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Kipriyanov's Fractional Calculus Prehistory and Legacy

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Abstract—This paper is partly a historical survey of various approaches and methods in the fractional calculus, partly a description of the Kipriyanov extraordinary theory in comparison with the classical one. The significance and outstanding methods in constructing the independent Kipriyanov fractional calculus theory are convexly stressed, also we represent modern results involving the Kipriyanov operator and corresponding generalization under the semigroup theory point of view.

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1. BRIEF HISTORICAL REVIEW

1.1. Birth of Kipriyanov's Fractional Calculus

Ivan Alexandrovich Kipriyanov was born on the Urals in the Chelyabinsk region. In 1945, Kipriyanov entered the Faculty of Physics and Mathematics of the Chelyabinsk Pedagogical Institute. In 1949 he was accepted to graduate school Steklov Mathematical Institute of the USSR Academy of Sciences, his supervisor was M.V. Keldysh In 1954 Kipriyanov defended his Ph.D. thesis "On summation of Fourier series and interpolation processes for functions of two variables". In this work, Kipriyanov found a class of functions of two variables, to which it is possible to completely transfer the results of S.M. Lozinsky that, in a sense, every theorem on convergence and summation of a one-dimensional Fourier series can be transferred to convergence and summation of the corresponding trigonometric interpolation process with equally spaced nodes. Moreover, these results remain valid for more general approximating processes. In his review of the dissertation, Professor Lozinsky noted the successful choice of a class of functions of two variables. This choice, as Lozinsky wrote, required analytical insight and provided the success of the work. In his review of the abstract of the dissertation, Professor L.V. Kantorovich noted that, having familiarized himself with the work of Kipriyanov on the abstract and two reports in Leningrad in November 1953 at the seminar on the theory of functions and functional analysis of Leningrad University, he formed a very positive opinion about it. The work made a very favorable impression on the other participants of the seminar, as its discussion showed. It should be noted that the seminar was attended by Professors Kantorovich, Lozinsky, I.P. Natanson, etc.

In the late 1950s and early 1960s, active work was carried out on the study of various kinds of functional spaces, important in themselves and also playing an important role in the modern theory of partial differential equations and probability theory.

In 1958, Kipriyanov's first work on fractional order derivatives appeared, in which the concept of fractional order partial derivatives for functions given in a cube was introduced, starting from the fractional integral in the sense of Marchaud and derivatives of the integer order in the sense of S.L. Sobolev,

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here we should note that the implemented approach was unique and distinctive in comparison with the regular fractional calculus theory of the time. The definition of the Kipriyanov fractional derivative is based upon some integral identity that relates to the Marchaud fractional derivative. The corresponding integral representation is given, the definition of two functional spaces is given, and embedding theorems and space completeness theorems are proved for them. One of these theorems in the one-dimensional case significantly complements the well-known theorem of Hardy and Littlewood on fractional integrals.

In the next cycle of works from 1959 to 1961, Kipriyanov studied the fractional derivatives in the direction introduced by him [1–4]. In 1960 Kipriyanov published a paper on the operator of fractional differentiation [2], which is a fractional order operator with respect to a second-order elliptic operator with smooth coefficients. It is remarkable that this operator allows us to study boundary value problems for differential equations containing, in addition to partial derivatives, fractional derivatives. Here, we should interrupt the description of Kipriyanov's other achievements and focus on the specific questions regarding his most significant scientific contribution—fractional calculus as an independently constructed axiomatic theory.

1.2. Branches in Generalizations of the Riemann–Liouville an Marchaud Operators

The central point of fractional calculus is a concept of fractional differentiation. In this regard, we should admit that the Riemann–Liouville operator of fractional differentiation is at the origin of the concept and plays a special role in the science. Such operators as Caputo and Marchaud certainly are worth mentioned in the context, the first one is not interesting for us since it is more like a reduction of the Riemann–Liouville operator on smooth functions disappearing at the initial point (if we consider the matter from the point of view that is of functional analysis), but the second one does completely reflect a true mathematical nature of fractional derivative as a notion, since it has a representation in terms of infinitesimal generator of the corresponding semigroup [5]. It is clear that considering such an approach we are forced to deal with more general notions of the operator theory and in this way the understanding of the notion of fractional derivative as a fractional power of infinitesimal generator is harmoniously completed, on the one hand.

On the other hand, for a harmony of the narrative, we should referee an extract of the paper [6] appealing to another generalization, if we interpret the fractional differential Riemann-Liouville operator as a particular case of the derivative of the convolution operator for which the so called Sonin condition holds [7]. We should note that the second direction in understanding the matter was developed by mathematicians such as Rubin [8–10], Vakulov [11], Samko [12, 13], Karapetyants [14, 15]. Let us remind that the so called mapping theorem for the Riemann–Liouville operator (the particular case of the Sonin operator) were firstly studied by Hardy and Littlewood [16] and nowadays is known as the Hardy–Littlewood theorem with limit index. However there was an attempt to extend this theorem on some class of weighted Lebesgue spaces defined as functional spaces endowed with the following norm

$$||f||_{L_p(I,\beta,\gamma)} := ||f||_{L_p(I,\mu)}, \quad \mu(x) = \omega^{\beta,\gamma}(x) := (x-a)^{\beta}(b-x)^{\gamma}, \quad \beta,\gamma \in \mathbb{R}, \quad I := (a,b).$$

In this direction the mathematicians such as Rubin and Karapetyants [14] had success, the following problem was considered $I^{\alpha}_{a+} : L_p(I,\beta,\gamma) \longrightarrow ?$ However the converse theorem was not! All these create the prerequisite to invent another approach for studying mapping properties of the Riemann–Liouville operator or, more generally, integral operators. Thus, trying to solve (at least in particular) more general problem, in the paper [17] we deal with mapping theorems for operators acting on Banach spaces in order to obtain afterwards the desired results applicable to integral operators. In this regard the following papers are worth noticing [18, 19], where in additional, a special technique based on the properties of the Jacobi polynomials was introduced. Based on this approach, in the paper [6] we offer a method of studying the Sonin operator [7], which is defined as a convolution operator ${}_sI^{\varrho}_{a+}\varphi := \varrho *_a \varphi$ under some conditions (the so called Sonin conditions) imposed on the kernel ϱ, i.e., there exists a function ϑ such that $\varrho * \vartheta = 1$. The particular case of the Sonin kernel is a kernel of the fractional integral Riemman–Liouville operator, many other examples can be found in papers [20, 21], the first one gives us a survey considering various types of kernels such as the Bessel-type function, the power-exponential function, the incomplete gamma function e.t.c., the main concept of the second one is to construct a widest class of functions being a Sonin kernel. Here, we can partly close the matter at this point having noted that there was a successful attempt to establish a criterion of the solvability of the Sonin–Abel Equation in the Weighted Lebesgue Space [6].

However, let us be back to the first understanding of the matter which is closely connected with the notion of the Kipriyanov operator.

1.3. The Semigroup Approach and the Spectral Theory

The idea discussed in this paragraph relates to a model that gives us a representation of a composition of fractional differential operators in terms of the semigroup theory. For instance we can represent a second order differential operator as a some kind of a transform of the infinitesimal generator of a shift semigroup. Continuing this line of reasonings we generalized a differential operator with a fractional integro-differential composition in final terms to some transform of the corresponding infinitesimal generator and introduced a class of transforms of m-accretive operators. Further, we used methods obtained in the papers [22, 23] to study spectral properties of non-selfadjoint operators acting in a complex separable Hilbert space, these methods alow us to obtain an asymptotic equivalence between the real component of the resolvent and the resolvent of the real component of an operator. Due to such an approach we obtain relevant results since an asymptotic formula for the operator real component can be established in many cases (see [24, 25]). Thus, a classification in accordance with resolvent belonging to the Schatten–von Neumann class was obtained, a sufficient condition of completeness of the root vectors system was formulated.

