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Varieties covered by affine spaces, uniformly rational varieties and their cones

I. Arzhantsev [a,*,1], S. Kaliman [b], M. Zaidenberg [c]

[a] *HSE University, Faculty of Computer Science, Pokrovsky Boulevard 11, Moscow, 109028, Russia*
[b] *Department of Mathematics, University of Miami, Coral Gables, FL 33124, USA*
[c] *Univ. Grenoble Alpes, CNRS, IF, 38000 Grenoble, France*

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ABSTRACT

It was shown in Kaliman snd Zaidenberg (2023) [26] that the affine cones over flag manifolds and rational smooth projective surfaces are elliptic in the sense of Gromov. The latter remains true after successive blowups of points on these varieties. In the present article we extend this to smooth projective spherical varieties (in particular, toric varieties) successively blown up along smooth subvarieties. The same holds, more generally, for uniformly rational projective varieties, in particular, for projective varieties covered by affine spaces. It occurs also that stably uniformly rational complete varieties are elliptic.

* Corresponding author.
E-mail addresses: arjantsev@hse.ru (I. Arzhantsev), kaliman@math.miami.edu (S. Kaliman), mikhail.zaidenberg@univ-grenoble-alpes.fr (M. Zaidenberg).

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In memory of Dmitri N. Akhiezer

1. Introduction

We work over algebraically closed field $\mathbb{K}$ of characteristic zero. All the varieties in this paper are algebraic varieties defined over $\mathbb{K}$; $\mathbb{P}^n$ and $\mathbb{A}^n$ stand for the projective resp. affine n-space over $\mathbb{K}$; $\mathbb{G}_{\mathrm{m}}$ and $\mathbb{G}_{\mathrm{a}}$ stand for the one-dimensional algebraic torus and the one-dimensional unipotent algebraic group over $\mathbb{K}$, respectively. All the notions, such as a neighborhood, a spray, etc. are considered in the algebraic category unless otherwise noted. “Ellipticity” below means “Gromov’s algebraic ellipticity”.

1.1. Gromov’s ellipticity

The notion of Gromov ellipticity appeared first in analytic geometry where it serves in order to establish the Oka-Grauert Principle in the most general form, see [21] and [16]. Besides, for an elliptic complex manifold X the following approximation property holds: every holomorphic map $f\colon K \to X$ from a neighborhood of a compact convex set $K \subset \mathbb{C}^n$ can be approximated by holomorphic maps $\mathbb{C}^n \to X$, see [21]. A manifold X with the latter property is called an *Oka manifold*, see the survey article [18]. If X is algebraic and elliptic in the algebraic sense, then f can be approximated by morphisms $\mathbb{C}^n \to X$, see [18, Corollary 6.5].

Gromov considered as well an analogous notion of ellipticity in the setup of algebraic varieties. It is known that an elliptic smooth algebraic variety X of dimension n admits a surjective morphism from $\mathbb{A}^{n+1}$ which also is smooth and surjective on an open subset of $\mathbb{A}^{n+1}$, see [31]. Assuming that X is also quasiaffine, this implies that the endomorphism monoid $\mathrm{End}(X)$ is highly transitive on X, see [26, Appendix A]. Furthermore, for $\mathbb{K} = \mathbb{C}$ the fundamental group $\pi_1(X)$ is finite, see [32].

Recall that a smooth algebraic variety X is called *elliptic* if it admits a dominating Gromov spray (E, p, S) where $p\colon E \to X$ is a vector bundle with zero section Z and $s\colon E \to X$ is a morphism such that $s|_Z = p|_Z$ and s is dominating at any point $x \in X$, that is, the restriction $s|_{E_x}$ to the fiber $E_x = p^{-1}(x)$ is dominant at the origin $0_x \in E_x$. The image $O_x = s(E_x) \subset X$ is called the *s-orbit of x*.

According to [21, 3.5.B] (see also [16, Proposition 6.4.2], [33, Remark 3] and [26, Appendix B]) if the ellipticity holds locally on an open covering of X, then it holds globally. Moreover, the ellipticity of X holds if X is subelliptic, that is, there is a dominating collection of sprays on X instead of a single spray, see [15, Definition 2.1], [16] and [25]. In other words, one can always replace a dominating collection of sprays on X with a single dominating spray.

1.2. Ellipticity of cones

Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n. The affine cone $\mathrm{cone}(X)$ in $\mathbb{A}^{N+1}$ blown up at the origin gives rise to a line bundle $F = \mathcal{O}_X(-1)$ on X whose zero section Z_F is the exceptional divisor of the blowup. The associated principal $\mathbb{G}_m$-fiber bundle $Y \to X$ with fiber $\mathbb{A}^1_* = \mathbb{A}^1 \setminus \{0\}$ is isomorphic to $F \setminus Z_F$ and so, to the punctured affine cone over X that is, the affine cone with its vertex removed:

$$Y = F \setminus Z_F \simeq_X \mathrm{cone}(X) \setminus \{0\}.$$

Our aim is to establish the ellipticity of Y provided X is elliptic, under certain additional assumptions on X. In [26] the second and the third authors suggested criteria of ellipticity of Y based on the so called *curve-orbit property* for some families of smooth rational curves and sprays on X. In particular, it was shown in [26] that the punctured affine cones over a flag variety G/P blown up in several points and infinitesimally near points are elliptic and the same holds for any rational smooth projective surface, see [26, Theorem 0.1]. In the present note we develop further the technique of [26]. This allows us to establish similar facts for uniformly rational varieties, in particular, for varieties of class $\mathcal{A}_0$, therefore, for smooth projective toric and, more generally, spherical varieties. Recall that a spherical variety is a normal G-variety which contains an open B-orbit, where G is a reductive algebraic group and B is a Borel subgroup in G. A flag variety G/P and a normal toric variety are spherical varieties. For G/P the latter follows from the Bruhat decomposition $G = BWB$, and for a toric T-variety it suffices to choose $G = B = T$.

1.3. Varieties of class $\mathcal{A}_0$

One says that a variety X belongs to class $\mathcal{A}_0$ if there is an open cover $\{A_i\}$ on X by affine cells $A_i \simeq \mathbb{A}^n$ where $n = \dim(X)$; see [15, Definition 2.3].[2] It is well known that a variety of class $\mathcal{A}_0$ is elliptic; see, e.g., [21, Sec. 3.5]. The blowup of a variety of class $\mathcal{A}_0$ in a point is again a variety of class $\mathcal{A}_0$, see [21, 3.5D]. More generally, suppose X is a variety of class $\mathcal{A}_0$ and $Z \subset X$ is a closed subvariety such that the pair $(A_i, Z \cap A_i)$ is isomorphic for any i to a pair $(\mathbb{A}^n, \mathbb{A}^k)$ with $n - k \geq 2$; in this case Z is called a *linear subvariety of* X. The blowup of a linear subvariety Z in X results again in a variety of class $\mathcal{A}_0$, see [3, Section 4, Statement 9].

