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Spaces of harmonic maps of the projective plane to the four-dimensional sphere

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Abstract. The spaces of harmonic maps of the projective plane to the four-dimensional sphere are investigated in this paper by means of twistor lifts. It is shown that such spaces are empty in case of even harmonic degree. In case of harmonic degree less than 6 it was shown that such spaces are path-connected and an explicit parameterization of the canonical representatives was found. In addition, the last section provides comparisons with the known results for harmonic maps of the two-dimensional sphere to the four-dimensional sphere of harmonic degree less than 6.

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1. Introduction

Let $\phi : M \to N$ be a map between Riemannian manifolds. We define its *energy* by the formula

$$E(\phi) = \frac{1}{2} \int\limits_M |d\phi(x)|^2 dx,$$

where $d\phi(x)$ is the differential of ϕ at the point $x \in M$; and dx is the volume element of M. Euler-Lagrange operator $\tau(\phi) = div(d\phi)$ associated with the functional E is called a *tension field of* ϕ. The map $\phi : M \to N$ is said to be *harmonic* if its tension field vanishes identically i.e. ϕ is a critical point of E.

Some particular cases of harmonic maps are well-known, i.e.

- If $dim\, M = 1$, then the harmonic maps are the geodesics of N.
- If $N = \mathbb{R}$, they are harmonic functions on M.

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- If $dim\, M = 2$, they include parametric representations of the minimal surfaces of N; the energy is the Dirichlet-Douglas integral.

The author's interest to harmonic maps is motivated by the relationship between harmonic maps and isoperimetric inequalities for the Laplace operator eigenvalues. It was discovered by Nadirashvili [19] and El Soufi and Ilias [11] that these inequalities are closely connected to minimal and harmonic maps from surfaces to spheres S^n. This connection permitted to completely solve recently the problem of isoperimetric inequalities for Laplace eigenvalues on the sphere S^n, see the paper [15] for the general case and the previous papers [13,21,22] for particular cases, and on the projective plane, see the paper [14] for the general case and the previous papers [17] and [20] for particular cases. More information on this subject could be found in surveys [23] and [24].

There are many papers on the theory of the harmonic maps. This paper is based on the results of the famous works of E. Calabi [6,7], J. Barbosa [1], R. L. Bryant [5], and essentially uses ideas developed in works of J. Bolton and L. M. Woodward [2–4]. We also refer to important results on the topology of spaces of harmonic maps from S^2 to S^4 obtained by B. Loo in [18], M. Kotani in [16], and M. Furuta, M. A. Guest, M. Kotani, and Y. Ohnita in [12]. We use the following proposition and fundamental theorems.

Proposition 1 [10]. *An isometric immersion $\phi : (M, g) \to (N, h)$ is minimal if and only if it is harmonic.*

Theorem 1 (*E. Calabi* [6]). *Let $X : S^2 \to \mathbb{R}^n$ be an immersion of a sphere S^2 into $\mathbb{R}^n$, whose image is a locally minimal surface in a sphere rS^{n-1} of radius r, and is not contained in any hyperplane of $\mathbb{R}^n$. Then the following conclusions hold.*

(1) *The area $A = A(S^2)$ of the image sphere S^2 is an integer multiple of $2\pi r^2$.*

(2) *The dimension n is odd.*

In this theorem one uses an assumption that the image of $X : S^2 \to \mathbb{R}^n$ is not contained in any hyperplane of $\mathbb{R}^n$. Such maps are called *linearly full*. It should be noted that if we induce metric on S^2 by X in the conditions of Theorem 1, then X is an isometric immersion, and hence harmonic due to Proposition 1. J. Barbosa clarifies paragraph (1) of Theorem 1 in the case of $r = 1$.

Theorem 2 (*J. Barbosa* [1]). *The area of a generalized minimal immersion $X : S^2 \to S^{2m}$ is $4\pi d$, for some $d \in \mathbb{N}$, $d \geq \frac{m(m+1)}{2}$.*

The word *generalized* in this theorem means that the immersion X may be branched at isolated points. The number d, appearing in this theorem, is called a *harmonic degree* (or *degree*) of a harmonic map X. The case $m = 2$ has been investigated by R. L. Bryant in [5] using the construction of twistor fibration $\pi : \mathbb{C}P^3 \to S^4$ introduced in Sect. 2. In the following theorem *horizontal* means orthogonal to the fibers of π (see Sect. 2).

Theorem 3 (*R. L. Bryant* [5]). (1) *Let $\psi : S^2 \to \mathbb{C}P^3$ be a linearly full horizontal holomorphic curve then $\phi = \pi\psi : S^2 \to S^4$ is a linearly full harmonic map. Conversely every linearly full harmonic map $\phi : S^2 \to S^4$ is equal to $\pm\pi\psi$ for some uniquely determined linearly full horizontal holomorphic curve ψ called the* twistor lift *of ϕ.*

(2) *Twistor lift of a linearly full harmonic map $\phi : S^2 \to S^4$ is an algebraic curve.*

(3) *Area of S^2 w.r.t. the metric induced by ϕ is equal to $4\pi d$, and d is a degree of its twistor lift.*

This paper investigates harmonic maps $\phi : \mathbb{R}P^2 \to S^4$ by using standard Riemannian double cover $S^2 \to \mathbb{R}P^2$ and twistor lifts to $\mathbb{C}P^3$ of *invariant* harmonic maps $S^2 \to S^4$ (i.e. harmonic maps that can be factorized through the double cover $S^2 \to \mathbb{R}P^2$). In this way we define harmonic degree of the map $\phi : \mathbb{R}P^2 \to S^4$ as a degree of its lift $\tilde{\phi} : S^2 \to S^4$ under double cover.

One of our aims in this paper is to show that there are two path-connected components $AHarm_d^{\pm}(S^4)$ in the spaces of harmonic maps from the projective plane to the four-dimensional sphere of low harmonic degrees. The similar question for spaces of harmonic maps from S^2 to S^4 of arbitrary degrees is addressed by B. Loo [18] and the more general case of maps to higher-dimensional spheres by M. Kotani [16]. Moreover, fundamental groups of the spaces of harmonic 2-spheres in the n-sphere are obtained by M. Furuta, M. A. Guest, M. Kotani, and Y. Ohnita in [12]. For the general overview on harmonic maps one may refer to the classical texts [8–10].

The structure of the paper is following. In Sect. 2 we give an overview of the twistor fibration and groups acting on it. Section 3 is devoted to the higher singularities that occur for the twistor lifts of harmonic maps. In Sect. 4 topologies on the sets of linearly full horizontal holomorphic curves and harmonic maps are defined, and their homeomorphism is described. Section 5 addresses general results on harmonic maps of arbitrary degree, for example, it shows that there are no harmonic maps from the projective plane to the four-dimensional sphere of even degree. In Sect. 6 the new facts obtained in previous section and similar ideas as used by J. Bolton and L. M. Woodward in [2] allow us to describe the spaces of harmonic maps of degree less than 6, and to show path-connectedness of these spaces and the “bubbling” phenomenon. We also compare throughout this section the results on path-connectedness for the projective plane with the similar results for the two-dimensional sphere.

2. The twistor fibration

We introduce the construction of the *twistor fibration* $\pi : \mathbb{C}P^3 \to S^4$. The *Hopf map* $\rho : \mathbb{C}P^3 \to \mathbb{H}P^1$ is given by

$$\rho([z_1, z_2, z_3, z_4]) = [z_1 + z_2 j, z_3 + z_4 j].$$

The canonical identification of $\mathbb{H}P^1$ and $S^4 \subset \mathbb{H} \oplus \mathbb{R} = \mathbb{R}^5$ is given by the stereographic projection from the south pole of S^4 onto the equatorial hyperplane $\mathbb{H}$ in $\mathbb{R}^5$ which is included in $\mathbb{H}P^1$ by $q \mapsto [1, q]$. This identification is given by

$$[q_1, q_2] \in \mathbb{H}P^1 \leftrightarrow \frac{(2\bar{q}_1 q_2, |q_1|^2 - |q_2|^2)}{|q_1|^2 + |q_2|^2} \in S^4.$$

Now if $\mathbb{H}^2$ is a left quaternionic vector space, then π is obtained by composing Hopf map ρ with the canonical identification of $\mathbb{H}P^1$ and S^4.

