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CLASSIFICATION OF MORSE–SMALE DIFFEOMORPHISMS WITH A FINITE SET OF HETEROCLINIC ORBITS ON SURFACES

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The article is dedicated to dear Yuliy Sergeevich Ilyashenko, a life guide and role model.

Abstract. In this paper, we consider orientation-preserving Morse-Smale diffeomorphisms on orientable closed surfaces. Such diffeomorphisms can have infinitely many heteroclinic orbits, which makes their topological classification very difficult. In fact, even in the case of a finite number of heteroclinic orbits, there are no exhaustive classification results. The main problem is that for all currently known complete topological invariants of such systems, the implementation is not described. In this paper, we present a complete topological classification of Morse-Smale diffeomorphisms with a finite number of heteroclinic orbits on surfaces, including a realization.

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1. Introduction and Formulation of Results

A discrete dynamical system (*cascade*) on an n-manifold M^n is the group of integer powers of some diffeomorphism $f\colon M^n \to M^n$. Diffeomorphisms $f\colon M^n \to M^n$, $f'\colon M'^n \to M'^n$ are called *topologically conjugate* if there exists a homeomorphism $h\colon M^n \to M'^n$ such that $hf = f'h$.

It is clear that the direct verification of the topological conjugacy of two cascades is an immense problem. An object or property of the system that is preserved under topological conjugacy is called *topological invariant* of the diffeomorphism that generates it. The search for topological invariants is a part of the *topological classification* of some set G of dynamical systems, which is understood as the solution of the following problems:

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1) searching for topological invariants of dynamical systems from the class G;
2) proof of the completeness of the set of found invariants, that is, proof that the coincidence of sets of topological invariants is a necessary and sufficient condition for the topological conjugacy of two dynamical systems from G;
3) *realization*, that is, description of admissible abstract invariants and the construction of a standard representative belonging to G according to the given invariant.

In this paper, we consider orientation-preserving Morse–Smale diffeomorphisms f on an orientable surface M^2. By definition, such a diffeomorphism has a hyperbolic non-wandering set consisting of a finite number of periodic orbits whose stable manifolds intersect transversally with unstable ones. Let $\mathcal{O}_i$, $\mathcal{O}_j$ be periodic orbits of the Morse–Smale diffeomorphism $f\colon M^2 \to M^2$. In [17] S. Smale introduced the notion of *partial order relation* $\prec$ for periodic orbits

$$\mathcal{O}_i \prec \mathcal{O}_j \iff W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} \neq \varnothing.$$

For saddle orbits $\mathcal{O}_i \prec \mathcal{O}_j$ any intersection point $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j}$ is called *heteroclinic*. A sequence of distinct saddle periodic orbits $\mathcal{O}_i = \mathcal{O}_{i_0}, \mathcal{O}_{i_1}, \ldots, \mathcal{O}_{i_k} = \mathcal{O}_j$ $(k \geqslant 1)$ such that $\mathcal{O}_{i_0} \prec \mathcal{O}_{i_1} \prec \cdots \prec \mathcal{O}_{i_k}$ is called a *chain of length k connecting the periodic orbits $\mathcal{O}_i$ and $\mathcal{O}_j$* (see Fig. 1).

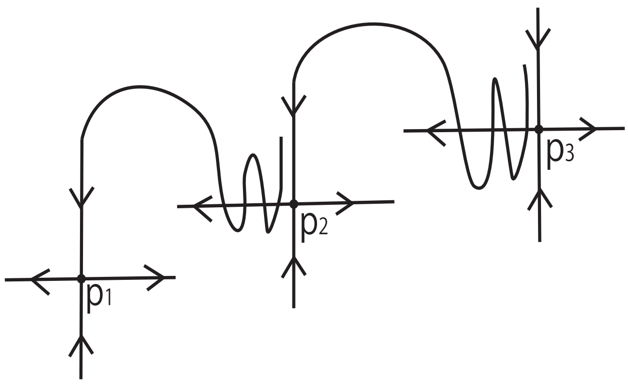

FIGURE 1. Chain of length 2: $\mathcal{O}_1 \prec \mathcal{O}_2 \prec \mathcal{O}_3$, for fixed saddle points $p_1 = \mathcal{O}_1$, $p_2 = \mathcal{O}_2$, $p_3 = \mathcal{O}_3$

The maximum length of the saddle chain connecting the orbits $\mathcal{O}_i$ and $\mathcal{O}_j$ is denoted as $\mathrm{beh}(\mathcal{O}_j|\mathcal{O}_i)$ (*beh* from *behaviounr*). We state $\mathrm{beh}(\mathcal{O}_j|\mathcal{O}_i) = 0$ if $W^u_{\mathcal{O}_j} \cap W^s_{\mathcal{O}_i} = \varnothing$. Let

$$\mathrm{beh}(f) = \max\{\mathrm{beh}(\mathcal{O}_j|\mathcal{O}_i)\}.$$

If $\mathrm{beh}(f) = 0$ for a Morse–Smale diffeomorphism $f\colon M^2 \to M^2$, then the diffeomorphism f is called a *gradient-like* diffeomorphism (f has no heteroclinic points).

Denote by G the class of orientation-preserving Morse–Smale diffeomorphisms f defined on orientable surfaces M^2 and satisfying the condition $\mathrm{beh}(f) \leqslant 1$. The complete topological invariants of such diffeomorphisms known today are described in the section 4. However, no realization is described for these invariants. The approach proposed in this paper for classifying diffeomorphisms of the set G differs from all currently available approaches. It allows to solve the implementation problem in the class under consideration.

The classification of Morse–Smale diffeomorphisms with $\mathrm{beh}(f) = 2$ is complicated by the fact that diffeomorphisms of this class have an infinite number of heteroclinic orbits. The authors suggest that the approach described in this article can be improved and used in the classification of this more complex class of diffeomorphisms.

Let us describe the invariants.

1.1. Diffeomorphism scheme $f \in G$. Let $f \in G$. Then the set Σ_f of periodic orbits of the diffeomorphism f can be divided into subsets Σ_f^i, $i \in \{\omega, s, u, \alpha\}$ as follows:

- Σ_f^ω is the set of all sink points;
- Σ_f^s is the set of saddle points, whose unstable manifolds do not contain heteroclinic points;
- Σ_f^u is the set of all other saddle points;
- Σ_f^α is the set of all source points.

It follows from the properties of the introduced order $\prec$ that every orbit $\mathcal{O}_i \in \Sigma_f^u$ is in the relation $\mathcal{O}_j \prec \mathcal{O}_i$ with some periodic orbit $\mathcal{O}_j \in \Sigma_f^s$. Let

$$\mathcal{A}_f = \Sigma_f^\omega \cup W^u_{\Sigma_f^s}, \quad \mathcal{R}_f = \Sigma_f^\alpha \cup W^s_{\Sigma_f^u}, \quad V_f = M^2 \setminus (\mathcal{A}_f \cup \mathcal{R}_f).$$

In the paper [10] it is shown that the sets $\mathcal{A}_f$, $\mathcal{R}_f$ are an attractor and a repeller of the system, respectively. Let

$$\hat{V}_f = V_f / f.$$

According to the paper [9], each connected component of the orbit space $\hat{V}_f$ is homeomorphic to a two-dimensional torus. Denote by $p_f \colon : V_f \to \hat{V}_f$ the natural projection, which is also a covering map for the space $\hat{V}_f$.

We denote by $\hat{V}_i$, $i \in \{1, 2, \dots, n\}$, the connected components of the orbit space $\hat{V}_f$. We set $V_i = p_f^{-1}(\hat{V}_i)$ and denote by $p_i \colon V_i \to \hat{V}_i$ the natural projection. The covering p_i induces a non-trivial homomorphism $\eta_i \colon \pi_1(\hat{V}_i) \to m_i\mathbb{Z}$ that associates $[\hat{c}] \in \pi_1(\hat{V}_i)$ with the number μm_i such that any lifting of the curve $\hat{c}$ connects the point $x \in V_i$ with the point $f^{\mu m_i}(x)$. We set

$$\hat{V}_f = \hat{V}_1 \sqcup \dots \sqcup \hat{V}_n$$

and denote by η_f the map composed of the homomorphisms $\eta_1, \dots, \eta_n$.

Denote by $m_f \in \mathbb{N}$ the smallest number such that all points of the non-wandering set of the diffeomorphism f^{m_f} are fixed and the diffeomorphism f^{m_f} preserves the orientation on W^u_σ for all $\sigma \in \Sigma_f$. Let us set $\tilde{f} = f^{m_f}$ and note that the sets $\mathcal{A}_f$ and $\mathcal{R}_f$ are also an attractor and repeller for the diffeomorphism $\tilde{f}$. We set

$$\tilde{V}_f = V_f / \tilde{f}.$$

Denote by $\tilde{p}_f \colon V_f \to \tilde{V}_f$ the natural projection, which is also a covering map.