The latter approach allows to construct an abstract model of a differential operator with a fractional Kipriyanov integro-differential operator composition in final terms, where modeling is understood as an interpretation of concrete differential operators in terms of the infinitesimal generator of a corresponding semigroup. Moreover, we can consider an approach in contracting the space of fractionally-differentiable functions which originates from the analog created by Kipriyanov and goes further up to the semigroup theory generalizations. In this paper we deal with a more general operator—a differential operator with a fractional integro-differential operator composition in final terms, which covers the corresponding one-dimensional operator. Various types of fractional integro-differential operator compositions were studied by such mathematicians as Prabhakar [26], Love [27], Erdelyi [28], McBride [29], Dimovskii and Kiryakova [30], Nakhushev [31]. In particular the aim of this paper is to represent a description of the previously obtained results under a specific point of view related with Kipriyanov's fractional calculus.

1.4. Evolution Equations with the Operator Function in the Second Term

Having created a direction of the spectral theory of non-selfadjoint operators, we can consider abstract theoretical results as a base for further research studying such mathematical objects as a Cauchy problem for evolution equation of fractional order in the abstract Hilbert space. We consider in the second term an operator function defined on a special operator class covering a generator transform considered in [5] and discussed in the previous paragraph, where a corresponding semigroup is supposed to be a C_0 semigroup of contractions. In its own turn the transform reduces to a linear composition of differential operators of real order in various senses such as the Riemann–Liouville fractional differential operator, the Kipriyanov operator, the Riesz potential, the difference operator [2, 5, 32]. Moreover, in the paper [33] we broadened the class of differential operators having considered the artificially constructed normal operator that cannot be covered by the Lidskii results [34]. It should be noted that the Kipriyanov operator is very useful in theoretical constructions as well as in applications since it covers Euclidean spaces and can be considered as a term in a perturbation of a differential operator of an arbitrary odd order acting in n-dimensional Euclidean space. This fact is based upon the the brilliant idea of Kipriyanov to consider directional coordinates in the n-dimensional Euclidean space, the latter approach is independent on dimension what is an enormous advantage for we can consider compositions of operators having various nature.

The application part of the theory involving fractional integro-differential constructions appeals to the results and problems which can be considered as particular cases of the abstract ones, the following papers a worth noting within the context [35–37]. At the same time, we should admit that abstract methods can be "clumsy" for some peculiarities can be considered only by a unique technique what forms a main contribution of the specialists dealing with concrete differential equations. Here, we should add that the relevance of the abstract problems can be expressed convexly by virtue of the application of the fractional integro-differential compositions with the Kipriyanov operator in physics and engineering sciences.

Apparently, in the paper [38] we realized the idea to broaden the class of fractional integro-differential compositions having considered a notion of operator function applicably to a Cauchy problem for an abstract fractional evolution equation with an operator function in the second term not containing the time variable, where the derivative in the first term is supposed to be of fractional order. Here, we should note that regarding to functional spaces we have that an operator function generates a variety of operators acting in a corresponding space. In this regard, even a power function gives us an interesting result [33]. In the context of the existence and uniqueness theorems, a significant refinement that is worth highlighting is the obtained formula for the solution represented by a series on the root vectors. In the absence of the norm convergence of the root vector series, we need to consider a notion of convergence in weaker Bari, Riesz, Abel–Lidskii senses [34, 39, 40].

In spite of the claimed rather applied objectives from the operator theory point of view, we admit that the problem of the root vectors expansion for a non-selfadjoint unbounded operator still remains relevant in the context of the paper. It is remarkable that the problem origins nearly from the first half of the last century [5, 22, 23, 34, 39, 41–47]. However, we have a particular interest when an operator is represented by a linear combination of operators where a so-called senior term is non-selfadjoint for a case corresponding to a selfadjoint operator was thoroughly studied in the papers [41–46]. In this regard the linear combination of the second order differential operator and the Kipriyanov operator represents a relevant model class for which the obtained spectral theory results [23] created a prerequisite for further abstract generalizations [5, 22].

2. ABSTRACT METHOD

2.1. Preliminaries

Let $C, C_i,\ i \in \mathbb{N}_0$ be positive constants. We assume that a value of C can be different in various formulas and parts of formulas but values of C_i are certain.

Denote by $\mathrm{Fr}\, M$ the set of boundary points of the set M. Everywhere further, if the contrary is not stated, we consider linear densely defined operators acting on a separable complex Hilbert space $\mathfrak{H}$.

Denote by $\mathcal{B}(\mathfrak{H})$ the set of linear bounded operators on $\mathfrak{H}$. Denote by $\tilde{L}$ the closure of an operator L. We establish the following agreement on using symbols $\tilde{L}^i := (\tilde{L})^i$, where i is an arbitrary symbol.

Denote by $\mathrm{D}(L)$, $\mathrm{R}(L)$, $\mathrm{N}(L)$ the *domain of definition*, the *range*, and the *kernel* or *null space* of an operator L, respectively. The deficiency (codimension) of $\mathrm{R}(L)$, dimension of $\mathrm{N}(L)$ are denoted by $\mathrm{def}\, L$, $\mathrm{nul}\, L$ respectively. Assume that L is a closed operator acting on $\mathfrak{H}$, $\mathrm{N}(L) = 0$, let us define a Hilbert space

$$\mathfrak{H}_L := \big\{ f, g \in \mathrm{D}(L),\ (f, g)_{\mathfrak{H}_L} = (Lf, Lg)_{\mathfrak{H}} \big\}.$$

Consider a pair of complex Hilbert spaces $\mathfrak{H}, \mathfrak{H}_+$, the notation $\mathfrak{H}_+ \subset\subset \mathfrak{H}$ means that $\mathfrak{H}_+$ is dense in $\mathfrak{H}$ as a set of elements and we have a bounded embedding provided by the inequality $\|f\|_{\mathfrak{H}} \le C_0 \|f\|_{\mathfrak{H}_+}$, $C_0 > 0$, $f \in \mathfrak{H}_+$, moreover, any bounded set with respect to the norm $\mathfrak{H}_+$ is compact with respect to the norm $\mathfrak{H}$.

Let L be a closed operator, for any closable operator S such that $\tilde{S} = L$, its domain $\mathrm{D}(S)$ will be called a core of L. Denote by $\mathrm{D}_0(L)$ a core of a closeable operator L.

Let $\mathrm{P}(L)$ be the resolvent set of an operator L and $R_L(\zeta)$, $\zeta \in \mathrm{P}(L)$, $[R_L := R_L(0)]$ denotes the resolvent of an operator L. Denote by $\lambda_i(L)$, $i \in \mathbb{N}$ the eigenvalues of an operator L.

