On homeomorphisms of three-dimensional manifolds with pseudo-Anosov attractors and repellers

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Abstract. The present paper is devoted to a study of orientation-preserving homeomorphisms on three-dimensional manifolds with a non-wandering set consisting of a finite number of surface attractors and repellers. The main results of the paper relate to a class of homeomorphisms for which the restriction of the map to a connected component of the non-wandering set is topologically conjugate to an orientation-preserving pseudo-Anosov homeomorphism. The ambient Ω-conjugacy of a homeomorphism from the class with a locally direct product of a pseudo-Anosov homeomorphism and a rough transformation of the circle is proved. In addition, we prove that the centralizer of a pseudo-Anosov homeomorphisms consists of only pseudo-Anosov and periodic maps. **Keywords:** pseudo-Anosov homeomorphism, two-dimensional attractor.

1 Introduction

In [3, 6] the dynamics of three-dimensional A-diffeomorphisms was studied under the assumption that their non-wandering set consists of surface two-dimensional basic sets. It is proved that diffeomorphisms of this class are ambiently Ω-conjugate to locally direct products of an Anosov diffeomorphism of a two-dimensional torus and a rough transformation of a circle. This work is a generalization of these results to a wider class $\mathcal{G}$ of maps, which we define as follows.

The set $\mathcal{G}$ consists of orientation-preserving homeomorphisms f of a closed orientable topological 3-manifold M^3 with the non-wandering set $NW(f)$ consisting of a finite number of connected components $B_0, \dots, B_{m-1}$ satisfying for any $i \in \{0, \dots, m-1\}$ the following conditions:

1. B_i is a cylindrical[1] embedding of a closed orientable surface of genus greater than 1;

[1]A subspace X of a topological space Y is called *a cylindrical embedding into Y of a topological space* $\bar{X}$ if there is a homeomorphism onto the image $h : \bar{X} \times [-1, 1] \to Y$ such that $X = h(\bar{X} \times \{0\})$.

2. there is a natural number k_i such that $f^{k_i}(B_i) = B_i$, $f^{\tilde{k}_i}(B_i) \neq B_i$ for any natural number $\tilde{k}_i < k_i$ and the restriction of the map $f^{k_i}|_{B_i}$ is topologically conjugate to an orientation-preserving pseudo-Anosov homeomorphism;

3. B_i is either an attractor[2] or a repeller for the homeomorphism f^{k_i}.

The simplest representatives of the class $\mathcal{G}$ are homeomorphisms of the set Φ which are constructed as follows.

Represent the circle as a subset of the complex plane $\mathbb{S}^1 = \{e^{i2\pi\theta} | 0 \leq \theta < 1\}$ and define a covering $p\colon \mathbb{R} \to \mathbb{S}^1$ so that $p(r) = s$, where $s = e^{i2\pi r}$.

Consider sets of numbers n, k, l such that $n, k \in \mathbb{N}$, $l \in \mathbb{Z}$, where $l = 0$ if $k = 1$, and $l \in \{1, \ldots, k-1\}$ is coprime to k if $k > 1$. For each set n, k, l we define a diffeomorphism $\bar{\varphi}_{n,k,l} : \mathbb{R} \to \mathbb{R}$ by the formula

$$\bar{\varphi}_{n,k,l}(r) = r + \frac{1}{4\pi nk} sin(2\pi nkr) + \frac{l}{k}.$$

Since $\bar{\varphi}_{n,k,l}(r) + 1 = \bar{\varphi}_{n,k,l}(r+1)$, it follows that the diffeomorphism $\bar{\varphi}_{n,k,l}$ is the lift of the circle map $\varphi_{n,k,l}(s) = p(\bar{\varphi}_{n,k,l}(p^{-1}(s)))$, where $p^{-1}(s)$ is the preimage of the point $s \in \mathbb{S}^1$ (see Statement 2.8).

Denote by S_g a closed orientable surface of genus $g > 1$ and by $Z(P)$ the centralizer $Z(P) = \{J\colon S_g \to S_g | \, PJ = JP\}$ of a homeomorphism $P\colon S_g \to S_g$.

Let us denote by $\mathcal{P}$ the set of all pseudo-Anosov homeomorphisms on the surface S_g.

Theorem 1. *A homeomorphism $J \in Z(P)$, where $P \in \mathcal{P}$, is either pseudo-Anosov or periodic*[3].

Consider orientation-preserving homeomorphisms $P \in \mathcal{P}$ and $J \in Z(P)$ such that the map $J^l P^k$ is a pseudo-Anosov homeomorphism. Let us represent the manifold M_J as the quotient space of the manifold $S_g \times \mathbb{R}$ by the action of the group $\Gamma = \{\gamma^i, i \in \mathbb{Z}\}$ of degrees of homeomorphism $\gamma\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$, given by the formula $\gamma(z, r) = (J(z), r-1)$, with natural projection $p_J\colon S_g \times \mathbb{R} \to M_J$.

[2]An invariant set B of a homeomorphism f is called an *attractor* if there is a closed neighborhood U of the set B such that $f(U) \subset int\,;U$, $\bigcap\limits_{j\geq 0} f^j(U) = B$. The attractor for the homeomorphism f^{-1} is called the *repeller* of the homeomorphism f.

[3]A homeomorphism f is called periodic if there exists $m \in \mathbb{N}$ such that $f^m = id$.

Define the map $\bar{\varphi}_{P,J,n,k,l}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ by the formula

$$\bar{\varphi}_{P,J,n,k,l}(z,r) = (P(z), \bar{\varphi}_{n,k,l}(r)).$$

It is readily verified that $\bar{\varphi}_{P,J,n,k,l}\gamma = \gamma\bar{\varphi}_{P,J,n,k,l}$. Then the orientation-preserving homeomorphism $\varphi_{P,J,n,k,l} : M_J \to M_J$ is correctly defined (see Statement 2.8) and given by the formula

$$\varphi_{P,J,n,k,l}(w) = p_{_J}(\bar{\varphi}_{P,J,n,k,l}(p_{_J}^{-1}(w))),$$

where $w \in M_J$ and $p_J^{-1}(w)$ is the preimage of the point $w \in M_J$. We call homeomorphisms of the form $\varphi_{P,J,n,k,l}$ *model maps*. Denote by Φ the set of all model maps.

Theorem 2. *Any homeomorphism from the class Φ belongs to the class $\mathcal{G}$.*

Theorem 3. *Any homeomorphism from the class $\mathcal{G}$ is ambiently Ω-conjugate*[4] *to a homeomorphism from the class Φ.*

2 Main definitions and auxiliary statements

2.1 Pseudo-Anosov homeomorphisms

Let M^n be a topological manifold of dimension n.

Family $\mathcal{F} = \{L_\alpha; \alpha \in A\}$ of path-connected subsets in M^n is called a *k-dimensional foliation* if it satisfies the following three conditions:

- $L_\alpha \cap L_\beta = \emptyset$ for any $\alpha, \beta \in A$ such that $\alpha \neq \beta$;
- $\bigcup\limits_{\alpha \in A} L_\alpha = M^n$;
- for any point $p \in M^n$ there is a local map (U, φ), $p \in U$, so that if $U \cap L_\alpha \neq \emptyset$, $\alpha \in A$, then the path-connected components of the set $\varphi(U \cap L_\alpha)$ have

[4] Recall that homeomorphisms $f_1\colon X \to X$ and $f_2\colon Y \to Y$ of topological manifolds X and Y are called *ambiently Ω-conjugated* if there is a homeomorphism $h\colon X \to Y$ such that $h(NW(f_1)) = NW(f_2)$ and $hf_1|_{NW(f_1)} = f_2h|_{NW(f_1)}$.

the form $\{(x_1, x_2, \ldots, x_n) \in \varphi(U);\ x_{k+1} = c_{k+1}, x_{k+2} = c_{k+2}, \ldots, x_n = c_n\}$, where the numbers $c_{k+1}, c_{k+2}, \ldots, c_n$ are constant on the linearly connected components.

A foliation $\mathcal{F}$ with a set of singularities S of M^n is a family of path-connected subsets of M^n such that the family of sets $\mathcal{F} \setminus S$ is a foliation of $M^n \setminus F$.

Let $q \in \mathbb{N}$. *The foliation W_q on $\mathbb{C}$ with the standard saddle singularity at the point O and q separatrices* is a family of path-connected subsets in $\mathbb{C}$ such that $W_q \setminus O$ is a foliation on $\mathbb{C} \setminus O$ and $Im\ z^{\frac{q}{2}} = const$ on leaves of $W_q \setminus O$. Rays $l_1, \ldots, l_q \in W_q$ satisfying equality $Im\ z^{\frac{q}{2}} = 0$ are called separatrices of the point O.

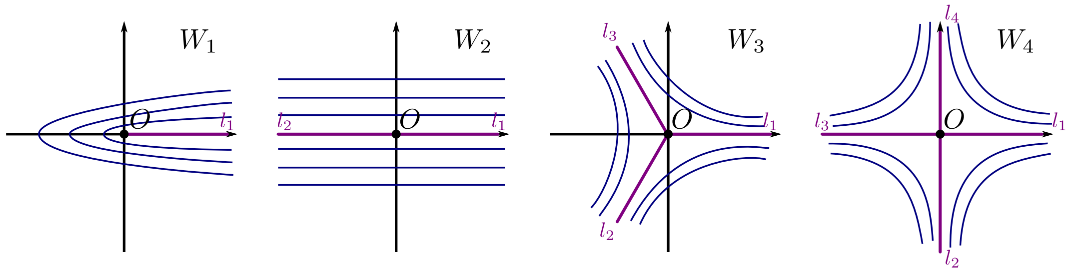

Figure 1: The foliation W_q on $\mathbb{C}$ with the standard saddle singularity at the point O and q separatrices for $q = 1, 2, 3, 4$.

A one-dimensional foliation $\mathcal{F}$ on M^2 is called a *foliation with saddle singularities* if the set S of singularities of the foliation $\mathcal{F}$ consists of a finite number of points $s_1, \ldots, s_c$ and for any point s_i $(i \in \{1, \ldots, c\})$ there is a neighborhood $U_i \subset M^2$, a homeomorphism $\psi_i \colon U_i \to \mathbb{C}$ and a number $q_i \in \mathbb{N}$ such that $\psi_i(s_i) = O$ and $\psi_i(\mathcal{F} \cap U_i) = W_{q_i} \setminus \{O\}$. The leaf containing the curve $\psi_i^{-1}(l_j)$, $j \in \{1, \ldots, q_i\}$, is called the separatrix of the point s_i. The point s_i is called *a saddle singularity with q_i separatrices.*

The transversal measure μ for a foliation $\mathcal{F}$ with saddle singularities on M^2 associates with each arc α transversal to $\mathcal{F}$ a non-negative Borel measure $\mu|_\alpha$ with the following properties:

1. if β is a subarc of the arc α, then $\mu|_\beta$ is a restriction of the measure $\mu|_\alpha$;

2. if α_0 and α_1 are two arcs transversal to $\mathcal{F}$ and connected by a homotopy $\alpha\colon [0,1]\times[0,1] \to M^2$ such that $\alpha([0,1]\times\{0\}) = \alpha_0$, $\alpha([0,1]\times\{1\}) = \alpha_1$ and $\alpha(\{t\}\times[0,1])$ for any $t \in [0,1]$ is contained in a leaf of $\mathcal{F}$ (see Fig. 2), then $\mu|_{\alpha_0} = \mu|_{\alpha_1}$.

