate symmetric homoclinic orbits the results by Härterich [22] and Champneys [11] can be applied which guarantee the existence of multi-round nondegenerate homoclinic orbits near the primary one and families of multi-round SPOs, respectively.

Applying these results to our case, we can assert that near any nondegenerate symmetric homoclinic orbit a family of SPOs exist which accumulate to this homoclinic orbit. The diameter of that neighborhood of the intersection point of a scroll Σ_1^u and a curve l_2^s, where these SPOs exist for sure, tends to zero as $k \to \infty$, here k numerates homoclinics which approach the heteroclinic orbit Γ_1.

Similarly, considering the intersection of the trace of the unstable manifold $W^u(p_1)$ and the scroll Σ_2^s in V_2^s, we obtain the second family of symmetric homoclinic orbits of p_1. In each neighborhood of such a homoclinic orbit we again find a family SPOs which accumulate at this homoclinic orbit and this neighborhood becomes thinner and thinner when a homoclinic orbit approaches the heteroclinic orbit Γ_1. Now we discuss the proof of Theorem 8, first of all, in relation to the genericity condition for the family. Let v_μ be a reversible unfolding of the field v_0 containing the connection C of the second type. All v_μ are reversible w.r.t. the same involution L.[2)] The connection C contains the heteroclinic orbit Γ_1 going, as time increases, from p_1 to p_2. Take a point $q \in \Gamma_1$ and a cross-section $N \ni q$ to the flow. For $|\mu|$ small enough all v_μ have two symmetric saddle-foci with their invariant stable and unstable manifolds, they smoothly depend on the parameter [22, 47]. For all v_μ the submanifold N remains a cross-section for their flows. Consider a four-dimensional manifold $N \times \mathbb{R}$, $\mu \in \mathbb{R}$. The continuations of $W^u(p_1)$ and $W^s(p_2)$ have their traces in $N \times \mathbb{R}$, giving two smooth two-dimensional submanifolds intersecting at the point $(q, 0) \in N \times \mathbb{R}$. We assume *these two submanifolds to be transverse* at $(q, 0)$. This is the genericity condition for the unfolding mentioned in Theorem 8. Geometrically this condition means that two smooth curves in N smoothly depending on a parameter cross each other noncollinearly at $\mu = 0$ and they diverge from each other for $\mu \neq 0$ with nonzero speed.

Again, as above, we remark that all cross-sections for C, constructed above, remain cross-sections for all vector fields of the unfolding for μ close enough to the critical $\mu = 0$. We assume this further. To prove the theorem, we need for a given neighborhood V of the initial connection C to find a set of parameters μ_n such that the vector field v_{μ_n} has a heteroclinic connection C_n that involves saddle-foci p_1, p_2 and two nonsymmetric nondegenerate heteroclinic orbits $G_1^{(n)}, G_2^{(n)}$, $G_2^{(n)} = L(G_1^{(n)})$, which belong to V and such that a closed loop $\overline{G_1^{(n)} \cup G_2^{(n)}}$ is homotopic to the go-around twice the initial closed loop $\bar{C}$.

To find such μ_n, we take the cross-section N_1^u (see Fig. 1, right panel, where $V_1^u \subset N_1^u$) containing the trace l_1^u of $W^u(p_1)$ (a piece of the closed curve $\xi_1 = \eta_1 = 0$ near the point $q_1(\mu) = \Gamma_1(\mu) \cap N_1^u$). The trace of $W^s(p_2)$ in N_1^u is the smooth segment $l_1^s(\mu)$ intersecting at $\mu = 0$ the curve $l_1^u(\mu)$ noncollinearly at the point $q_1(\mu)$. Due to the genericity condition on the unfolding, for $\mu \neq 0$ these curves diverge at the distance of order $|\mu|$. Using the transition map $S_1(\mu) : N_1^u \to N_2^s$, which is defined near the point $q_1(\mu)$, we transfer the curves $l_1^u(\mu), l_1^s(\mu)$ to the cross-section N_2^s. Here the curve $l_1^s(\mu)$ transforms to the segment of the closed curve $W^s(p_2) \cap N_2^s$, but $l_1^u(\mu)$ transforms to the curve $l_2^u(\mu)$ which is noncollinear to the trace of $W^s(p_2)$ at $\mu = 0$, but these two curves diverge for $\mu \neq 0$.