Let us introduce the following notation:

- $\mathcal{L}^s_f$, $\mathcal{L}^u_f$ are sets of all stable, unstable, respectively, saddle separatrices of the diffeomorphism f and $\mathcal{L}_f = \mathcal{L}^s_f \cup \mathcal{L}^u_f$;
- $L^s_f = \bigcup_{l\in\mathcal{L}^s_f} l$, $L^u_f = \bigcup_{l\in\mathcal{L}^u_f} l$ and $L_f = L^s_f \cup L^u_f$;
- $\hat{\mathcal{L}}^s_f = \{\hat{l} = p_f(l) \,|\, l \in \mathcal{L}^s_f\}$, $\hat{\mathcal{L}}^u_f = \{\hat{l} = p_f(l) \,|\, l \in \mathcal{L}^u_f\}$ and $\hat{\mathcal{L}}_f = \hat{\mathcal{L}}^s_f \cup \hat{\mathcal{L}}^u_f$;
- $\hat{L}^s_f = \bigcup_{\hat{l}\in\hat{\mathcal{L}}^s_f} \hat{l}$, $\hat{L}^u_f = \bigcup_{\hat{l}\in\hat{\mathcal{L}}^u_f} \hat{l}$ and $\hat{L}_f = \hat{L}^s_f \cup \hat{L}^u_f$;
- $\hat{\mathcal{P}}_f$ is an involution on the set $\hat{\mathcal{L}}_f$ such that for any element $\hat{l} \in \hat{\mathcal{L}}_f$ the equality $\hat{l} \cup \hat{\mathcal{P}}_f(\hat{l}) = p_f(W^\delta_\sigma \setminus \sigma)$ is holds for some $\delta \in \{s, u\}$ and $\sigma \in (\Sigma^s_f \cup \Sigma^u_f)$;
- $\tilde{\mathcal{L}}^s_f = \{\tilde{l} = \tilde{p}_f(l),\ |\, l \in \mathcal{L}^s_f\}$, $\tilde{\mathcal{L}}^u_f = \{\tilde{l} = \tilde{p}_f(l) \,|\, l \in \mathcal{L}^u_f\}$ and $\tilde{\mathcal{L}}_f = \tilde{\mathcal{L}}^s_f \cup \tilde{\mathcal{L}}^u_f$;
- $\tilde{L}^s_f = \bigcup_{\tilde{l}\in\tilde{\mathcal{L}}^s_f} \tilde{l}$, $\tilde{L}^u_f = \bigcup_{\tilde{l}\in\tilde{\mathcal{L}}^u_f} \tilde{l}$ and $\tilde{L}_f = \tilde{L}^s_f \cup \tilde{L}^u_f$;
- $\tilde{\mathcal{P}}_f$ is an involution on the set $\tilde{\mathcal{L}}_f$ such that for any element $\tilde{l} \in \tilde{\mathcal{L}}_f$ the equality $\tilde{l} \cup \tilde{\mathcal{P}}_f(\tilde{l}) = \tilde{p}_f(W^\delta_\sigma \setminus \sigma)$ is holds for some $\delta \in \{s, u\}$ and $\sigma \in (\Sigma^s_f \cup \Sigma^u_f)$.

For any diffeomorphism $f \in G$ we set

$$\hat{S}_f = (\hat{V}_f, \hat{\mathcal{L}}_f, \hat{\mathcal{P}}_f), \quad \tilde{S}_f = (\tilde{V}_f, \tilde{\mathcal{L}}_f, \tilde{\mathcal{P}}_f).$$

Definition 1 (Diffeomorphism scheme)**.** We will call the pair

$$\mathcal{S}_f = (\hat{S}_f, \tilde{S}_f)$$

the *scheme* of the diffeomorphism $f \in G$.

1.2. The set $\hat{S}_f$. Sets $\hat{S}_f$, $\hat{S}_{f'}$ for diffeomorphisms $f, f' \in G$ are called *equivalent* (see Fig. 2) if there exists a homeomorphism $\hat{\varphi}\colon \hat{V}_f \to \hat{V}_{f'}$ such that $\eta_f = \eta_{f'}\hat{\varphi}_*$ and

(1) $\hat{L}^s_{f'} = \hat{\varphi}(\hat{L}^s_f)$, $\hat{L}^u_{f'} = \hat{\varphi}(\hat{L}^u_f)$;

(2) the homeomorphism $\hat{\varphi}$ induces a one-to-one correspondence $\hat{\varphi}_\star\colon \hat{\mathcal{L}}_f \to \hat{\mathcal{L}}_{f'}$ by the formula $\hat{\varphi}_\star(\hat{l}) = \hat{\varphi}(\hat{l})$ and holds the property $\hat{\varphi}_\star\hat{\mathcal{P}}_f = \hat{\mathcal{P}}_{f'}\hat{\varphi}_\star$.

Figure 2 shows the diffeomorphisms $f, f' \in G$. On the left a diffeomorphism f of a two-dimensional sphere whose non-wandering set consists of fixed source points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, fixed saddle points $\sigma_1, \sigma_2, \sigma_3$ and a fixed sink point ω_1 is shown. The separatrices $l_1, \ldots, l_6 \in \mathcal{L}_f$ are also marked. The projections of the separatrices $\hat{\mathcal{L}}_f = \{\hat{l}_1, \ldots, \hat{l}_6\}$ on the set $\hat{V}_f = \hat{V}_1 \sqcup \hat{V}_2$. The involution $\hat{\mathcal{P}}_f$ acts as follows:

$$\hat{\mathcal{P}}_f(\hat{l}_1) = \hat{l}_2, \quad \hat{\mathcal{P}}_f(\hat{l}_3) = \hat{l}_4, \quad \hat{\mathcal{P}}_f(\hat{l}_5) = \hat{l}_6.$$

The figure on the right shows a diffeomorphism f' of a two-dimensional torus whose non-wandering set consists of fixed source points α'_1, α'_2, fixed saddle points $\sigma'_1, \sigma'_2, \sigma'_3$and a fixed sink point ω'_1. The separatrices $l'_1, \ldots, l'_6 \in \mathcal{L}_{f'}$ are also marked. the projections of the separatrices $\hat{\mathcal{L}}_{f'} = \{\hat{l}'_1, \ldots, \hat{l}'_6\}$ on the set $\hat{V}_{f'} = \hat{V}'_1 \sqcup \hat{V}'_2$. The involution $\hat{\mathcal{P}}_{f'}$ acts as follows:

$$\hat{\mathcal{P}}_{f'}(\hat{l}'_1) = \hat{l}'_3, \quad \hat{\mathcal{P}}_f(\hat{l}'_2) = \hat{l}'_4, \quad \hat{\mathcal{P}}_f(\hat{l}'_5) = \hat{l}'_6.$$

For $\hat{S}_f$, $\hat{S}_{f'}$ there exists a homeomorphism $\hat{\varphi}\colon \hat{V}_f \to \hat{V}_{f'}$ such that $\eta_f = \eta_{f'}\hat{\varphi}$ and point (1) of the sets equivalence holds. But any such homeomorphism does not

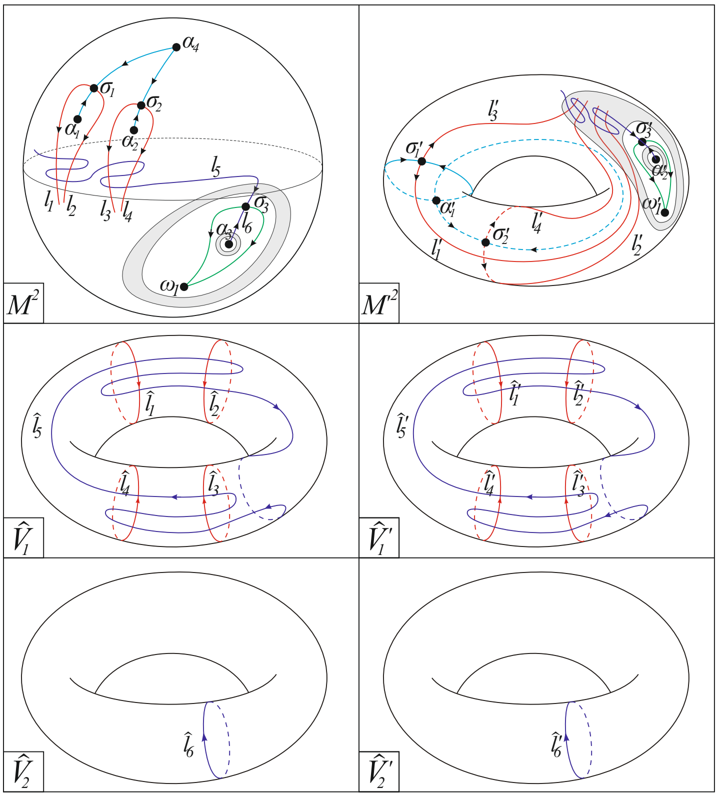

FIGURE 2. Phase portraits of topologically non-conjugate diffeomorphisms f, $f' \in G$ with non-equivalent sets $\hat{S}_f$, $\hat{S}_{f'}$.

satisfy condition (2) $\hat{\varphi}_\star \hat{\mathcal{P}}_f = \hat{\mathcal{P}}_{f'} \hat{\varphi}_\star$. The topological non-conjugacy of the diffeomorphisms f, f' obviously follows from non-homeomorphic supporting surface.

1.3. The set $\tilde{S}_f$. Sets $\tilde{S}_f$, $\tilde{S}_{f'}$ of the diffeomorphisms f, $f' \in G$ are called *equivalent* (see Fig. 3) if there exists a homeomorphism $\tilde{\varphi}\colon \hat{V}_f \to \hat{V}_{f'}$ such that:

(1) $\tilde{L}_{f'} = \tilde{\varphi}(\tilde{L}_f)$;

(2) the homeomorphism $\tilde{\varphi}$ induces a one-to-one correspondence $\tilde{\varphi}_\star(\tilde{l}) = \tilde{\varphi}(\tilde{l})$ by the formula $\tilde{\varphi}_\star\colon \tilde{\mathcal{L}}_f \to \tilde{\mathcal{L}}_{f'}$ and holds the property $\tilde{\varphi}_\star \tilde{\mathcal{P}}_f = \tilde{\mathcal{P}}_{f'} \tilde{\varphi}_\star$.