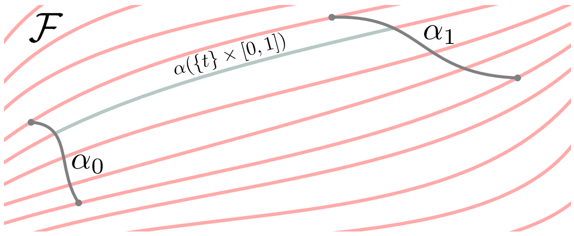

Figure 2: Curves α_0 and α_1 are connected by homotopy α.

An orientation-preserving homeomorphism $P\colon S_g \to S_g$ of a closed orientable surface of genus $g > 1$ is called a *pseudo-Anosov map (pA-homeomorphism)* with *dilatation* $\lambda > 1$ if on surface S_g there is a pair of P-invariant transversal foliations $\mathcal{F}^s_P$, $\mathcal{F}^u_P$ with a set of saddle singularities S and transversal measures μ_s, μ_u such that :

- each saddle singularity from S has at least three separatrices;
- $\mu_s(P(\alpha)) = \lambda\mu_s(\alpha)$ ($\mu_u(P(\alpha)) = \lambda^{-1}\mu_u(\alpha)$) for any arc α transversal to $\mathcal{F}^s_P$ ($\mathcal{F}^u_P$).

Let $P\colon S_g \to S_g$ be a pseudo-Anosov homeomorphism. Define the stable (unstable) manifold $W^s(x) = \{y \in M^3 : d(P^n(x), P^n(y)) \to 0, n \to +\infty\}$ ($W^u(x) = \{y \in M^3 : d(P^n(x), P^n(y)) \to 0, n \to -\infty\}$) of $x \in S_g$, where d is a metric on S_g. Note that the stable (unstable) manifold of the point $x \notin S$ is a leaf of the foliation $\mathcal{F}^s_P$ ($\mathcal{F}^u_P$) and a stable (unstable) manifold of the point $x \in S$ is the union of a finite number of separatrices belonging to the foliation $\mathcal{F}^s_P$ ($\mathcal{F}^u_P$) and the point x.

A rectangle is a subset $\Pi \subset S_g$ that is the image of a continuous map v of the square $[0,1]\times[0,1]$ into S_g with the following properties: v is one-to-one on the interior of the square and maps segments of its horizontal partition into arcs of leaves $\mathcal{F}^s_P$, and segments of its vertical partition into arcs of leaves $\mathcal{F}^u_P$. Denote by

$\dot{\Pi}$ the image of the interior of the square. We will call the images of the horizontal and vertical sides contracting and stretching sides of the rectangle Π.

A Markov partition for a pseudo-Anosov homeomorphism P is a finite family of rectangles $\tilde{\Pi} = \{\Pi_1, \dots, \Pi_n\}$ for which the following conditions are satisfied:

- $\bigcup\limits_i \Pi_i = S_g$; $\dot{\Pi}_i \cap \dot{\Pi}_j = \emptyset$ for $i \neq j$;
- let $\partial^s\tilde{\Pi}$ ($\partial^u\tilde{\Pi}$) be the union of all contracting (stretching) sides of rectangles $\Pi_1, \dots, \Pi_n$, then $P(\partial^s\tilde{\Pi}) \subset \partial^s\tilde{\Pi}$; $P(\partial^u\tilde{\Pi}) \supset \partial^u\tilde{\Pi}$.

Statement 2.1 ([1], Proposition 10.17)**.** *A pseudo-Anosov homeomorphism has a Markov partition.*

A foliation $\mathcal{F}$ is called *uniquely ergodic* if there exists a single $\mathcal{F}$-invariant measure up to multiplication by a scalar.

Statement 2.2 ([1], Theorem 12.1)**.** *The foliations $\mathcal{F}_P^s$ and $\mathcal{F}_P^u$ of the pseudo-Anosov homeomorphism P are uniquely ergodic.*

Statement 2.3 ([1], Theorem 12.5)**.** *Two homotopic pseudo-Anosov diffeomorphisms are conjugate by a diffeomorphism isotopic to the identity.*

Statement 2.4 ([8], Lemma 3.1)**.** *A homeomorphism that is topologically conjugate to a pseudo-Anosov homeomorphism is also pseudo-Anosov.*

Statement 2.5 ([8], Theorem 3.2)**.** *The set of periodic points of a pseudo-Anosov homeomorphism is dense everywhere on the surface.*

Statement 2.6 ([8], Note 3.6)**.** *Every leaf of foliations $\mathcal{F}_P^s$ and $\mathcal{F}_P^u$ of the pseudo-Anosov homeomorphism P is everywhere dense on S_g.*

2.2 Group action on a topological space

Let us recall some facts related to the action of a group on a topological space (for more details, see [4]).

For a continuous mapping $h\colon X \to Y$ of a topological space X into a topological space Y, denote by $h^{-1}(V)$ the preimage of the set $V \subset Y$, that is, $h^{-1}(V) = \{x \in X | h(x) \in V\}$.

Let the action of a group G be free and discontinuous on a Hausdorff space X and let the orbits space X/G be connected. The definition of the projection $p_{X/G}\colon X \to X/G$ implies that $p_{X/G}^{-1}(x)$ is an orbit of some point $\bar{x} \in p_{X/G}^{-1}(x)$. Let c be a path in X/G for which $c(0) = c(1) = x$. The monodromy theorem implies that there is the unique path $\bar{c}$ in X starting from $\bar{x}$ ($\bar{c}(0) = \bar{x}$) which is a lift of the path$c$. Therefore, there is an element $g \in G$ for which $\bar{c}(1) = g(\bar{x})$. Hence, the map $\eta_{X/G,\bar{x}}\colon \pi_1(X/G, x) \to G$ defined by $\eta_{X/G,\bar{x}}([c]) = g$ is well defined, i.e. it is independent of the choice of the path in the class $[c]$.

Statement 2.7 ([4], Statement 10.32)**.** *The map $\eta_{X/G,\bar{x}}\colon \pi_1(X/G, x) \to G$ is a nontrivial homomorphism. It is called the homomorphism induced by the cover $p_{X/G}\colon X \to X/G$.*

Let G be an abelian group and let $\bar{c}'$ be the lift of a path $c \in \pi_1(X/G, x)$ starting from a point $\bar{x}' = \bar{c}'(0)$ distinct from the point $\bar{x}$ and let $g'(\bar{x}') = \bar{c}'(1)$. Since there is the unique element $g'' \in G$ for which $g''(\bar{x}) = \bar{x}'$ the monodromy theorem implies $g''(\bar{c}) = \bar{c}'$. Then $g''g = g'g''$ and, therefore, $g' = g$. Thus $\eta_{X/G,\bar{x}} = \eta_{X/G,\bar{x}'}$ and from now on we omit the index $\bar{x}$ in the notation of the epimorphism $\eta_{X/G,\bar{x}}$ and we write $\eta_{X/G}$ if G is an abelian group.

Statement 2.8 ([4], Statement 10.35)**.** *Let cyclic groups G, G' act freely and discontinuously on G, G'- space X and let g, g' be their respective generators. Then*

1. *if $\bar{h}\colon X \to X$ is a homeomorphism for which $\bar{h}(g(\bar{x})) = g'(\bar{h}(\bar{x}))$ for every $\bar{x} \in X$ then the map $h\colon X/G \to X/G'$ defined by $h = p_{X/G'}(\bar{h}(p_{X/G}^{-1}(x)))$ is a homeomorphism and $\eta_{X/G} = \eta_{X/G'}h_*$;*
2. *if $h\colon X/G \to X/G'$ is a homeomorphism for which $\eta_{X/G} = \eta_{X/G'}h_*$ then there is the unique homeomorphism $\bar{h}\colon X \to X$ which is a lift of h and such that $\bar{h}(g(\bar{x})) = g'(\bar{h}(\bar{x}))$, $\bar{h}(\bar{x}) = \bar{x}'$ for $\bar{x} \in X$ and $\bar{x}' \in p_{X/G'}^{-1}(x')$, where $x' = h(p_{X/G}(\bar{x}))$.*

3 On the centralizer of a pseudo-anosov map

In this section we prove that a homeomorphism $J \in Z(P)$, where $P \in \mathcal{P}$, is either a pseudo-Anosov homeomorphism or a periodic homeomorphism.

Proof. Let $P \in \mathcal{P}$ and $J \in Z(P)$. Since $P = JPJ^{-1}$, it follows that J maps stable manifolds of P into stable ones, and unstable ones into unstable ones. Therefore, $J(\mathcal{F}^s_P) = \mathcal{F}^s_P$ and $J(\mathcal{F}^u_P) = \mathcal{F}^u_P$. The foliations $\mathcal{F}^s_P$, $\mathcal{F}^u_P$ have transversal measures μ_s, μ_u. Let us define for the foliation $\mathcal{F}^s_P$ ($\mathcal{F}^u_P$) a transversal measure $\tilde{\mu}_s(\alpha_s) = \mu_s(J(\alpha_s))$ ($\tilde{\mu}_u(\alpha_u) = \mu_u(J(\alpha_u))$), where α_s (α_u) is the arc transversal to the foliation $\mathcal{F}^s_P$ ($\mathcal{F}^u_P$). Since foliations $\mathcal{F}^s_P$, $\mathcal{F}^u_P$ are uniquely ergodic (Proposition 2.3), there exist numbers $\nu_s, \nu_u \in \mathbb{R}_+$ such that $\tilde{\mu}_s = \nu_s\mu_s$ and $\tilde{\mu}_u = \nu_u\mu_u$. Thus, $\mu_s(J(\alpha_s)) = \nu_s\mu_s(\alpha_s)$, $\mu_u(J(\alpha_u)) = \nu_u\mu_u(\alpha_u)$ for arc α_s transversal to $\mathcal{F}^s_P$ and the arc α_u transversal to $\mathcal{F}^u_P$.

Since the pseudo-Anosov homeomorphism P has a Markov partition (see Statement 2.1) consisting of n rectangles $\Pi_1, \dots, \Pi_n$, it follows that on each rectangle Π_i ($i \in \{1, \dots, n\}$) the measure $\mu_s \otimes \mu_u$ is defined by the formula $\mu_s \otimes \mu_u(\Pi_i) = \mu_s(\alpha_{s,i})\mu_u(\alpha_{u,i}) = \mu_i$, where $\alpha_{s,i}$ is the stretching side of the rectangle Π_i and $\alpha_{u,i}$ is the contracting side. Since the foliations $\mathcal{F}^s_P$, $\mathcal{F}^u_P$ are invariant under J, it follows that the set $J(\Pi_i)$ ($i \in \{1, \dots, n\}$) is also a rectangle with measure $\mu_s \otimes \mu_u(J(\Pi_i)) = \mu_s(J(\alpha_{s,i}))\mu_u(J(\alpha_{u,i})) = \nu_s\nu_u\mu_i$. Thus, $\mu_s \otimes \mu_u(S_g) = \mu_s \otimes \mu_u(\bigcup\limits_i \Pi_i) = \bigcup\limits_i \mu_i$ and $\mu_s \otimes \mu_u(J(S_g)) = \mu_s \otimes \mu_u(\bigcup\limits_i(J(\Pi_i))) = \nu_s\nu_u(\bigcup\limits_i \mu_i)$. Since $J(S_g) = S_g$, it follows that $\nu_s\nu_u = 1$. Let $\nu = \nu_s$.