1.4. Uniformly rational varieties

This class of varieties strictly contains the class $\mathcal{A}_0$, see Example 4.8.

Definition 1.1. An algebraic variety X is called *uniformly rational*[3] if for each $x \in X$ there is an open neighborhood X_0 of x in X isomorphic to an open subset of $\mathbb{A}^n$.

[2] A variety of class $\mathcal{A}_0$ is also said to be *A-covered*, see [3, Definition 4].
[3] In other terms, regular, plain or locally flattenable, see [21, 35.D], [6] and [39], respectively.

In [21, 3.5.E‴] Gromov asked whether a smooth complete rational variety is uniformly rational. It seems that this question is still open, see [7, Question 1.1] and [9, p. 41]. On the other hand, not every complete uniformly rational variety belongs to class $\mathcal{A}_0$. For instance, none of the smooth rational cubic fourfolds in $\mathbb{P}^5$ and none of the smooth threefold intersections of a pair of quadrics in $\mathbb{P}^5$ contains a Zariski open set isomorphic to an affine space, see [37], and [41]. However, these varieties are uniformly rational, see [7] and Example 4.8 below.

For the following property of uniformly rational varieties see [21, Proposition 3.5E], [6, Theorem 4.4] and [7, Proposition 2.6].

Theorem 1.2. *Let X be a uniformly rational variety and $\tilde{X} \to X$ be the blowing of X up along a smooth subvariety of codimension at least 2. Then $\tilde{X}$ is uniformly rational.*

Notice that the total space of a locally trivial fiber bundle over a uniformly rational variety with a uniformly rational general fiber also is uniformly rational.

1.5. Main results

We prove the following theorem.

Theorem 1.3 *(Theorem 3.3). Let X be a complete uniformly rational variety of positive dimension. Then X is elliptic. Let further X be projective, D be an ample divisor on X and $Y = F \setminus Z_F$ be the principal $\mathbb{G}_{\mathrm{m}}$-fiber bundle associated with a line bundle F where either $F = \mathcal{O}_X(-D)$ or $F = \mathcal{O}_X(D)$. Then Y is elliptic.*

Remarks 1.4. 1. Up to isomorphism over X which inverses the $\mathbb{G}_{\mathrm{m}}$-action, the $\mathbb{G}_{\mathrm{m}}$-variety $Y = F \setminus Z_F$ stays the same under replacing D by $-D$.

2. For a trivial line bundle F on X the variety $Y \simeq_X X \times (\mathbb{A}^1 \setminus \{0\})$ is not elliptic and $\pi_1(Y)$ is infinite for $\mathbb{K} = \mathbb{C}$.

3. If $\mathrm{Pic}(X) = \mathbb{Z}$ then any non-principal divisor D on X is either ample or anti-ample.

4. As a simple example, consider $X = \mathbb{P}^1$ and let D be a point of $\mathbb{P}^1$. Then $F = \mathcal{O}_{\mathbb{P}^1}(-1)$ is the tautological line bundle on $\mathbb{P}^1$ and $Y = \mathbb{A}^2 \setminus \{0\}$, which is elliptic. For $D = 0$ we obtain $Y = \mathbb{P}^1 \times (\mathbb{A}^1 \setminus \{0\})$, which is not elliptic.

From Theorems 1.2 and 1.3 we deduce the following fact.

Corollary 1.5. *The variety X' resulting from a sequence of blowups of a complete uniformly rational variety X along smooth subvarieties is elliptic.*

A closely related result in [33, Corollary 2] says that the blowup X' with a smooth center of codimension at least 2 in a variety X of class $\mathcal{A}_0$ is subelliptic. Hence, X' is elliptic by [25, Theorem 0.1]. See also [24] for a similar result.

Notice also that due to Theorem 1.3 and to Gromov's theorem mentioned above, any uniformly rational compact complex algebraic variety X is an Oka manifold.

Smooth complete spherical varieties and smooth complete rational T-varieties of complexity one belong to class $\mathcal{A}_0$, see [8] and [3]. Hence, these varieties are uniformly rational. So, we have the following corollary.

Corollary 1.6 *(cf. Corollary 4.6). Let a smooth projective variety X be either spherical or a rational T-variety of complexity one. Then the conclusions of Theorem 1.3 hold for X successively blown up along smooth subvarieties. In particular, this holds if X is a toric variety or a flag variety.*

The next result concerns complete unirational varieties.

Theorem 1.7. *Let X be a complete unirational variety of dimension n. Then there exists a smooth complete uniformly rational variety $\tilde{X}$ of dimension n and surjective morphisms $\mathbb{A}^{n+1} \to \tilde{X} \to X$. If X is rational then the morphism $\tilde{X} \to X$ can be chosen to be birational. Furthermore, if the base field $\mathbb{K}$ is $\mathbb{C}$, then there are surjective morphisms $\mathbb{A}^n \to \tilde{X} \to X$.*

Proof. By Chow's Lemma there exists a projective variety X' and a birational surjective morphism $X' \to X$, see [22, Ch. II. Exercise 4.10]. Clearly, replacing X by X' we may assume that X is projective.

Choose a generically finite dominant rational map $h\colon \mathbb{P}^n \dashrightarrow X$, which is birational if X is rational. By Hironaka's theorem on elimination of indeterminacy there exists a commutative diagram

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbb{P}^n \arrow[rr, dashed, "h"] & & X \end{tikzcd}$$

where f is a composition of blowups with smooth irreducible centers and g is a generically finite morphism, which is birational if h is, see [23] and [29, Corollary 3.18 and Theorem 3.21]. By Theorem 1.2, $\tilde{X}$ is uniformly rational, hence elliptic, see Theorem 1.3. This allows to apply a theorem of Kusakabe [31] which says that there is a surjective morphism $\mathbb{A}^{n+1} \to \tilde{X}$. Moreover, if $\mathbb{K} = \mathbb{C}$, then there is a surjective morphism $\mathbb{A}^n \to \tilde{X}$ by a result of Forstnerič, see [17, Theorem 1.6]. □

From Theorem 1.7 we deduce the following characterization of unirationality:

Corollary 1.8. *A complete variety X over $\mathbb{K}$ of dimension n is unirational if and only if X admits a surjective morphism from $\mathbb{A}^{n+1}$ (resp., from $\mathbb{A}^n$ if the base field is $\mathbb{C}$).*

Remarks 1.9. 1. The assumptions of completeness in Theorems 1.3 and 1.7 are important, as the following simple example shows: take the complement T of the coordinate cross $xy = 0$ in $\mathbb{C}^2$. See, however, [1] and [4] for certain classes of affine and quasiaffine varieties, respectively, that are images of affine spaces.