Figure 3 shows the diffeomorphisms f, $f' \in G$. On the left a diffeomorphism f of a two-dimensional sphere whose non-wandering set consists of fixed source points α_1, α_2, α_3, α_4, α_5, fixed saddle points σ_1, σ_2, σ_3, σ_4 and a fixed sink point ω_1 is shown. The separatrices $l_1, \dots, l_8 \in \mathcal{L}_f$ are also marked. The projections of the separatrices $\hat{\mathcal{L}}_f = \{\hat{l}_1, \dots, \hat{l}_4\}$ are marked on the set $\hat{V}_f = \hat{V}_1 \sqcup \hat{V}_2$. The involution $\hat{\mathcal{P}}_f$ acts as follows:

$$\hat{\mathcal{P}}_f(\hat{l}_1) = \hat{l}_2, \quad \hat{\mathcal{P}}_f(\hat{l}_3) = \hat{l}_4.$$

The separatrices $\tilde{\mathcal{L}}_f = \{\tilde{l}_1, \ldots, \tilde{l}_8\}$ are also marked on the set $\tilde{V}_f = \tilde{V}_1 \sqcup \tilde{V}_2$. The involution $\tilde{\mathcal{P}}_f$ acts as follows:

$$\tilde{\mathcal{P}}_f(\tilde{l}_1) = \tilde{l}_2, \quad \tilde{\mathcal{P}}_f(\tilde{l}_3) = \tilde{l}_4, \quad \tilde{\mathcal{P}}_f(\tilde{l}_5) = \tilde{l}_6, \quad \tilde{\mathcal{P}}_f(\tilde{l}_7) = \tilde{l}_8.$$

The figure on the right shows a diffeomorphism f' of a two-dimensional torus whose non-wandering set consists of fixed source points $\alpha'_1, \alpha'_2, \alpha'_3$, fixed saddle points $\sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4$ and the fixed sink point ω'_1. The separatrices $l'_1, \ldots, l'_8 \in \mathcal{L}_{f'}$ are also marked. The projections of the separatrices $\hat{\mathcal{L}}_{f'} = \{\hat{l}'_1, \ldots, \hat{l}'_4\}$ are marked on the set $\hat{V}_{f'} = \hat{V}'_1 \sqcup \hat{V}'_2$. The involution $\hat{\mathcal{P}}_{f'}$ acts as follows:

$$\hat{\mathcal{P}}_{f'}(\hat{l}'_1) = \hat{l}'_2, \quad \hat{\mathcal{P}}_f(\hat{l}'_3) = \hat{l}'_4.$$

The projections of the separatrices $\tilde{\mathcal{L}}'_f = \{\tilde{l}'_1, \ldots, \tilde{l}'_8\}$ are marked on the set $\tilde{V}_{f'} = \tilde{V}'_1 \sqcup \tilde{V}'_2$. The involution $\tilde{\mathcal{P}}_{f'}$ acts as follows:

$$\tilde{\mathcal{P}}_{f'}(\tilde{l}'_1) = \tilde{l}_4, \quad \tilde{\mathcal{P}}_{f'}(\tilde{l}'_2) = \tilde{l}'_5, \quad \tilde{\mathcal{P}}_{f'}(\tilde{l}'_3) = \tilde{l}'_6, \quad \tilde{\mathcal{P}}_{f'}(\tilde{l}'_7) = \tilde{l}'_8.$$

For sets $\hat{S}_f, \hat{S}_{f'}$ there is a homeomorphism $\hat{\varphi}\colon \hat{V}_f \to \hat{V}_{f'}$ such that $\eta_f = \eta_{f'}\hat{\varphi}$ and points (1) and (2) of the set equivalence hold. There is also a homeomorphism $\tilde{\varphi}\colon \tilde{V}_f \to \tilde{V}_{f'}$ for the sets $\tilde{S}_f, \tilde{S}_{f'}$ such that point (1) of the equivalence of sets is satisfied. But any such homeomorphism does not satisfy condition (2) $\tilde{\varphi}_\star\tilde{\mathcal{P}}_f = \tilde{\mathcal{P}}_{f'}\tilde{\varphi}_\star$. The topological non-conjugacy of the diffeomorphisms f, f' obviously follows from the non-homeomorphic of the supporting manifold.

1.4. Scheme equivalence class is a complete invariant

Definition 2 (Scheme equivalence)**.** Schemes $\mathcal{S}_f, \mathcal{S}_{f'}$ of diffeomorphisms $f, f' \in G$ will be called *equivalent* if there exists a homeomorphism $\hat{\varphi}\colon \hat{V}_f \to \hat{V}_{f'}$, implementing the equivalence of the sets $\hat{S}_f, \hat{S}_{f'}$ and lifting to the homeomorphism $\tilde{\varphi}\colon \tilde{V}_f \to \tilde{V}_{f'}$, which implements the equivalence of the sets $\tilde{S}_f, \tilde{S}_{f'}$.

Theorem 1. *Two diffeomorphisms $f, f' \in G$ are topologically conjugate if and only if their schemes $\mathcal{S}_f, \mathcal{S}_{f'}$ are equivalent.*

1.5. Abstract scheme. To solve the realization problem, we introduce the concept of an abstract scheme.

Let $m \in \mathbb{N}$, $m_i \in \mathbb{N}$, $i = 1, \ldots, n$, be the divisors of m and $V_i = (\mathbb{R}^2 \setminus (0, 0)) \times \mathbb{Z}_{m_i}$. Let us define a diffeomorphism $\phi_i\colon V_i \to V_i$ by the formula

$$\phi_i(x, y, k) = \begin{cases} (x, y, k+1), & k \in \{0, 1, \ldots, m_i - 2\}; \\ (\frac{x}{2}, \frac{y}{2}, 0), & k = m_i - 1. \end{cases}$$

Then $\hat{V}_i = V_i/\phi_i$ is a torus. Let $a_i = \{(x, y, 0) \in V_i\colon x > 0,\, y = 0\}$ be a ray with orientation to the initial, $b_i = \{(x, y, 0) \in V_i\colon x^2 + y^2 = 1\}$ be a circle with the clockwise orientation and $p_i\colon V_i \to \hat{V}_i$ be a natural projection. Then the curves $\hat{a}_i = p_i(a_i)$, $\hat{b}_i = p_i(b_i)$ are generators on the torus $\hat{V}_i$ (see Fig. 4). Hence any knot $c \subset \hat{V}_i$ with respect to these generators has a homotopy type $[c] = \langle \mu_c, \nu_c \rangle$. Let us define an epimorphism $\eta_i\colon \pi_1(\hat{V}_i) \to m_i\mathbb{Z}$ by the formula

$$\eta_i([c]) = \mu_c \cdot m_i.$$

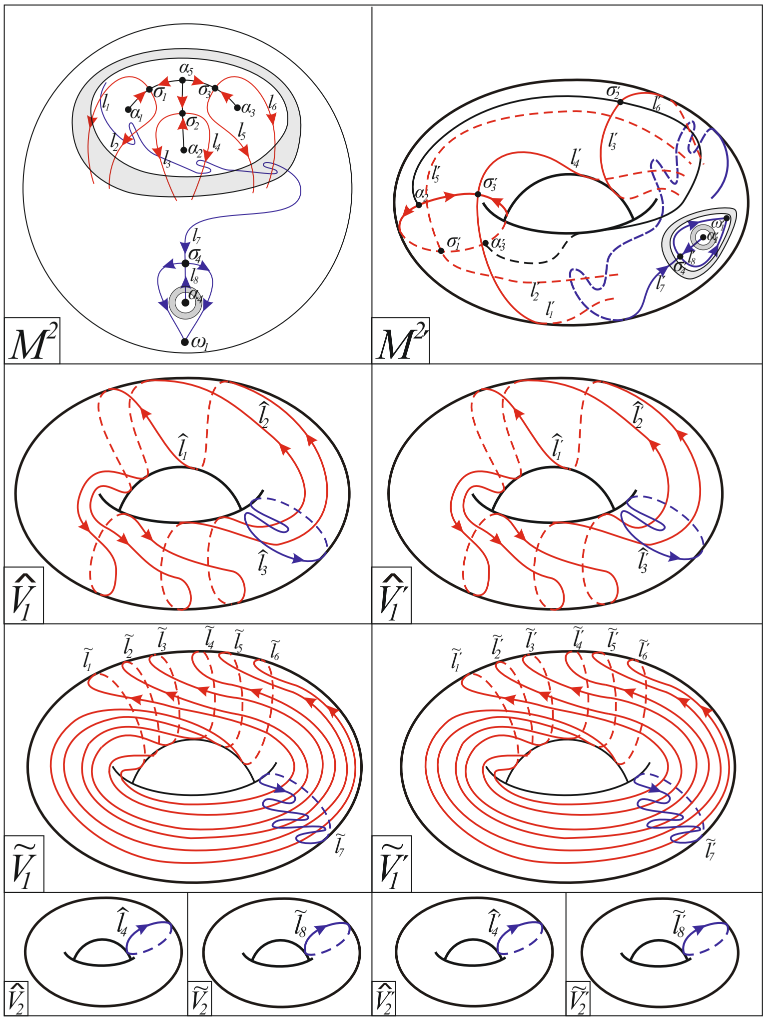

FIGURE 3. Phase portraits of topologically non-conjugate diffeomorphisms $f,\ f' \in G$ with non-equivalent sets $\tilde{S}_f,\ \tilde{S}_{f'}$