Consider the case $\nu \neq 1$. The homeomorphism J has a pair of invariant transversal foliations $\mathcal{F}^s_P$, $\mathcal{F}^u_P$ with a common set of saddle singularities having at least three separatrices, and transversal measures μ_s, μ_u such that that $\mu_s(J(\alpha)) = \nu\mu_s(\alpha)$ ($\mu_u(J(\alpha)) = \nu^{-1}\mu_u(\alpha)$) for any arc α transversal to $\mathcal{F}^s_P$ ($\mathcal{F}^u_P$). Consequently, for $\nu > 1$ ($\nu < 1$) the homeomorphism J is a pseudo-Anosov map with dilatation $\nu > 1$ ($\frac{1}{\nu} > 1$).

Consider the case $\nu = 1$. Since the foliation $\mathcal{F}^s_P$ is invariant under J, it follows that separatrices of saddle singularities under the action of J are mapped into separatrices of saddle singularities. Since the set of separatrices is finite, there exists $m \in \mathbb{N}$ such that $J^m(s_i) = s_i$ and $J^m(l) = l$ for some separatrix l of the saddle singularity s_i of the foliation $\mathcal{F}^s_P$.

Let us prove that $J^m(x) = x$ for any point $x \in l$. Let $[s_i, x]$ be the arc of the curve l bounded by points s_i and x. Since $\mu_u(J^m[s_i, x]) = \mu_u([s_i, x])$, it follows that $J^m([s_i, x]) = [s_i, x]$. Therefore, $J^m(x) = x$.

Since the leaf l is dense everywhere on S_g (see Statement 2.6) and $J^m|_l = id$, it follows that $J^m(z) = z$ for any $z \in S_g$.

Consequently, the map J is a periodic homeomorphism for $\nu = 1$ and is pseudo-Anosov for $\nu \neq 1$. □

4 On the model maps

In this section we prove Theorem 2 and auxiliary lemmas.

Recall that a map $f_2 \colon Y \to Y$ of a topological space Y is called a *factor* of a map $f_1 \colon X \to X$ of a topological space X if there is a surjective continuous map $h \colon X \to Y$ such that $hf_1 = f_2 h$. The map h is called *semiconjugacy.*

Lemma 4.1. *Let $f_1 \colon X \to X$, $f_2 \colon Y \to Y$ be homeomorphisms of topological spaces X and Y such that f_2 is a factor of f_1 with semiconjugacy $h \colon X \to Y$. Then:*

1. *$h(NW(f_1)) \subset NW(f_2)$;*
2. *if $f_2^k(V_y) = V_y$ for some $k \in \mathbb{N}$, $V_y \subset Y$, then $f_1^k(V_x) \subset V_x$;*
3. *if $f_1^k(V_x) = V_x$ for some $k \in \mathbb{N}$, $V_x \subset X$, then $f_2^k(V_y) = V_y$, where $V_y = h(V_x)$.*

Proof. Let $f_1 \colon X \to X$, $f_2 \colon Y \to Y$ be homeomorphisms of topological spaces X and Y such that f_2 is a factor of f_1 with semiconjugacy $h \colon X \to Y$, that is, $hf_1 = f_2 h$. Let us prove each point of the lemma separately.

1. Consider the point $x \in NW(f_1)$ and the point $y = h(x)$ with an arbitrary open neighborhood U_y. Let $U_x = h^{-1}(U_y)$. Since h is a continuous map, the inverse image U_x of the open set U_y is also open. Then, by the definition of a non-wandering point x, there exists $n \in \mathbb{N}$ such that $f_1^n(U_x) \cap U_x \neq \emptyset$. Let $f_1^n(U_x) \cap U_x = \hat{U}_x$ and $\hat{U}_y = h(\hat{U}_x)$. Since $\hat{U}_x \subset U_x$, then $h(\hat{U}_x) \subset h(U_x)$,

that is, $\hat{U}_y \subset U_y$. Note that $hf_1^n = f_2^n h$. Since $\hat{U}_y \subset h(f_1^n(U_x))$, then $\hat{U}_y \subset f_2^n(h(U_x)) = f_2^n(U_y)$. Therefore, $f_2^n(U_y) \cap U_y \neq \emptyset$. Thus, $y = h(x) \in NW(f_2)$.

2. Let $f_2^k(V_y) = V_y$, where $k \in \mathbb{N}$, $V_y \subset Y$, $V_x = h^{-1}(V_y)$ and $f_1^k(V_x) = V_x'$. Then $f_2^k(h(V_x)) = f_2^k(V_y) = V_y$ and $h(f_1^k(V_x)) = h(V_x')$. Since $hf_1^k = f_2^k h$, it follows that $h(V_x') = V_y$. Therefore, $V_x' \subset V_x$, that is, $f_1^k(V_x) \subset V_x$.

3. Let $f_1^k(V_x) = V_x$, where $k \in \mathbb{N}$, $V_x \subset X$ and $V_y = h(V_x)$. Then $h(f_1^k(V_x)) = h(V_x) = V_y$. Since $hf_1^k = f_2^k h$, then $f_2^k(h(V_x)) = f_2^k(V_y) = V_y$. Therefore, $f^k(V_y) = V_y$.

□

We will call a set of numbers n, k, l *correct* if $n, k \in \mathbb{N}$, $l \in \mathbb{Z}$, where $l = 0$ for $k = 1$ and $l \in \{1, \dots, k-1\}$ is coprime to k for $k > 1$. Everywhere else in this section the set of numbers n, k, l is correct. Let us recall main notation and formulas.

- The manifold M_J is the quotient space of $S_g \times \mathbb{R}$ under the action of the group $\Gamma = \{\gamma^i, i \in \mathbb{Z}\}$ of degrees of homeomorphism $\gamma\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ given by the formula $\gamma(z, r) = (J(z), r-1)$, where $J\colon S_g \to S_g$ is an orientation-preserving homeomorphism;

- $p_J\colon S_g \times \mathbb{R} \to M_J$ is the natural projection inducing the homomorhisms $\eta_{M_J}\colon M_J \to \mathbb{Z}$;

- $\bar{\varphi}_{n,k,l}\colon \mathbb{R} \to \mathbb{R}$ is the diffeomorphism given by the formula

$$\bar{\varphi}_{n,k,l}(r) = r + \frac{1}{4\pi nk} sin(2\pi nkr) + \frac{l}{k}; \tag{1}$$

- $\mathbb{S}^1 = \{e^{i2\pi\theta} | 0 \leq \theta < 1\}$, $p\colon \mathbb{R} \to \mathbb{S}^1$ is the covering, given by the formula $p(r) = s$, where $s = e^{i2\pi r}$;

- $\varphi_{n,k,l}\colon \mathbb{S}^1 \to \mathbb{S}^1$is the diffeomorphism given by the formula

$$\varphi_{n,k,l}(s) = p(\bar{\varphi}_{n,k,l}(p^{-1}(s))); \tag{2}$$

- $\bar{\varphi} = \bar{\varphi}_{P,J,n,k,l}(z,r)\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ is the homeomorphism given by the formula

$$\bar{\varphi}(z,r) = (P(z), \bar{\varphi}_{n,k,l}(r)), \tag{3}$$

  where $P\colon S_g \to S_g$ is an orientation-preserving pseudo-Anosov homeomorphism such that $J \in Z(P)$;

- model homeomorphism $\varphi = \varphi_{P,J,n,k,l}\colon M_J \to M_J$ is given by the formula

$$\varphi(w) = p_{_J}(\bar{\varphi}(p_{_J}^{-1}(w))); \tag{4}$$

- Φ is a set of model homeomorphisms.

Let us introduce the following notation:

- $\mathcal{B}_i = p_{_J}(S_g \times \{\frac{i}{2nk}\}) \in M_J$ $(i \in \{0,\dots,2nk-1\})$;
- $b_i = p(\frac{i}{2nk}) \in \mathbb{S}^1$ $(i \in \{0,\dots,2nk-1\})$;
- $p_{J,r}\colon S_g \times \{r\} \to p_{_J}(S_g \times \{r\})$ is the homeomorphism given by the formula

$$p_{J,r} = p_J|_{S_g\times\{r\}},\ r \in \mathbb{R}; \tag{5}$$

- $\rho\colon S_g \times \mathbb{R} \to S_g$ is the canonical projection given by the formula

$$\rho(z,r) = z; \tag{6}$$

- $\rho_r\colon S_g \times \{r\} \to S_g$ is the homeomorphism given by the formula

$$\rho_r = \rho|_{S_g\times\{r\}},\ r \in \mathbb{R}. \tag{7}$$

Note that the Eq. (4) is obtained from the relation

$$p_{_J}\bar{\varphi} = \varphi p_{_J}, \tag{8}$$

and Eq. (2) is obtained from the relation

$$p\bar{\varphi}_{n,k,l} = \varphi_{n,k,l}p. \qquad (9)$$

Since $p_{_J}: S_g \times \mathbb{R} \to M_J$ is a natural projection, it follows that

$$p_{_J}\gamma = p_{_J}. \qquad (10)$$

Denote by $h_J\colon M_J \to \mathbb{S}^1$ the continuous surjective map given by the formula

$$h_J(w) = p(r), \text{ where } w = p_J(z,r) \in M_J. \qquad (11)$$

It is readily verified that $h_J\varphi = \varphi_{n,k,l}p_J$. Thus, the following lemma is true.

Lemma 4.2. *The homeomorphism $\varphi_{n,k,l}\colon \mathbb{S}^1 \to \mathbb{S}^1$ is the factor of the homeomorphism $\varphi\colon M_J \to M_J$ with semiconjugacy $h_J\colon M_J \to \mathbb{S}^1$.*

It is directly verified (see Eqs. (1) and (2)) that the non-wandering set of the diffeomorphism $\varphi_{n,k,l}$ consists of $2nk$ points $b_0, \dots, b_{2nk-1}$ of period k such that points with odd indices i are sinks and points with even indices are source.

Let us prove Theorem 2, that is, we prove the inclusion $\Phi \subset \mathcal{G}$.

Proof. Consider the model homeomorphism $\varphi = \varphi_{P,J,n,k,l}\colon M_J \to M_J$. Since the homeomorphism J preserves orientation, it follows that the manifold M_J is orientable. Preserving orientation of homeomorphisms P and $\varphi_{n,k,l}$ implies preserving orientation by homeomorphism φ inducing by map $\bar{\varphi}(z,r) = (P(z), \bar{\varphi}_{n,k,l}(r))$.