2. Recall Gromov's question (see [21, 3.5″]): is every unirational smooth complete variety elliptic?

3. By the Lefschetz Principle, the theorem of Forstnerič cited above holds over any algebraically closed field of characteristic zero which has infinite transcendence degree over $\mathbb{Q}$, see [14].

4. The version of Kusakabe's theorem used in the above proof says that a smooth complete elliptic variety $\tilde{X}$ admits a morphism $\mathbb{A}^{n+1} \to \tilde{X}$. Moreover, this morphism could be chosen so that its restriction to an open subset of $\mathbb{A}^{n+1}$ is smooth and surjective. For the reader's convenience we sketch a short argument close to the original one in [31]. Notice that the original Kusakabe's theorem works for a not necessarily complete smooth elliptic variety.

Fix a dominating spray (E, p, s) on $\tilde{X}$ of rank $r \geq n$. For any point $x \in \tilde{X}$ the restriction $s|_{E_x} \colon E_x \to \tilde{X}$ has surjective differential at the origin $0_x \in E_x$. Hence, one can choose a vector subspace $F_x \subset E_x$ of dimension n such that $s|_{F_x} \colon F_x \to \tilde{X}$ is étale at $0_x \in F_x$. It follows that F_x contains an open neighborhood V_x of $0_x \in F_x$ such that the differential of $s|_{V_x}$ has rank n at any point $v \in V_x$ and $s(V_x)$ contains a Zariski open dense neighborhood U_x of x in $\tilde{X}$. Choosing a finite open covering $\{U_{x_i}\}$ of $\tilde{X}$, $i = 1, \ldots, k$ one has $s(V) = \tilde{X}$ where $V = \cup_i V_{x_i}$.

Since $\tilde{X}$ is unirational, hence rationally connected, there exists a rational curve C in $\tilde{X}$ which passes through $x_1, \ldots, x_k$, see [28, Ch. IV, Theorem 3.9]. The normalization morphism $\eta \colon \mathbb{P}^1 \to C$ induces a vector bundle $\eta^*E \to \mathbb{P}^1$. Choose an affine chart $A \simeq \mathbb{A}^1$ in $\mathbb{P}^1$ which contains $\eta^{-1}(x_i)$ for $i = 1, \ldots, k$. The restriction of $\eta^*E|_A$ is trivial: $\eta^*E|_A \cong_A \mathbb{A}^1 \times \mathbb{A}^r = \mathbb{A}^{r+1}$. Let $\tilde{V}$ resp. $\tilde{F}$ be the preimage of V resp. of $F = \cup_i F_{x_i}$ in $\eta^*E|_A$. Identifying $\eta^*E|_A$ with the trivial vector bundle $\mathbb{A}^1 \times \mathbb{A}^r \to \mathbb{A}^1$ one can find an automorphism φ of the latter identical on $\mathbb{A}^1$ which sends every $\mathbb{A}^n$-component of $\tilde{F}$ to the fixed $\mathbb{A}^n$-subspace of $\mathbb{A}^r$. Thus, φ sends $\tilde{F}$ into $\mathbb{A}^1 \times \mathbb{A}^n = \mathbb{A}^{n+1}$. Since $s(V) = \tilde{X}$, letting $\tilde{s} = s \circ \eta_* \circ \varphi^{-1}$ one has $\tilde{s}(\tilde{V}) = \tilde{X}$, where $\eta_* \colon \eta^*E|_A \to E|_A$ is the natural surjective morphism induced by η. Moreover, there exists an open neighborhood Ω of $\tilde{V}$ in $\mathbb{A}^{n+1}$ such that $\tilde{s}(\Omega) = \tilde{X}$ and $\tilde{s}|_\Omega \colon \Omega \to \tilde{X}$ is smooth. $\square$

1.6. Generalized affine cones

By Theorem 1.3 the punctured affine cones over uniformly rational projective varieties are elliptic. Let us mention further examples.

Recall that an effective $\mathbb{G}_m$-action $\lambda \colon \mathbb{G}_m \times \bar{Y} \to \bar{Y}$ on a normal affine variety $\bar{Y}$ is called *good* if there exists a point $y_0 \in \bar{Y}$ which belongs to the closure of any λ-orbit. The structure algebra $A = \mathcal{O}_{\bar{Y}}(\bar{Y})$ of such a $\mathbb{G}_m$-variety $\bar{Y}$ is positively graded: $A = \bigoplus_{k \geq 0} A_k$ where $A_0 = \mathbb{K}$ and A_k for $k > 0$ consists of λ-homogeneous elements of weight k. Let

$Y = \bar{Y} \setminus \{y_0\}$ and $X = \mathrm{Proj}(A) = Y/\lambda$. Then X is a normal projective variety, see [12, Proposition 3.3]. According to [12, Theorem 3.5], see also [13, Theorem 3.3.4], there exists an ample $\mathbb{Q}$-Cartier divisor $D = \sum_i p_i/q_i D_i$ on X, where the D_i are prime divisors and the integers p_i, q_i are coprime, such that

$$A_k = H^0(X, \mathcal{O}_X(\lfloor kD \rfloor)) \quad \text{for every} \quad k \geq 0.$$

Furthermore, the $\mathbb{G}_{\mathrm{m}}$-action λ on Y is free if and only if D is a Cartier divisor that is, $q_i = 1\ \forall i$, see [12, Corollaire 2.8.1].

Conversely, given a smooth projective variety X and an ample Cartier divisor D on X one can consider the generalized affine cone

$$\bar{Y} = \mathrm{Spec}\left(\bigoplus_{n=0}^{\infty} H^0\left(X, \mathcal{O}_X(nD)\right)\right).$$

This is a normal affine variety equipped with a good $\mathbb{G}_{\mathrm{m}}$-action, see [12, Sec. 3] or [13, Proposition 3.3.5]. Letting $Y = \bar{Y} \setminus \{y_0\}$ where $y_0 \in \bar{Y}$ is the unique $\mathbb{G}_{\mathrm{m}}$-fixed point, one gets a morphism $\pi\colon Y \to X = Y/\mathbb{G}_{\mathrm{m}}$. Every fiber of π is reduced, irreducible and isomorphic to $\mathbb{A}^1_*$, see [12, Proposition 2.8] and [13, Proposition 3.4.5]. Furthermore, the $\mathbb{G}_{\mathrm{m}}$-action on Y is free and π is locally trivial, see [12, the proof of Proposition 2.8].

Consider also the line bundle $F = \mathcal{O}_X(-D) \to X$ equipped with the associated $\mathbb{G}_{\mathrm{m}}$-action. We have a birational morphism $F \to \bar{Y}$ contracting the zero section Z_F to a normal point $y_0 \in \bar{Y}$, cf. [12, 3.4]. It restricts to an equivariant isomorphism of smooth quasiaffine varieties

$$F \setminus Z_F \simeq \bar{Y} \setminus \{y_0\}$$

equipped with free $\mathbb{G}_{\mathrm{m}}$-actions, see [12, Corollaire 2.9], [13, Sec. 3.4, p. 49] and [27, Sec. 1.15]; cf. [36, p. 183].