Suppose L is a compact operator and $N := (L^*L)^{1/2}$, $r(N) := \dim \mathrm{R}(N)$; then the eigenvalues of the operator N are called the *singular numbers* (*s-numbers*) of the operator L and are denoted by $s_i(L)$, $i = 1, 2, ..., r(N)$. If $r(N) < \infty$, then we put by definition $s_i = 0$, $i = r(N) + 1, 2, ...$

Let $\nu(L)$ denotes the sum of all algebraic multiplicities of an operator L. Denote by $n(r)$ a function equals to the quantity of the elements of the sequence $\{a_n\}_1^\infty$, $|a_n| \uparrow \infty$ within the circle $|z| < r$. Let A be a compact operator, denote by $n_A(r)$, *counting function* a function $n(r)$ corresponding to the sequence $\{s_i^{-1}(A)\}_1^\infty$.

Let $\mathfrak{S}_p(\mathfrak{H})$, $0 < p < \infty$ be a Schatten–von Neumann class and $\mathfrak{S}_\infty(\mathfrak{H})$ be the set of compact operators.

Denote by $\tilde{\mathfrak{S}}_\rho(\mathfrak{H})$ the class of the operators such that $A \in \tilde{\mathfrak{S}}_\rho(\mathfrak{H}) \Rightarrow \{A \in \mathfrak{S}_{\rho+\varepsilon},\ A \overline{\in} \mathfrak{S}_{\rho-\varepsilon},\ \forall \varepsilon > 0\}$. In accordance with [47], we will call it *Schatten–von Neumann class of the convergence exponent*.

Suppose L is an operator with a compact resolvent and $s_n(R_L) \leq C\, n^{-\mu}$, $n \in \mathbb{N}$, $0 \leq \mu < \infty$; then we denote by $\mu(L)$ order of the operator L (see [45]).

Denote by

$$\mathfrak{Re} L := (L + L^*)/2, \quad \mathfrak{Im} L := (L - L^*)/2i$$

the real and imaginary Hermitian components of an operator L respectively. In accordance with the terminology of the monograph [48], the set

$$\Theta(L) := \{z \in \mathbb{C} : z = (Lf, f)_{\mathfrak{H}},\ f \in \mathrm{D}(L),\ \|f\|_{\mathfrak{H}} = 1\}$$

is called the *numerical range* of an operator L.

An operator L is called *sectorial* if its numerical range belongs to a closed sector

$$\mathfrak{L}_\iota(\theta) := \{\zeta : |\arg(\zeta - \iota)| \leq \theta < \pi/2\},$$

where ι is the vertex and θ is the semi-angle of the sector $\mathfrak{L}_\iota(\theta)$. If we want to stress the correspondence between ι and θ, then we will write θ_ι. An operator L is called *bounded from below*, if the following relation holds

$$\mathrm{Re}(Lf, f)_{\mathfrak{H}} \geq \gamma_L \|f\|^2_{\mathfrak{H}}, \quad f \in \mathrm{D}(L), \quad \gamma_L \in \mathbb{R},$$

where γ_L is called a lower bound of L.

An operator L is called *accretive* if $\gamma_L = 0$. An operator L is called *strictly accretive* if $\gamma_L > 0$. An operator L is called *m-accretive*, if the next relation holds

$$(A + \zeta)^{-1} \in \mathcal{B}(\mathfrak{H}), \quad \|(A + \zeta)^{-1}\| \leq (\mathrm{Re}\zeta)^{-1}, \quad \mathrm{Re}\zeta > 0.$$

An operator L is called *symmetric*, if one is densely defined and the following equality holds

$$(Lf, g)_{\mathfrak{H}} = (f, Lg)_{\mathfrak{H}}, \quad f, g \in \mathrm{D}(L).$$

Consider a sesquilinear form $s[\cdot,\cdot]$ (see [48]) defined on a linear manifold of the Hilbert space $\mathfrak{H}$. Let $\mathfrak{h} = (s + s^*)/2$, $\mathfrak{k} = (s - s^*)/2i$ be a real and imaginary component of the form s respectively, where $s^*[u, v] = \overline{s[v, u]}$, $\mathrm{D}(s^*) = \mathrm{D}(s)$. Denote by $s[\cdot]$ the quadratic form corresponding to the sesquilinear form $s[\cdot,\cdot]$. According to these definitions, we have $\mathfrak{h}[\cdot] = \mathrm{Re}\, s[\cdot]$, $\mathfrak{k}[\cdot] = \mathrm{Im}\, s[\cdot]$. Denote by $\tilde{s}$ the closure of a form s. The range of a quadratic form $s[f]$, $f \in \mathrm{D}(s)$, $\|f\|_{\mathfrak{H}} = 1$ is called the *range* of the sesquilinear form s and is denoted by $\Theta(s)$. A form s is called *sectorial* if its range belongs to a sector having a vertex ι situated at the real axis and a semi-angle $0 \leq \theta < \pi/2$. Due to Theorem 2.7 [48, p. 323] there exist unique m-sectorial operators $T_s, T_{\mathfrak{h}}$ associated with the closed sectorial forms $s, \mathfrak{h}$ respectively. The operator $T_{\mathfrak{h}}$ is called a *real part* of the operator T_s and is denoted by $\mathrm{Re}\, T_s$.

Assume that T_t, $(0 \leq t < \infty)$ is a semigroup of bounded linear operators on $\mathfrak{H}$, by definition put

$$Af = -\lim_{t \to +0} \left(\frac{T_t - I}{t} \right) f,$$

where $\mathrm{D}(A)$ is a set of elements for which the last limit exists in the sense of the norm $\mathfrak{H}$. In accordance with definition [49, p. 1] the operator $-A$ is called the *infinitesimal generator* of the semigroup T_t.

Let $f_t : I \to \mathfrak{H}$, $t \in I := [a, b]$, $-\infty < a < b < \infty$. The following integral is understood in the Riemann sense as a limit of partial sums

$$\sum_{i=0}^{n} f_{\xi_i} \Delta t_i \xrightarrow{\mathfrak{H}} \int_I f_t dt, \quad \lambda \to 0,$$

where $(a = t_0 < t_1 < ... < t_n = b)$ is an arbitrary splitting of the segment I, $\lambda := \max\limits_i (t_{i+1} - t_i)$, ξ_i is an arbitrary point belonging to $[t_i, t_{i+1}]$. The sufficient condition of the last integral existence is a continuous property (see [50, p. 248]), i.e., $f_t \xrightarrow{\mathfrak{H}} f_{t_0}, t \to t_0, \forall t_0 \in I$. The improper integral is understood as a limit

$$\int\limits_a^b f_t dt \xrightarrow{\mathfrak{H}} \int\limits_a^c f_t dt, \quad b \to c, \quad c \in [-\infty, \infty].$$

Using notations of the paper [1], we assume that Ω is a convex domain of the n-dimensional Euclidean space $\mathbb{E}^n$, P is a fixed point of the boundary $\partial\Omega$, $Q(r, \mathbf{e})$ is an arbitrary point of Ω. Let $\mathrm{d} := \mathrm{diam}\Omega$, we denote by $\mathbf{e}$ a unit vector having a direction from P to Q, denote by $r = |P - Q|$ the Euclidean distance between the points P, Q, and use the shorthand notation $T := P + \mathbf{e}t$, $t \in \mathbb{R}$. We consider the Lebesgue classes $L_p(\Omega)$, $1 \le p < \infty$ of complex valued functions. For the function $f \in L_p(\Omega)$, we have

$$\int\limits_\Omega |f(Q)|^p dQ = \int\limits_\omega d\chi \int\limits_0^{d(\mathbf{e})} |f(Q)|^p r^{n-1} dr < \infty, \tag{1}$$

where $d\chi$ is an element of solid angle of the unit sphere surface (the unit sphere belongs to $\mathbb{E}^n$) and ω is a surface of this sphere, $d := d(\mathbf{e})$ is the length of the segment of the ray going from the point P in the direction $\mathbf{e}$ within the domain Ω. Without loss of generality, we consider only those directions of $\mathbf{e}$ for which the inner integral on the right-hand side of equality (1) exists and is finite. It is the well-known fact that these are almost all directions. We use a shorthand notation $P \cdot Q = P^i Q_i = \sum_{i=1}^n P_i Q_i$ for the inner product of the points $P = (P_1, P_2, ..., P_n)$, $Q = (Q_1, Q_2, ..., Q_n)$ which belong to $\mathbb{E}^n$.