Hence, we obtain an explicit formula for the twistor fibration

$$\pi([z_1, z_2, z_3, z_4]) = \\ = \frac{(2(\bar{z}_1 z_3 + z_2 \bar{z}_4), 2(\bar{z}_1 z_4 - z_2 \bar{z}_3), |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2)}{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2}.$$

If $\mathbb{C}P^3$ is endowed with the Fubini-Study metric of constant holomorphic sectional curvature 1 then π is a Riemannian submersion and the *horizontal distribution* on $\mathbb{C}P^3$ consists of those tangent vectors to $\mathbb{C}P^3$ which are orthogonal to the fibers of π.

Let

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{1}$$

Then the projectivization $PSp(2; \mathbb{C})$ of the complexified symplectic group

$$Sp(2; \mathbb{C}) = \{A \in SL(4; \mathbb{C}) | A^t J A = J\}$$

acts on $\mathbb{C}P^3$ in the standard way as the group of holomorphic diffeomorphisms, which preserve the horizontal distribution, with $PSp(2) = PSp(2; \mathbb{C}) \cap PU(4)$ the subgroup of holomorphic isometries which preserve the horizontal distribution [3].

It should be noted that all of these groups act transitively on $\mathbb{C}P^3$ and the groups $PSp(2, \mathbb{C})$ and $PSp(2)$ are path-connected [26]. It should be noted also that the elements of $Sp(2)$ are matrices $(\mathbf{u}, J\bar{\mathbf{u}}, \mathbf{v}, J\bar{\mathbf{v}})$ such that $\mathbf{u}, \mathbf{v}$ are unit vectors in $\mathbb{C}^4$, and $\mathbf{v}$ unitarily orthogonal to $\mathbf{u}$ and $J\bar{\mathbf{u}}$.

Let S be a Riemann surface and $\psi : S \to \mathbb{C}P^3$ be a holomorphic curve. Then ψ is *horizontal* if at each point it is tangent to the horizontal distribution on $\mathbb{C}P^3$, or equivalently if it intersects each fiber of π orthogonally. It was proved in [5] that the condition for a holomorphic curve $\psi = [\mathbf{f}] = [f_1, f_2, f_3, f_4]$ to be horizontal is

$$(\mathbf{f}', J\mathbf{f}) = 0, \tag{2}$$

where (,) denotes the complex bilinear extension to $\mathbb{C}^4$ of the standard real scalar product on $\mathbb{R}^4$. The horizontality condition may be written as

$$f_1' f_2 - f_1 f_2' + f_3' f_4 - f_3 f_4' = 0. \tag{3}$$

Here and throughout the rest of the paper we write holomorphic curves $\psi : S \to \mathbb{C}P^n$ locally in terms of a local complex coordinate z in the form $\psi = [\mathbf{f}] = [f_1, f_2, \ldots, f_{n+1}]$, where $f_1, \ldots, f_{n+1}$ are holomorphic functions of z without common zeros. We will call such a representation a *reduced form* of ψ.

3. Higher singularities for horizontal holomorphic curves

Let us recall the definition of singularity type of holomorphic curves in $\mathbb{C}P^n$ [2]. Let S be a Riemann surface and suppose that $\psi : S \to \mathbb{C}P^n$ is a linearly full holomorphic curve. We write ψ locally in reduced form $\psi = [\mathbf{f}] = [f_1, f_2, \ldots, f_{n+1}]$, and let $\mathbf{f}^{(j)}$ denote the j-th derivative of $\mathbf{f}$ with respect to z. For each $k = 0, \ldots, n-1$ we define *k-th osculating curve* as $[\mathbf{f} \wedge \ldots \wedge \mathbf{f}^{(k)}]$. A *higher singularity* of ψ is a point $p \in S$ where the derivative of k-th osculating curve is equal to zero for some $k = 0, \ldots, n-1$. The set $Z(\psi)$ of higher singularities of ψ is therefore given by

$$Z(\psi) = \{p \in S \,|\, \mathbf{f} \wedge \cdots \wedge \mathbf{f}^{(n)}(p) = 0\}.$$

If z is a local complex coordinate with $z(p) = 0$, then $\mathbf{f}$ can be written in the *normal form*

$$\mathbf{f}(z) = h_0(z)\mathbf{a}_0 + z^{r_0(p)+1} h_1(z)\mathbf{a}_1 + \cdots + z^{r_0(p)+\cdots r_{n-1}(p)+n} h_n(z)\mathbf{a}_n$$

for some suitable choice of basis $\mathbf{a}_0, \ldots, \mathbf{a}_n$ of $\mathbb{C}^{n+1}$, non-negative integers $r_0(p), \ldots, r_{n-1}(p)$, and holomorphic functions $h_0(z), \ldots, h_n(z)$, each non-zero at $z = 0$. If $r_k(p) > 0$ then the derivative of the k-th osculating curve has a zero of order $r_k(p)$ at p and the *singularity type* of ψ is defined to be the set

$$\{(p; r_0(p), \ldots, r_{n-1}(p)) \,|\, p \in Z(\psi)\}.$$

Lemma 1 [2]. *Let $g \in PGL(n+1; \mathbb{C})$ and ω be a conformal diffeomorphism of S. If ψ has singularity type $\{(p; r_0(p), \ldots, r_{n-1}(p)) \,|\, p \in Z(\psi)\}$ then $g\psi\omega^{-1}$ has singularity type $\{(\omega(p); r_0(p), \ldots, r_{n-1}(p)) \,|\, p \in Z(\psi)\}$.*

We now give a criterion for determining the higher singularities of a linearly full horizontal holomorphic curve $\psi : S \to \mathbb{C}P^3$.

Lemma 2 [2]. *Let $\psi : S \to \mathbb{C}P^3$ be a linearly full horizontal holomorphic curve written locally in terms of a complex coordinate z in the reduced form as $\psi(z) = [\mathbf{f}(z)] = [f_1(z), f_2(z), f_3(z), f_4(z)]$. Then $z = a$ is a higher singularity of ψ if and only if*

$$(f_1'' f_2' - f_1' f_2'' + f_3'' f_4' - f_3' f_4'')(a) = 0, \tag{4}$$

or equivalently, $(\mathbf{f}'', J\mathbf{f}')(a) = 0$.

For a linearly full holomorphic curve $\psi : S^2 \to \mathbb{C}P^n$ and $k = 0, \ldots, n-1$ we define

$$r_k = \sum_{p \in Z(\psi)} r_k(p).$$

Proposition 2. *Let $\psi : S^2 \to \mathbb{C}P^3$ be a linearly full horizontal holomorphic curve and $p \in S^2$. Then $r_0(p) = r_2(p)$ and*

$$2r_0 + r_1 = 2d - 6, \tag{5}$$

where d is the degree of ψ.

Proofs of the lemmas and proposition of this section can be found in [2].

4. A canonical form

Due to Theorem 3, the fact that linearly full harmonic map $\phi : S^2 \to S^4$ has a twistor lift which is, moreover, an algebraic curve, gives us a good opportunity for investigating harmonic maps. Namely, let $HHol_d^{LF}(\mathbb{C}P^3)$ denote the set of linearly full horizontal holomorphic maps of S^2 to $\mathbb{C}P^3$ of degree d, and let $Harm_d^{LF}(S^4)$ denote the set of linearly full harmonic maps of S^2 to S^4 with induced area $4\pi d$. Then $Harm_d^{LF}(S^4)$ is divided into two components $Harm_d^+(S^4)$ and $Harm_d^-(S^4)$, which can be interchanged by the antipodal involution of S^4, and there is a bijective correspondence

$$\pi_*^\pm : HHol_d^{LF}(\mathbb{C}P^3) \to Harm_d^\pm(S^4) \tag{6}$$

defined by $\pi_*^\pm(\psi) = \pm\pi \circ \psi$.

Clearly, $p \in S^2$ is a branch point of ψ if and only if $r_0(p) > 0$ while as is shown in Section 7 of [3], if $r_0(p) = 0$ then p is an umbilic of ψ if and only if $r_1(p) > 0$. Further, it follows from the paper [25] that the branch points and umbilics of ϕ coincide with those of ψ. Thus the higher singularities of ψ occur at the branch points or umbilics of ϕ.