Notice that while $\bar{Y}$ above is normal, for $X \subset \mathbb{P}^N$ the affine cone $\mathrm{cone}(X)$ is normal if and only if the embedding $X \hookrightarrow \mathbb{P}^N$ is projectively normal, that is, for every $d \geq 1$ the linear system cut out on X by the hypersurfaces of degree d is complete, see [22, Chap. II, Example 7.8.4]. Thus, if D is a hyperplane section of $X \subset \mathbb{P}^N$ then $\bar{Y}$ as above is the normalization of the affine cone $\mathrm{cone}(X)$. In particular, the punctured cone $\mathrm{cone}(X)\setminus\{0\}$ is $\mathbb{G}_{\mathrm{m}}$-equivariantly isomorphic to $\bar{Y} \setminus \{y_0\}$.

2. The curve-orbit property

We need the following more general analog of the curve-orbit property $(*)$ for $\mathbb{G}_{\mathrm{a}}$-sprays defined in [26, Definition 2.7].

Definition 2.1. Given a smooth variety B of dimension $n-1$ consider the $\mathbb{G}_{\mathrm{a}}$-action on the cylinder $V = B \times \mathbb{A}^1$ by shifts on the second factor:

$$s_V : \mathbb{G}_a \times V \to V, \quad (t,(b,v)) \mapsto (b, v+t)$$

along with the associated $\mathbb{G}_a$-*spray* (E_V, p_V, s_V) on V where $E_V = V \times \mathbb{A}^1$ and $p_V : E_V \to V$ is the first projection.

Let X be a smooth variety of dimension n. Assume that X admits a birational morphism $\psi : V \to X$ biregular on an open dense subset $V_0 \subset V$ with image $X_0 \subset X$. Consider the spray (E_0, p_0, s_0) on X_0 with values in X conjugate to $(E_V, p_V, s_V)|_{V_0}$ via ψ. That is, $E_0 = X_0 \times \mathbb{A}^1$, $p_0 : E_0 \to X_0$ is the first projection and

$$s_0 : E_0 \to X, \quad (x,t) \mapsto \psi(s_V(t, \psi^{-1}(x))).$$

Extend (E_0, p_0, s_0) to a rank 1 spray (E, p, s) on X; the latter spray exists due to Gromov's Extension Lemma, see [21, 3.5B], [16, Propositions 6.4.1-6.4.2] or [26, Proposition 8.1]. We call (E, p, s) a $\mathbb{G}_a$-*like spray on* X. This spray is associated with the birational $\mathbb{G}_a$-action on X conjugate via ψ to the standard $\mathbb{G}_a$-action on the cylinder V, see [11, Chap. 1].

Remark 2.2. The closure $C_x = \overline{O_x}$ in X of the s_0-orbit O_x of a point $x \in X_0$ is a rational curve. By construction, the intersection $C_x \cap X_0$ is smooth and (E, p, s) restricts to a dominating spray on $O_x \cap X_0$, cf. Lemma 2.4 below. Moreover, the morphism $s : E_x \simeq \mathbb{A}^1 \to C_x$ admits a lift to the normalization $\mathbb{P}^1$ of C_x, and the latter morphism $\mathbb{A}^1 \to \mathbb{P}^1$ is an embedding.

However, the curve C_x can have singularities off X_0. Thus, the setup of Definition 2.1 does not guarantee that X verifies on X_0 either the curve-orbit property $(*)$ of [26, Definition 2.7], or the enhanced curve-orbit property $(**)$ of [26, Proposition 3.1]. Indeed, the latter properties postulate the smoothness of C_x for $x \in X_0$, which occurs to be a rather restrictive condition for our purposes. The question arises whether any smooth complete variety of class $\mathcal{A}_0$, or even every complete uniformly rational variety, verifies the curve-orbit property $(*)$ of [26] with smooth rational curves; cf. Corollary 2.7 and Lemma 3.1 below. Notice that this property holds for smooth complete rational surfaces and for flag varieties G/P, see [26].

In order to use the criterion of ellipticity of cones over projective varieties from [26, Corollary 2.9] we introduce the following objects.

Definition 2.3. Let X and B be as in Definition 2.1. Consider a $\mathbb{P}^1$-cylinder $W = B \times \mathbb{P}^1$ with base B. Assume that X admits a birational morphism $\varphi : W \to X$ biregular on an open dense subset $W_0 \subset W$ with image $X_0 \subset X$. Given a point $u \in \mathbb{P}^1$ consider the cylinder $V_u := B \times (\mathbb{P}^1 \setminus \{u\}) \simeq B \times \mathbb{A}^1$, the birational morphism $\psi_u = \varphi|_{V_u} : V_u \to X$ and the open dense subsets $V_u \cap W_0 \subset V_u$ and $X_u = \psi_u(V_u \cap W_0) \subset X_0$. Thus, the data (W, φ, X_0) yields a one-parameter family of $\mathbb{G}_a$-like sprays (E_u, p_u, s_u) on X where $u \in \mathbb{P}^1$, see Definition 2.1.

Lemma 2.4. *Under the setup of Definition 2.3 let for $x \in X_0$,*

$$w = \varphi^{-1}(x) = (b, u_x) \in W_0 \quad \textit{and} \quad C_x = \varphi(\{b\} \times \mathbb{P}^1) \subset X. \tag{1}$$

Then C_x is a complete rational curve in X through x such that $C_x \cap X_0$ is smooth. If $x \in X_u$, that is $\varphi(b,u) \neq x$, then the s_u-orbit $O_{u,x}$ of x is one-dimensional and (E_u, p_u, s_u) restricts to a spray on C_x dominating at x and such that $s_u|_{E_{u,x}} : E_{u,x} \to O_{u,x}$ is a birational morphism étale over x.

Proof. The $\mathbb{G}_\mathrm{a}$-like spray (E_u, p_u, s_u) inherits a kind of the composition property of a $\mathbb{G}_\mathrm{a}$-action. Namely, for any $x' \in O_{u,x} \cap X_0$ the s_u-orbits $O_{u,x'}$ and $O_{u,x}$ coincide. This implies that (E_u, p_u, s_u) restricts to a spray on C_x. The rest of the proof is easy and is left to the reader. □

Definition 2.5. Modifying [26, Definition 2.7] we say that a complete rational curve C on a smooth variety X verifies the *strengthened two-orbit property* at a smooth point $x \in C$ if

(∗) there exists a pair of rank 1 sprays (E_i, p_i, s_i) $(i = 1, 2)$ on X such that C is covered by the one-dimensional s_i-orbits $O_{i,x}$, $s_i : E_{i,x} \to O_{i,x}$ is a birational morphism étale over x and (E_i, p_i, s_i) restricts to a spray on $O_{i,x}$ dominating at x.