Denote by $D_i f$ a weak partial derivative of the function f with respect to a coordinate variable with index $1 \le i \le n$. We assume that all functions have a zero extension outside of $\bar{\Omega}$. Everywhere further, unless otherwise stated, we use notations of the papers [1, 2, 32, 40, 48].

Below, we represent the conditions of Theorem 1 [5] that gives us a description of spectral properties, in terms of the real part order, of a non-selfadjoint operator L acting in $\mathfrak{H}$.

(H1) There exists a Hilbert space $\mathfrak{H}_+ \subset\subset \mathfrak{H}$ and a linear manifold $\mathfrak{M}$ that is dense in $\mathfrak{H}_+$. The operator L is defined on $\mathfrak{M}$.

(H2) $|(Lf, g)_\mathfrak{H}| \le C_1 ||f||_{\mathfrak{H}_+} ||g||_{\mathfrak{H}_+}$, $\mathrm{Re}(Lf, f)_\mathfrak{H} \ge C_2 ||f||^2_{\mathfrak{H}_+}$, $f, g \in \mathfrak{M}$, $C_1, C_2 > 0$.

Here, we should remark that since there is no general statement claiming that the intersection of the domain of definitions of an operator and its adjoint is a dense set, then we cannot restrict the reasonings considering Hermitian real component but compelled to involve the notion of the operator real part. This is why it is rather reasonable to suggest the the issue should be undergone to a comprehensive analysis.

Consider a condition $\mathfrak{M} \subset \mathrm{D}(W^*)$, in this case the real Hermitian component $\mathcal{H} := \mathfrak{Re}\, W$ of the operator is defined on $\mathfrak{M}$, the fact is that $\tilde{\mathcal{H}}$ is selfadjoint, bounded from bellow (see Lemma 3 [22]), where $H = ReW$. Hence a corresponding sesquilinear form (denote this form by h) is symmetric and bounded from bellow also (see Theorem 2.6 [48], p. 323). It can be easily shown that $h \subset \mathfrak{h}$, but using this fact we cannot claim in general that $\tilde{\mathcal{H}} \subset H$ (see [48], p. 330). We just have an inclusion $\tilde{\mathcal{H}}^{1/2} \subset H^{1/2}$ (see [48], p. 332).

Note that the fact $\tilde{\mathcal{H}} \subset H$ follows from a condition $\mathrm{D}_0(\mathfrak{h}) \subset \mathrm{D}(h)$ (see Corollary 2.4 [48], p. 323). However, it is proved (see proof of Theorem 4 [22]) that relation H2 guaranties that $\tilde{\mathcal{H}} = H$.

Note that the last relation is very useful in applications, since in most concrete cases we can find a concrete form of the operator $\mathcal{H}$.

2.2. Intrinsic Properties of the Kipriyanov Operator

Here, we study a case $\alpha \in (0,1)$. Assume that $\Omega \subset \mathbb{E}^n$ is a convex domain, with a sufficient smooth boundary (C^3 class) of the n-dimensional Euclidian space. For the sake of the simplicity we consider that Ω is bounded, but the results can be extended to some type of unbounded domains. In accordance with the definition given in the paper [51], we consider the directional fractional integrals. By definition, put

$$(\mathfrak{I}^{\alpha}_{0+}f)(Q) := \frac{1}{\Gamma(\alpha)}\int\limits_{0}^{r}\frac{f(P+t\mathbf{e})}{(r-t)^{1-\alpha}}\left(\frac{t}{r}\right)^{n-1}dt, \quad (\mathfrak{I}^{\alpha}_{d-}f)(Q) := \frac{1}{\Gamma(\alpha)}\int\limits_{r}^{d}\frac{f(P+t\mathbf{e})}{(t-r)^{1-\alpha}}\,dt,$$
$$f \in L_p(\Omega), \quad 1 \le p \le \infty.$$

The properties of these operators are described in detail in the papers [51, 52]. Similarly to the monograph [32] we consider left-side and right-side cases. For instance, $\mathfrak{I}^{\alpha}_{0+}$ is called a left-side directional fractional integral. We suppose $\mathfrak{I}^{0}_{0+} = I$. Nevertheless, this fact can be easily proved dy virtue of the reasonings corresponding to the one-dimensional case and given in [32]. We also consider integral operators with a weighted factor (see [32, p. 175]) defined by the following formal construction

$$\left(\mathfrak{I}^{\alpha}_{0+}\mu f\right)(Q) := \frac{1}{\Gamma(\alpha)}\int\limits_{0}^{r}\frac{(\mu f)(P+t\mathbf{e})}{(r-t)^{1-\alpha}}\left(\frac{t}{r}\right)^{n-1}dt,$$

where μ is a real-valued function.

We introduce the classes of functions representable by the directional fractional integrals.

$$\mathfrak{I}^{\alpha}_{0+}(L_p) := \left\{u:\ u(Q) = (\mathfrak{I}^{\alpha}_{0+}g)(Q)\right\}, \quad \mathfrak{I}^{\alpha}_{d-}(L_p) = \left\{u:\ u(Q) = (\mathfrak{I}^{\alpha}_{d-}g)(Q)\right\},$$
$$g \in L_p(\Omega), \quad 1 \le p \le \infty.$$

Define the following auxiliary operators acting in $L_p(\Omega)$ and depended on the parameter $\varepsilon > 0$. In the left-side case

$$(\psi^{+}_{\varepsilon}f)(Q) = \begin{cases} \int\limits_{0}^{r-\varepsilon}\frac{f(Q)r^{n-1}-f(T)t^{n-1}}{(r-t)^{\alpha+1}r^{n-1}}dt, & \varepsilon \le r \le d, \\ \frac{f(Q)}{\alpha}\left(\frac{1}{\varepsilon^{\alpha}}-\frac{1}{r^{\alpha}}\right), & 0 \le r < \varepsilon. \end{cases} \tag{2}$$

In the right-side case

$$(\psi^{-}_{\varepsilon}f)(Q) = \begin{cases} \int\limits_{r+\varepsilon}^{d}\frac{f(Q)-f(T)}{(t-r)^{\alpha+1}}dt, & 0 \le r \le d-\varepsilon, \\ \frac{f(Q)}{\alpha}\left(\frac{1}{\varepsilon^{\alpha}}-\frac{1}{(d-r)^{\alpha}}\right), & d-\varepsilon < r \le d. \end{cases}$$

Using the definitions of the monograph [32, p. 181], we consider the following operators. In the left-side case

$$(\mathfrak{D}^{\alpha}_{0+,\varepsilon}f)(Q) = \frac{1}{\Gamma(1-\alpha)}f(Q)r^{-\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}(\psi^{+}_{\varepsilon}f)(Q).$$

In the right-side case

$$(\mathfrak{D}^{\alpha}_{d-,\varepsilon}f)(Q) = \frac{1}{\Gamma(1-\alpha)}f(Q)(d-r)^{-\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}(\psi^{-}_{\varepsilon}f)(Q).$$