It is clear that the group $PSp(2)$ acts freely on $HHol_d^{LF}(\mathbb{C}P^3)$ by postcomposition via its standard action on $\mathbb{C}P^3$ and it follows from Lemma 1 that the singularity type is preserved. The identifications determined by $\pi_*^\pm$ determine an action of $PSp(2)$ on $Harm_d^\pm(S^4)$ which preserves induced area, branch points, umbilics and antipodal invariance. Similarly the rotation group $SO(3) \cong PSU(2)$ also acts on $HHol_d^{LF}(\mathbb{C}P^3)$ and $Harm_d^\pm(S^4)$ by precomposition via its standard action on $S^2 = \mathbb{C}P^1$ and also preserves antipodal invariance. Note that the actions of $PSp(2)$ and $PSU(2)$ commute and that the maps $\pi_*^\pm$ are $PSp(2)$-equivariant and $PSU(2)$-equivariant.

Now we consider the vector space $\mathbb{C}[z]_d$ of polynomials of degree at most d, and let V be the subset of $(\mathbb{C}[z]_d)^4$ consisting of those quadruples of coprime polynomials with maximum degree d for which the map $z \to [f_1(z), f_2(z), f_3(z), f_4(z)]$ is linearly full in $\mathbb{C}P^3$. Then V is a projective subset of $(\mathbb{C}[z]_d)^4$, and we identify its projectivisation $P(V)$ with the space of linearly full holomorphic maps of degree d from S^2 to $\mathbb{C}P^3$ in the usual way via

$$[f_1, f_2, f_3, f_4] \leftrightarrow (z \to [f_1(z), f_2(z), f_3(z), f_4(z)]).$$

Here, and subsequently, we use the complex coordinate on S^2 defined by the stereographic projection from the south pole of S^2 onto the equatorial plane which means, in the usual sense, we may identify S^2 with $\mathbb{C} \cup \{\infty\}$. According to this identification antipodal involution interchanges z and $-\frac{1}{\bar{z}}$.

We endow $(\mathbb{C}[z]_d)^4$ with its natural topology as a vector space, and $P((\mathbb{C}[z]_d)^4)$ the quotient topology. Then V is an open subset of $(\mathbb{C}[z]_d)^4$, and $P(V)$ is an open subset of $P((\mathbb{C}[z]_d)^4)$. Subsets of any of these spaces are then endowed with the induced topology. We denote by $V^H \subset V$ the subset of horizontal maps. Next we give $Harm_d^{LF}(S^4)$ the compact-open topology and rewrite a lemma from [4].

Lemma 3. *$Harm_d^{\pm}(S^4)$ are closed subsets of $Harm_d^{LF}(S^4)$, and the maps $\pi_*^{\pm}$: $P(V^H) \to Harm_d^{\pm}(S^4)$ are homeomorphisms.*

For convenience, we introduce *coefficient matrix* (or *matrix*) of a holomorphic curve $\mathbf{f}(z) = (f_1(z), f_2(z), f_3(z), f_4(z)) \in (\mathbb{C}[z]_d)^4$. It is given by

$$F = \begin{pmatrix} a_1^0 & a_1^1 & \dots & a_1^{d-1} & a_1^d \\ a_2^0 & a_2^1 & \dots & a_2^{d-1} & a_2^d \\ a_3^0 & a_3^1 & \dots & a_3^{d-1} & a_3^d \\ a_4^0 & a_4^1 & \dots & a_4^{d-1} & a_4^d \end{pmatrix},$$

where a_i^j is a j-th coefficient of the polynomial $f_i(z) = a_i^0 + a_i^1 z + \dots + a_i^j z^j + \dots + a_i^d z^d$, and $1 \leq i \leq 4$, $0 \leq j \leq d$. It is clear from definition that $(\mathbf{f}(z))^t = F \cdot (1, z, z^2, \dots, z^d)^t$ and also that $\mathbf{f}(z)$ is linearly full if the coefficient matrix F has a full rank.

Note that an element $A \in Sp(2)$, acting on $\mathbf{f}(z) \in (\mathbb{C}[z]_d)^4$ by postcomposition (i.e. $\mathbf{f}(z) \to A \circ \mathbf{f}(z)$), acts on coefficient matrix F by the left multiplication (i.e. $F \to AF$). Also, as noted earlier, the group $PSU(2) \cong SO(3)$ acts on S^2 as a group of rotations. This action defines an action of $PSU(2)$ on $\mathbb{R}P^2$ since it preserves the antipodal points. It should be noted that the group $PSU(2)$ is path-connected as well as the group of Möbius transformations $PSL(2, \mathbb{C})$ [26].

In the following sections we investigate linearly full harmonic maps of S^2 to S^4 of degrees $d = 3$, 4, 5 and 6, which are the lifts of harmonic maps of $\mathbb{R}P^2$ to S^4 through the double cover, i.e. maps invariant w.r.t. the antipodal map (we will call them *invariant maps* for brevity). Let $AHarm_d^{LF}(S^4) \subset Harm_d^{LF}(S^4)$ denote the space of invariant linearly full harmonic maps of S^2 to $S^4$of degree d, with $AHarm_d^{\pm}(S^4) \subset Harm_d^{\pm}(S^4)$. And let us denote the twistor lifts of this spaces as $AHHol_d^{LF}(\mathbb{C}P^3) \subset HHol_d^{LF}(\mathbb{C}P^3)$. Our main idea is the deformation of elements of $AHarm_d^{\pm}(S^4) \subset Harm_d^{\pm}(S^4)$ to some canonical form by the action by some appropriate elements of $PSU(2)$ and $PSp(2)$. Note that these groups act continuously on $AHHol_d^{LF}(\mathbb{C}P^3)$ and, due to their path-connectedness, we can recover some information about path-connectedness of $AHHol_d^{LF}(\mathbb{C}P^3)$.

5. Harmonic maps of arbitrary degree

In this section we will exclude the condition of antipodal invariance by finding the general form of invariant maps. In this way we will need some addition to Theorem 3.

Theorem 4 (*R. L. Bryant* [5]). *Let $\hat{\psi} : S^2 \to \mathbb{C}P^3$ be a linearly full horizontal antiholomorphic curve then $\phi = \pi\hat{\psi} : S^2 \to S^4$ is a linearly full harmonic map. Conversely every linearly full harmonic map $\phi : S^2 \to S^4$ is equal to $\pm\pi\hat{\psi}$ for some uniquely determined linearly full horizontal antiholomorphic curve $\hat{\psi}$ called the* antiholomorphic twistor lift *of ϕ.*

Moreover if $\psi = [\mathbf{f}] = [f_1, f_2, f_3, f_4]$ is a twistor lift of ϕ then $\hat{\psi} = \bar{\psi}J = [\bar{\mathbf{f}}J] = [\bar{f}_2, -\bar{f}_1, \bar{f}_4, -\bar{f}_3]$ is an antiholomorphic twistor lift of ϕ [3].

For $\mathbf{f}(z) = (f_1(z), f_2(z), f_3(z), f_4(z))$ of degree n we introduce

$$\begin{aligned}\tilde{\mathbf{f}}(z) &= (\tilde{f}_1(z), \tilde{f}_2(z), \tilde{f}_3(z), \tilde{f}_4(z)) = \\ &= (\bar{z}^n f_1(-1/\bar{z}), \bar{z}^n f_2(-1/\bar{z}), \bar{z}^n f_3(-1/\bar{z}), \bar{z}^n f_4(-1/\bar{z})).\end{aligned}$$

Then $[\tilde{f}(z)]$ is an antiholomorphic curve in $\mathbb{C}P^3$.