If for any $x \in X$ there exists a curve $C = C_x$ as above, then we say that X verifies the *strengthened curve-orbit property*.

Remark 2.6. Following the lines of the proof of Proposition 2.8 one can establish the ellipticity of the punctured affine cone over an elliptic smooth projective variety with an ample polarization under the following weaker assumption:

(∗′) for each $x \in X$ there exists a rational curve C_x in X and a pair of sprays $\{(E_i, p_i, s_i)\}_{i=1,2}$ on X such that
- x is a smooth point of C_x;
- $s_i|_{E_{i,x}} : E_{i,x} \to O_{i,x}$ is a birational morphism étale over x;
- $C_x = O_{1,x} \cup O_{2,x}$.

However, in the concrete setup of the present paper the strengthened curve-orbit property (∗) holds as well.

We have the following corollary.

Corollary 2.7. *Let X be a smooth projective variety, and let (W, φ, X_0) be a data as in Definition 2.3. For $x \in X_0$ let C_x be a curve as in (1). Then C_x verifies the strengthened two-orbit property (∗) at x with a pair of $\mathbb{G}_\mathrm{a}$-like sprays. If for each $x \in X$ there exists a* *data (W, φ, X_0) such that $x \in X_0$ then the strengthened curve-orbit property holds on X with pairs of $\mathbb{G}_a$-like sprays.*

Proof. Let $\varphi^{-1}(x) = (b, u_x) \in W_0$. Pick two distinct points $u_1, u_2 \in \mathbb{P}^1$ different from u_x and consider the corresponding $\mathbb{G}_a$-like sprays $(E_i, p_i, s_i) = (E_{u_i}, p_{u_i}, s_{u_i})$, $i = 1, 2$, see Definition 2.3. Due to Lemma 2.4 these sprays fit in Definition 2.5 of the strengthened two-orbit property. This yields the first assertion. Now the second is immediate. □

Using Definitions 2.1 and 2.3 we can generalize the ellipticity criterion for cones in [26, Corollary 2.9] as follows. The proof repeats verbatim the proof of Corollary 2.9 in [26] with minor changes.

Proposition 2.8. *Let X be a smooth projective variety and $\varrho\colon F \to X$ be an ample line bundle. Suppose that X is elliptic and for any point $x \in X$ there exists a data (W, φ, X_0) as in Definition 2.3 such that $x \in X_0$. Then $Y = F \setminus Z_F$ is elliptic.*

Proof. A dominating spray (E, p, s) on X lifts to a spray $(\hat{E}, \hat{p}, \hat{s})$ of rank $n = \dim(X)$ on Y where $\hat{E}$ fits in the commutative diagrams

$$\begin{array}{ccc} \hat{E} & \xrightarrow{\hat{p}} & Y \\ \downarrow^{\hat{\varrho}} & & \downarrow^{\varrho|_Y} \\ E & \xrightarrow{p} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} \hat{E} & \xrightarrow{\hat{s}} & Y \\ \downarrow^{\hat{\varrho}} & & \downarrow^{\varrho|_Y} \\ E & \xrightarrow{s} & X \end{array} \tag{2}$$

see [26, Lemma 2.3].

Given $y \in Y$ we let $x = \varrho(y) \in X$ and let C_x be a rational curve on X passing through x which is smooth at x and such that (C_x, x) satisfies the strengthened two-orbit property with a pair of $\mathbb{G}_a$-like sprays (E_i, p_i, s_i) on X, $i = 1, 2$, see Definition 2.5 and Corollary 2.7. Due to [26, Lemma 2.3] these sprays admit a lift to a pair of rank 1 sprays $(\hat{E}_i, \hat{p}_i, \hat{s}_i)$ on Y.

We will show, following the lines of the proof of Proposition 2.6 in [26], that the triplet of sprays $(\hat{E}, \hat{p}, \hat{s})$ and $(\hat{E}_i, \hat{p}_i, \hat{s}_i)$, $i = 1, 2$ is dominating at y. This implies the assertion.

Notice that the tangent space at y to the $\hat{s}$-orbit is a hyperplane $H \subset T_yY$ such that $d\varrho(H) = T_xX$. We claim that the pair of tangent vectors at y to the $\hat{s}_i$-orbits $\hat{O}_{i,y}$ on Y span a plane P in T_yY such that $d\varrho(P) = T_xC_x$. Accepting this claim, there exists a nonzero vector $v \in P$ such that $d\varrho(v) = 0$. Hence $v \notin H$ and so, $\operatorname{span}(H, P) = T_yY$, which gives the desired domination.

To show the claim, consider a morphism of normalization $\varphi_x\colon \mathbb{P}^1 \to C_x$ and the pullback line bundle $\tilde{F} = \tilde{F}(x) = \varphi_x^*(F|_{C_x}) \to \mathbb{P}^1$. Since $F \to X$ is ample one has $[F] \cdot [C_x] \neq 0$. Hence, $\tilde{F} \to \mathbb{P}^1$ is nontrivial and so is the associated principal $\mathbb{G}_m$-fiber bundle $\tilde{Y} = \tilde{Y}(x) = \varphi_x^*(Y|_{C_x}) \to \mathbb{P}^1$, see, e.g., [35, Proposition 4.1]. For $i = 1, 2$ consider the pullback $\tilde{p}_i\colon \tilde{E}_i \to \tilde{Y}$ of the line bundle $\hat{p}_i\colon \hat{E}_i \to Y$ via the induced morphism $\tilde{Y} \to Y|_{C_x}$.

According to the second diagram in (2) we have

$$\varrho(\hat{O}_{i,y}) = \varrho \circ \hat{s}_i(\hat{E}_{i,y}) = s_i(E_{i,x}) = O_{i,x} \subset C_x.$$

The birational morphism $s_i|_{E_{i,x}}: E_{i,x} = \mathbb{A}^1 \to C_x$ admits a lift to normalization $\tilde{s}_{i,x}: \mathbb{A}^1 \to \mathbb{P}^1$. In fact, $\tilde{s}_{i,x}$ is a birational morphism smooth at 0. Hence, $\tilde{s}_{i,x}$ sends $\mathbb{A}^1$ isomorphically onto $U_i := \mathbb{P}^1 \setminus \{u_i\}$, the notation being as in the proof of Corollary 2.7.