The left-side and right-side fractional derivatives are understood respectively as the following limits

$$\mathfrak{D}^{\alpha}_{0+}f = \lim_{\substack{\varepsilon\to 0\\ (L_p)}}\mathfrak{D}^{\alpha}_{0+,\varepsilon}f, \quad \mathfrak{D}^{\alpha}_{d-}f = \lim_{\substack{\varepsilon\to 0\\ (L_p)}}\mathfrak{D}^{\alpha}_{d-,\varepsilon}f, \quad 1 \le p < \infty.$$

Consider the Kipriyanov fractional differential operator defined in the paper [2] by the formal expression

$$\mathfrak{D}^{\alpha}(Q)=\frac{\alpha}{\Gamma(1-\alpha)}\int\limits_{0}^{r}\frac{[f(Q)-f(T)]}{(r-t)^{\alpha+1}}\left(\frac{t}{r}\right)^{n-1}dt+C_{n}^{(\alpha)}f(Q)r^{-\alpha},\quad P\in\partial\Omega,$$

where $C_{n}^{(\alpha)}=(n-1)!/\Gamma(n-\alpha)$. It is remarkable that Theorem 2 [2] establishes the mapping properties of the Kipriyanov operator, here we represent its statement in the explicit form: under the assumptions

$$lp\leq n,\quad 0<\alpha<l-\frac{n}{p}+\frac{n}{q},\quad q>p,$$

we have that for sufficiently small $\delta>0$ the following inequality holds

$$||\mathfrak{D}^{\alpha}f||_{L_q(\Omega)}\leq\frac{K}{\delta^{\nu}}||f||_{L_p(\Omega)}+\delta^{1-\nu}||f||_{L_p^l(\Omega)},\quad f\in\dot{W}_p^{\,l}(\Omega),$$

where

$$\nu=\frac{n}{l}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{\alpha+\beta}{l}.$$

The constant K does not depend on $\delta,\ f$; the point $P\in\partial\Omega$; β is an arbitrarily small fixed positive number. It is remarkable that Lemma 2.5 [51] establishes the connection between the fractional differential operators, more precisely it establishes the following relation

$$(\mathfrak{D}^{\alpha}f)(Q)=\left(\mathfrak{D}^{\alpha}_{0+}f\right)(Q),\quad f\in\dot{W}_p^{\,l}(\Omega),$$

what leads us to the inclusion $\mathfrak{D}^{\alpha}\subset\mathfrak{D}^{\alpha}_{0+}$.

The following theorem [51] establishes the mapping properties of directional fractional integral operators.

Theorem 1. *The following estimates hold*

$$||\mathfrak{I}^{\alpha}_{0+}u||_{L_p(\Omega)}\leq C_{\alpha,\mathrm{d}}||u||_{L_p(\Omega)},\quad ||\mathfrak{I}^{\alpha}_{d-}u||_{L_p(\Omega)}\leq C_{\alpha,\mathrm{d}}||u||_{L_p(\Omega)},\quad C_{\alpha,\mathrm{d}}=\mathrm{d}^{\alpha}/\Gamma(\alpha+1),\quad 1\leq p<\infty.$$

The proof of the following so-called representation theorem given in [51] implements the scheme of the proof corresponding to the one-dimensional case invented by B.S. Rubin [53, 54]. The author's own merit is a creation of the adopted version applicable to the Kipriyanov operator, we represent it in the expanded form since it may be treated as the intersection of the classical fractional calculus with the theory invented by Kipriyanov.

Theorem 2. *Suppose $f\in L_p(\Omega)$, there exists $\lim\limits_{\varepsilon\to0}\psi^{+}_{\varepsilon}f$ or $\lim\limits_{\varepsilon\to0}\psi^{-}_{\varepsilon}f$ with respect to the norm $L_p(\Omega)$, $1\leq p<\infty$, then $f\in\mathfrak{I}^{\alpha}_{0+}(L_p)$ or $f\in\mathfrak{I}^{\alpha}_{d-}(L_p)$ respectively.*

Proof. Let $f\in L_p(\Omega)$ and $\lim\limits_{\substack{\varepsilon\to0\\(L_p)}}\psi^{+}_{\varepsilon}f=\psi$. Consider the function

$$(\varphi^{+}_{\varepsilon}f)(Q)=\frac{1}{\Gamma(1-\alpha)}\left\{\frac{f(Q)}{r^{\alpha}}+\alpha(\psi^{+}_{\varepsilon}f)(Q)\right\}.$$

Taking into account (2), we can easily prove that $\varphi^{+}_{\varepsilon}f\in L_p(\Omega)$. Obviously, there exists the limit $\varphi^{+}_{\varepsilon}f\to\varphi\in L_p(\Omega),\quad\varepsilon\to0$. Taking into account Theorem 1, we can complete the proof, if we show that

$$\mathfrak{I}^{\alpha}_{0+}\varphi^{+}_{\varepsilon}f\xrightarrow{L_p}f,\quad\varepsilon\to0.\tag{3}$$

In the case $\varepsilon\leq r\leq d$, we have

$$(\mathfrak{I}^{\alpha}_{0+}\varphi^{+}_{\varepsilon}f)(Q)\cdot\frac{\pi r^{n-1}}{\sin\alpha\pi}=\int\limits_{\varepsilon}^{r}\frac{f(P+y\mathbf{e})y^{n-1-\alpha}}{(r-y)^{1-\alpha}}dy$$

$$+\,\alpha\int\limits_{\varepsilon}^{r}(r-y)^{\alpha-1}dy\int\limits_{0}^{y-\varepsilon}\frac{f(P+y\mathbf{e})y^{n-1}-f(T)t^{n-1}}{(y-t)^{\alpha+1}}dt$$

$$+\,\frac{1}{\varepsilon^{\alpha}}\int\limits_{0}^{\varepsilon}f(P+y\mathbf{e})(r-y)^{\alpha-1}y^{n-1}dy=I.$$

By direct calculation, we obtain

$$I=\frac{1}{\varepsilon^{\alpha}}\int\limits_{0}^{r}f(P+y\mathbf{e})(r-y)^{\alpha-1}y^{n-1}dy-\alpha\int\limits_{\varepsilon}^{r}(r-y)^{\alpha-1}dy\int\limits_{0}^{y-\varepsilon}\frac{f(T)}{(y-t)^{\alpha+1}}t^{n-1}dt. \tag{4}$$

Changing the variable in the second integral, we have

$$\alpha\int\limits_{\varepsilon}^{r}(r-y)^{\alpha-1}dy\int\limits_{0}^{y-\varepsilon}\frac{f(T)}{(y-t)^{\alpha+1}}t^{n-1}dt=\alpha\int\limits_{0}^{r-\varepsilon}(r-y-\varepsilon)^{\alpha-1}dy\int\limits_{0}^{y}\frac{f(T)}{(y+\varepsilon-t)^{\alpha+1}}t^{n-1}dt$$

$$=\alpha\int\limits_{0}^{r-\varepsilon}f(T)t^{n-1}dt\int\limits_{t}^{r-\varepsilon}\frac{(r-y-\varepsilon)^{\alpha-1}}{(y+\varepsilon-t)^{\alpha+1}}dy=\alpha\int\limits_{0}^{r-\varepsilon}f(T)t^{n-1}dt\int\limits_{t+\varepsilon}^{r}(r-y)^{\alpha-1}(y-t)^{-\alpha-1}dy. \tag{5}$$