Let us prove that the connected component $\mathcal{B}_i$ ($i \in \{0, \dots, 2nk-1\}$) is a cylindrical embedding of the surface S_g. For $i \in \{0, \dots, 2nk-1\}$ we set $\bar{U}_i = S_g \times [\frac{i}{2nk} - \frac{i}{4nk}, \frac{i}{2nk} + \frac{i}{4nk}]$ and $U_i = p_J(\bar{U}_i)$. Since $p_{_J}: S_g \times \mathbb{R} \to M_J$ is a covering, it follows that for any $i \in \{0, \dots, 2nk-1\}$ its restriction $p_{_J}|_{\bar{U}_i}: \bar{U}_i \to U_i$ is a homeomorphism. In addition, $p_{_J}|_{\bar{U}_i}(S_g \times \{\frac{i}{2nk}\}) = \mathcal{B}_i$. Therefore, $\mathcal{B}_i$ ($i \in \{0, \dots, 2nk-1\}$) is a cylindrical embedding of S_g.

Let us prove that $\varphi^k(\mathcal{B}_i) = \mathcal{B}_i$, $\varphi^{\tilde{k}_i}(\mathcal{B}_i) \neq \mathcal{B}_i$ ($i \in \{0, \dots, 2nk-1\}$) for any natural number $\tilde{k}_i < k$. In accordance with Lemma 4.2, the map $\varphi_{n,k,l}$ is the factor of a homeomorphism φ with semiconjugacy h_J. Note that $h_J^{-1}(b_i) = \mathcal{B}_i$

$(i \in \{0, \dots, 2nk-1\})$, where $b_i \in \mathbb{S}^1$ is a point of period k. It follows from Lemma 4.1 that $\varphi^k(\mathcal{B}_i) \subset \mathcal{B}_i$. Since the map φ^k is a homeomorphism and the component $\mathcal{B}_i$ is homeomorphic to S_g, it follows that $\varphi^k(\mathcal{B}_i) = \mathcal{B}_i$. Suppose that $\varphi^{\tilde{k}}(\mathcal{B}_i) = \mathcal{B}_i$ for some natural number $\tilde{k} < k$. Then Lemma 4.1 implies that $\varphi^{\tilde{k}}_{n,k,l}(b_i) = b_i$. We come to contradiction that point b_i has period k.

Let us prove that the map $\varphi^k|_{\mathcal{B}_i}$ $(i \in \{0, \dots, 2nk-1\})$ is topologically conjugate to the orientation-preserving pseudo-Anosov homeomorphism. Since

$$\gamma^l\Big(\bar{\varphi}^k\Big(z, \frac{i}{2nk}\Big)\Big) = \Big(J^l\Big(P^k(z)\Big), \frac{i}{2nk}\Big), \tag{12}$$

it follows that

$$\rho_{\frac{i}{2nk}}\Big(\gamma^l\Big(\bar{\varphi}^k\Big(\rho^{-1}_{\frac{i}{2nk}}(z)\Big)\Big)\Big) = J^l\Big(P^k(z)\Big). \tag{13}$$

For any point $w \in \mathcal{B}_i$ we get $\varphi^k(w) \overset{(4)}{=} p_{_J}(\bar{\varphi}^k(p_{_J}^{-1}(w))) \overset{(10)}{=} p_{_J}(\gamma^l(\bar{\varphi}^k(p_{_J}^{-1}(w)))) \overset{(12)}{=} p_{J,\frac{i}{2nk}}(\gamma^l(\bar{\varphi}^k(p^{-1}_{J,\frac{i}{2nk}}(w)))) \overset{(13)}{=} p_{J,\frac{i}{2nk}}(\rho^{-1}_{\frac{i}{2nk}}(J^l(P^k(\rho_{\frac{i}{2nk}}(p^{-1}_{J,\frac{i}{2nk}}(w))))))$. Consequently, the homeomorphism $\varphi^k|_{\mathcal{B}_i}$ is topologically conjugate to the orientation-preserving pseudo-Anosov homeomorphism J^lP^k via the homeomorphism $p_{J,\frac{i}{2nk}}\rho^{-1}_{\frac{i}{2nk}}$.

Lemmas 4.1 and 4.2 imply that $NW(\varphi) \subset (\mathcal{B}_0 \cup \dots \cup \mathcal{B}_{2nk-1})$.

Since the set of periodic points of a pseudo-Anosov homeomorphism is dense everywhere on the surface (Proposition 2.5) and $\varphi^k(\mathcal{B}_i) = \mathcal{B}_i$ $(i \in \{0, \dots, 2nk-1\})$, it follows that $NW(\varphi) = \mathcal{B}_0 \cup \dots \cup \mathcal{B}_{2nk-1}$.

Let us prove that the connected components $\mathcal{B}_i$ with odd indices i belong to the set of attractors of the homeomorphism φ. Points b_i with odd indices i are sink points of the diffeomorphism $\varphi^k_{n,k,l}$. Therefore, $\varphi^k(u_i) \subset int\ u_i$ and $\bigcap\limits_{j\geq 0} \varphi^{jk}_{n,k,l}(u_i) = b_i$ for the neighborhood $u_i = h_J(U_i) = p([\frac{i}{2nk} - \frac{i}{4nk}, \frac{i}{2nk} + \frac{i}{4nk}])$ of point b_i with odd index i. Since $h_J^{-1}(p[a,b]) = p_J(S_g \times [a,b])$ for any $a, b \in \mathbb{R}$, $h_J\varphi^{jk} = \varphi^{jk}_{n,k,l}h_J$ and $h_J^{-1}(b_i) = \mathcal{B}_i$, it follows that $\varphi^k(U_i) \subset int\ U_i$, $\bigcap\limits_{j\geq 0} \varphi^{jk}(U_i) = \mathcal{B}_i$. Consequently, connected components $\mathcal{B}_i$ with odd indices i are attractors of the map φ^k.

Analogously one proves that connected components $\mathcal{B}_i$ with even indices i belong to the set of repellers.

Thus $\varphi \in \mathcal{G}$. □

5 The ambient Ω-conjugacy of a homeomorphism $f \in \mathcal{G}$ to a model map

Recall that the set Φ consists of model homeomorphisms of the form $\varphi_{P,J,n,k,l}$. This section contains a proof of Ω-conjugacy of homeomorphisms of the class $\mathcal{G}$ with homeomorphisms of the set Φ and auxiliary lemmas. We will also use the notation introduced in the Section 3 below.

Let us denote by $\mathcal{H}$ the set of all homeomorphisms f satisfying the following conditions:

1. there exists an orientation-preserving homeomorphism $J\colon S_g \to S_g$ such that $f\colon M_J \to M_J$;

2. f preserves the orientation of M_J;

3. there exists $m \in \mathbb{N}$ such that the non-wandering set $NW(f)$ of the homeomorphism f consists of $2m$ connected components $\mathcal{B}_0 \cup \dots \cup \mathcal{B}_{2m-1}$;

4. for any $i \in \{0, \dots, 2m-1\}$ there is a natural number k_i such that $f^{k_i}(\mathcal{B}_i) = \mathcal{B}_i$, $f^{\tilde{k}_i}(\mathcal{B}_i) \neq \mathcal{B}_i$ for any natural $\tilde{k}_i < k_i$ and the map $f^{k_i}|_{\mathcal{B}_i}$ preserves the orientation of $\mathcal{B}_i$;

5. $f(\mathcal{B}_i) = \mathcal{B}_j$, where the numbers $i, j \in \{0, \dots, 2m-1\}$ are either even or odd at the same time.

Note that homeomorphisms of the set Φ belong to the class $\mathcal{H}$.

For $m \in \mathbb{N}$ we denote by $\mathcal{T}_m$ the set $\mathcal{T}_m = \{\frac{i}{2m}, i \in \mathbb{Z}\}$. Then $p_J^{-1}(NW(f)) = S_g \times \mathcal{T}_m$, where $f \in \mathcal{H}$.

Lemma 5.1. *For any homeomorphism $f \in \mathcal{H}$ with non-wandering set consisting of $2m$ connected components, there exist and unique correct set of numbers n, k, l and a lift $\bar{f}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ such that*

$$\bar{f}(z, r) = \left(f_r(z), r + \frac{l}{k}\right), \ \forall r \in \mathcal{T}_{nk},$$

where $nk = m$ *and* $f_r\colon S_g \to S_g$ *is an orientation-preserving homeomorphism given by*

$$f_r = \rho_{r+\frac{l}{k}} \bar{f} \rho_r^{-1}.$$

Proof. Let $f\colon M_J \to M_J$ be a homeomorphism from the class $\mathcal{H}$.

Let us prove that there is a lift $\bar{f}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ of the homeomorphism f. By Statement 2.8 it sufficies to show that $\eta_{M_J} = \eta_{M_J} f_*$.

Consider the loop $c \in M_J$ which is the projection of the curve $\bar{c} \in S_g \times \mathbb{R}$ ($p_J(\bar{c}) = c$), bounded by points $\bar{c}(0) = (z, 1)$, $\bar{c}(1) = \gamma(\bar{c}(0)) = (J(z), 0)$ and intersecting each set $S_g \times \{\frac{i}{2m}\}$, $i \in \{0, \dots, 2m-1\}$ at exactly one point. By construction, the curve c intersects each connected component $\mathcal{B}_0, \dots, \mathcal{B}_{2m-1}$ at exactly one point and $\eta_{M_J}([c]) = 1$. We set $C = f(c)$ and $C(0) = f(c(0))$. Since f is a homeomorphism such that $f(\mathcal{B}_i) = \mathcal{B}_{i'}$, $i, i' \in \{0, \dots, 2m-1\}$, it follows that the curve $C = f(c)$ also intersects each component of $\mathcal{B}_0, \dots, \mathcal{B}_{2m-1}$ at exactly one point. We set $\mathcal{B}_j = f(\mathcal{B}_0)$. Choosing a point $\bar{C}(0) \in p_J^{-1}(C(0))$ such that $\bar{C}(0) \in S_g \times \{\frac{j}{2m} + 1\}$ by the monodromy theorem there is a unique lift $\bar{C}$ of the path C starting at the point $\bar{C}(0)$. Since the loop C intersects each component $\mathcal{B}_0, \dots, \mathcal{B}_{2m-1}$ at exactly one point, it follows that there are 2 cases: 1) $\bar{C}(1) = \gamma^{-1}(\bar{C}(0))$, 2) $\bar{C}(1) = \gamma(\bar{C}(0))$.

Let us show that the case 1) is not realized.

Consider the case $m = 1$. Then $f(\mathcal{B}_0) = \mathcal{B}_0$. Since the homeomorphism f preserves the orientation M_J and the orientation $\mathcal{B}_0$, it follows that the curve $C(t)$ must be parameterized in one direction with the parameterization of the curve $c(t)$ with respect to the surface $\mathcal{B}_0$. Thus $\bar{C}(1) = \gamma(\bar{C}(0))$.