By Lemma 2.4 (E_i, p_i, s_i) restricts to a $\mathbb{G}_{\mathrm{a}}$-like spray on C_x. It is easily seen that there is a pullback $\varphi_x^*((E_i, p_i, s_i)|_{C_x})$ to a $\mathbb{G}_{\mathrm{a}}$-like spray (E_i', p_i', s_i') on $\mathbb{P}^1$ dominating at x and such that the s_i'-orbit of x coincides with U_i. Now, the pullback of (E_i', p_i', s_i') to $\tilde{Y}$ via $\tilde{Y} \to \mathbb{P}^1$ gives a $\mathbb{G}_{\mathrm{a}}$-like spray $(\tilde{E}_i, \tilde{p}_i, \tilde{s}_i)$ on $\tilde{Y}$ with $\tilde{p}_i: \tilde{E}_i \to \tilde{Y}$ as above.

There are trivializations $\tilde{Y}|_{U_i} \cong_{U_i} U_i \times \mathbb{A}^1_*$, $i = 1, 2$. The standard $\mathbb{G}_{\mathrm{a}}$-action on $U_i \simeq \mathbb{A}^1$ lifts to a $\mathbb{G}_{\mathrm{a}}$-action on $\tilde{Y}|_{U_i}$ whose orbits are the constant sections. Since any morphism $\mathbb{A}^1 \to \mathbb{A}^1_*$ is constant, the one-dimensional $\tilde{s}_i$-orbits in $\tilde{Y}$ also are constant sections. Since the $\mathbb{G}_{\mathrm{m}}$-fiber bundle $\tilde{Y} \to \mathbb{P}^1$ is nontrivial, it admits no global section. Hence, in an appropriate affine coordinate z on $U_1 \cap U_2$ the transition function equals z^k with $k \neq 0$. It follows that the constant sections over U_1 meet transversally the ones over U_2.

The normalization morphism $\varphi: \mathbb{P}^1 \to C_x$ is étale over x. Hence, also the morphism $\tilde{Y} \to Y|_{C_x}$ is étale over y. Finally, the $\hat{s}_i$-orbits $\hat{O}_{1,y}$ and $\hat{O}_{2,y}$ meet transversally at y. This proves our claim. $\square$

3. Uniformly rational varieties

3.1. The ellipticity and the curve-orbit property of complete uniformly rational varieties

In order to apply Proposition 2.8 to complete (e.g., projective) varieties we need the following Lemmas 3.1–3.2.

Lemma 3.1. *Let X be a smooth complete variety of dimension $n \geq 2$. Then X is uniformly rational if and only if for any point $x \in X$ there exists a data (W, φ, X_0) where $W = B \times \mathbb{P}^1$ is a cylinder over an open set $B \subset \mathbb{A}^{n-1}$ and $\varphi: W \dashrightarrow X$ is a birational map which sends biregularly an open subset $W_0 \subset B \times \mathbb{A}^1 \subset W$ onto a neighborhood X_0 of x in X; cf. Definitions 2.1 and 2.3. Furthermore, if X is uniformly rational, then one can choose a $\mathbb{G}_{\mathrm{a}}$-like spray (E_u, p_u, s_u) as in Definition 2.3 so that the differential $\mathrm{d}s_u$ sends $T_{0_x}E_{u,x}$ to a general line in T_xX.*

Proof. The "if" part is immediate. To show the "only if" part, suppose X is uniformly rational. Then for any $x \in X$ there is an open subset $V_0 \subset \mathbb{A}^n$ and an isomorphism h_0 of V_0 onto a neighborhood of x in X.

Embed $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$ and let $h: \mathbb{P}^n \dashrightarrow X$ be the birational extension of h_0. Let $v = h_0^{-1}(x) \in V_0$. Fix a general point $P \in H = \mathbb{P}^n \setminus \mathbb{A}^n$ and let $\mathcal{L}$ be the family of projective lines in $\mathbb{P}^n$ which pass through P. The projection $\pi_P\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ with center P restricts to a linear projection $\tau\colon \mathbb{A}^n \to \mathbb{A}^{n-1}$. The latter defines a decomposition $\mathbb{A}^n \cong \mathbb{A}^{n-1} \times \mathbb{A}^1$ and a family of parallel affine lines $l_y = \{y\} \times \mathbb{A}^1$ in $\mathbb{A}^n$ where $y \in \mathbb{A}^{n-1}$ and $\bar{l}_y \in \mathcal{L}$. We may assume that $\bar{l}_0 \in \mathcal{L}$ passes through v.

The embedding $\mathbb{A}^n = \mathbb{A}^{n-1} \times \mathbb{A}^1 \hookrightarrow \mathbb{P}^n$ extends to a birational morphism $\psi\colon \mathbb{A}^{n-1} \times \mathbb{P}^1 \to \mathbb{P}^n$ which contracts $\mathbb{A}^{n-1} \times \{\infty\}$ to P. The birational map $\tilde{\varphi} := h \circ \psi\colon \mathbb{A}^{n-1} \times \mathbb{P}^1 \dashrightarrow X$ fits in the lower triangle of the commutative diagram

$$\begin{tikzcd}
& \hat{X} \arrow[dl, "f"'] \arrow[dr, "g"] & \\
\mathbb{P}^n \arrow[rr, dashed, "h"] & & X \\
& \mathbb{A}^{n-1} \times \mathbb{P}^1 \arrow[ul, "\psi"] \arrow[ur, dashed, "\tilde{\varphi}"'] &
\end{tikzcd}$$

In the upper triangle, $\hat{X}$ stands for the resolution of the closure Γ_h of the graph of h in $\mathbb{P}^n \times X$, while f and g stand for the standard projections of Γ_h to the factors composed with the resolution morphism $\hat{X} \to \Gamma_h$. Thus, $\hat{X}$ is a smooth projective variety and f and g are birational morphisms.

We claim that one can choose an open neighborhood B of $\tau(v) = 0 \in \mathbb{A}^{n-1}$ and an open subset $W_0 \subset W := B \times \mathbb{P}^1$ which contains v so that the restriction $\varphi = \tilde{\varphi}|_{W_0}\colon W_0 \to X$ is biregular onto its image $X_0 = \varphi(W_0) \ni x$, as desired.

To show the claim notice that the inverse f^{-1} of the birational morphism $f\colon \hat{X} \to \mathbb{P}^n$ between smooth projective varieties is a blowup of $\mathbb{P}^n$ whose center is an ideal sheaf supported on a closed subvariety $Z \subset \mathbb{P}^n$ of codimension at least 2, see [22, Ch. II, Theorem 7.17]. On the other hand, Z is the indeterminacy locus of $h = g \circ f^{-1}$ ([22, Ch. II, Proposition 7.13(b)]). Thus, h is regular on $\mathbb{P}^n \setminus Z$ and so, $\tilde{\varphi} = h \circ \psi$ is regular on $(\mathbb{A}^{n-1} \times \mathbb{P}^1) \setminus \psi^{-1}(Z)$.