Let us first describe the picture for $\mu = 0$. The curve $l_2^u(0)$ is divided by the point $q_2 = \Gamma_1(0) \cap N_2^s$ into two halves to which Lemma 2 is applicable. Thus, we get in the cross-section $N_2^u = L(N_2^s)$ two infinite spirals winding at the closed curve $W^u(p_2) \cap N_2^u$. The intersection of these spirals with the neighborhood $V_2^u \subset N_2^u$ (see Fig. 1, right panel) of the point $q_2' = L(q_2)$ gives two infinite sets of segments approaching in C^1-topology the segment $W^u(p_2) \cap V_2^u$. Using the transition map $S_2 : N_2^u \to N_1^s,\ S_2 = L \circ S_1^{-1} \circ L^{-1}$, we transfer these sets of segments to the cross-section N_1^s. In N_1^s these segments tend to the smooth segment $W^u(p_2) \cap V_1^s$ which are noncollinear to the trace of the smooth segment $W^s(p_1) \cap V_1^s$. Again, by Lemma 2, the T_1-preimage of the segment $l_1^s = W^s(p_2) \cap V_1^u$ gives two infinite spirals in N_1^s winding at the closed curve $W_1^s \cap N_1^s$ and their intersection with V_1^s gives the infinite set of segments approaching in C^1-topology the segment $W^s(p_1) \cap V_1^s$. The intersections of segments from these two infinite families give (if they exist) the heteroclinic orbit going from p_1 to p_2 as time increases, but generally speaking, these two sets of segments do not intersect, since their basic curves are noncollinear.

Now we shall vary μ near zero. Both saddle-foci and their stable/unstable manifolds smoothly depend on a parameter if v_μ depends smoothly on μ. But the trace $l_1^s(\mu) \subset V_1^u$ for $\mu \neq 0$ does not intersect the curve $l_1^u(\mu)$ due to the genericity assumption. Therefore, the preimage of $l_1^s(\mu)$ under the map $T_1(\mu)$ for μ small enough is a smooth curve that makes many revolutions around the closed curve $W^s(p_1) \cap N_1^s$ approaching it, but after a large number of revolutions (their number depends

[2)]Similarly, one may consider the involution L_μ smoothly depending on μ.

on the smallness $|\mu|$: the smaller μ, the larger the number of revolutions and the closer the sharp tip of the curve to the trace $W^s(p_2) \cap N_1^s$), it makes a sharp turn and unwinds in the backward direction along the θ_1-coordinate in N_1^s.

Remark 3. The situation described here is very similar to that encountered in Hamiltonian systems near a transverse homoclinic orbit of a saddle-focus [37] or a heteroclinic contour with two saddle-foci, when both of them belong to the same level set of the Hamiltonian [36]. When passing through the singular level set of the Hamiltonian, the local map has discontinuity along the trace of the stable manifold and any transversal segment to this trace behaves under the local map similarly to what we see for the reversible case. There the role of a parameter is played by the value of the Hamiltonian for the case of a homoclinic orbit or the detuning parameter μ which transfers saddle-foci to different level sets of the Hamiltonian.

The same behavior takes place for the curve $l_1^u(\mu)$. Namely, at $\mu \neq 0$ this curve transforms by the map $S_1(\mu)$ to the curve $l_2^u(\mu) \subset N_2^s$ which does not intersect the trace of $W^s(p_2)$ and after the action of the map $T_2(\mu)$ it transforms to the smooth curve in N_2^u that behaves similar to what is said above. We work with its pieces which belong to the neighborhood V_2^u. The transition map $S_2(\mu)$ transforms these curves into V_1^s. Thus, again, we have two sets of curves. One set consists of finitely many (though large enough) segments which are C^1-close to the curve $l_1^s : \xi_1 = \eta_1 = 0$, the other set consists of finitely many (though also large enough) segments which are C^1-close to the curve $W^u(p_2) \cap V_1^s$. When μ varies from $-\mu_0$ to μ_0, in view of the genericity condition, for some μ_n some pairs of segments from different families necessarily intersect, giving for the related value of μ_n a heteroclinic orbit $G_1(\mu_n)$ going from p_1 to p_2. By symmetry, we have in this case a pairing heteroclinic orbit $G_2(\mu_n)$. These two heteroclinic orbits together form a heteroclinic connection which is 2-round w.r.t. the initial one, C. This completes the proof of Theorem 8.

6. CONCLUSION

In this paper we studied two types of heteroclinic connections involving saddle-foci, which for the first type form a pair of nonsymmetric equilibria, being permuted by the involution and for the second type a pair of symmetric saddle-foci. In both cases these equilibria are connected by a pair of nondegenerate heteroclinic orbits, making up, along with the saddle-foci, an orientable closed curve. The focus was on the existence of families of symmetric periodic orbits, heteroclinic connections of higher roundness and existence of homoclinic orbits of saddle-foci. The investigation of the orbit structure near the connection has shown that the orbit behavior is very complicated, so we restricted our attention to these classes of orbits. It is well known, however, that near the homoclinic orbit of a saddle-focus with a positive saddle value many hyperbolic sets are contained. But the situation in a reversible system is more delicate [24, 25, 29, 30] and requires a separate study. In particular, here the phenomena of switching for homoclinic networks are observed [2, 27]. We hope to examine them elsewhere.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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