Let $\hat{\mathcal{L}}_i^s$ and $\hat{\mathcal{L}}_i^u$ be families of pairwise disjoint knots such that $\eta_i([l]) > 0$ for any $\hat{l} \in (\hat{\mathcal{L}}_i^s \cup \hat{\mathcal{L}}_i^u)$ and the sets $\hat{L}_i^s = \bigcup_{\hat{l} \in \hat{\mathcal{L}}_i^s} \hat{l}$, $\hat{L}_i^u = \bigcup_{\hat{l} \in \hat{\mathcal{L}}_i^u} \hat{l}$ intersect transversally. Let

$$\hat{\mathcal{L}}^s = \bigcup_{i=1}^{n} \hat{\mathcal{L}}_i^s, \quad \hat{\mathcal{L}}^u = \bigcup_{i=1}^{n} \hat{\mathcal{L}}_i^u, \quad \hat{\mathcal{L}} = \hat{\mathcal{L}}^s \cup \hat{\mathcal{L}}^u,$$

$$\hat{L}^s = \bigcup_{i=1}^{n} \hat{L}_i^s, \quad \hat{L}^u = \bigcup_{i=1}^{n} \hat{L}_i^u, \quad \hat{L} = \hat{L}^s \cup \hat{L}^u,$$

$$V = V_1 \sqcup \cdots \sqcup V_n, \quad \hat{V} = \hat{V}_1 \sqcup \cdots \sqcup \hat{V}_n.$$

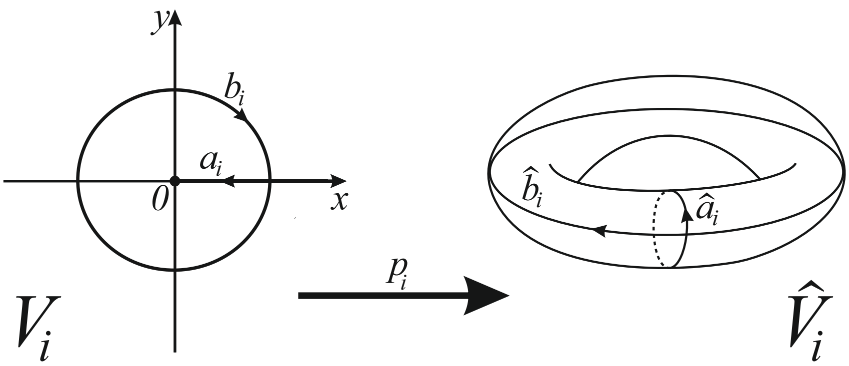

FIGURE 4. Generators $\hat{a}_i$, $\hat{b}_i$ on the torus $\hat{V}_i$.

Denote by $\phi\colon V \to V$ the map composed of the diffeomorphisms $\phi_1, \ldots, \phi_n$, by $p\colon V \to \hat{V}$ the mapping composed of the projections $p_1, \ldots, p_n$ and by η the map composed of the homomorphisms $\eta_1, \ldots, \eta_n$.

Let us define on the set $\hat{\mathcal{L}}$ an involution $\hat{\mathcal{P}}\colon \hat{\mathcal{L}} \to \hat{\mathcal{L}}$ such that:

- $\hat{\mathcal{P}}(\hat{\mathcal{L}}^s) = \hat{\mathcal{L}}^s$, $\hat{\mathcal{P}}(\hat{\mathcal{L}}^u) = \hat{\mathcal{L}}^u$;
- $\eta([\hat{\mathcal{P}}(\hat{l})]) = \eta([\hat{l}])$ for any $\hat{l} \in \hat{\mathcal{L}}$;
- the set $\hat{l} \cup \hat{\mathcal{P}}(\hat{l})$ has a non-empty intersection with the set $\hat{L}^s$ for any node $\hat{l} \in \hat{\mathcal{L}}^u$.

We set $\tilde{\phi} = \phi^m\colon V \to V$, $\tilde{V} = V/\tilde{\phi}$ and denote by $\tilde{p}\colon V \to \tilde{V}$ natural projection. We set $q = p\tilde{p}^{-1}\colon \tilde{V} \to \hat{V}$, $\tilde{L}^s = q^{-1}(\hat{L})$, $\tilde{L}^u = q^{-1}(\hat{L})$, $\tilde{L} = q^{-1}(\hat{L})$ and denote by $\tilde{\mathcal{L}}$ the set of connected components of the set $q^{-1}(\hat{\mathcal{L}})$. Note that the diffeomorphism ϕ induces a substitution $\tilde{\Phi}\colon \tilde{\mathcal{L}} \to \tilde{\mathcal{L}}$ by the formula

$$\tilde{\Phi}(\tilde{l}) = \tilde{p}\phi\tilde{p}^{-1}(\tilde{l}).$$

Let us define on the set $\tilde{\mathcal{L}}$ an involution $\tilde{\mathcal{P}}\colon \tilde{\mathcal{L}} \to \tilde{\mathcal{L}}$ such that:

- $\tilde{\Phi}\tilde{\mathcal{P}} = \tilde{\mathcal{P}}\tilde{\Phi}$, $q\tilde{\mathcal{P}} = \hat{\mathcal{P}}q$;
- the set obtained from $\tilde{V}$ by identifying the knots $\tilde{l}$, $\tilde{\mathcal{P}}(\tilde{l})$, $\tilde{l} \in \tilde{\mathcal{L}}$, is connected.

We state

$$\hat{S} = (\hat{V}, \hat{\mathcal{L}}, \hat{\mathcal{P}}), \quad \tilde{S} = (\tilde{V}, \tilde{\mathcal{L}}, \tilde{\mathcal{P}}).$$

Definition 3 (Abstract scheme)**.** A pair $\mathcal{S} = (\hat{S}, \tilde{S})$ is called an *abstract scheme.*

It is directly verified that the scheme of any diffeomorphism $f \in G$ satisfies all the properties of the abstract scheme. Moreover, the following realization theorem holds.

Theorem 2. *For any abstract scheme $\mathcal{S}$, there exists a diffeomorphism $f \in G$ whose scheme $\mathcal{S}_f$ is equivalent to the scheme $\mathcal{S}$.*

Given any abstract scheme $\mathcal{S} = (\hat{S}, \tilde{S})$, one can determine the genus of the surface on which the diffeomorphism $f \in G$ is realized by the given scheme. To do this, we construct two sets of circles S_ω, S_α from the set $\tilde{S}$ as follows.

By [8, Lemma 3.2.1], for each connected component $\tilde{V}_i$ of the set $\tilde{V}_f$ there exists a node β_i^s that intersects with every node $\tilde{l} \in \tilde{\mathcal{L}}^s \cap \tilde{V}_i$ at a single point (see Fig. 5), let's call such a knot *equator*. Denote by B^s the union of all such equators. Points $\tilde{l} \cap B^s$, $\tilde{\mathcal{P}}(\tilde{l}) \cap B^s$ are called *paired*. Denote by S_ω the set of knots obtained from B^s by taking a connected sum along pairwise disjoint neighbourhoods of paired points (see Fig. 5).

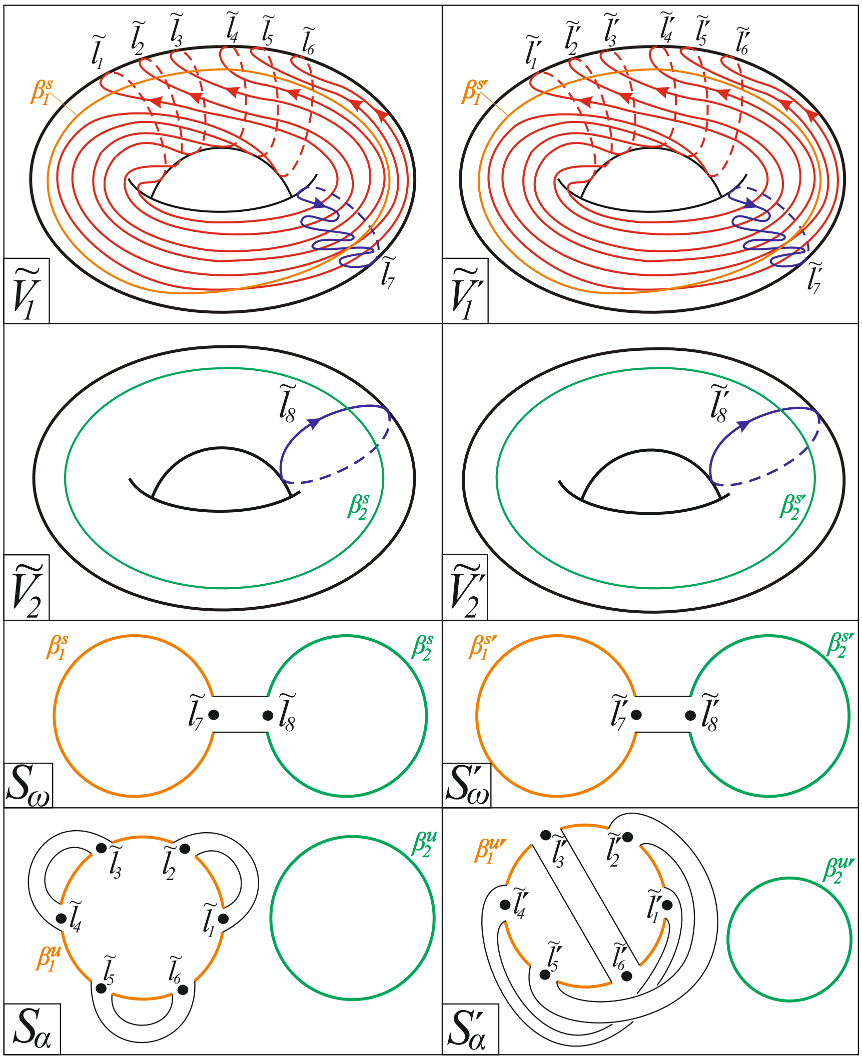

FIGURE 5. Sets $\tilde{S}$, $\tilde{S}'$ with marked knots $\beta_1^s = \beta_1^u$, $\beta_2^s = \beta_2^u$, $\beta_1'^s = \beta'^u_1$, $\beta_2'^s = \beta'^u_2$ and knots sets S_ω, S_α, S'_ω, S'_α, with $k^s = k'^s = 2$, $k^u = k'^u = 6$, $k^\omega = k'^\omega = 1$, $k^\alpha = 5$, $k'^\alpha = 3$.