Since $P \in H$ is a general point, we may assume that $P \notin Z$. Thus, h is regular in P and $\tilde{\varphi} = h \circ \psi$ is regular on $\mathbb{A}^{n-1} \times \{\infty\}$ (recall that $\psi(\mathbb{A}^{n-1} \times \{\infty\}) = P$). Since $P \notin Z$ the image $\pi_P(Z)$ is a closed subvariety of $\mathbb{P}^{n-1}$. Since $\operatorname{codim}_{\mathbb{P}^n} Z \geq 2$ the image $\Delta := \tau(Z \setminus H) = \pi_P(Z) \setminus \pi_P(H)$ is a proper closed subvariety of $\mathbb{A}^{n-1} = \mathbb{P}^{n-1} \setminus \pi_P(H)$. Since P is a general point of H and $\operatorname{codim}_{\mathbb{P}^n} Z \geq 2$ we may suppose that $\bar{l}_0 \cap Z = \emptyset$ and so, $\tau(v) \notin \Delta$. Since by our construction h is regular on V_0 we have $Z \cap V_0 = \emptyset$.

Let $B = \mathbb{A}^{n-1} \setminus \Delta$. It follows from the preceding that $\tilde{\varphi}$ is regular on $B \times \mathbb{A}^1$. Furthermore, $\tilde{\varphi}$ is regular on $W = B \times \mathbb{P}^1$ and sends biregularly the neighborhood $W_0 := \psi^{-1}(V_0) \subset W$ of $\psi^{-1}(v)$ onto the neighborhood $X_0 := \tilde{\varphi}(W_0)$ of x in X. Letting $\varphi = \tilde{\varphi}|_W$ the claim follows. This proves the first assertion of the lemma.

To show the second assertion we let $u = \infty \in \mathbb{P}^1$ so that $v \in V_u = B \times \mathbb{A}^1$, see Definition 2.3. Via the conjugation with φ, the $\mathbb{G}_\mathrm{a}$-action on V_u by shifts on the factor $\mathbb{A}^1$ yields a $\mathbb{G}_\mathrm{a}$-like spray (E_u, p_u, s_u) on X as in Definition 2.1. The restriction of s_u to $E_{u,x}$ is an immersion at the origin $0_x \in E_{u,x}$, see Lemma 2.4.

Since $\tau\colon \mathbb{A}^n \to \mathbb{A}^{n-1}$ is a general linear projection, the tangent vector at v to the orbit of v under the $\mathbb{G}_a$-action on V_u is a general vector in $T_v V_0$. It follows that the differential $\mathrm{d}s_u$ sends $T_{0_x} E_{u,x}$ to a general line in $T_x X$. □

Lemma 3.2. *A complete uniformly rational variety X is elliptic.*

Proof. By Lemma 3.1 one can find n different rank 1 sprays (E_i, p_i, s_i) on X, $i = 1, \ldots, n$ where $n = \dim(X)$ such that the lines $\mathrm{d}s_i(T_{0_{i,x}} E_{i,x})$ span the tangent space $T_x X$. The composition of these sprays gives a rank n spray on X dominating at x, see [25, Corollary 2.2]. This implies that X is locally elliptic, hence elliptic, see [25, Theorem 1.1]. □

3.2. The main results

We can now deduce our main result.

Theorem 3.3. *For a complete uniformly rational variety X the following holds.*

(i) *X is elliptic.*
(ii) *Assume that X is projective and let $F \to X$ be an ample or anti-ample line bundle on X with zero section Z_F. Then $Y = F \setminus Z_F$ is elliptic.*

Proof. Statement (i) follows from Lemma 3.2 and (ii) follows from Proposition 2.8 due to Lemma 3.1. □

In Corollary 3.7 below we slightly generalize statement (i). Let us introduce the following notion.

Definition 3.4. We say that a variety X is *stably uniformly rational* if for some $k \geq 0$ the variety $X \times \mathbb{A}^k$ is uniformly rational.

Remark 3.5. There exist non-rational stably rational varieties, see [5]. However, we do not know whether every stably uniformly rational variety is uniformly rational. One may also ask whether there exists a non-rational stably uniformly rational variety.

On the other hand, we have the following lemma suggested by Lárusson, see [24, Proposition 1.9].

Lemma 3.6. *If the product $X = X_1 \times X_2$ of smooth varieties X_1 and X_2 is elliptic, then the X_i are elliptic.*

For the reader's convenience we sketch the proof.

Proof. Let (E, p, s) be a dominating spray on X. Pick a point $P \in X_2$ and consider the restriction $p_1 \colon E_1 \to X_1$ of $p \colon E \to X$ to $X_1 \times \{P\} \subset X$. Letting now

$$s_1 = \mathrm{pr}_1 \circ s|_{E|_{X_1 \times \{P\}}} \colon E_1 \to X_1$$

yields a desired dominating spray (E_1, p_1, s_1) on X_1, cf. the proof of Proposition 1.9 in [24]. □

Corollary 3.7. *A complete stably uniformly rational variety X is elliptic.*

Proof. Let $X \times \mathbb{A}^k$ be uniformly rational. Then also $\hat{X} = X \times \mathbb{P}^k$ is. By Theorem 3.3(i) $\hat{X}$ is elliptic. By Lárusson's Lemma 3.6 X is elliptic too. □

The ampleness assumption in Theorem 3.3(ii) is not a necessary one. Indeed, we have the following version of this theorem.

Theorem 3.8. *Let $\pi \colon X \to B$ be a locally trivial fiber bundle with the base B and the fiber V being uniformly rational smooth complete varieties. Let D be a relatively ample divisor on X, that is, $D \cdot X_b$ is an ample divisor on the fiber $X_b = \pi^{-1}(b) \simeq V$ for all $b \in B$. Let $F = \mathcal{O}_X(\pm D)$. Then X and $Y = F \setminus Z_F$ are elliptic.*

Proof. The ellipticity of X follows from Lemma 3.2 and the remark after Theorem 1.2. For every $b \in B$ the conclusion of Lemma 3.1 holds for the fiber X_b. It follows that X_b verifies the strengthened curve-orbit property $(*)$, see Lemma 2.7, that is, for any $x \in X_b$ there is a rational curve C_x on X_b smooth at x and a pair of $\mathbb{G}_\mathrm{a}$-like spays (E_i, p_i, s_i), $i = 1, 2$ on X_b as in Remark 2.6. Due to the local triviality of π and Gromov's Localization Lemma these sprays can be extended on X. Now the argument from the proof of Proposition 2.8 applies and yields the ellipticity of Y. □

Observe that a relatively ample divisor D as in Theorem 3.8 does not need to be ample on X; cf. e.g., [26, Example 2.11].

4. Examples

We start with the following well known example. For the reader's convenience we provide an argument.