Theorem 5. *There is no harmonic maps of real projective plane to the four-dimensional sphere of even degree, i.e. spaces $AHarm_n^{LF}(S^4)$ are empty for even n.*

Proof. Let us suppose that such maps exist. Then for each map $\phi \in AHarm_n^{LF}(S^4)$ we will compose its twistor lift $\psi = [\mathbf{f}(z)] = [f_1(z), f_2(z), f_3(z), f_4(z)]$ with antipodal involution of S^2 and get an antiholomorphic twistor lift $\tilde{\psi} = [\tilde{\mathbf{f}}(z)]$. Using Theorem 4 we get $\psi = -\bar{\tilde{\psi}}J$. Then

$$\begin{aligned} g(z) \cdot \mathbf{f}(z) &= g(z) \cdot (f_1(z), f_2(z), f_3(z), f_4(z)) \\ &= -\overline{\tilde{\mathbf{f}}(z)}J = (-\overline{\tilde{f}_2(z)}, \overline{\tilde{f}_1(z)}, -\overline{\tilde{f}_4(z)}, \overline{\tilde{f}_3(z)}),\end{aligned} \tag{7}$$

for some function $g(z)$ on S^2. Now we will prove that $g(z)$ is a holomorphic function. From (7) we have

$$g(z) = \frac{-\overline{\tilde{f}_2(z)}}{f_1(z)} = \frac{\overline{\tilde{f}_1(z)}}{f_2(z)} = \frac{-\overline{\tilde{f}_4(z)}}{f_3(z)} = \frac{\overline{\tilde{f}_3(z)}}{f_4(z)}. \tag{8}$$

It is clear that $\overline{\tilde{f}_i(z)}$ $(i = 1, 2, 3, 4)$ are polynomials of z of degree less or equal to n, so they are holomorphic functions on $\mathbb{C}$. At the same time $f_i(z)$ $(i = 1, 2, 3, 4)$ don't have common roots and therefore $g(z)$ is holomorphic on $\mathbb{C}$. By the definition there is polynomial of degree n among $f_i(z)$ $(i = 1, 2, 3, 4)$ and from (8) we conclude that $g(z)$ is holomorphic at ∞. Thus $g(z)$ is a holomorphic function on S^2 and therefore $g(z) = \frac{1}{\beta} = const \neq 0$.

Since $f_i(z) = a_i^0 + a_i^1 z + \cdots + a_i^j z^j + \cdots + a_i^n z^n$, $(i = 1, 2, 3, 4)$ we have $\overline{\tilde{f}_i(z)} = (-1)^n \overline{a_i^n} + (-1)^{n-1}\overline{a_i^{n-1}} z + \cdots + (-1)^{n-j}\overline{a_i^{n-j}} z^j + \cdots + \overline{a_i^0} z^n$, $(i = 1, 2, 3, 4)$. Considering (8) we can write

$$\begin{cases} a_1^n = -\beta\overline{a_2^0}, & a_2^0 = (-1)^n\beta\overline{a_1^n}, \\ a_2^n = \beta\overline{a_1^0}, & a_1^0 = (-1)^{n+1}\beta\overline{a_2^n}, \\ a_3^n = -\beta\overline{a_4^0}, & a_4^0 = (-1)^n\beta\overline{a_3^n}, \\ a_4^n = \beta\overline{a_3^0}, & a_3^0 = (-1)^{n+1}\beta\overline{a_4^n}. \end{cases} \tag{9}$$

From these relations we can conclude for leading coefficients $a_i^n = (-1)^{n+1}|\beta|^2 a_i^n$ and since n is even one has $a_i^n = -|\beta|^2 a_i^n$. But it can only be when $a_i^n = 0$ $(i = 1, 2, 3, 4)$ and thus the degree of the curve is less than n. So we have a contradiction which proves this theorem. $\square$

Remark 1. In the case of odd n we have $a_i^n = |\beta|^2 a_i^n$ for leading coefficients, therefore $|\beta| = 1$ and $\beta = \exp(i\tilde{\beta})$ for some $\tilde{\beta} \in \mathbb{R}$. In this way any curve of odd degree is of the form $\psi = [\mathbf{f}(z)] = [f_1(z), \beta\overline{\tilde{f}_1(z)}, f_3(z), \beta\overline{\tilde{f}_3(z)}]$. Note that if we act on the curve by rotation $\omega(z) = \exp(-\mathrm{i}\frac{\tilde{\beta}}{\mathrm{n}})z$, we get the curve with $\beta = 1$. (Note also that this rotation leaves points 0 and ∞ unchanged). Thus any curve of odd degree can be rotated to a curve of form $\psi = [\mathbf{f}(z)] = [f_1(z), \overline{\tilde{f}_1(z)}, f_3(z), \overline{\tilde{f}_3(z)}]$.

We need the following useful lemmas in the next sections.

Lemma 4. *Any curve $\psi \in HHol_d^{LF}(\mathbb{C}P^3)$ by an appropriate element $g \in PSp(2) \subset PSp(2, \mathbb{C})$ can be reduced to the curve $g\psi$ with the coefficient matrix of the form*

$$\begin{pmatrix} * & * & * & * & \dots & * & * & * & * \\ 0 & 0 & 0 & * & \dots & * & * & * & * \\ 0 & * & * & * & \dots & * & * & * & * \\ 0 & 0 & * & * & \dots & * & * & * & * \end{pmatrix}.$$

In addition, if $\psi \in AHHol_d^{LF}(\mathbb{C}P^3) \subset HHol_d^{LF}(\mathbb{C}P^3)$ then by the action of an appropriate element $g \in PSp(2)$ it can be reduced to the curve $g\psi$ with the coefficient matrix of the form

$$\begin{pmatrix} * & * & * & * & \dots & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & \dots & * & * & * & * \\ 0 & * & * & * & \dots & * & * & 0 & 0 \\ 0 & 0 & * & * & \dots & * & * & * & 0 \end{pmatrix}.$$

Proof. First, the transitivity of the action of $PSp(2)$ on $\mathbb{C}P^3$ allows us to change the first column to the announced form. Next, we notice that the elements of $Sp(2)$ that preserve the vector $(1, 0, 0, 0)^t$ have the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & v_1 & -\bar{v}_2 \\ 0 & 0 & v_2 & \bar{v}_1 \end{pmatrix},$$

and, since we have a matrix from $U(2)$ in the right bottom corner, we can eliminate the last term of the second column. Other two zeros in the second row is due to the horizontality condition (3). Finally, we get zeros in the right side of the coefficient matrix due to Remark 1. □

Lemma 5 [2]. *The group $PSp(2,\mathbb{C})$ acts transitively on $\mathbb{C}P^3$ and the stabilizer of the point $[1,0,0,0]^t$ consists of the matrices of the form*

$$\begin{pmatrix} \xi & \eta & \xi(\gamma\lambda-\alpha\mu) & \xi(\delta\lambda-\beta\mu) \\ 0 & \zeta & 0 & 0 \\ 0 & \lambda & \alpha & \beta \\ 0 & \mu & \gamma & \delta \end{pmatrix}, \quad \alpha\delta-\beta\gamma=\xi\zeta=1.$$

6. Spaces of linearly full harmonic maps of degree less than 6

In previous sections we have shown that there are no linearly full harmonic maps of $\mathbb{R}P^2$ to S^4 of even degree. Also it is clear (due to Theorem 2) that there are no linearly full maps of degree less than 3. Let us describe spaces of harmonic maps of degree 3, 4 and 5, and compare the results with similar ones for maps from S^2 due to Bolton and Woodward [2], Kotani [16], and Loo [18].

Theorem 6. *Every linearly full horizontal holomorphic curve $\psi \in AHHol_3^{LF}(\mathbb{C}P^3)$ can be reduced by the action of appropriate elements $g \in PSp(2)$ and $\omega \in PSU(2)$ to the canonical form*

$$\Psi^3(z) = g\psi\omega(z) = [1, -z^3, \sqrt{3}z, \sqrt{3}z^2].$$

Proof. Using Lemma 4 and Remark 1 we get the coefficient matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a & 0 & 0 \\ 0 & 0 & -\bar{a} & 0 \end{pmatrix}.$$

Then, acting on this curve with

$$\omega(z) = -z,\ g = diag\{1, 1, \exp(-i\mathrm{arg(a)}), \exp(i\mathrm{arg(a)})\}, \tag{10}$$

we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 \end{pmatrix},$$

with $a \in \mathbb{R}$. Checking horizontality condition (3) we conclude $a^2 = 3$. Thus we have two cases

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}$$

which can be interchanged by $g = diag\{1, 1, -1, -1\}$. This completes the proof. $\square$

Corollary 1. *Spaces $AHHol_3^{LF}(\mathbb{C}P^3)$ and $AHarm_3^{\pm}(S^4)$ are path-connected.*

Proof. follows from Theorem 6, Lemma 3 and path-connectedness and continuity of actions of $PSp(2)$ and $PSU(2)$. $\square$

Thus, the spaces of linearly full harmonic maps of $\mathbb{R}P^2$ to S^4 of area 6π in the induced metric have two path-connected components $AHarm_3^+(S^4)$ and $AHarm_3^-(S^4)$, which are homeomorphic through the antipodal involution of S^4. It is worth mentioning that all induced metrics by these maps are canonical since $PSp(2)$ and $PSU(2)$ act by isometries on the respective Riemannian manifolds.

We have the similar results for harmonic two-spheres.