A set of knots S_α is constructed similarly from the equators B^u to the knots $\tilde{\mathcal{L}}^s$. Denote by k^s, k^u, k^ω, k^α the number of knots in the sets $\tilde{\mathcal{L}}^s$, $\tilde{\mathcal{L}}^u$, S_ω, S_α, respectively.

The following lemma connects the genus of the supporting surface of the diffeomorphism $f \in G$ with scheme invariant $\mathcal{S}$.

Lemma 1. *The genus g of the supporting surface for the diffeomorphism $f \in G$ with the scheme $\mathcal{S}$ can be calculated by the formula*

$$2 - 2g = k^\alpha + k^\omega - \frac{1}{2}(k^s + k^u). \tag{1}$$

2. Necessary and Sufficient Conditions for Topological Conjugacy of Diffeomorphisms of Class G

Let $f\colon M^2 \to M^2$, $f'\colon M'^2 \to M'^2$ be diffeomorphisms from the class G and let $\mathcal{S}_f$, $\mathcal{S}_{f'}$ be the corresponding schemes. Let us prove that diffeomorphisms f, f' are topologically conjugate if and only if their schemes are equivalent.

2.1. Necessity. ($\Rightarrow$) Assume that there exists a homeomorphism $h\colon M^2 \to M'^2$ such that $hf = f'h$. We set $\hat{\varphi} = p_{_{f'}} h p_{_f}^{-1}\colon \hat{V}_f \to \hat{V}_{f'}$ and $\tilde{\varphi} = \tilde{p}_{f'} h \tilde{p}_f^{-1}\colon \tilde{V}_f \to \tilde{V}_{f'}$. Let us show that the homeomorphisms $\hat{\varphi}$, $\tilde{\varphi}$ satisfy the equivalence conditions for the schemes $\mathcal{S}_f$, $\mathcal{S}_{f'}$.

Let us show that the condition $\eta_f = \eta_{f'}\hat{\varphi}$ is satisfied. Let $\hat{V}_i$ be an element of the set $\hat{V}_f$ and $\hat{c} \in \hat{V}_i$ be a simple closed curve. We state $\hat{V}'_i = \hat{\varphi}(\hat{V}_i)$ and $\hat{c}' = \hat{\varphi}(\hat{c})$. Let $\hat{x} \in \hat{c}$. The curve $\hat{p}_f^{-1}(\hat{c})$ connects the points x, $f^{\eta_f m_i}(x) \in p_f^{-1}(\hat{x})$. Let $\hat{x}' = \hat{\varphi}(\hat{x})$, then the curve $\hat{p}_{f'}^{-1}(\hat{c}')$ connects the points x', $f'^{\eta_{f'}([\hat{c}])m_i}(x') \in p_f^{-1}(\hat{x}')$. Then $h(f^{\eta_f([\hat{c}])m_i}(x)) = f'^{\eta_{f'}([\hat{c}'])m_i}(h(x))$ and hence $\eta_f([\hat{c}]) = \eta_{f'}([\hat{\varphi}(\hat{c})])$.

2.1.1. *Equivalence of sets $\hat{S}_f$, $\hat{S}_{f'}$.* Let us prove the conditions (1), (2) of the equivalence of the sets $\hat{S}_f$, $\hat{S}_{f'}$ for the homeomorphism $\hat{\varphi}$.

(1) Since the conjugating homeomorphism h takes the invariant manifolds of periodic points of the diffeomorphism f into invariant manifolds of periodic points of the diffeomorphism f' with stability and dimension preserved, then

$$L^s_{f'} = h(L^s_f), \quad L^u_{f'} = h(L^u_f).$$

Then, since $\hat{\varphi} = p_{_{f'}} h p_f^{-1}$, then

$$\hat{L}^s_{f'} = p_{_{f'}}(L^s_{f'}) = p_{_{f'}}(h(L^s_f)) = p_{_{f'}}(h(p_f^{-1}(\hat{L}^s_f))) = \hat{\varphi}(\hat{L}^s_f).$$

Similarly $\hat{L}^u_{f'} = \hat{\varphi}(\hat{L}^u_f)$.

(2) Let $\hat{l}_1, \hat{l}_2 \in \hat{\mathcal{L}}_f$ be such that $\hat{l}_1 \cup \hat{l}_2 = p_f(W^\delta_\sigma)$ for some $\delta \in \{s, u\}$ and $\sigma \in (\Sigma^s_f \cup \Sigma^u_f)$. Then there exist $\hat{l}'_1 \cup \hat{l}'_2 \in \hat{\mathcal{L}}_{f'}$ such that $\hat{l}'_1 \cup \hat{l}'_2 = p_{f'}(h(W^\delta_\sigma))$ and hence

$$\hat{\varphi}_\star \hat{\mathcal{P}}_f(\hat{l}_1) = \hat{\varphi}_\star(\hat{l}_2) = \hat{\varphi}(\hat{l}_2) = \hat{l}'_2 = \hat{\mathcal{P}}_{f'}(\hat{l}'_1) = \hat{\mathcal{P}}_{f'}\hat{\varphi}(\hat{l}_1) = \hat{\mathcal{P}}_{f'}\hat{\varphi}_\star(\hat{l}_1).$$

2.1.2. *Set equivalence* $\tilde{S}_f$, $\tilde{S}_{f'}$. Let us prove the conditions (1), (2) of the equivalence of the sets $\tilde{S}_f$, $\tilde{S}_{f'}$ for the homeomorphism $\tilde{\varphi}$.

(1) Since h is a homeomorphism conjugating the diffeomorphisms f, f', one has $m_f = m_{f'}\,(= m)$. Then h also conjugates the diffeomorphisms $\tilde{f} = f^m$, $\tilde{f}' = f'^m$ and, consequently, the manifolds $\tilde{V}_f$, $\tilde{V}_{f'}$ are m-layered covers of the manifolds $\hat{V}_f$, $\hat{V}_{f'}$, respectively, and the homeomorphism $\tilde{\varphi}$ is a lift of the homeomorphism $\hat{\varphi}$. Moreover, the homeomorphism h maps invariant manifolds of periodic points of the diffeomorphism $\tilde{f}$ into invariant manifolds of periodic points of the diffeomorphism $\tilde{f}'$ with stability and dimension preserved. Whence it follows that

$$L^s_{\tilde{f}'} = h(L^s_{\tilde{f}}), \quad L^u_{\tilde{f}'} = h(L^u_{\tilde{f}}).$$

Since $\tilde{p}_f = p_{\tilde{f}}$, $\tilde{p}_{f'} = p_{\tilde{f}'}$ and $\tilde{\varphi} = p_{\tilde{f}'} h p^{-1}_{\tilde{f}}$, then

$$\tilde{L}^s_{f'} = p_{\tilde{f}'}(L^s_{\tilde{f}'}) = p_{\tilde{f}'}(h(L^s_{\tilde{f}})) = p_{\tilde{f}'}(h(p^{-1}_{\tilde{f}}(L^s_{\tilde{f}}))) = \tilde{\varphi}(\tilde{L}^s_f).$$

Similarly $\tilde{L}^u_{f'} = \tilde{\varphi}(\tilde{L}^u_f)$.

(2) Let $\tilde{l}_1, \tilde{l}_2 \in \tilde{\mathcal{L}}_f$ such that $\tilde{l}_1 \cup \tilde{l}_2 = \tilde{p}_f(W^\delta_\sigma)$ for some $\delta \in \{s, u\}$ and $\sigma \in (\Sigma^s_f \cup \Sigma^u_f)$. Then there exist $\tilde{l}'_1 \cup \tilde{l}'_2 \in \tilde{\mathcal{L}}_{f'}$ such that $\tilde{l}'_1 \cup \tilde{l}'_2 = \tilde{p}_{f'}(h(W^\delta_\sigma))$ and hence,

$$\tilde{\varphi}_\star \tilde{\mathcal{P}}_f(\tilde{l}_1) = \tilde{\varphi}_\star(\tilde{l}_2) = \tilde{\varphi}(\tilde{l}_2) = \tilde{l}'_2 = \tilde{\mathcal{P}}_{f'}(\tilde{l}'_1) = \tilde{\mathcal{P}}_{f'}\tilde{\varphi}(\tilde{l}_1) = \tilde{\mathcal{P}}_{f'}\tilde{\varphi}_\star(\tilde{l}_1).$$

2.2. Sufficiency. ($\Leftarrow$) Let schemes $\mathcal{S}_f$ and $\mathcal{S}_{f'}$ of diffeomorphisms $f, f' \in G$ are equivalent. Let us show that the diffeomorphisms f, f' are topologically conjugated. Let us construct a conjugating homeomorphism $h\colon M^2 \to M'^2$ step by step.

Step 1. The construction of the diffeomorphism $H\colon V_f \to V_{f'}$. Since schemes $\mathcal{S}_f$ and $\mathcal{S}_{f'}$ are equivalent, then there are homeomorphisms $\hat{\varphi}\colon \hat{V}_f \to \hat{V}_{f'}$ and $\tilde{\varphi}\colon \tilde{V}_f \to \tilde{V}_{f'}$ satisfying the conditions of equivalence of sets $\hat{S}_f$, $\hat{S}_{f'}$ and $\tilde{S}_f$, $\tilde{S}_{f'}$ respectively. Due to homeomorphism $\hat{\varphi}$ and condition $\eta_f = \eta_{f'}\hat{\varphi}$ it follows that $\hat{\varphi}(\hat{V}_i) = \hat{V}'_i$, which means $\forall i \in \{1, \ldots, n\}\ \exists!\, i' \in \{1, \ldots, n'\}$ and consequently $n = n'$ and $m_i = m'_{i'}$. Without loss of generality, we will assume that $i' = i$, since this can always be achieved by renumbering the components.