Consider the case $m > 1$. Let us denote by $\xi_c\colon \mathbb{S}^1 \to c$, $\xi_C\colon \mathbb{S}^1 \to C$ homeomorphisms such that $\xi_c(b_i) = \mathcal{B}_i \cap c$, $\xi_C(b_i) = \mathcal{B}_i \cap C$, where $i \in \{0, \dots, 2m-1\}$. Define the homeomorphism $\psi\colon \mathbb{S}^1 \to \mathbb{S}^1$ by the formula $\psi = \xi_C^{-1} f \xi_c$. Let us prove that the homeomorphism ψ preserves orientation. Assume the converse. Let us prove that there exists $q \in \{0, \dots, 2m-1\}$ such that $\psi(b_q) = b_q$. Let $\mathcal{B}_j = f(\mathcal{B}_0)$. Then $\psi(b_0) = b_j$. If $j = 0$, then $q = 0$. Let $j \neq 0$. By the condition of the class $\mathcal{H}$, the number j is even. Since ψ by assumption changes the orientation of $\mathbb{S}^1$ and the set $b_0 \cup \dots \cup b_{2m-1}$ is invariant, it follows that the arc of the circle (b_0, b_j) is mapped into itself and $\psi(b_i) = b_{j-i}$, $i \in \{0, \dots, \frac{j}{2}\}$. Thus $\psi(b_{\frac{j}{2}}) = b_{\frac{j}{2}}$ and $q = \frac{j}{2}$.

Therefore, $f(\mathcal{B}_q) = \mathcal{B}_q$. Since ψ changes orientation, it follows that the curve $C(t)$ is parameterized in the direction opposite to the parameterization of the curve $c(t)$ with respect to the surface $\mathcal{B}_q$ (see Fig. 3). Since the homeomorphism f preserves the orientation M_J and the orientation $\mathcal{B}_q$, then the parameterization of the curve $C(t)$ must be parameterized in one direction with the parameterization of the curve $c(t)$ with respect to the surface $\mathcal{B}_q$. We got a contradiction. Consequently, the homeomorphism ψ preserves the orientation of $\mathbb{S}^1$. Then $\bar{C}(1) = \gamma(\bar{C}(0))$.

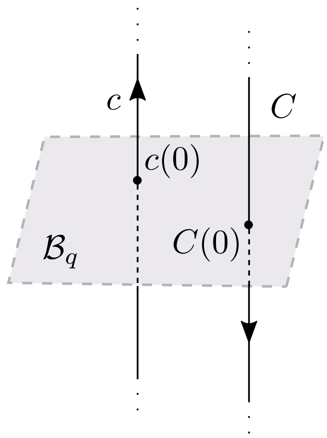

Figure 3: Direction of increasing parameter $t \in [0, 1]$ on curves c and C.

Thus $\bar{C}(1) = \gamma(\bar{C}(0))$ and $\eta_{M_J}(f_*([c])) = 1$. Consequently, $\eta_{M_J} = \eta_{M_J} f_*$ and there is a unique lift $\bar{f}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ of the homeomorphism f such that $\bar{f}(\bar{c}(1)) = \bar{C}(1)$ and

$$\bar{f}\gamma = \gamma\bar{f}. \tag{14}$$

Let us find the correct set of numbers n, k, l for the homeomorphism f. The case $m = 1$ corresponds to the correct set of numbers $n = 1$, $k = 1$ and $l = 0$. Consider the case $m > 1$. Since the homeomorphism ψ is orientation preserving, it follows that it has a rational rotation number $\frac{l}{k}$, where $k \in \mathbb{N}$, $l \in \{0, \ldots, k-1\}$ and $(l, k) = 1$ (see [7, Theorem 4.1]). From [7, Theorem 4.2] it follows that all periodic points of the homeomorphism ψ have period k. Since point b_i with even (odd) index i is mapped to point $b_{i'}$ with even (odd) index i', it follows that $2m$ points $b_0, \ldots, b_{2m-1}$ are divided into 2 invariant sets of equal power, each of which consists of points of period k. Therefore, m is divisible by k. We set $n = \frac{m}{k}$. Thus

n, k, l is the required correct set of numbers.

Since the rotation number of ψ is equal to $\frac{l}{k}$, it follows that $\psi(b_0) = b_{2nl}$, that is, $f(\mathcal{B}_0) = \mathcal{B}_{2nl}$.

Let us find a formula that defines the map $\bar{f}$ for the point $(z, r) \in S_g \times \mathcal{T}_{nk}$. Since $\bar{C}(1) = \gamma(\bar{C}(0))$, it follows that $\bar{C}(1) \in S_g \times \{\frac{2nl}{2nk}\} = S_g \times \{\frac{l}{k}\}$. Invariance of the set $p_J^{-1}(NW(f)) = S_g \times \mathcal{T}_{nk}$ under $\bar{f}$ implies that $\bar{f}(S_g \times [0,1]) = S_g \times [\frac{l}{k}, 1+\frac{l}{k}]$, where $\bar{f}(S_g \times \{0\}) = S_g \times \{\frac{l}{k}\}$. From here we get that $\bar{f}(S_g \times \{\frac{i}{2nk}\}) = S_g \times \{\frac{i}{2nk} + \frac{l}{k}\}$ for any $i \in \{0, \ldots, 2nk-1\}$. Using Eq. (14) we obtain that $\bar{f} = \gamma^m \bar{f} \gamma^{-m}$ for any $m \in \mathbb{Z}$. Then $\bar{f}(S_g \times \{r\}) = \gamma^{[r]}(\bar{f}(\gamma^{-[r]}(S_g \times \{r\})))$, where $[r]$ is the integer part of the number $r \in \mathbb{R}$. Thus it is readily verified that $\bar{f}(S_g \times \{r\}) = S_g \times \{r + \frac{l}{k}\}$ for $r \in \mathcal{T}_{nk}$. Then for any $r \in \mathcal{T}_{nk}$ the homeomorphism $f_r \colon S_g \to S_g$ is correctly defined and given by the formula $f_r = \rho_{r+\frac{l}{k}} \bar{f} \rho_r^{-1}$. Thus $\bar{f}(z, r) = (f_r(z), r + \frac{l}{k})$ for any $r \in \mathcal{T}_{nk}$.

It remains to prove that f_r preserves the orientation of S_g, where $r \in \mathcal{T}_{nk}$. Preserving orientation of M_J by f implies preserving orientation of $S_g \times \mathbb{R}$ by its lift $\bar{f}$. Since $\bar{f}(S_g \times \{r\} = f_r(S_g) \times \{r + \frac{l}{k}\}$ for any $r \in \mathcal{T}_{nk}$, it follows that the homeomorphism $\bar{f}$ preserves the orientation of $\mathbb{R}$. Therefore, $\bar{f}$ preserves the orientation of S_g, that is, f_r preserves the orientation of S_g. □

Note that in the case $f = \varphi_{P,J,n,k,l}$ the equality $f_r(z) = P(z)$ holds for any $r \in \mathcal{T}_{nk}$ and $\bar{f} = \bar{\varphi}_{P,J,n,k,l}$.

Lemma 5.2. *Let $f \in \mathcal{H}$. Then f_r is isotopic to f_0 for any $r \in \mathcal{T}_{nk}$.*

Proof. Let $f \in \mathcal{H}$. Let us prove that f_r is isotopic to f_0 for any $r \in \mathcal{T}_{nk}$.

Define a family of continuous maps $F_{r,t} \colon S_g \to S_g$ by the formula $F_{r,t}(z) = \rho(\bar{f}(z, rt))$, where $t \in [0,1]$, $r \in \mathcal{T}_{nk}$. Then $F_{r,t}$ defines a homotopy connecting the maps $F_{r,0} = f_0$ and $F_{r,1} = f_r$. Thus, homeomorphisms f_0 and f_r are homotopic. It follows from [9, p. 5.15] that they are isotopic for any $r \in \mathcal{T}_{nk}$. □

Lemma 5.3. *Let $f \colon M^3 \to M^3$ be a homeomorphism from the class $\mathcal{G}$. Then there exists a homeomorphism $f' \in \mathcal{H}$ is topologically conjugate to f.*

Proof. Let $f \colon M^3 \to M^3$ be a homeomorphism from the class $\mathcal{G}$ with non-wandering set consisting of q connected components $B_0, \ldots, B_{q-1}$.

In accordance with [2, Lemma 2.1], the set $M^3 \setminus (B_0 \cup \cdots \cup B_{q-1})$ consists of q connected components $V_0, \ldots, V_{q-1}$, bounded by one connected component of an attractor and one connected component of a repeller. Therefore, $q = 2m$, where $m \in \mathbb{N}$. Without loss of generality, for $m > 1$ we can assume that $cl\ V_i \cap cl\ V_{i-1} = B_{i-1}$, where $i \in \{1, \ldots, 2m-2\}$ and $cl\ V_0 \cap cl\ V_{2m-1} = B_{2m-1}$.

In accordance with [2, Lemma 2.2], each connected component V_i, $i \in \{0, \ldots, 2m-1\}$ of the set $M^3 \setminus (B_0 \cup \cdots \cup B_{2m-1})$ is homeomorphic to $S_g \times [0, 1]$. It follows from [5, Lemma 2] that there exists a continuous surjective map $H\colon S_g \times [0, 1] \to M^3$ (see Fig. 4) such that maps $H|_{S_g \times \{\frac{i}{m}\}}\colon S_g \times \{\frac{i}{m}\} \to B_i$ ($i \in \{0, \ldots, 2m-1\}$), $H|_{S_g \times \{1\}}\colon S_g \times \{1\} \to B_0$ and $H|_{S_g \times (0,1)}\colon S_g \times (0, 1) \to M^3 \setminus B_0$ are homeomorphisms.

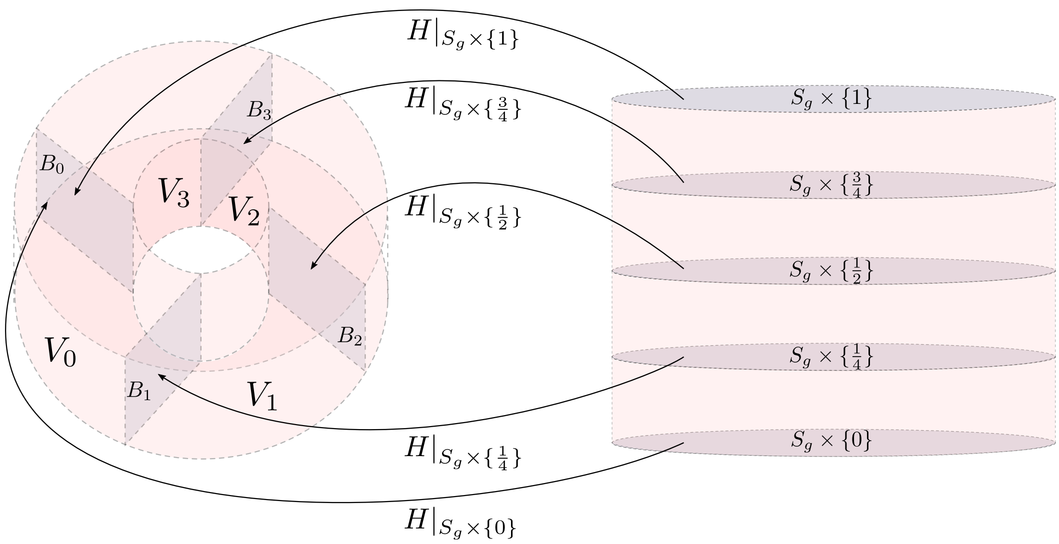

Figure 4: Action of the homeomorphism H in the case $m = 2$.