Applying formula (13.18) [32, p. 184], we get

$$\int\limits_{t+\varepsilon}^{r}(r-y)^{\alpha-1}(y-t)^{-\alpha-1}dy=\frac{1}{\alpha\varepsilon^{\alpha}}\,\frac{(r-t-\varepsilon)^{\alpha}}{r-t}. \tag{6}$$

Combining relations (4), (5), and (6), using the change of the variable $t=r-\varepsilon\tau$, we get

$$(\mathfrak{I}^{\alpha}_{0+}\varphi^{+}_{\varepsilon}f)(Q)\,\frac{\pi r^{n-1}}{\sin\alpha\pi}=\frac{1}{\varepsilon^{\alpha}}\left\{\int\limits_{0}^{r}f(P+y\mathbf{e})(r-y)^{\alpha-1}y^{n-1}dy-\int\limits_{0}^{r-\varepsilon}\frac{f(T)(r-t-\varepsilon)^{\alpha}}{r-t}t^{n-1}dt\right\}$$

$$=\frac{1}{\varepsilon^{\alpha}}\int\limits_{0}^{r}\frac{f(T)\left[(r-t)^{\alpha}-(r-t-\varepsilon)^{\alpha}_{+}\right]}{r-t}t^{n-1}dt=\int\limits_{0}^{r/\varepsilon}\frac{\tau^{\alpha}-(\tau-1)^{\alpha}_{+}}{\tau}f(P+[r-\varepsilon\tau]\mathbf{e})(r-\varepsilon\tau)^{n-1}d\tau,$$

$$\tau_{+}=\begin{cases}\tau, & \tau\geq 0;\\ 0, & \tau<0.\end{cases} \tag{7}$$

Consider the auxiliary function $\mathcal{K}$ defined in the paper [32, p. 105]

$$\mathcal{K}(t)=\frac{\sin\alpha\pi}{\pi}\,\frac{t^{\alpha}_{+}-(t-1)^{\alpha}_{+}}{t},\qquad\int\limits_{0}^{\infty}\mathcal{K}(t)dt=1;\quad\mathcal{K}(t)>0. \tag{8}$$

Combining (7), (8) and taking into account that f has the zero extension outside of $\bar{\Omega}$, we obtain

$$(\mathfrak{I}^{\alpha}_{0+}\varphi^{+}_{\varepsilon}f)(Q)-f(Q)=\int\limits_{0}^{\infty}\mathcal{K}(t)\left\{f(P+[r-\varepsilon t]\mathbf{e})(1-\varepsilon t/r)^{n-1}_{+}-f(P+r\mathbf{e})\right\}dt. \tag{9}$$

Consider the case $0\leq r<\varepsilon$. Taking into account (2), we get

$$(\mathfrak{I}^{\alpha}_{0+}\varphi^{+}_{\varepsilon}f)(Q)-f(Q)=\frac{\sin\alpha\pi}{\pi\varepsilon^{\alpha}}\int\limits_{0}^{r}\frac{f(T)}{(r-t)^{1-\alpha}}\left(\frac{t}{r}\right)^{n-1}dt-f(Q)$$

$$= \frac{\sin \alpha\pi}{\pi \varepsilon^{\alpha}} \int\limits_{0}^{r} \frac{f(P+[r-t]\mathbf{e})}{t^{1-\alpha}} \left(\frac{r-t}{r}\right)^{n-1} dt - f(Q). \tag{10}$$

Consider the domains

$$\Omega_{\varepsilon} := \{Q \in \Omega,\ d(\mathbf{e}) \geq \varepsilon\}, \quad \tilde{\Omega}_{\varepsilon} = \Omega \setminus \Omega_{\varepsilon}.$$

In accordance with this definition we can divide the surface ω into two parts ω_{ε} and $\tilde{\omega}_{\varepsilon}$, where ω_{ε} is the subset of ω such that $d(\mathbf{e}) \geq \varepsilon$ and $\tilde{\omega}_{\varepsilon}$ is the subset of ω such that $d(\mathbf{e}) < \varepsilon$. Using (9) and (10), we get

$$||(\mathfrak{I}^{\alpha}_{0+}\varphi^{+}_{\varepsilon} f) - f||^{p}_{L_p(\Omega)} = \int\limits_{\omega_{\varepsilon}} d\chi \int\limits_{\varepsilon}^{d} \left| \int\limits_{0}^{\infty} \mathcal{K}(t)[f(Q-\varepsilon t\mathbf{e})(1-\varepsilon t/r)^{n-1}_{+} - f(Q)]dt \right|^{p} r^{n-1}dr$$

$$+ \int\limits_{\omega_{\varepsilon}} d\chi \int\limits_{0}^{\varepsilon} \left| \frac{\sin \alpha\pi}{\pi \varepsilon^{\alpha}} \int\limits_{0}^{r} \frac{f(P+[r-t]\mathbf{e})}{t^{1-\alpha}} \left(\frac{r-t}{r}\right)^{n-1} dt - f(Q) \right|^{p} r^{n-1}dr$$

$$+ \int\limits_{\tilde{\omega}_{\varepsilon}} d\chi \int\limits_{0}^{d} \left| \frac{\sin \alpha\pi}{\pi \varepsilon^{\alpha}} \int\limits_{0}^{r} \frac{f(P+[r-t]\mathbf{e})}{t^{1-\alpha}} \left(\frac{r-t}{r}\right)^{n-1} dt - f(Q) \right|^{p} r^{n-1}dr = I_1 + I_2 + I_3.$$

Consider I_1, using the generalized Minkowski inequality, we get

$$I_1^{\frac{1}{p}} \leq \int\limits_{0}^{\infty} \mathcal{K}(t) \left(\int\limits_{\omega_{\varepsilon}} d\chi \int\limits_{\varepsilon}^{d} |f(Q-\varepsilon t\mathbf{e})(1-\varepsilon t/r)^{n-1}_{+} - f(Q)|^{p} r^{n-1}dr \right)^{\frac{1}{p}} dt.$$

Let us define the function

$$h(\varepsilon, t) := \mathcal{K}(t) \left(\int\limits_{\omega_{\varepsilon}} d\chi \int\limits_{\varepsilon}^{d} |f(Q-\varepsilon t\mathbf{e})(1-\varepsilon t/r)^{n-1}_{+} - f(Q)|^{p} r^{n-1}dr \right)^{\frac{1}{p}} dt.$$

It can easily be checked that the following inequalities hold

$$|h(\varepsilon, t)| \leq 2\mathcal{K}(t)||f||_{L_p(\Omega)}, \quad \forall \varepsilon > 0; \tag{11}$$

$$|h(\varepsilon, t)| \leq \left(\int\limits_{\omega_{\varepsilon}} d\chi \int\limits_{\varepsilon}^{d} \left|(1-\varepsilon t/r)^{n-1}_{+}[f(Q-\varepsilon t\mathbf{e}) - f(Q)]\right|^{p} r^{n-1}dr \right)^{\frac{1}{p}} dt$$

$$+ \left(\int\limits_{\omega_{\varepsilon}} d\chi \int\limits_{0}^{d} \left|f(Q)[1-(1-\varepsilon t/r)^{n-1}_{+}]\right|^{p} r^{n-1}dr \right)^{\frac{1}{p}} dt = I_{11} + I_{12}.$$