We define a homeomorphism $H_i\colon V_i \to V_i$ by the formula

$$H_i(x) = \tilde{p}^{-1}_{f'}(\tilde{\varphi}(\tilde{p}_f(x))).$$

and denote by $H\colon V_f \to V_{f'}$ the composition of diffeomorphisms $H_1, \ldots, H_n$.

Step 2. Extension of the homeomorphism to the set of saddle points. We extend by continuity the homeomorphism H to the homeomorphism $H\colon V_f \cup \Sigma^s_f \cup \Sigma^u_f \to V_{f'} \cup \Sigma^s_{f'} \cup \Sigma^u_{f'}$ as follows. For every curve $l^s_1 = \tilde{p}^{-1}_f(\tilde{l}^s_1)$ there is unique curve $l^s_2 = \tilde{p}^{-1}_f(\tilde{l}^s_2)$ such that $\tilde{l}^s_1 = \tilde{\mathcal{P}}_f(\tilde{l}^s_2)$ and $l^s_1 \cup l^s_2 = W^s_\sigma \setminus \sigma$, where $\sigma \in \Sigma^s_f$. It follows from the properties of the constructed homeomorphism H that, for the curves $l'^s_1 = H(l^s_1)$, $l'^s_2 = H(l^s_2)$, one has $l'^s_1 \cup l'^s_2 = W^s_{\sigma'} \setminus \sigma'$, where $\sigma' \in \Sigma^s_{f'}$. Let

$$H(\sigma) = \sigma'.$$

The equality $H(f^k(\sigma)) = f'^k(\sigma')$ extends the homeomorphism H to the orbit of the saddle point σ and, therefore, to the entire set Σ^s_f. Similarly, the homeomorphism H can be extended to the set Σ^u_f.

Similarly to the proof of Theorem 1 in [12] the homeomorphism H is modified to a homeomorphism $h\colon M^2 \to M'^2$ conjugating the diffeomorphisms f, f'.

3. Realization of Diffeomorphisms of the Class G by an Abstract Scheme

In this section, for any abstract scheme $\mathcal{S} = (\hat{S}, \tilde{S})$ we construct a diffeomorphism $f \in G_1$ such that the scheme $\mathcal{S}$ and $\mathcal{S}_f$ are equivalent. Moreover, we will prove Lemma 1 by establishing relation (1) for the constructed diffeomorphism. We will use the notation introduced in Section 1.

From each abstract scheme $\mathcal{S}$, a diffeomorphism $\phi\colon V \to V$ and the sets $L^s = \tilde{p}^{-1}(\tilde{L}^s)$, $L^u = \tilde{p}^{-1}(\tilde{L}^u)$ can be recovered. The sets L^s, L^u consisting of oriented curves divided into pairs $l = \tilde{p}^{-1}(\tilde{l})$, $l' = \tilde{p}^{-1}(\tilde{\mathcal{P}}(\tilde{l}))$. Moreover, each such pair corresponds to a natural number $k_l = \eta([p(l)])$. We choose pairwise disjoint ϕ-invariant tubular neighbourhoods N_l for pairs l, l'. We state $\phi_l = \phi|_{N_l}$.

Let $\mathcal{N} = \{(x, y) \in \mathbb{R}^2\colon |xy| \leqslant 1\}$, $\mathcal{N}^s = \mathcal{N} \setminus Oy$ and $\mathcal{N}^u = \mathcal{N} \setminus Ox$. We define on the set $\mathcal{N} \times \mathbb{Z}_k$ the following diffeomorphisms:

$$a_{k,+}(x, y, \kappa) = \begin{cases} (x, y, \kappa + 1) & \forall \kappa \in \{0, 1, \dots, k-1\}, \\ (\frac{x}{2}, 2y, 0) & \text{for } \kappa = k; \end{cases}$$

$$a_{k,-}(x, y, \kappa) = \begin{cases} (x, y, \kappa + 1) & \forall \kappa \in \{0, 1, \dots, k-1\}, \\ (-\frac{x}{2}, -2y, 0) & \text{for } \kappa = k. \end{cases}$$

Let $l \subset L^\delta$, $\delta \in \{s, u\}$. If $\phi^k(l) = l'$ for some $k \in \{1, \dots, k_l - 1\}$ then the orbit space of the action of the diffeomorphism ϕ_l on N_l is isomorphic to the two-dimensional annulus, which entails the existence of a diffeomorphism $\vartheta_l\colon N_l \to \mathcal{N}^\delta \times \mathbb{Z}_{k_l/2}$ such that $\vartheta_l(l \cup l') = Ox \setminus \{O\}$ and $\vartheta_l \circ \phi_l = a_{k_l/2,-} \circ \vartheta_l$. In this case let $\mathcal{N}_l = \mathcal{N} \times \mathbb{Z}_{k_l/2}$ and $a_l = a_{k_l/2,-}$. In the oppsite case the orbit space of the action of the diffeomorphism ϕ_l on N_l is isomorphic to a pair of two dimensional annulus, which entails the existence of a diffeomorphism $\vartheta_l\colon N_l \to \mathcal{N}^\delta \times \mathbb{Z}_{k_l}$ such that $\vartheta_l(l \cup l') = Ox \setminus \{O\}$ and $\vartheta_l \circ \phi_l = a_{k_l,+} \circ \vartheta_l$. In this case let $\mathcal{N}_l = \mathcal{N} \times \mathbb{Z}_{k_l}$ and $a_l = a_{k_l,+}$. In both cases, if the set $N_l \cap L^u$ is not empty, then we choose a diffeomorphism ϑ_l so that each connected component of the set $\vartheta_l(N_l \cap L^u)$ is parallel to axis Oy for $\delta = s$ and parallel to axis Ox for $\delta = u$.

Denote by N^δ the disjunct union of the sets $\mathcal{N}_l$, $l \subset L^\delta$, by a_δ the mapping composed of diffeomorphisms a_l and by ϑ_δ is a map composed of diffeomorphisms of ϑ_l. Denote by $W = V \sqcup N^s \sqcup N^u$ and introduce on the set W a minimal equivalence relation $\sim$, satisfying the following conditions: $x \sim y$, if $y = \vartheta_s(x)$; $x \sim y$, if $y = \vartheta_u(x)$. Denote by $\dot{M}^2$ the set of equivalence classes and by $p_{_W}\colon W \to \dot{M}^2$ the natural projection associating the point $x \in \tilde{W}$ its equivalent class. By [12, Theorem 3], $\dot{M}^2$ is a non-compact surface without boundary on which the diffeomorphisms ϕ, a_s, a_u induce a Morse–Smale diffeomorphism with $k^u + k^s$ saddle points, which allows compactification of exactly k^ω sink and k^α source points satisfying the relation (1). The diffeomorphism $f \in G$ constructed in this way on the closed surface M^2 is the desired one.

4. Historical Background

4.1. Classification of gradient-like diffeomorphisms on a surface. In this subsection, we describe known approaches to the classification of gradient-like diffeomorphisms on surfaces.

4.1.1. *Directed graph equipped with data.* Let f be a gradient-like diffeomorphism of an orientable surface M^2. Denote by σ the saddle point of the diffeomorphism f of period m_σ. Let ν_σ be the *orientation type of* σ, which is equal to 1 if the diffeomorphism $f^{m_\sigma}|_{W^u_\sigma}$ is orientation-preserving and -1 otherwise. Let l^s_σ (l^u_σ) be the stable (unstable) separatrix of the saddle point σ, i.e., l^s_σ (l^u_σ) is a connected component of the set $W^s_\sigma \setminus \sigma$ ($W^u_\sigma \setminus \sigma$). Since l^s_σ (l^u_σ) does not intersect any saddle point of the unstable (stable) manifold, there exists a sink point ω (source point α) such that $\mathrm{cl}(l^u_\sigma) = l^u_\sigma \cup \sigma \cup \omega$ ($\mathrm{cl}(l^s_\sigma) = l^s_\sigma \cup \sigma \cup \alpha$) [8, Lemma 3.2.1]. For $\delta \in \{s, u\}$, we assume that the separatrix l^δ_σ is directed towards the saddle point σ for $\delta = s$ and away from the saddle point for $\delta = u$.

We will say that the *oriented graph* Γ_f *is a diffeomorphism graph* f (see Fig. 6) if

1) *vertices* of the graph Γ_f correspond to periodic points of the non-wandering set Ω_f; the vertex corresponding to the periodic saddle point σ is equipped with the weight ν_σ;
2) *oriented edges* of the graph Γ_f correspond to directed separatrices of saddle points.

The diffeomorphism f induces an automorphism f_* of the graph Γ_f. Let Γ_f, $\Gamma_{f'}$ be the diffeomorphism graphs f, f'. For topologically conjugated diffeomorphisms f, f', it is necessary to have an isomorphism between the graphs Γ_f and $\Gamma_{f'}$ conjugating the automorphisms f_* and f'_*. Unfortunately, in the general case, the existence of a graph isomorphism is not enough for the diffeomorphisms f, f' to be conjugate, even if every periodic point is fixed and every separatrix is f-invariant. Indeed, consider the diffeomorphisms f and f' whose phase portraits are shown in the figure 6. Although these diffeomorphisms have isomorphic graphs, they are not topologically conjugate. To verify this, note that any conjugating homeomorphism necessarily maps the sink basin ω of the diffeomorphism f to the sink basin ω' of the diffeomorphism f'. However, such a homeomorphism cannot be extended to the entire sphere in such a way that it takes the invariant manifolds of saddle points of the diffeomorphism f into the invariant manifolds of saddle points of the diffeomorphism f'.