Let $J(z) = \rho_0((H|_{S_g \times \{0\}})^{-1}(H|_{S_g \times \{1\}}(\rho_1^{-1}(z))))$ (see Fig. 5).

Denote by $[r]$ the integer part of the number $r \in \mathbb{R}$. Define a continuous map $h\colon S_g \times \mathbb{R} \to M^3$ by the formula $h(z, r) = H(\gamma^{[r]}(z, r))$.

Let the homeomorphism $\xi\colon M^3 \to M_J$ be given by the formula $\xi = p_J(h^{-1}(w))$. Set $f' = \xi f \xi^{-1}$.

Let us prove that the homeomorphism f' satisfies all 5 conditions of the class $\mathcal{H}$. Since M^3 is orientable and homeomorphic to M_J, it follows that J preserves

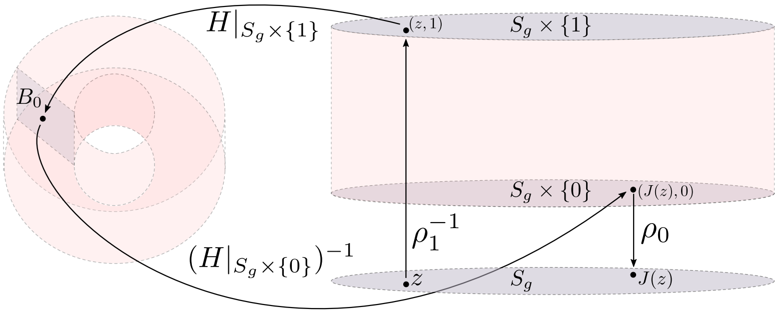

Figure 5: Homeomorphism $J\colon S_g \to S_g$.

the orientation of S_g and condition 1 is satisfied. Since f preserves the orientation of M^3, it follows that f' preserves the orientation of M_J and condition 2 is satisfied. Since $\xi(NW(f)) = NW(f')$ and $h^{-1}(NW(f)) = S_g \times \mathcal{T}_{nk}$, it follows that $NW(f') = p_J(h^{-1}(NW(f))) = p_J(S_g \times \mathcal{T}_{nk}) = \mathcal{B}_0 \cup \cdots \cup \mathcal{B}_{2m-1}$. Therefore, condition 3 is satisfied. Since for any B_i ($i \in \{0,\dots,2m-1\}$) there is a natural number k_i such that $f^{k_i}(B_i) = B_i$, $f^{\tilde{k}_i}(B_i) \neq B_i$ for any natural $\tilde{k}_i < k_i$ and the map $f^{k_i}|_{B_i}$ preserves the orientation of B_i, it follows that the same is true for the connected component $\mathcal{B}_i$ of the non-wandering set $NW(f')$, that is, condition 4 is satisfied. The connected components of the non-wandering set $NW(f)$ of the homeomorphism f are numbered in such a way that if B_i is the connected component of an attractor of the homeomorphism f, then $B_{i+1 \pmod{2m}}$ is the connected component of a repeller of the homeomorphism f. Therefore, $f(B_i) = B_j$, where $i, j \in \{0,\dots,2m-1\}$ are either even or odd at the same time. Since $\xi(B_i) = \mathcal{B}_i$ ($i \in \{0,\dots,2m-1\}$), it follows that $f'(\mathcal{B}_i) = \mathcal{B}_j$, where $i, j \in \{0,\dots,2m-1\}$ are simultaneously either even or odd, that is, condition 5 is satisfied. Thus, $f' \in \mathcal{H}$. □

Everywhere below in this section we mean by $\bar{f}$, f_r and n, k, l the lift of the homeomorphism $f \in \mathcal{H}$, the homeomorphism $f_r\colon S_g \to S_g$, $r \in \mathcal{T}_{nk}$, and the correct set of numbers n, k, l from Lemma 5.1.

Lemma 5.4. *Let $f \in \mathcal{H} \cap \mathcal{G}$. Then f_0 is isotopic either to some periodic homeomorphism or to some pseudo-Anosov homeomorphism.*

Proof. Let $f \in \mathcal{H} \cap \mathcal{G}$.

Let us prove that f_0 is isotopic either to some periodic homeomorphism or to some pseudo-Anosov homeomorphism.

Since $\bar{f}$ is a lift of a homeomorphism f, it follows that

$$p_J \bar{f} = f p_J. \tag{15}$$

Therefore,

$$f(w) = p_J(\bar{f}(p_J^{-1}(w))). \tag{16}$$

For $r \in \mathcal{T}_{nk}$ denote by $\phi_r \colon S_g \to S_g$ the homeomorphism given by the formula

$$\phi_r = J^l f_{r+\frac{(k-1)l}{k}} \cdots f_{r+\frac{l}{k}} f_r. \tag{17}$$

Then it is readily verified that

$$\gamma^l(\bar{f}^k|_{S_g \times \mathcal{T}_{nk}}(z,r)) = (\phi_r(z), r), \quad \text{where } r \in \mathcal{T}_{nk}. \tag{18}$$

Therefore,

$$\phi_r = \rho_r \gamma^l \bar{f}^k \rho_r^{-1}. \tag{19}$$

Thus, $f^k|_{\mathcal{B}_0}(w) \overset{(16)}{=} p_J(\bar{f}^k(p_J^{-1}(w))) \overset{(10)}{=} p_J(\gamma^l(\bar{f}^k(p_J^{-1}(w)))) \overset{(18)}{=} p_{J,0}(\gamma^l(\bar{f}^k(p_{J,0}^{-1}(w)))) \overset{(19)}{=}$
$p_{J,0}(\rho_0^{-1}(\phi_0(\rho_0(p_{J,0}^{-1}(w)))))$, that is,

$$f^k|_{\mathcal{B}_0} = p_{J,0}\rho_0^{-1}\phi_0\rho_0 p_{J,0}^{-1}. \tag{20}$$

Therefore, the homeomorphism ϕ_0 is topologically conjugate to the homeomorphism $f^k|_{\mathcal{B}_0}$ via the map $p_{J,0}\rho_0^{-1}$. Since the homeomorphism $f^k|_{\mathcal{B}_0}$ is topologically conjugate to the pseudo-Anosov homeomorphism, it follows that the homeomorphism ϕ_0 is also a pseudo-Anosov map (see Statement 2.4).

Eq. (14) implies that $(J(f_r(z)), r + \frac{l}{k} - 1) = (f_{r-1}(J(z)), r - 1 + \frac{l}{k})$ and

$$J f_r = f_{r-1} J \text{ for any } r \in \mathcal{T}_{nk}. \tag{21}$$

Therefore, $f_0 J^l = J^l f_l$. Then $f_0(J^l f_{\frac{(k-1)l}{k}} \cdots f_{\frac{l}{k}} f_0) = (J^l f_l f_{\frac{(k-1)l}{k}} \cdots f_{\frac{l}{k}}) f_0$, that is,

$$\phi_0 = f_0^{-1} \phi_{\frac{l}{k}} f_0. \tag{22}$$

It follows from Eq. (22) and Statement 2.4 that $\phi_{\frac{l}{k}}$ is also a pseudo-Anosov homeomorphism.

Since f_r is isotopic to f_0 for any $r \in \mathcal{T}_{nk}$ by Lemma 5.2, it follows that $J^l f_{\frac{(k-1)l}{k}} \cdots f_{\frac{l}{k}} f_0$ is isotopic to $J^l f_l f_{\frac{(k-1)l}{k}} \cdots f_{\frac{l}{k}}$, that is, ϕ_0 is isotopic to $\phi_{\frac{l}{k}}$. Then, according to Statement 2.3, there exists an isotopic to the identity homeomorphism $h \colon S_g \to S_g$ such that

$$\phi_0 = h \phi_{\frac{l}{k}} h^{-1}. \tag{23}$$

Putting Eq. (23) in Eq. (22), we obtain that $\phi_0 = f_0^{-1}(h^{-1}\phi_0 h) f_0$, that is, $(hf_0)\phi_0 = \phi_0(hf_0)$.

Since $\phi_0 \in \mathcal{P}$ and $hf_0 \in Z(\phi_0)$, it follows that the homeomorphism hf_0 is either periodic or pseudo-Anosov by Theorem 1. Isotopicity to the identity of h implies that f_0 is isotopic either to some periodic homeomorphism or to some pseudo-Anosov homeomorphism. $\square$

Lemma 5.5. *Let $f \in \mathcal{H} \cap \mathcal{G}$ and f_0 be isotopic to some periodic homeomorphism. Then there exists a homeomorphism $f' \in \mathcal{H}$ such that f' is topologically conjugate to f and f'_0 is isotopic to some pseudo-Anosov homeomorphism.*

Proof. Let $f \colon M_J \to M_J$ be a homeomorphism from the class $\mathcal{H} \cap \mathcal{G}$ with non-wandering set consisting of $2nk$ connected components of period k, and f_0 is isotopic to some periodic homeomorphism.

Let us show that $k \neq 1$. Assume the converse. Then $l = 0$ and the homeomorphism ϕ_0 has the form $\phi_0 = f_0$ (see Eq. (17)). According to Eq. (20), the homeomorphism ϕ_0 is topologically conjugate to the pseudo-Anosov homeomorphism $f^k|_{\mathcal{B}_0}$. We come to contradiction with the fact that $k = 1$. Therefore, $k > 1$.

Define the homeomorphisms $\bar{h}, \gamma' \colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ by the formulas $\bar{h}(z,r) = (z, -r)$, $\gamma'(z,r) = (J^{-1}(z), r-1)$. Recall that $\gamma(z,r) = (J(z), r-1)$. Since $(J(z), -(r-1)) = (J(z), (-r)+1)$, it follows that $\bar{h}\gamma = (\gamma')^{-1}\bar{h}$. Therefore, the homeomorphism $\bar{h}$ projects into the homeomorphism $h \colon M_J \to M_{J^{-1}}$ (see

Statement 2.8), given by the formula $h = p_{J^{-1}}(\bar{h}(p_J^{-1}(w)))$, where $p_{J^{-1}} \colon S_g \times \mathbb{R} \to M_{J^{-1}}$ is a natural projection.