Theorem 7. *Every linearly full horizontal holomorphic curve $\psi \in HHol_3^{LF}(\mathbb{C}P^3)$ can be reduced by the action of appropriate elements $g \in PSp(2, \mathbb{C})$ and $\omega \in PSL(2, \mathbb{C})$ to the canonical form*

$$\tilde{\Psi}^3(z) = g\psi\omega(z) = [1, -z^3, \sqrt{3}z, \sqrt{3}z^2].$$

Proof. Using Lemma 4 we get coefficient matrix of the form

$$\begin{pmatrix} 1 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & r & p & q \\ 0 & 0 & s & t \end{pmatrix}. \tag{11}$$

Next, we use Lemma 5 several times. After the each time, we rename all non-zero coefficients as they were before. So, our sequence of transformation matrices from Lemma 5 is following

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & \frac{t}{d} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d^{-1} & 0 \\ 0 & -t & 0 & d \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & -\frac{q}{d} \\ 0 & 1 & 0 & 0 \\ 0 & -q & d & 0 \\ 0 & 0 & 0 & d^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s & -p \\ 0 & 0 & 0 & s^{-1} \end{pmatrix}, \\ &\begin{pmatrix} d & -c & 0 & 0 \\ 0 & d^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{-d} & 0 & 0 & 0 \\ 0 & \sqrt{-d}^{-1} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{s}{r}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{r}{s}} \end{pmatrix}. \end{aligned} \tag{12}$$

This sequence of transformations gives us the following sequence of coefficient matrices

$$\begin{pmatrix} 1 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & r & p & 0 \\ 0 & 0 & s & t \end{pmatrix} \to \begin{pmatrix} 1 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & r & p & 0 \\ 0 & 0 & s & 0 \end{pmatrix} \to \begin{pmatrix} 1 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & r & 0 & 0 \\ 0 & 0 & s & 0 \end{pmatrix} \to$$
$$\to \begin{pmatrix} 1 & a & b & 0 \\ 0 & 0 & 0 & d \\ 0 & r & 0 & 0 \\ 0 & 0 & s & 0 \end{pmatrix} \to \begin{pmatrix} 1 & a & b & 0 \\ 0 & 0 & 0 & -1 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \end{pmatrix}. \tag{13}$$

Now we use the horizontality condition (3) and conclude that $a = b = 0$, $\frac{s^2}{3} = 1$. We have $s = \pm\sqrt{3}$ and these $\pm$ cases can be interchanged by $g = diag\{1, 1, -1, -1\}$. This completes the proof. $\square$

Corollary 2. *Spaces $HHol_3^{LF}(\mathbb{C}P^3)$ and $Harm_3^{\pm}(S^4)$ are path-connected.*

Proof. follows from Theorem 7, Lemma 3 and path-connectedness and continuity of actions of $PSp(2, \mathbb{C})$ and $PSL(2, \mathbb{C})$. $\square$

Here we see that the canonical form $\tilde{\Psi}^3(z)$ for S^2 is equal to the canonical form $\Psi^3(z)$ for $\mathbb{R}P^2$.

We continue with the case of harmonic maps of degree 4. We already know that there are no such maps of the projective plane. As for the maps of two-sphere we conclude from (5) that such maps can have either 1 or 2 higher singularities. We will see later that the case of one higher singularity is impossible.

Theorem 8. *Every linearly full horizontal holomorphic curve $\psi \in HHol_4^{LF}(\mathbb{C}P^3)$ can be reduced by the action of appropriate elements $g \in PSp(2, \mathbb{C})$ and $\omega \in PSL(2, \mathbb{C})$ to the canonical form*

$$\tilde{\Psi}_a^4(z) = g\psi\omega(z) = [1 + az, -z^4, \sqrt{2}z(1 + \frac{3}{2}az), \sqrt{2}z^3],\ a \in \mathbb{R}.$$

Proof. We move one of the higher singularities to zero using an appropriate element $\omega \in PSU(2)$. Next we use Lemma 4 and get the coefficient matrix of form

$$\begin{pmatrix} 1 & a & b & e & c \\ 0 & 0 & 0 & f & d \\ 0 & r & m & p & q \\ 0 & 0 & l & s & t \end{pmatrix}. \tag{14}$$

We conclude that $rl = 0$ because of (4) and higher singularity at zero. It is clear that without loss of generality we may assume that $l = 0$. Horizontality condition (3) gives us $f = -\frac{rl}{3} = 0$. So, in fact, our coefficient matrix is of the form

$$\begin{pmatrix} 1 & a & b & e & c \\ 0 & 0 & 0 & 0 & d \\ 0 & r & m & p & q \\ 0 & 0 & 0 & s & t \end{pmatrix}. \tag{15}$$

Next we use sequence of transformations (12) and get the coefficient matrix

$$\begin{pmatrix} 1 & a & b & e & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & s & m & 0 & 0 \\ 0 & 0 & 0 & s & 0 \end{pmatrix}. \tag{16}$$

For this matrix horizontality condition (3) implies $b = e = 0$, $3a = ms$, $s^2 = 2$. So $s = \pm\sqrt{2}$ and these two cases can be interchanged by $g = diag\{1, 1, -1, -1\}$. We choose $s = \sqrt{2}$, then $m = \frac{3}{\sqrt{2}}a$. Now let $x = \arg(a)$, then using multiplication by e^{i2x} and actions of $g = \{e^{-i2x}, e^{i2x}, e^{-ix}, e^{ix}\}$, $\omega(z) = e^{-ix}z$ we get $a \in \mathbb{R}$. This completes the proof. □

Remark 2. In this theorem we see that the parametric family $\tilde{\Psi}^4_a(z)$ is continuously dependent on a, therefore we can deform each of its elements to the form

$$\tilde{\Psi}^4_0(z) = [1, -z^4, \sqrt{2}z, \sqrt{2}z^3]. \tag{17}$$

Equation (4) describing the higher singularities for these curves is $z(az+2) = 0$, therefore the second singularity is at the point $z_0 = -\frac{2}{a}$. Note that the second singularity for $\tilde{\Psi}^4_0(z)$ is at infinity. On the other hand, if we want to move the second singularity to zero, we must move a to infinity, but then degree of the curve is decreasing to 1. So we conclude that such curves cannot have one higher singularity. Therefore, in fact, we can deform every curve of degree 4 to $\tilde{\Psi}^4_0(z)$ by some element $g \in PSp(2,\mathbb{C})$ and Moebius transformation $\omega \in PSL(2,\mathbb{C})$ which moves higher singularities to the points 0 and ∞. In fact, we have here a bubbling phenomenon. An a-parametric family of curves degenerates to a map of degree 1 and the bubble grows as "a" tends to ∞.

This remark immediately leads us to the following.

Corollary 3. *Spaces $HHol^{LF}_4(\mathbb{C}P^3)$ and $Harm^{\pm}_4(S^4)$ are path-connected.*

Proof. follows from Theorem 8, Lemma 3, Remark 2 and path-connectedness and continuity of actions of $PSp(2,\mathbb{C})$ and $PSL(2,\mathbb{C})$. □

Note that canonical forms for cases of degrees 3 and 4 coincide with the following theorem from [2].

Theorem 9 [2]. *Let $\psi : S^2 \to \mathbb{C}P^3$ be a linearly full horizontal holomorphic curve of degree d which has at most two higher singularities. Then there exist elements $g \in PSp(2,\mathbb{C})$ and $\omega \in PSL(2,\mathbb{C})$ such that $\Psi(z) = g\psi\mu(z) = [1, k_2 z^{2k_1+k_2}, -(2k_1 + k_2)z^{k_1}, z^{k_1+k_2}]$, for some positive integers k_1, k_2 with $2k_1 + k_2 = d$. The higher singularities of $\Psi(z)$, if any, occur at points 0 and ∞.*

The last harmonic degree we consider in this paper is 5.

Theorem 10. *Every linearly full horizontal holomorphic curve* $\psi \in AHHol_5^{LF}(\mathbb{C}P^3)$ *can be reduced by the action of appropriate elements* $g \in PSp(2)$ *and* $\omega \in PSU(2)$ *to the canonical form*

$$\Psi_\eta^5(z) = g\psi\omega(z) = [1 + qz, -qz^4 + z^5, \mu z + \eta z^2, \eta z^3 - \mu z^4],$$

where $q = \frac{\eta}{\sqrt{3}}\sqrt{\frac{\eta^2+5}{\eta^2+4}}$, $\mu = \frac{2}{\sqrt{3}}\sqrt{\frac{\eta^2+5}{\eta^2+4}}$, $\eta \in [0, +\infty)$.