Thus, the directed graph Γ_f of the diffeomorphism f does not define the topological conjugacy class f, so the graph Γ_f must contain additional information. In order to obtain a complete classification of gradient-like diffeomorphisms on surfaces, in 1985 A. Bezdenezhnykh and V. Grines [1], [2] introduced *equipped graphs* similar to M. Peixoto graphs [16] for gradient-like streams. The equipment included information about the boundaries of the cells of the diffeomorphism. Below we present a modification of this equipment, proposed later by V. Grines and O. Pochinka [8].

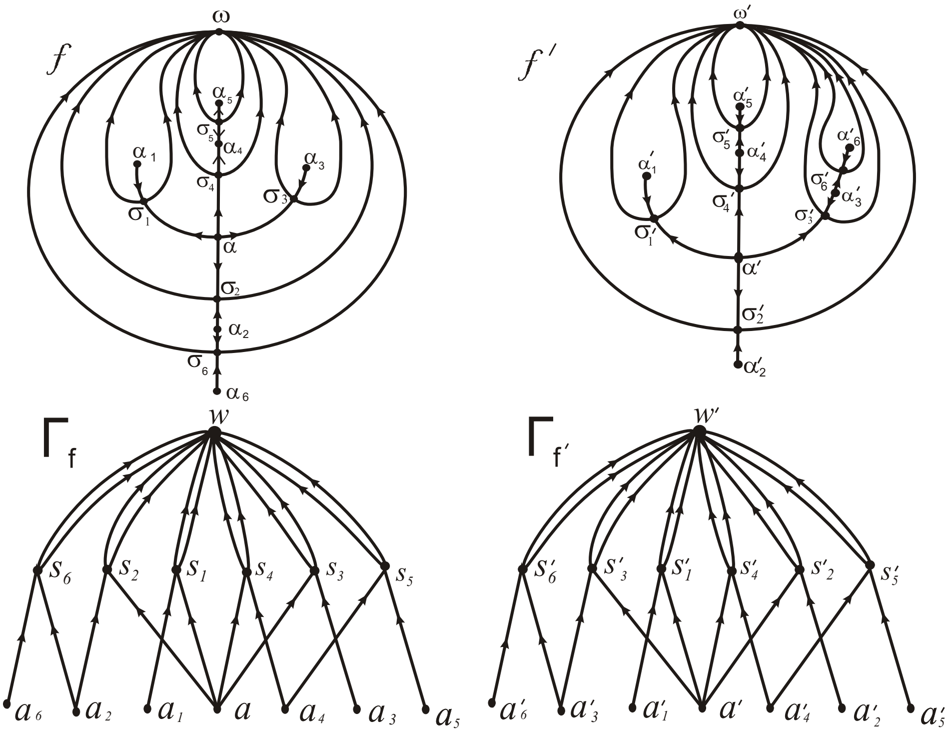

FIGURE 6. Diffeomorphisms f, $f'\colon \mathbb{S}^2 \to \mathbb{S}^2$ have isomorphic graphs, but they are not topologically conjugate.

Let ω be the sink of the diffeomorphism f and let L_ω be a subset of the manifold M^2 consisting of separatrices whose closures contain the sink ω. Then there exists a smooth 2-disk B_ω such that $\omega \in B_\omega$ and each separatrix $l \subset L_\omega$ intersects the set ∂B_ω at a single point [8, Proposition 2.1.3]. Such equipped graphs can be defined as follows. Let ω be a sink of f and let L_ω be the subset of the manifold M^2 that consists of the separatrices having ω in their closures. Then there exists a smooth 2-disk B_ω such that $\omega \in B_\omega$ and each separatrix $l \subset L_\omega$ intersects ∂B_ω at an unique point; see, for example, [8, Proposition 2.1.3]. For the vertex w corresponding to the periodic sink point ω, let E_w denote the set of edges of the directed graph Γ_f incident to w. Let N_w denote the cardinality of the set E_w. We enumerate the edges of the set E_w in the following way. First we pick in the basin of the sink ω a 2-disk B_ω and set $c_\omega = \partial B_\omega$. We define a pair of vectors $(\vec{\tau}, \vec{n})$ at some point of the curve c_ω in such a way that the vector $\vec{n}$ is directed inside the disk B_ω, the vector $\vec{\tau}$ is tangent to the curve c_ω and induces a counter-clockwise orientation on c_ω with respect to B_ω (we call this orientation positive). Enumerate the edges $e_1, \ldots, e_{N_w}$ from E_w according to the ordering of the corresponding separatrices as we move along c_ω starting from some point on c_ω. This enumeration of the edges of the set E_w is said to be *compatible* with the embedding of the separatrices.

Denote such a graph by Γ_f^*.

Let Γ_f^* and $\Gamma_{f'}^*$ be equipped graphs of diffeomorphisms f and f' respectively and let Γ_f^* and $\Gamma_{f'}^*$ be isomorphic by an isomorphism ξ. Let a vertex w of the graph Γ_f^* correspond to a sink and let $w' = \xi(w)$. Then the isomorphism ξ induces the permutation $\Theta_{w,w'}$ on $\{1, \dots, N\}$ (where $N = N_w = N_{w'}$) defined by $\Theta_{w,w'}(i) = j \Leftrightarrow \xi(e_i) = e'_j$.

Two equipped graphs Γ_f^*, $\Gamma_{f'}^*$ of diffeomorphisms f, f' are said to be *isomorphic* if there exists an isomorphism ξ of the graphs Γ_f, $\Gamma_{f'}$ such that

1) ξ sends the vertices into the vertices and preserves the values of the vertices corresponding to the saddle periodic points; it sends the edges into the edges and preserves their direction;
2) the permutation $\Theta_{w,w'}$ induced by ξ is a power of a cyclic permutation[1] for each vertex w corresponding to a sink;
3) $f'_* = \xi f_* \xi^{-1}$.

The isomorphism of equipped graphs is a complete invariant of the topological conjugacy of gradient-like Morse–Smale diffeomorphisms on closed surfaces. Let us show that the framed graphs Γ_f^*, $\Gamma_f'^*$ of the diffeomorphisms f, f' shown in the figure 6 are not isomorphic. To do this, suppose that the vertex w (w') of the graph corresponds to the sink ω (ω'). It is directly verified that any isomorphism ξ induces a permutation $\Theta_{w,w'}$ which is not a power of a cyclic permutation and, therefore, framed graphs Γ_f^*, $\Gamma_{f'}^*$ are not isomorphic.

4.1.2. *Tricolour graph.* Let us describe an alternative approach to the classification of gradient-like diffeomorphisms on surfaces proposed by V. Grines, S. Kapkaeva, and O. Pochinka [7] and similar to the approach of A. Oshemkov and V. Sharko for gradient-like flows [15].

Let, as before, f be a gradient-like Morse–Smale diffeomorphism on a closed surface M^2. The non-wandering set Ω_f can be represented as $\Omega_f = \Omega_f^0 \cup \Omega_f^1 \cup \Omega_f^2$, where Ω_f^0, Ω_f^1, Ω_f^2 denote the set of sinks, saddles, and sources of the diffeomorphism f, respectively. For the remainder of this subsection, we assume that f has at least one saddle point[2].

Remove from the surface M^2 the closure of the union of stable and unstable manifolds of all saddle points of the diffeomorphism f and denote the resulting set by $\tilde{M}$, i.e., $\tilde{M} = M^2 \setminus (\Omega_f^0 \cup W^u_{\Omega_f^1} \cup W^s_{\Omega_f^1} \cup \Omega_f^2)$. The set $\tilde{M}$ is represented as a union of areas (*cells*) homeomorphic to the open two-dimensional disk, the boundary of each of which has one of three types (see Fig. 7) and contains exactly one source, one sink, one or two saddle points and some of their separatrices.

Let A be any cell from the set $\tilde{M}$, and α and ω be the source and sink contained in its boundary. A simple curve $\tau \subset A$ whose boundary points are the source α

[1] It is directly checkable that the property of the permutation to be a power of a cyclic permutation is independent of the choice of the curves c_ω and $c_{\omega'}$.

[2] If a Morse–Smale diffeomorphism $f\colon M^n \to M^n$ has no saddle points, then its non-wandering set consists of one source and one sink. All "source-sink" diffeomorphisms are topologically conjugate; the proof of this fact is given, for example, in [8] (Theorem 2.2.1).

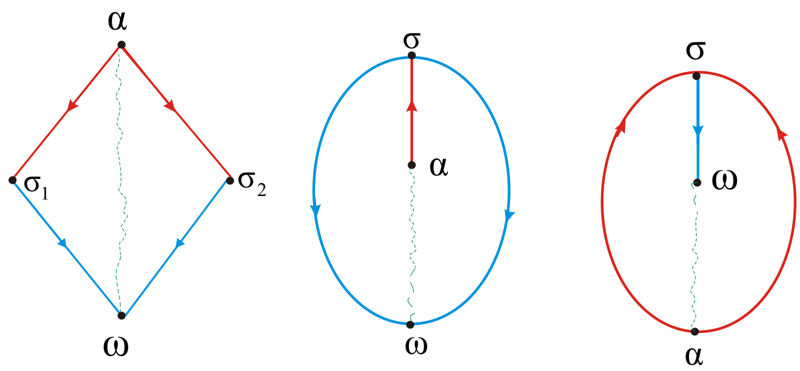

FIGURE 7. Cell types with t-curves

and the sink ω is called a *t-curve* (see Fig. 7). Denote by $\mathcal{T}$ the f-invariant set and which consists of t-curves taken one in each cell.