Lemma 4.1. *A smooth complete toric variety X belongs to class $\mathcal{A}_0$.*

Proof. Let N be a lattice of rank $n = \dim(X)$ and Σ be the complete fan in $N_\mathbb{Q} = N \otimes_\mathbb{Z} \mathbb{Q}$ associated with X. Since X is smooth every n-cone $\sigma \in \Sigma$ is simplicial generated by vectors $v_1, \ldots, v_n \in N$ which form a base of N. The corresponding affine variety X_σ is isomorphic to $\mathbb{A}^n$ and the affine charts X_σ cover X, see, e.g., [19, Sec. 1.4]. $\square$

For the proof of the next lemma we address the original paper.

Lemma 4.2 *([3, Theorem 5]). A smooth complete rational variety X with a torus action of complexity 1 belongs to class $\mathcal{A}_0$.*

Remark 4.3. As a weak analog of the last result for affine varieties, let us mention that every smooth rational affine variety with a torus action of complexity 0 or 1 is uniformly rational, see [34]. See also [38] for examples of smooth contractible uniformly rational affine threefolds with a torus action of complexity 2 non-isomorphic to $\mathbb{A}^3$. These include, in particular, the famous Koras-Russell cubic.

Lemma 4.1 extends to complete spherical varieties. For the reader's convenience we sketch a proof.

Lemma 4.4 *([8, Sec. 1.5, Corollaire]). A smooth complete spherical variety X belongs to class $\mathcal{A}_0$.*

Proof. Recall first the Local Structure Theorem, see [8, Théorème 1.4]. Consider a normal G-variety Z, where G is a connected reductive algebraic group. Let $z \in Z$ be a point such that the orbit Gz is a projective variety. Then the stabilizer G_z is a parabolic subgroup. Let $P \subset G$ be the opposite parabolic subgroup with the Levi decomposition $P = LP^u$ where P^u is the unipotent radical of P and $L = P \cap G_z$ is the Levi subgroup in P. The Local Structure Theorem asserts that there is a locally closed affine subset $V \subset Z$ such that $z \in V$, $LV = V$, P^uV is open in Z and the action of P^u on Z defines an isomorphism $P^uV \cong P^u \times V$.

Returning to Lemma 4.4 we let $Z = X$ and we apply the notation above. It suffices to show that any point $x \in X$ whose orbit Gx is closed in X admits a neighborhood in X isomorphic to $\mathbb{A}^n$. Indeed, for any $y \in X$ the closure of the orbit Gy contains such a point x.

Since X is spherical, in the setup of the Local Structure Theorem the reductive Levi subgroup L of $P = G_x^-$ acts on the corresponding smooth affine variety V with an open orbit and with an L-fixed point x. Applying Luna's Étale Slice theorem, see, e.g., [40, Corollary of Theorem 6.7], we deduce that V is equivariantly isomorphic to an L-module. It follows that x has a neighborhood in X isomorphic to $P^u \times V$, which is an affine space. $\square$

Remark 4.5. A similar argument proves the following fact, see [39, Theorem 3]. Let G be a connected reductive algebraic group and X be a smooth affine G-variety. Assume that $\mathcal{O}_X(X)^G = \mathbb{K}$ and the unique closed G-orbit O in X is rational. Then X is uniformly rational.

Due to Lemmas 4.1, 4.2 and 4.4 the following corollary of Theorem 3.3 is immediate.

Corollary 4.6. *The punctured affine and generalized affine cones over a smooth projective spherical variety equipped with an ample polarization are elliptic. This remains true after successively blowing up such a variety in smooth subvarieties. The same holds for a smooth projective rational variety X with a torus action of complexity* 1*.*

Remark 4.7. Recall that the normalization of the affine cone cone(X) over a smooth toric variety X in $\mathbb{P}^n$ is a normal affine toric variety with no torus factor. It is known that such a variety is flexible, see [2, Theorem 0.2(2)]. It follows that $Y = \operatorname{cone}(X) \setminus \{0\}$ is elliptic. However, it is not clear whether the flexibility of cone(X) survives under blowing-up a point in X.

Let us mention some known examples of uniformly rational smooth Fano varieties.

Examples 4.8. 1. It is known that a smooth rational cubic hypersurface in $\mathbb{P}^{n+1}$, $n \geq 2$ and a smooth intersection of two smooth quadric hypersurfaces in $\mathbb{P}^{n+2}$, $n \geq 3$ are uniformly rational, cf. [21, 3.5.E‴] and [7, Examples 2.4 and 2.5]. According to Theorem 3.3, such a variety X is elliptic and the punctured (generalized) affine cone Y over X equipped with an ample polarization is elliptic.

Notice that any smooth cubic hypersurface of dimension $n \geq 2$ in $\mathbb{P}^{n+1}$ is unirational, see [30, Theorem 1.38]. All smooth cubic surfaces in $\mathbb{P}^3$ are rational. By contrast, no smooth cubic threefold in $\mathbb{P}^4$ is rational, see [10]. It is unknown (but is plausible) whether for $n \geq 4$ the general cubic hypersurface in $\mathbb{P}^{n+1}$ is irrational. However, for any $k \geq 1$ there are smooth cubic hypersurfaces in $\mathbb{P}^{2k+1}$ which contain two disjoint linear k-subspaces. Any such cubic hypersurface is rational (hence also uniformly rational), see [30, 1.33–1.35].

A smooth intersection X of two smooth quadric hypersurfaces in $\mathbb{P}^{n+2}$, $n \geq 3$ is always rational, see, e.g., [20, p. 796]. For $n = 3$ no smooth quartic threefold X as above belongs to class $\mathcal{A}_0$. The latter follows from the classification of smooth Fano threefolds with Picard number 1 which contain an open subset isomorphic to $\mathbb{A}^3$, see [9, Theorem 4.31].

2. It is shown in [7, Proposition 3.2] that the moduli space $\overline{\mathcal{M}}_{0,n}$ of stable n-pointed rational curves is a complete uniformly rational variety. By Theorem 3.3 it is elliptic, hence is the image of an affine space under a surjective morphism.

3. The same holds for a small algebraic resolution of a nodal cubic threefold in $\mathbb{P}^4$, see [7, Example 2.4 and Theorem 3.5]. See also [7, Section 3] for further examples.

4. Any smooth Fano-Mukai fourfold with Picard number 1, genus 10 and index 2 is rational. The moduli space $\mathcal{M}$ of such fourfolds is one-dimensional. With one exception, every such fourfold X_t belongs to class $\mathcal{A}_0$, see [42, Theorem 2]. The exceptional Fano-Mukai fourfold X_0 is covered by open $\mathbb{A}^2$-cylinders $Z_i \times \mathbb{A}^2$ where the Z_i are smooth rational affine surfaces, see [42, Proposition 7.3]. Since the Z_i are uniformly rational,

so is X_0. By Theorem 3.3 the punctured affine cones and generalized affine cones over $X_t \in \mathcal{M}$ are elliptic. In fact, all these cones over X_t, without any exception, are even flexible, see [42, Theorem 1].

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