Proof. First, we can conclude from (5) that such curves have four higher singularities. Certainly, they are two pairs of antipodal points. We choose one of these points and rotate it to 0 by an appropriate element of $PSU(2)$. Next, using Lemma 4 and Remark 1 we obtain the coefficient matrix

$$\begin{pmatrix} 1 & q & p & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{p} & -\bar{q} & 1 \\ 0 & \mu & \eta & \nu & 0 & 0 \\ 0 & 0 & -\bar{\nu} & \bar{\eta} & -\bar{\mu} & 0 \end{pmatrix}.$$

Using horizontality condition (3), we get a system of equations

$$\begin{cases} -3p + \nu\bar{\mu} = 0, \\ 2q - p\bar{q} - \eta\bar{\mu} = 0, \\ -3q\bar{q} + p\bar{p} + 5 - 3\mu\bar{\mu} + \eta\bar{\eta} + \nu\bar{\nu} = 0. \end{cases} \tag{18}$$

Using (4) and taking into account (18), we get the equation of higher singulari ties

$$pz^4 + 2qz^3 + (5 - q\bar{q} - \mu\bar{\mu})z^2 - 2\bar{q}z + \frac{\mu\bar{\nu}}{3} = 0. \tag{19}$$

Since we have higher singularity at 0, we get $\mu\bar{\nu} = 0$. Then the first equation of (6) implies follows $p = 0$. Let us suppose that $\mu = 0$, then the second equation of (18) implies $q = 0$, but this contradicts last equation of (18). Thus, we have $\nu = 0$.

The next step is transforming non-zero coefficients into the reals. Let $q = \tilde{q}e^{ix}$, then using multiplication by $\exp i\frac{5}{2}x$ and actions of $\omega(z) = \exp(-ix)z$, $g = diag\{\exp -\frac{5}{2}x, \exp\frac{5}{2}x, 1, 1\}$ we get $q \in \mathbb{R}$. Further, we use the action of $g = diag\{1, 1, e^{-i\mathrm{Arg}(\mu)}, e^{i\mathrm{Arg}(\mu)}\}$ and get $\mu \in \mathbb{R}$. We conclude from the second equation of (18) that also $\eta \in \mathbb{R}$. So, we have the coefficient matrix

$$\begin{pmatrix} 1 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q & 1 \\ 0 & \mu & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta & -\mu & 0 \end{pmatrix}$$

and equations

$$\begin{cases} 2q - \eta\mu = 0, \\ -3q^2 + 5 - 3\mu^2 + \eta^2 = 0, \end{cases} \tag{20}$$

with real coefficients. The solutions of these equations are

$$\mu = \pm\frac{2}{\sqrt{3}}\sqrt{\frac{\eta^2+5}{\eta^2+4}}, \; q = \pm\frac{\eta}{\sqrt{3}}\sqrt{\frac{\eta^2+5}{\eta^2+4}}, \; \eta \in \mathbb{R}.$$

This pair of families of the solutions can be interchanged by multiplication on $-i$ and actions of $\omega(z)=-z$, $g=diag\{i,-i,i,-i\}$. The last step is to notice that multiplication by $-i$ and actions of $\omega(z)=-z$, $g=diag\{i,-i,-i,i\}$ send η, q, μ to $-\eta, -q, \mu$, respectively. This completes the proof. $\square$

Corollary 4. *The spaces $AHHol_5^{LF}(\mathbb{C}P^3)$ and $AHarm_5^{\pm}(S^4)$ are path -connected.*

Proof. We can see that the canonical form $\Psi_\eta^5(z)$ depends continuously on η. Thus, there is a path connecting any canonical form $\Psi_\eta^5(z)$ with $\Psi_0^5(z)$. Then our statement follows from Theorem 10, Lemma 3 and path-connectedness and continuity of actions of $PSp(2)$ and $PSU(2)$. $\square$

Hence the space of linearly full harmonic maps of $\mathbb{R}P^2$ to S^4 of area 10π in the induced metric has two path-connected components $AHarm_5^+(S^4)$ and $AHarm_5^-(S^4)$, which are homeomorphic through the antipodal involution of S^4.

Remark 3. Under the conditions of Theorem 10 the coefficient matrix of the canonical form for $\eta=0$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\sqrt{\frac{5}{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{3}} & 0 \end{pmatrix},$$

and herewith four umbilic points with $r_1(p)=1$ are glued in two umbilic points with $r_1(0)=2$, $r_1(\infty)=2$.

On the other hand if we put $\eta \to +\infty$ then we get the following coefficient matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 \end{pmatrix}.$$

This matrix gives us a map from $\mathbb{C}P^1 = S^2$ to $\mathbb{C}P^3$ which is not defined in $z=0$, and coincides with the canonical curve of degree 3 from Theorem 6 at all other points. It means that we can define this limit map by continuity at the point $z=0$, and this map will be the same as the canonical curve of degree 3 with coefficient matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}.$$

It looks like we have here an annihilation of pairs of umbilic points at 0 and ∞. We look closer on this further.

It is also worth mentioning here that η parameterizes the angle between diameters connecting pairs of antipodal singularities on S^2, where $\eta = 0$ corresponds to the angle equal to 0, and $\eta = +\infty$ to the angle equal to π, and as we seen before this two cases are different. We see that curves in Theorem 10 with different values of η belong to different orbits of the action of $PSp(2)$ and $PSU(2)$ since the action of $PSp(2)$ does not change higher singularities and $PSU(2)$ does not change the angle between any pair of diameters.

One can ask "What about the area? Why it decreases discontinuously?" The answer to this question is that we have a bubbling phenomenon. We will find the weak limit of the sequence of conformal factors of metrics induced on $\mathbb{R}P^2$ to see it. In fact, the bubble occur right at the converging point of higher singularities. For this reason we need a more appropriate parameterization in which the pair of higher singularities is real, symmetric with respect to zero and converges to zero. Same calculations as in the proof of Theorem 10 give us following 1-parametric family

$$\Psi^5_m(z) = [1 + pz^2, pz^3 + z^5, mz + nz^3, -nz^2 - mz^4],$$

with $n = 3\sqrt{\frac{3m^2-5}{m^2+9}}$ and $p = \frac{nm}{3} = m\sqrt{\frac{3m^2-5}{m^2+9}}$, $|m| \geq \sqrt{\frac{5}{3}}$.

Let us look at the limit when m tends to $+\infty$. In this parameterization

$$\lim_{m\to+\infty} \Psi^5_m(z) = [z, z^2, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}z^3]$$

is of the third degree (in the sense of 3).

It is well known fact that there is just one conformal class of metrics on S^2 and on $\mathbb{R}P^2$. This fact allows us to compute a conformal factor $h(z)$ of each induced metric with respect to the canonical metric on S^2 of area 4π ($g_{ind} = h(z)g_{can}$). Let us apply both induced and canonical metrics to the reification X of the vector field $\sqrt{2}\frac{\partial}{\partial z}$ and find

$$h(z) = \frac{g_{ind}(X, X)}{g_{can}(X, X)}. \tag{21}$$

The canonical metric on S^2 in terms of the isothermal coordinate $z = x + iy$ is $g_{can} = 4\frac{dx^2+dy^2}{(1+|z|^2)^2}$. Hence, $g_{can}(X, X) = \frac{4}{(1+|z|^2)^2}$. We endowed $\mathbb{C}P^3$ with the Fubini-Study metric

$$g_{FS} = 4\frac{|\mathbf{Z}|^2|d\mathbf{Z}|^2 - (\bar{\mathbf{Z}}, d\mathbf{Z})(\mathbf{Z}, d\bar{\mathbf{Z}})}{|\mathbf{Z}|^4} = 4\frac{Z_\alpha \bar{Z^\alpha} dZ_\beta d\bar{Z^\beta} - \bar{Z^\alpha} Z_\beta dZ_\alpha d\bar{Z^\beta}}{(Z_\alpha \bar{Z^\alpha})^2}$$

which has a reification $\tilde{g}_{FS} = Re\, g_{FS}$. It is easy to check that

$$\tilde{g}_{FS}(Re\,\Xi, Re\,\Xi) = \frac{1}{2}g_{FS}(\Xi, \Xi) \tag{22}$$

for any holomorphic vector field Ξ. It suffices now to compute $\tilde{g}_{FS}((\Psi^5_m)_*X, (\Psi^5_m)_*X) = g_{ind;m}(X, X)$ for finding the denominator of (21).