Set $f' = hfh^{-1}$. Recall that for a homeomorphism $f \in \mathcal{H}$ there is a unique lift $\bar{f} \colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ such that $\bar{f}_{S_g \times \mathcal{T}_{nk}}(z, r) = (f_r(z), r + \frac{l}{k})$, where n, k, l is the correct set of numbers. Consider the lift $\bar{f}'$ of the homeomorphism f' given by the formula $\bar{f}' = \gamma^{-1}\bar{h}\bar{f}\bar{h}^{-1}$. Then for any $r \in \mathcal{T}_{nk}$ we have $\bar{f}'(z, r) = (J(f_r(z)), r + \frac{k-l}{k})$. Since $k \neq 1$, it follows that $l \in \{1, \dots, k-1\}$. Therefore, $(k-l) \in \{1, \dots, k-1\}$ and coprime to k. Thus, $n, k, (k-l)$ is the correct set of numbers and $f'_r = Jf_r$.

Let us prove that the homeomorphism f'_0 is isotopic to some pseudo-Anosov homeomorphism. By Lemma 5.4 the homeomorphism f'_0 is isotopic either to some periodic map or to some pseudo-Anosov map. Suppose that the homeomorphism $f'_0 = Jf_0$ is isotopic to a periodic homeomorphism. Then the homeomorphism $J = f'_0 f_0^{-1}$ is also isotopic to a periodic homeomorphism. Since J and f_0 are isotopic to periodic homeomorphisms and, according to Lemma 5.2, f_0 is isotopic to f_r for any $r \in \mathcal{T}_{nk}$, it follows tha the homeomorphism $\phi_0 = J^l f_{\frac{(k-1)l}{k}} \cdots f_{\frac{l}{k}} f_0$ is also isotopic to periodic homeomorphism. We come to contradiction with the fact that ϕ_0 is topologically conjugate to the pseudo-Anosov homeomorphism $f^k|_{\mathcal{B}_0}$ (see Eq. (20)). Consequently, the homeomorphism f'_0 is isotopic to the pseudo-Anosov homeomorphism. Thus, $f' \in \mathcal{H}$ is topologically conjugate to f and f'_0 is isotopic to some pseudo-Anosov homeomorphism. □

Lemma 5.6. *Let $f \in \mathcal{H} \cap \mathcal{G}$ and f_0 be isotopic to some pseudo-Anosov homeomorphism P. Then there is a homeomorphism $f' \colon M_{J'} \to M_{J'}$ from the class $\mathcal{H}$ such that f' is topologically conjugate to f, $J'P = PJ'$ and f'_0 is isotopic to P.*

Proof. Let $f \colon M_J \to M_J$ be a homeomorphism from the class $\mathcal{H} \cap \mathcal{G}$ and P be a pseudo-Anosov homeomorphism of the surface S_g, isotopic to f_0.

Let us construct a homeomorphism $J' \colon S_g \to S_g$. Set

$$P' = J^{-1}PJ. \tag{24}$$

Denote by F_t the isotopy connecting the homeomorphisms $F_0 = f_0$ and $F_1 = P$. Then the family of maps $J^{-1}F_tJ$ defines an isotopy connecting the maps $J^{-1}F_0J = J^{-1}f_0J = f_1$ and $J^{-1}F_1J = J^{-1}PJ = P'$. Since f_0 is isotopic to f_1 (see Lemma

5.2) and to P, f_1 is isotopic to P', it follows that P is isotopic to P'. Homeomorphism P is topologically conjugate to the pseudo-Anosov homeomorphism P', P is isotopic to P'. Then by Statement 2.3 there exists an isotopic to the identity homeomorphism ξ such that

$$P' = \xi P \xi^{-1}. \tag{25}$$

Set

$$J' = J\xi,\ \gamma' = (J'(z), r-1). \tag{26}$$

Note that $J'P \overset{(26)}{=} J\xi P \overset{(25)}{=} JP'\xi \overset{(24)}{=} PJ\xi \overset{(26)}{=} PJ'$.

Let us construct a homeomorphism $Y\colon M_J \to M_{J'}$. Denote by ξ_t the isotopy connecting the homeomorphism $\xi_0 = \xi$ and the identity map $\xi_1 = id$. Define the homeomorphism $y_r\colon S_g \to S_g$ by the formula

$$y_r = \begin{cases} \xi_{6nk(1-r)} & \text{for } r \in [1 - \frac{1}{6nk}, 1]; \\ id & \text{for } r \in [0.1 - \frac{1}{6nk}]. \end{cases}$$

Define the homeomorphism $y\colon S_g \times [0,1] \to S_g \times [0,1]$ by the formula $y(z,r) = (y_r(z), r)$. Note that

$$y(z,0) = (z,0) \text{ and } y\Big(z, \frac{l}{k}\Big) = \Big(z, \frac{l}{k}\Big). \tag{27}$$

Denote by $[r]$ the integer part of the number $r \in \mathbb{R}$. Define the homeomorphism $\bar{Y}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ by the formula

$$\bar{Y}(z,r) = (\gamma')^{-[r]}(y(\gamma^{[r]}(z,r))). \tag{28}$$

Since $\gamma'\bar{Y} = \bar{Y}\gamma$, it follows that the homeomorphism $\bar{Y}$ projects into the homeomorphism $Y\colon M_J \to M_{J'}$ (see Statement 2.8), given by the formula $Y = p_{J'}(\bar{Y}(p_J^{-1}(w)))$, where $p_J\colon S_g \times \mathbb{R} \to M_J$, $p_{J'}\colon S_g \times \mathbb{R} \to M_{J'}$ are natural projections.

Set $f' = YfY^{-1}\colon M_{J'} \to M_{J'}$. By construction $f' \in \mathcal{H}$. Let us prove that f'_0 is isotopic to P. Consider the lift

$$\bar{f}' = \bar{Y}\bar{f}\bar{Y}^{-1} \tag{29}$$

of the homeomorphism f. It is readily verified that $\bar{f}'(z,r)=(f'_r(z), r+\frac{l}{k})$, where $r\in\mathcal{T}_{nk}$ and f'_r is a homeomorphism of S_g. Let us show that $f'_0=f_0$. Indeed, $\bar{f}'(z,0)\overset{(29)}{=}\bar{Y}(\bar{f}(\bar{Y}^{-1}(z,0)))\overset{(28)}{=}\bar{Y}(\bar{f}(y^{-1}(z,0)))\overset{(27)}{=}\bar{Y}(\bar{f}(z,0))=\bar{Y}(f_0(z),\frac{l}{k})\overset{(28)}{=}y_{\frac{l}{k}}(f_0(z),\frac{l}{k})\overset{(27)}{=}(f_0(z),\frac{l}{k})$. Thus, f'_0 is also isotopic to P. $\square$

Let us prove that any homeomorphism from the class $\mathcal{G}$ is ambiently Ω-conjugate to a homeomorphism from the class Φ.

Proof. Let $f\in\mathcal{G}$.

According to Lemma 5.3, wihout loss of generality, we may assume that f is defined on $M_J=S_g\times\mathbb{R}/\Gamma$ with natural projection $p_J\colon S_g\times\mathbb{R}\to M_J$, where J is a orientation-preserving homeomorphism of the surface S_g and $\Gamma=\{\gamma^i | i\in\mathbb{Z}\}$ is a group of degrees of the homeomorphism $\gamma\colon S_g\times\mathbb{R}\to S_g\times\mathbb{R}$ given by the formula $\gamma(z,r)=(J(z),r-1)$. It follows from Lemma 5.1 that the non-wandering set of the homeomorphism f consists of $2nk$ connected components $\mathcal{B}_0,\dots,\mathcal{B}_{2nk-1}$ and there is a lift $\bar{f}$ of the homeomorphism f such that $\bar{f}(z,r)=(f_r(z),r+\frac{l}{k})$ for any $r\in\mathcal{T}_{nk}$, where $f_r\colon S_g\to S_g$ is an orientation preserving homeomorphism of the surface and n,k,l is the correct set of numbers.

According to Lemmas 5.2,5.4,5.5,5.6, without loss of generality we may assume that f_r is isotopic to some orientation-preserving pseudo-Anosov homeomorphism P for any $r\in\mathcal{T}_{nk}$ and $J\in Z(P)$. Since J preserves the orientation of S_g, it follows that the homeomorphism J^lP^k also preserves the orientation of S_g.

Let us prove that the homeomorphism J^lP^k is a pseudo-Anosov homeomorphism. Using Eqs. (18) and (19), we obtain

$$f^k|_{p_J(S_g\times\{r\})}=p_{J,r}\rho_r^{-1}\phi_r\rho_r p_{J,r}^{-1},\ r\in\mathcal{T}_{nk}, \tag{30}$$

that is, the homeomorphism ϕ_r ($r\in\mathcal{T}_{nk}$) is topologically conjugate to the pseudo-Anosov homeomorphism $f^k|_{p_J(S_g\times\{r\})}$. Since by Lemma 5.2 the homeomorphism f_r for any $r\in\mathcal{T}_{nk}$ is isotopic to P, it follows that the homeomorphism $\phi_r=J^lf_{r+\frac{(k-1)l}{k}}\cdots f_{r+\frac{l}{k}}f_r$ is isotopic to J^lP^k, that is, the homeomorphism J^lP^k is isotopic to the pseudo-Anosov homeomorphism. According to Theorem 1, we obtain that the homeomorphism J^lP^k is a pseudo-Anosov map.

Note that homeomorphisms J^lP^k and ϕ_r are isotopic for any $r \in \mathcal{T}_{nk}$ and are pseudo-Anosov homeomorphisms. Then, according to Statement 2.3, maps ϕ_r and J^lP^k are topologically conjugate for any $r \in T$ via some isotopic to the identity homeomorphism. Denote such a homeomorphism by h_r. Then for any $r \in \mathcal{T}_{nk}$ we obtain that

$$J^lP^k = h_r(\phi_r)h_r^{-1}. \tag{31}$$

Thus, each homeomorphism $f \in \mathcal{G}$ corresponds to the correct set of numbers n, k, l and orientation-preserving homeomorphisms $P\colon S_g \to S_g$, $J\colon S_g \to S_g$ such that the homeomorphisms P, J^lP^k are pseudo-Anosov and $J \in Z(P)$. Therefore, there is correctly defined model map $\varphi_{P,J,n,k,l} \in \Phi$.

Let us prove that the homeomorphism f is ambiently Ω-conjugate to $\varphi_{P,J,n,k,l}$. We construct a homeomorphism $f'\colon M_J \to M_J$, topologically conjugate to f and coinciding with the homeomorphism $\varphi_{P,J,n,k,l}$ on the non-wandering set ($f'|_{NW(f')} = \varphi_{P,J,n,k,l}|_{NW(\varphi_{P,J,n,k,l})}$).

We divide the construction into steps.

Step 1. Construct a homeomorphism $x\colon S_g \times U \to S_g \times U$, where $U = \bigcup\limits_{j\in\{0,\ldots,k-1\}} U_j$, $U_j = [-\frac{1}{4nk} - j\frac{l}{k}, \frac{1}{k} - \frac{1}{4nk} - j\frac{l}{k})$.