By virtue of the average continuity property of the functions belonging to $L_p(\Omega)$, we have $\forall t > 0:\ I_{11} \to 0, \varepsilon \to 0$. Consider I_{12} and let us define the function

$$h_1(\varepsilon, t, r) := |f(Q)| \left|1-(1-\varepsilon t/r)^{n-1}_{+}\right|.$$

Apparently, the following relations hold almost everywhere in Ω

$$\forall t > 0, \quad h_1(\varepsilon, t, r) \leq |f(Q)|, \quad h_1(\varepsilon, t, r) \to 0, \quad \varepsilon \to 0.$$

Applying the Lebesgue dominated convergence theorem, we get $I_{12} \to 0, \varepsilon \to 0$. It implies that

$$\forall t > 0, \quad \lim_{\varepsilon \to 0} h(\varepsilon, t) = 0. \tag{12}$$

Taking into account (11), (12) and applying the Lebesgue dominated convergence theorem again, we obtain $I_1 \to 0$, $\varepsilon \to 0$. Consider I_2, using the Minkowski inequality, we get

$$I_2^{\frac{1}{p}} \leq \frac{\sin\alpha\pi}{\pi\varepsilon^{\alpha}} \left(\int\limits_{\omega_\varepsilon} d\chi \int\limits_0^{\varepsilon} \left| \int\limits_0^r \frac{f(Q-t\mathbf{e})}{t^{1-\alpha}} \left(\frac{r-t}{r}\right)^{n-1} dt \right|^p r^{n-1} dr \right)^{\frac{1}{p}} + \left(\int\limits_{\omega_\varepsilon} d\chi \int\limits_0^{\varepsilon} |f(Q)|^p r^{n-1} dr \right)^{\frac{1}{p}}$$

$$= I_{21} + I_{22}.$$

Applying the generalized Minkowski inequality, we obtain

$$I_{21} \frac{\pi}{\sin\alpha\pi} = \varepsilon^{-\alpha} \left(\int\limits_{\omega_\varepsilon} d\chi \int\limits_0^{\varepsilon} \left| \int\limits_0^r \frac{f(Q-t\mathbf{e})}{t^{1-\alpha}} \left(\frac{r-t}{r}\right)^{n-1} dt \right|^p r^{n-1} dr \right)^{\frac{1}{p}}$$

$$\leq \varepsilon^{-\alpha} \left\{ \int\limits_{\omega_\varepsilon} \left[\int\limits_0^{\varepsilon} t^{\alpha-1} \left(\int\limits_t^{\varepsilon} |f(Q-t\mathbf{e})|^p \left(\frac{r-t}{r}\right)^{(p-1)(n-1)} (r-t)^{n-1} dr \right)^{\frac{1}{p}} dt \right]^p d\chi \right\}^{\frac{1}{p}}$$

$$\leq \varepsilon^{-\alpha} \left\{ \int\limits_{\omega_\varepsilon} \left[\int\limits_0^{\varepsilon} t^{\alpha-1} \left(\int\limits_t^{\varepsilon} |f(P+[r-t]\mathbf{e})|^p (r-t)^{n-1} dr \right)^{\frac{1}{p}} dt \right]^p d\chi \right\}^{\frac{1}{p}}$$

$$\leq \varepsilon^{-\alpha} \left\{ \int\limits_{\omega_\varepsilon} \left[\int\limits_0^{\varepsilon} t^{\alpha-1} \left(\int\limits_0^{\varepsilon} |f(P+r\mathbf{e})|^p r^{n-1} dr \right)^{\frac{1}{p}} dt \right]^p d\chi \right\}^{\frac{1}{p}} = \alpha^{-1} ||f||_{L_p(\Delta_\varepsilon)},$$

where $\Delta_\varepsilon := \{Q \in \Omega_\varepsilon,\ r < \varepsilon\}$. Note that mess $\Delta_\varepsilon \to 0$, $\varepsilon \to 0$, therefore, $I_{21}, I_{22} \to 0$, $\varepsilon \to 0$. It follows that $I_2 \to 0$, $\varepsilon \to 0$. In the same way, we obtain $I_3 \to 0$, $\varepsilon \to 0$. Since we proved that $I_1, I_2, I_3 \to 0$, $\varepsilon \to 0$, then relation (3) holds. This completes the proof corresponding to the left-side case. The proof corresponding to the right-side case is absolutely analogous. □

The following theorem proved in [51] establishes the strictly accretive property (see [48]) of the Kipriyanov operator what gives us an opportunity to establish the numerical range of values of the operator, the latter notion plays a significant role in the spectral theory. Denote by $\mathrm{Lip}\,\lambda$, $0 < \lambda \leq 1$ the set of functions satisfying the Hölder–Lipschitz condition

$$\mathrm{Lip}\,\lambda := \left\{ \rho(Q) :\ |\rho(Q) - \rho(P)| \leq M r^{\lambda},\ P, Q \in \bar{\Omega} \right\}.$$

Theorem 3. *Suppose $\rho(Q)$ is a real non-negative function, $\rho \in Lip\,\lambda$, $\lambda > \alpha$; then the following inequality holds*

$$\mathrm{Re}(f, \mathfrak{D}^{\alpha} f)_{L_2(\Omega,\rho)} \geq C_{\alpha,\rho} ||f||^2_{L_2(\Omega,\rho)}, \quad f \in H_0^1(\Omega),$$

where

$$C_{\alpha,\rho} = \frac{1}{2\mathrm{d}^{\alpha}} \left\{ \frac{1}{\Gamma(1-\alpha)} + \frac{(n-1)!}{\Gamma(n-\alpha)} - \frac{\alpha M \mathrm{d}^{\lambda}}{2\Gamma(1-\alpha)(\lambda-\alpha)\inf\rho} \right\}.$$

Moreover, if we have in additional that for every fixed direction $\mathbf{e}$ *the function ρ is monotonically non-increasing, then*

$$C_{\alpha,\rho} = \frac{1}{2\mathrm{d}^{\alpha}} \left\{ \frac{1}{\Gamma(1-\alpha)} + \frac{(n-1)!}{\Gamma(n-\alpha)} \right\}.$$

Consider a linear combination of the uniformly elliptic operator, which is written in the divergence form, and a composition of the fractional integro-differential operator, where the fractional differential operator is understood as the adjoint operator regarding the Kipriyanov operator (see [1, 2, 23])

$$L := -\mathcal{T} + \mathfrak{I}^{\sigma}_{0+} \rho\, \mathfrak{D}^{\alpha}_{d-},\ \sigma \in [0,1), \quad \mathrm{D}(L) = H^2(\Omega) \cap H^1_0(\Omega),$$

where $\mathcal{T} := D_j(a^{ij}D_i\cdot)$, $i,j=1,2,...,n$, under the following assumptions regarding coefficients

$$a^{ij}(Q)\in C^2(\bar{\Omega}),\quad \mathrm{Re}a^{ij}\xi_i\xi_j\geq\gamma_a|\xi|^2,\quad \gamma_a>0,\quad \mathrm{Im}\,a^{ij}=0\ (n\geq 2),\quad \rho\in L_\infty(\Omega). \tag{13}$$

Note that in the one-dimensional case the operator $\mathfrak{I}^{\sigma}_{0+}\rho\,\mathfrak{D}^{\alpha}_{d-}$ is reduced to a weighted fractional integro-differential operator composition, which was studied properly by many researchers (see introduction, [32], p. 175).