Indeed, twistor fibration is a Riemannian submersion and Ψ^5_m is orthogonal to the fibers as it mentioned before. We have

$$\tilde{g}_{FS}((\Psi^5_m)_*X, (\Psi^5_m)_*X) = 4\frac{|\Psi^5_m|^2|\frac{\partial}{\partial z}\Psi^5_m|^2 - |(\overline{\Psi^5_m}, \frac{\partial}{\partial z}\Psi^5_m)|^2}{|\Psi^5_m|^4} =$$

$$= 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\ln|\Psi^5_m|^2 = \triangle \ln|\Psi^5_m|^2$$

according to (22). We do further calculations in the polar coordinates $z = r(\cos\varphi + i\sin\varphi)$, $\triangle = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$. Let $h_m(z)$ be the conformal factor for the metric induced by Ψ^5_m.

Proposition 3. *Conformal factors $h_m(z)$ converge as $m \to +\infty$ to the $4\pi\delta(x, y)$ $+ 3$ in the weak sense ($\delta(x, y)$ is the Dirac delta).*

Proof. First, we find out that $|\Psi^5_m|^2 = (1 + r^2)^3 w_m(r, \varphi)$ with

$$w_m(r, \varphi) = (2pr^2\cos 2\varphi + r^4 + (m^2 - 3)r^2 + 1).$$

Then, clearly,

$$\triangle \ln|\Psi^5_m|^2 = \triangle \ln w_m(r, \varphi) + 3\triangle \ln(1 + r^2) = \triangle \ln w_m(r, \varphi) + \frac{12}{(1 + r^2)^2}$$

and

$$h_m(z) = \frac{(1 + r^2)^2}{4}\triangle \ln w_m(r, \varphi) + 3.$$

Thus, we have to show that $\tilde{h}_m(r, \varphi) = \frac{(1+r^2)^2}{4}\triangle \ln w_m(r, \varphi)$ converges weakly to $4\pi\delta(x, y)$. Since $\int_0^{2\pi} d\varphi \int_0^1 h_m(z)\frac{4rdr}{(1+r^2)^2} = 10\pi$ is an area of $\mathbb{R}P^2$ it sufficient to show that

$$\lim_{m\to+\infty}\int\limits_0^{2\pi} d\varphi \int\limits_{\frac{1}{m}}^1 \tilde{h}_m(r, \varphi)\frac{4rdr}{(1 + r^2)^2} = \lim_{m\to+\infty}\int\limits_0^{2\pi} d\varphi \int\limits_{\frac{1}{m}}^1 r\triangle \ln w_m(r, \varphi)dr = 0.$$

First, we note that $\int_0^{2\pi} \frac{\partial^2}{\partial\varphi^2}\ln w_m(r, \varphi)d\varphi = \frac{\partial}{\partial\varphi}\ln w_m(r, \varphi)|_0^{2\pi} = 0$. Next,

$$\begin{aligned}
\tilde{w}_m(r, \varphi) &= r\frac{\partial}{\partial r}\ln w_m(r, \varphi) = 2 + 2\frac{r^4 - 1}{w_m(r, \varphi)},\\
\tilde{w}_m(0, \varphi) &= 0,\\
\tilde{w}_m(1, \varphi) - \tilde{w}_m\left(\frac{1}{m}, \varphi\right) &= \frac{1}{p}\left(1 - \frac{1}{m^2}\right)\frac{1}{\cos 2\varphi + \frac{1}{2pm^2} + \frac{m^2-2}{2p}} < \frac{2}{m^2 - 2p - 2},
\end{aligned}$$

and thus

$$\int_0^{2\pi} d\varphi \int_{\frac{1}{m}}^{1} r\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\ln w_m(r,\varphi)dr = \int_0^{2\pi} d\varphi\left(r\frac{\partial}{\partial r}\ln w_m(r,\varphi)\right)|_{\frac{1}{m}}^{1} =$$

$$= \int_0^{2\pi} d\varphi\left(\tilde{w}_m(1,\varphi) - \tilde{w}_m\left(\frac{1}{m},\varphi\right)\right) < \int_0^{2\pi} \frac{2}{m^2-2p-2}d\varphi \xrightarrow[m\to+\infty]{} 0.$$

$\square$

This means that we have here the bubbling phenomenon, i.e. the “bubble” of area 4π is growing at the point 0 whilst the metric on $\mathbb{R}P^2$ converges to canonical metric of area 6π. The word “bubble” means S^2 with canonical metric. And actually, we observe that the area is preserved. We see here that “bubble” appear right at the point 0 where we defined the limit map by continuity in 3. In fact, pairs of higher singularities can annihilate in any point (for example, we can just send 0 to other point by the action of $PSU(2)$). This means that there are other copies of $AHHol_3^{LF}(\mathbb{C}P^3)$ in the closure of $AHHol_5^{LF}(\mathbb{C}P^3)$ in the space $P((\mathbb{C}[z]_5)^4)$ and they differ by the point where “bubble” appears, i.e. regular parts of conformal factors of induced metrics are the same but singular parts are different.

Let us now look at the closure of $AHHol_d^{LF}(\mathbb{C}P^3)$ in the space $P((\mathbb{C}[z]_d)^4)$. We call the relative complement of $AHHol_d^{LF}(\mathbb{C}P^3)$ in the closure of $AHHol_d^{LF}(\mathbb{C}P^3)$ in $P((\mathbb{C}[z]_d)^4)$ the *boundary* of $AHHol_d^{LF}(\mathbb{C}P^3)$ in the space $P((\mathbb{C}[z]_d)^4)$. We have the following lemma for the boundary of $AHHol_d^{LF}(\mathbb{C}P^3)$.

Lemma 6. *A point in the boundary of $AHHol_d^{LF}(\mathbb{C}P^3)$ could be identified with a linearly full invariant horizontal holomorphic curve of degree less than d.*

Proof. Recall that $AHHol_d^{LF}(\mathbb{C}P^3)$ is the subspace in the space $P((\mathbb{C}[z]_d)^4)$ defined by the number of conditions. Let us figure out which of these conditions are held by the closure points of $AHHol_d^{LF}(\mathbb{C}P^3)$. Horizontality and invariance conditions are held since they are defined by algebraic equalities. It is also easy to show that if a limiting curve (a point of the closure) are not linearly full then all the coefficients of such curve must vanish. Hence, the closure points represent linearly full maps. We see that there are two conditions which could not be preserved by closure points, - the condition of degree, and the condition of coprime polynomials. For polynomials that are not coprime we can define the required curve by dividing all these polynomials by greatest common divisor, which is the same as to define the limit map by continuity at the points where greatest common divisor vanishes. $\square$

Remark 4. It is now fair to assume that “bubbles” occur at the zeros of greatest common divisor, i.e. we have here the same phenomenon as in Proposition 3.

Now we need to define the space $AHHol^{LF}_{\leq 5}(\mathbb{C}P^3)$ of linearly full invariant horizontal holomorphic curves of degree less or equal to 5. We define it as the closure of $AHHol^{LF}_5(\mathbb{C}P^3)$ in the space $P((\mathbb{C}[z]_5)^4)$.