Let $T = \{0, \frac{1}{2nk}, \ldots, \frac{2n-1}{2nk}\}$. Note that $T = \mathcal{T}_{nk} \cap U_0$ and $r \in \mathcal{T}_{nk} \cap U_j$ has the form $r = i - j\frac{l}{k}$, where $j \in \{0, \ldots, k-1\}$ and the number $i \in T$ is uniquely determined. For $i \in T$ and $j \in \{0, \ldots, k-1\}$ we define the homeomorphism $\xi_{i,j}\colon S_g \to S_g$ by the formula

$$\xi_{i,j} = P^{-j}h_i \underbrace{f_{i-j\frac{l}{k}+(j-1)\frac{l}{k}} \cdots f_{i-j\frac{l}{k}}}_{j \text{ maps}}. \tag{32}$$

Since the homeomorphism $f_{i-j\frac{l}{k}+(j-1)\frac{l}{k}} \cdots f_{i-j\frac{l}{k}+\frac{l}{k}} f_{i-j\frac{l}{k}}$ is isotopic to P^j for $j \in \{1, \ldots, k-1\}$ and the homeomorphism h_i is isotopic to the identity, it follows that the homeomorphism $\xi_{i,j}$ is isotopic to the identity for any $j \in \{0, \ldots, k-1\}$. Let $\xi_{i,j,t}$ denote the isotopy connecting the homeomorphism $\xi_{i,j,0} = \xi_{i,j}$ and the identity map $\xi_{i,j,1} = id$.

For $r \in U$ we define the homeomorphism $x_r\colon S_g \to S_g$ by the formula

$$x_r = \begin{cases} \xi_{i,j,6nk|r-(i-j\frac{l}{k})|} & \text{for } |r-(i-j\frac{l}{k})| \le \frac{1}{6nk}; \\ id & \text{for others } r \in U. \end{cases}$$

Define the homeomorphism $x\colon S_g \times U \to S_g \times U$ by the formula

$$x(z,r) = (x_r(z), r).$$

Note that

$$x\Big(z, i - j\frac{l}{k}\Big) = \Big(\xi_{i,j}(z), i - j\frac{l}{k}\Big). \tag{33}$$

Step 2. Let us extend the homeomorphism $x\colon S_g \times U \to S_g \times U$ to the homeomorphism $\bar{X}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$.

Let us prove that for any point $r \in \mathbb{R}$ there is a unique integer $m \in \mathbb{Z}$ such that $(r-m) \in U$.

Divide the half-interval $[-\frac{1}{4nk}, 1-\frac{1}{4nk})$ into k half-intervals: $[-\frac{1}{4nk}, 1-\frac{1}{4nk}) = [-\frac{1}{4nk}, \frac{1}{k}-\frac{1}{4nk}) \cup [-\frac{1}{4nk}+\frac{1}{k}, \frac{2}{k}-\frac{1}{4nk}) \cup \dots \cup [-\frac{1}{4nk}+\frac{k-1}{k}, 1-\frac{1}{4nk})$. Obviously, for any $r \in \mathbb{R}$ there is a unique number $a \in \mathbb{Z}$ such that $r-a \in [-\frac{1}{4nk}, 1-\frac{1}{4nk})$. Let $r-a \in [-\frac{1}{4nk}+\frac{j}{k}, \frac{j+1}{k}-\frac{1}{4nk})$, where $j \in \{0,\dots,k-1\}$. Since j runs through the complete system of residues $\{0,1,\dots,k-1\}$ modulo k and l is coprime with k, it follows that $(-jl)$ also runs through a complete system of residues $\{0,-l,\dots,-l(k-1)\}$ modulo k [10, page 46]. Consequently, there are integers $i \in \{0,-l,\dots,-l(k-1)\}$ and b such that $j+bk = i$. Then $(r-a+b) \in [-\frac{1}{4nk}+\frac{j+bk}{k}, \frac{j+1+bk}{k}-\frac{1}{4nk}) = [-\frac{1}{4nk}+\frac{i}{k}, \frac{1}{k}+\frac{i}{k}-\frac{1}{4nk}) \subset U$. Thus, $m = a-b$ is the required integer such that $(r-m) \in U$.

Let $\varrho(r)$ denotes an integer $\varrho(r) \in \mathbb{Z}$ such that $(r-\varrho(r)) \in U$. Define the map $\bar{X}\colon S_g \times \mathbb{R} \to S_g \times \mathbb{R}$ by the formula $\bar{X}(z,r) = \gamma^{-\varrho(r)}(x(\gamma^{\varrho(r)}(z,r)))$ for $(z,r) \in S_g \times \mathbb{R}$. Then $\bar{X}\gamma = \gamma\bar{X}$.

Step 3. Construct a homeomorphism $f'\colon M_J \to M_J$.

Let us set $\bar{f}' = \bar{X}\bar{f}\bar{X}^{-1}$. Since $\bar{X}\gamma = \gamma\bar{X}$ and $\bar{f}\gamma = \gamma\bar{f}$, it follows that $\bar{f}'\gamma = \gamma\bar{f}'$ and homeomorphisms $\bar{X}$ and $\bar{f}'$ project into homeomorphisms $f'\colon M_J \to M_J$, $X\colon M_J \to M_J$ (see Statement 2.8), given by the formulas $f' = p_J(\bar{f}'(p_J^{-1}(w)))$, $X = p_J(\bar{X}(p_J^{-1}((w)))$ and $f' = XfX^{-1}$.

Let us prove that $\bar{f}'|_{S_g\times\mathcal{T}_{nk}} = \bar{\varphi}_{P,J,n,k,l}|_{S_g\times\mathcal{T}_{nk}}$. Since $\bar{X}(S_g\times\{r\}) = S_g\times\{r\}$ and $\bar{f}(S_g\times\{r\}) = S_g\times\{r+\frac{l}{k}\}$ for any $r\in\mathcal{T}_{nk}$, it follows that $\bar{f}'(S_g\times\{r\}) = \bar{X}(\bar{f}(\bar{X}^{-1}(S_g\times\{r\}))) = S_g\times\{r+\frac{l}{k}\}$. Then for any $r\in\mathcal{T}_{nk}$ the homeomorphisms $f'_r\colon S_g\to S_g$, $X_r\colon S_g\to S_g$ are correctly defined by $f'_r = \rho_{r+\frac{l}{k}}\bar{f}'\rho_r^{-1}$, $X_r = \rho_{r+\frac{l}{k}}\bar{X}\rho_r^{-1}$ and

$$f'_r = X_{r+\frac{l}{k}} f_r X_r^{-1}. \tag{34}$$

Then

$$X_r = J^{-m(r)} x_r J^{m(r)}. \tag{35}$$

By construction, $\bar{\varphi}_{P,J,n,k,l}(z,r) = (P(z), r+\frac{l}{k})$ and $\bar{f}'(z,r) = (f'_r(z), r+\frac{l}{k})$ for any $r\in\mathcal{T}_{nk}$.

Let us prove that $f'_r = P$ for any $r\in\mathcal{T}_{nk}$. Let us represent $r\in\mathcal{T}_{nk}$ in the form $r = i - j\frac{l}{k} + m$, where $i\in T$, $j\in\{0,\dots,k-1\}$ and $m\in\mathbb{Z}$.

Let $k = 1$. Then $f'_r = f'_{i+m} \overset{(34)}{=} X_{i+m} f_{i+m} X_{i+m}^{-1} \overset{(35)}{=}$
$J^{-m} x_i J^m f_{i+m} J^{-m} x_i^{-1} J^m \overset{(33)}{=}$
$J^{-m}\xi_{i,0} J^m f_{i+m} J^{-m}\xi_{i,0}^{-1} J^m \overset{(21)}{=} J^{-m}\xi_{i,0} f_i \xi_{i,0}^{-1} J^m \overset{(32)}{=} J^{-m} h_i f_i h_i^{-1} J^m \overset{(17)}{=}$
$J^{-m} h_i \phi_i h_i^{-1} J^m \overset{(31)}{=}$
$J^{-m} P J^m = P$.

Let $k>1$. We consider the cases 1) $j\ge 1$ and 2) $j=0$ separately.

1) If $j\ge 1$, then $j-1\in\{0,\dots,k-2\}$ and the homeomorphism $\xi_{i,j-1}$ is correctly defined. We obtain that $f'_r = f'_{i-j\frac{l}{k}+m} \overset{(34)}{=} X_{i-(j-1)\frac{l}{k}+m} f_{i-j\frac{l}{k}+m} X^{-1}_{i-j\frac{l}{k}+m} \overset{(35)}{=}$
$J^{-m} x_{i-(j-1)\frac{l}{k}} J^m f_{i-j\frac{l}{k}+m} J^{-m} x^{-1}_{i-j\frac{l}{k}} J^m$
$\overset{(33)}{=} J^{-m}\xi_{i,j-1} J^m f_{i-j\frac{l}{k}+m} J^{-m}\xi_{i,j}^{-1} J^m \overset{(21)}{=} J^{-m}\xi_{i,j-1} f_{i-j\frac{l}{k}} \xi_{i,j}^{-1} J^m \overset{(32)}{=}$
$J^{-m} P^{-j+1} h_i f_{i-(j-1)\frac{l}{k}+(j-2)\frac{l}{k}} \cdots f_{i-(j-1)\frac{l}{k}} f_{i-j\frac{l}{k}} f^{-1}_{i-j\frac{l}{k}} \cdots f^{-1}_{i-j\frac{l}{k}+(j-1)\frac{l}{k}} h_i^{-1} P^j J^m =$
$J^{-m} P^{-j+1} h_i h_i^{-1} P^j J^m = P$.

2) If $j=0$, then $r+\frac{l}{k} = i+\frac{l}{k}+m = i-(k-1)\frac{l}{k}+(m+l)$. We obtain that $f'_r = f'_{i+m} \overset{(34)}{=} X_{i-(k-1)\frac{l}{k}+(m+l)} f_{i+m} X_{i+m}^{-1} \overset{(35)}{=} J^{-m-l}\xi_{i,k-1} J^{m+l} f_{i+m} J^{-m}\xi_{i,0}^{-1} J^m \overset{(21)}{=}$
$J^{-m-l}\xi_{i,k-1} J^l f_i \xi_{i,0}^{-1} J^m \overset{(32)}{=}$
$J^{-m-l} P^{-k+1} h_i f_{i-(k-1)\frac{l}{k}+(k-2)\frac{l}{k}} \cdots f_{i-(k-1)\frac{l}{k}} J^l f_i h_i^{-1} J^m$
$\overset{(21)}{=} J^{-m-l} P^{-k+1} h_i J^l f_{i+(k-1)\frac{l}{k}} \cdots f_{i-\frac{l}{k}} f_i h_i^{-1} J^m \overset{(17)}{=} J^{-m-l} P^{-k+1} h_i \phi_i h_i^{-1} J^m$
$\overset{(31)}{=} J^{-m-l} P^{-k+1} J^l P^k J^m = P$.

We obtain that $\bar{f}'(p_J^{-1}(NW(f')) = \bar{\varphi}_{P,J,n,k,l}(p_J^{-1}(\varphi_{P,J,n,k,l})$.

Consequently, $f'|_{NW(f')} = \varphi_{P,J,n,k,l}|_{NW(\varphi_{P,J,n,k,l})}$ and the homeomorphism f is ambiently Ω-conjugate to the homeomorphism $\varphi_{P,J,n,k,l}$ via the map X. □

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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