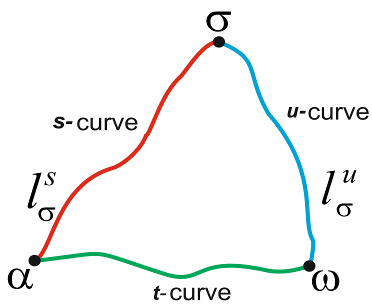

FIGURE 8. Triangular domain

Any connected component of the set $M_\Delta = \tilde{M} \setminus \mathcal{T}$ is called a *triangular domain*. Let Δ_f denote the set of all triangular domains of the diffeomorphism f. The boundary of each triangular domain $\delta \in \Delta_f$ contains three periodic points: source α, saddle σ, sink ω. It also contains the stable separatrix l_σ^s (we will call it *s-curve*) with boundary points α and σ, the unstable separatrix l_σ^u (we will call it *u-curve*) with boundary points ω and σ and the curve τ (we will call it t-curve) with boundary points α and ω (see Fig. 8). *Triangular area* bounded by s-, u- and t-curves. We will say that two triangular domains *have a common side* if this side belongs to the closures of both domains. Construct a *three-color* (s, t, u) graph T_f corresponding to a gradient-like Morse–Smale diffeomorphism f as follows (see Fig. 9):

1) T_f vertices are in one-to-one correspondence with triangular domains of Δ sets;
2) two vertices of a graph are incident to an edge of color s, t, and u if the corresponding triangular domains have a common s, t, and u-curve.

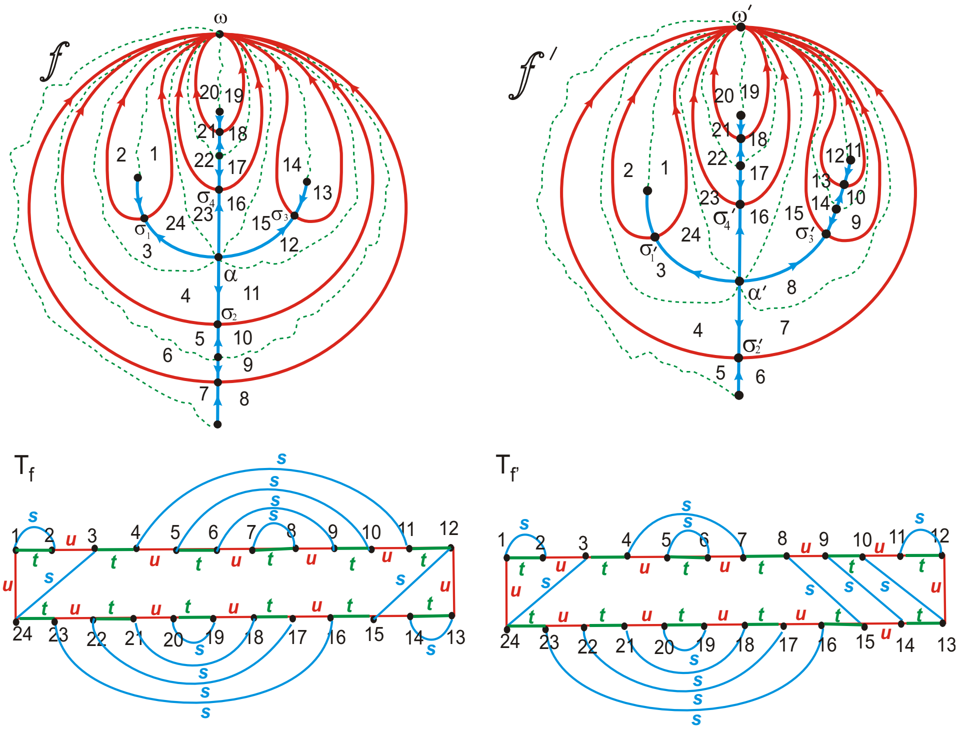

FIGURE 9. Non-isomorphic three-color graphs T_f, $T_{f'}$ are associated with non-conjugate gradient-like diffeomorphisms f, f' shown in the figure 6

By construction, tricolour graphs obtained from different partitions into triangular domains (depending on the choice of t -curve) are isomorphic. The diffeomorphism f induces an automorphism $f_* = \pi_f f \pi_f^{-1}$ on the vertex set of the graph T_f. Two three-color graphs with automorphisms (T_f, f_*) and $(T_{f'}, f'_*)$ of diffeomorphisms f, f' are called *isomorphic* if there exists a one-to-one correspondence ξ between the sets of their vertices which preserves the incidence and color relations, as well as the conjugating automorphisms of f_* and f'_* (that is, $f'_* = \xi f_* \xi^{-1}$). The isomorphism class of a three-color graph with substitution (T_f, f_*) is a complete invariant of the topological conjugacy of a gradient-like diffeomorphism f defined on a closed surface.

4.2. Classification of diffeomorphisms $\mathbf{beh}(f) = 1$. In this section, we present results known to the authors on the classification of Morse–Smale diffeomorphisms with a finite number of heteroclinic orbits.

4.2.1. *Heteroclinic substitution.* In 1993, V. Grines [6] proved that the invariant for some subset[3] of such diffeomorphisms is a graph similar to the Peixoto graph, equipped with a *heteroclinic permutation* describing the scheme of intersection of invariant manifolds, as in Figure 10. The isomorphism of such graphs is a necessary and sufficient condition for the topological conjugacy of the considered diffeomorphisms.

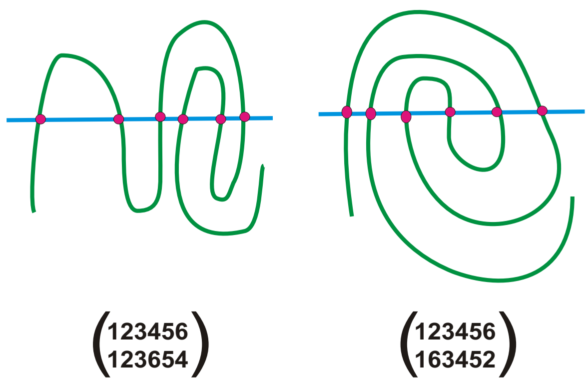

FIGURE 10. Heteroclinic substitution

4.2.2. *The scheme.* In 1993, R. Langevin [11] proposed to consider the *orbit space* of the basin sink and the projections of unstable separatrices of saddle points onto the resulting orbit space. This approach was generalized and successfully applied by Ch. Bonatti, V. Grines, V. Medvedev, E. Pecu and O. Pochinka in [3], [4] for the topological classification of Morse–Smale diffeomorphisms f with $\mathrm{beh}(f) \leqslant 1$ on 3-manifolds. In 2010, T. Mitryakova and O. Pochinka [13] applied this method in the topological classification of Morse–Smale diffeomorphisms f with $\mathrm{beh}(f) \leqslant 1$ on an orientable surface, under the assumption that all periodic the points of the diffeomorphism f are fixed. They constructed a topological invariant, called the "*scheme*", consisting of a finite number of two-dimensional tori corresponding to the orbit space of the sink and source basins, together with a set of simple closed curves corresponding to the orbit spaces of the separatrices (see Fig. 11). They also proved that this invariant is complete for the considered class of diffeomorphisms and in [14] they solved the problem of their realization using an admissible *abstract scheme*.

4.2.3. *Markov partitions.* In 1998, a different approach was taken by H. Bonatti and P. Langevin [5], who considered Smale diffeomorphisms of compact surfaces, that is, structurally stable diffeomorphisms with zero-dimensional basis sets. That is, the

[3]In [6], the class of Morse–Smale diffeomorphisms with a finite number of heteroclinic orbits was considered under the assumption that the orbits of the intersection points of connected fundamental segments of separatrices exhaust all heteroclinic orbits belonging to the intersection of these separatrices.

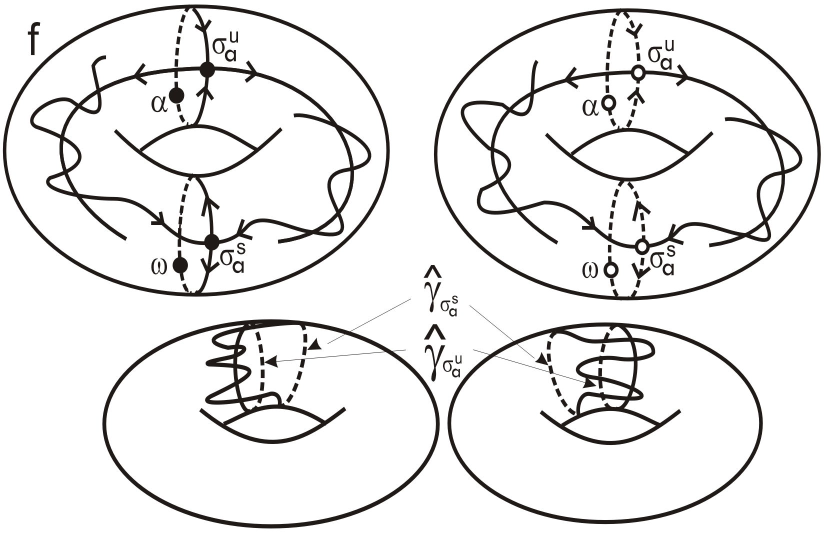

FIGURE 11. Scheme for a Morse–Smale diffeomorphism with $\mathrm{beh}(f) = 1$

classification of Morse–Smale diffeomorphisms was obtained as part of the classification of structurally stable diffeomorphisms with zero-dimensional basis sets. They proved that each Smale diffeomorphism corresponds to a finite combinatorial object, which is a set of geometric types of Markov partitions. However, Morse–Smale diffeomorphisms were not singled out for separate consideration, and therefore such a classification turned out to be unreasonably laborious for them.

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