Proposition 4. *The boundary of $AHHol^{LF}_5(\mathbb{C}P^3)$ in the space $P((\mathbb{C}[z]_5)^4)$ is the orbit of the point with $\eta = +\infty$ in* Remark 3 *under the action of $PSp(2)$ and $PSU(2)$.*

Proof. Let us denote by $L \subset AHHol^{LF}_5(\mathbb{C}P^3)$ the subspace of canonical forms from Theorem 10, i.e. the subspace consisting of points $\Psi^5_\eta \in AHHol^{LF}_5(\mathbb{C}P^3)$, with $\eta \in [0, +\infty)$. Then the only limit point of L in $P((\mathbb{C}[z]_5)^4)$ which does not lie in L is that with $\eta = +\infty$ in Remark 3. The groups $PSp(2)$ and $PSU(2)$ act on the larger space $P((\mathbb{C}[z]_5)^4) \supset AHHol^{LF}_5(\mathbb{C}P^3)$ and this action is continuous, in the sense that the action map

$$\mathscr{A} : PSp(2) \times P((\mathbb{C}[z]_5)^4) \times PSU(2) \to P((\mathbb{C}[z]_5)^4),\ \mathscr{A} : (g, \psi, \omega) \mapsto g\psi\omega$$

is continuous. Let us look at a boundary point ψ. It is the limit of the sequence $\{\psi_n\} \subset AHHol^{LF}_5(\mathbb{C}P^3)$. By the Theorem 10 we have sequences $\{(g_n, \omega_n)\} \subset PSp(2) \times PSU(2)$ and $\{\Psi^5_{\eta_n}\} \subset L \subset AHHol^{LF}_5(\mathbb{C}P^3)$, such that $\psi_n = g_n\Psi^5_{\eta_n}\omega_n$. Since $PSp(2) \times P((\mathbb{C}[z]_5)^4) \times PSU(2)$ is compact and metrizable, we have a subsequence $\{(g_k, \Psi^5_{\eta_k}, \omega_k)\}$, which converges to a point $(g, \Psi^5_\infty, \omega)$, with $\Psi^5_\infty = \lim_{k\to+\infty} \Psi^5_{\eta_k}$. Note that the subsequence $\{\psi_k = g_k\Psi^5_{\eta_k}\omega_k\}$ converges to the point ψ. Since $\mathscr{A}$ is continuous, we have $\mathscr{A}(g, \Psi^5_\infty, \omega) = g\psi\omega$. Clearly, the point $\Psi^5_\infty \in P((\mathbb{C}[z]_5)^4)$ does not lie in L, but it is a limit point of L. This completes the proof of the inclusion *boundary* $\subset$ *orbit*. The proof of the inverse inclusion is trivial. □

Theorem 11. *The space $AHHol^{LF}_{\leq 5}(\mathbb{C}P^3)$ is path-connected.*

Proof. follows immediately from the definition of $AHHol^{LF}_{\leq 5}(\mathbb{C}P^3)$, Corollary 4, Proposition 4, and path-connectedness of the closure of $L \subset P((\mathbb{C}[z]_5)^4)$. □

As for the harmonic maps of the two-sphere we start the description with the simple case when ψ has at most two higher singularities.

Theorem 12. *Let $\psi : S^2 \to \mathbb{C}P^3$ be a linearly full horizontal holomorphic curve of degree 5 which has at most two higher singularities. Then there exist elements $g \in PSp(2, \mathbb{C})$ and $\omega \in PSL(2, \mathbb{C})$ reducing ψ to the one of the following form*

$$\tilde{\Psi}^5_1(z) = \left[1, z^5, 2z^2, -\frac{5}{2}z^3\right], \tag{23}$$

$$\tilde{\Psi}^5_2(z) = \left[1, z^5, z, -\frac{5}{3}z^4\right]. \tag{24}$$

Proof. From Theorem 9 we get two possible cases $\Psi(z) = [1, z^5, -5z^2, z^3]$ and $\Psi(z) = [1, 3z^5, -5z, z^4]$ which can be reduced to (23) and (24) by elements $g_1 = diag\{1, 1, -\frac{2}{5}, -\frac{5}{2}\}$ and $g_2 = diag\{\sqrt{3}, \frac{1}{\sqrt{3}}, -\frac{\sqrt{3}}{5}, -\frac{5}{\sqrt{3}}\}$ respectively. □

It is worth mentioning that, in fact, both of forms from this theorem have two higher singularities.

So, we can consider the case of at least two higher singularities. In this most general case we have the following result.

Theorem 13 [2]. *Let $\psi : S^2 \to \mathbb{C}P^3$ be a linearly full horizontal holomorphic curve of degree 5. Then there exist elements $g \in PSp(2, \mathbb{C})$ and $\omega \in PSL(2, \mathbb{C})$ such that*

$$\tilde{\Psi}^5(z) = g\psi\mu(z) = [1 + az, (h + z)z^4, (r + lz)z, (m + sz)z^3] \tag{25}$$

canonical linearly full horizontal holomorphic curve, where a, h, r, l, m, s satisfy

$$2a + ls = 0,\ 5 + 3ah + lm + 3rs = 0,\ 2h + rm = 0. \tag{26}$$

Remark 5. Degree of curve from parametric family $\{(25), (26)\}$ decreases iff $ah = 1$, $rs = lm = -2$, $hl = r$ or, equivalently, $ah = 1$, $rs = lm = -2$, $ar = l$. In this case degree of the curve is 4 and we have a bubbling phenomenon here. Two of higher singularities are at points 0 and ∞, and another two are meeting at point -1 and are annihilating with the birth of bubble of area 4π.

Lemma 7. *Every canonical form $\tilde{\Psi}^5(z)$ from* Theorem 13 *can be deformed to canonical form $\tilde{\Psi}^5_0(z) = [1 + z, (2 + z)z^4, (1 + 2z)z, -(4 + z)z^3]$.*

Proof. The first step is in showing that there exists deformation of $\tilde{\Psi}^5_1(z)$ to $\tilde{\Psi}^5_2(z)$ from Theorem 12. We will do it by deformation both of $\tilde{\Psi}^5_1(z)$ and $\tilde{\Psi}^5_2(z)$ to $\tilde{\Psi}^5_0(z)$. These deformations are given by

$$\tilde{\Psi}^5_1(z, t) = \left[1 + tz, (2t + z)z^4, (t + 2z)z, \left(\frac{-5 - 3t^2}{2} + tz\right) z^3\right], t \in [0, 1] \tag{27}$$

and

$$\tilde{\Psi}^5_2(z) = \left[1 + tz, (2t + z)z^4, (1 + 2tz)z, \left(-4t + \frac{2t^2 - 5}{3} z\right) z^3\right], t \in [0, 1] \tag{28}$$

respectively.

The second step is in deforming every curve $\tilde{\Psi}^5(z)$ to one of the two curves $\tilde{\Psi}^5_1(z)$ and $\tilde{\Psi}^5_2(z)$. Here we have three cases

1. $rs = 0$, $lm = -5$ and the deformation to $\tilde{\Psi}^5_1(z)$ is given by

$$\begin{aligned}\tilde{\Psi}^5_1(z, t) = [&1 + a(1 - t)z, (h(1 - t) + z)z^4,\\ &(r(1 - t) + lz)z, (m + s(1 - t)z)z^3], t \in [0, 1]\end{aligned} \tag{29}$$

and then applying $g_1 = diag\{1, 1, \frac{2}{l}, -\frac{5}{2m}\}$.

2. $lm = 0$, $rs = -\frac{5}{3}$ and the deformation to $\tilde{\Psi}_2^5(z)$ is given by

$$\begin{aligned}\tilde{\Psi}_2^5(z,t) = [&1 + a(1-t)z, (h(1-t)+z)z^4,\\ &(r + l(1-t)z)z, (m(1-t)+sz)z^3],\ t \in [0,1]\end{aligned} \tag{30}$$

   and then applying $g_2 = diag\{1, 1, \frac{1}{r}, -\frac{5}{3s}\}$.

3. $r, l, m, s \neq 0$ and the deformation to $\tilde{\Psi}_2^5(z)$ is given by

$$\begin{aligned}\tilde{\Psi}_2^5(z,t) = [&1 + a(1-t)z, (h(1-t)+z)z^4, (r + l(1-t)z)z,\\ &(m(1-t) - \frac{5 + (3ah + lm)(1-t)^2}{3r}z)z^3],\ t \in [0,1]\end{aligned} \tag{31}$$

   and then applying $g_2 = diag\{1, 1, \frac{1}{r}, r\}$.

It is clear from Remark 5 that the degree is preserved throughout all of these deformations. This completes the proof. $\square$

Corollary 5. *Spaces $HHol_5^{LF}(\mathbb{C}P^3)$ and $Harm_5^{\pm}(S^4)$ are path-connected.*

Proof. follows from Theorem 13, Lemmas 3, 7 and path-connectedness and continuity of actions of $PSp(2,\mathbb{C})$ and $PSL(2,\mathbb{C})$. $\square$

For the closures of spaces $HHol_d^{LF}(\mathbb{C}P^3)$ in $P((\mathbb{C}[z]_d)^4)$ we can see that the limit points may represent not linearly full maps. As an example, one can just take any $\psi = [f_1, f_2, f_3, f_4] \in HHol_d^{LF}(\mathbb{C}P^3)$ and construct 1-parametric family $\psi_t = [f_1, tf_2, f_3, tf_4] \in HHol_d^{LF}(\mathbb{C}P^3)$ for all positive t, which obviously converges to not linearly full map as $t \to 0$.

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