2.3. The Semi-Group Model

Bellow, we explore a special operator class for which a number of spectral theory theorems can be applied. Further we construct an abstract model of a differential operator in terms of m-accretive operators and call it an m-accretive operator transform, we find such conditions that being imposed guaranty that the transform belongs to the class. As an application of the obtained abstract results we study a differential operator with a fractional integro-differential operator composition in final terms on a bounded domain of the n-dimensional Euclidean space. One of the central points is a relation connecting fractional powers of m-accretive operators and fractional derivative in the most general sense. By virtue of such an approach we express fractional derivatives in terms of infinitesimal generators, in this regard the Kipriyanov operator is considered.

We represent propositions devoted to properties of accretive operators and related questions. For the reader convenience, we would like to establish well-known facts of the operator theory under an appropriate point of view.

Lemma 1. *Assume that A is a closed densely defined operator, the following condition holds*

$$||(A+\lambda)^{-1}||_{\mathrm{R}\to\mathfrak{H}}\leq\frac{1}{\lambda},\quad \lambda>0,$$

where a notation $R := R(A+\lambda)$ is used. Then, the operators A, A^* are m-accretive.

In accordance with the definition given in [50], we can define a positive and negative fractional powers of a positive operator A as follows

$$A^{\alpha}:=\frac{\sin\alpha\pi}{\pi}\int\limits_0^\infty\lambda^{\alpha-1}(\lambda+A)^{-1}A\,d\lambda;\quad A^{-\alpha}:=\frac{\sin\alpha\pi}{\pi}\int\limits_0^\infty\lambda^{-\alpha}(\lambda+A)^{-1}\,d\lambda,\quad \alpha\in(0,1). \tag{14}$$

This definition can be correctly extended on m-accretive operators, the corresponding reasonings can be found in [48]. Thus, further we define positive and negative fractional powers of m-accretive operators by formula (14). The following lemma reflects the property of the fractional powers of m-accretive operators what gives us the invaluable technique to deal with the infinitesimal generators.

Lemma 2. *Assume that $\alpha\in(0,1)$, the operator J is m-accretive, J^{-1} is bounded, then*

$$||J^{-\alpha}f||_{\mathfrak{H}}\leq C_{1-\alpha}||f||_{\mathfrak{H}},\quad C_{1-\alpha}=2(1-\alpha)^{-1}||J^{-1}||+\alpha^{-1},\quad f\in\mathfrak{H}. \tag{15}$$

Consider a transform of an m-accretive operator J acting in $\mathfrak{H}$

$$Z^{\alpha}_{G,F}(J):=J^*GJ+FJ^{\alpha},\quad \alpha\in[0,1), \tag{16}$$

where symbols G,F denote operators acting in $\mathfrak{H}$. Further, using a relation $L=Z^{\alpha}_{G,F}(J)$ we mean that there exists an appropriate representation for the operator L. The following theorem gives us a tool to describe spectral properties of transform (16), as it will be shown further it has an important application in fractional calculus since allows to represent fractional differential operators as a transform of the infinitesimal generator of a semigroup.

Theorem 4. *Assume that the operator J is m-accretive, J^{-1} is compact, G is bounded, strictly accretive, with a lower bound $\gamma_G>C_\alpha||J^{-1}||\,||F||$, $D(G)\supset R(J)$, $F\in\mathcal{B}(\mathfrak{H})$, where C_α is a constant* (15). *Then, $Z^{\alpha}_{G,F}(J)$ satisfies conditions H1–H2.*

Consider the shift semigroup in a direction acting on $L_2(\Omega)$ and defined as follows $T_tf(Q)=f(Q+\mathbf{e}t)$, where $Q\in\Omega$, $Q=P+\mathbf{e}r$. Bellow, we represent the complete proof of the lemma proved in [5] to show the reader some techniques related to the shift semigroup.

Lemma 3. *The semigroup T_t is a C_0 semigroup of contractions.*

Proof. By virtue of the continuous in average property, we conclude that T_t is a strongly continuous semigroup. It can be easily established due to the following reasonings, using the Minkowski inequality, we have

$$\left\{\int_{\Omega}|f(Q+\mathbf{e}t)-f(Q)|^2dQ\right\}^{\frac{1}{2}}\leq\left\{\int_{\Omega}|f(Q+\mathbf{e}t)-f_m(Q+\mathbf{e}t)|^2dQ\right\}^{\frac{1}{2}}$$

$$+\left\{\int_{\Omega}|f(Q)-f_m(Q)|^2dQ\right\}^{\frac{1}{2}}+\left\{\int_{\Omega}|f_m(Q)-f_m(Q+\mathbf{e}t)|^2dQ\right\}^{\frac{1}{2}}=I_1+I_2+I_3<\varepsilon,$$

where $f\in L_2(\Omega)$, $\{f_n\}_1^\infty\subset C_0^\infty(\Omega)$; m is chosen so that $I_1,I_2<\varepsilon/3$ and t is chosen so that $I_3<\varepsilon/3$. Thus, there exists such a positive number t_0 that $||T_tf-f||_{L_2}<\varepsilon,\quad t<t_0$, for arbitrary small $\varepsilon>0$. Using the assumption that all functions have the zero extension outside $\bar{\Omega}$, we have $||T_t||\leq 1$. Hence we conclude that T_t is a C_0 semigroup of contractions (see [49]). □

The following theorem represented in [51] is formulated in terms of the infinitesimal generator $-A$ of the semigroup T_t. It is a central point in the application of the spectral theory methods to the abstract integro-differential constructions.

Theorem 5. *We claim that $L=Z^\alpha_{G,F}(A)$. Moreover, if γ_a is sufficiently large in comparison with $||\rho||_{L_\infty}$, then L satisfies conditions H1–H2, where we put $\mathfrak{M}:=C_0^\infty(\Omega)$, if we additionally assume that $\rho\in Lip\lambda$, $\lambda>\alpha$, then $\tilde{\mathcal{H}}=H$.*

The meaning of the following lemma is rather significant since it establishes a very useful property of the infinitesimal generator $-A$ of the semigroup T_t, using which we can construct a Hilbert space corresponding to the operator A let alone the secondary fact establishing the core of the operator A.

Lemma 4. *We claim that $A=\tilde{A}_0$, $N(A)=0$, where A_0 is a restriction of A on the set $C_0^\infty(\Omega)$.*

In the following paragraph, we study generalized constructions originated from the shift semigroup, they may be also interesting due to the applications related to the multidimensional case as well as being themselves non-standard constructions demonstrating one more class for which hypotheses H1 and H2 hold.

2.4. Further Generalizations

Consider a linear space $\mathbb{L}_2^n(\Omega):=\{f=(f_1,f_2,...,f_n),\quad f_i\in L_2(\Omega)\}$, endowed with the inner product

$$(f,g)_{\mathbb{L}_2^n}=\int_{\Omega}(f,g)_{\mathbb{E}^n}dQ,\quad f,g\in\mathbb{L}_2^n(\Omega).$$

It is clear that this pair forms a Hilbert space and let us use the same notation $\mathbb{L}_2^n(\Omega)$ for it. Consider a sesquilinear form

$$t(f,g):=\sum_{i=1}^n\int_{\Omega}(f,\mathbf{e_i})_{\mathbb{E}^n}\overline{(g,\mathbf{e_i})}_{\mathbb{E}^n}dQ,\quad f,g\in\mathbb{L}_2^n(\Omega),$$

where $\mathbf{e_i}$ corresponds to $P_i\in\partial\Omega$, $i=1,2,...,n$ (i.e., $Q=P_i+\mathbf{e_i}r$). The proofs of the propositions represented in this paragraph are given in [55].

Lemma 5. *The points $P_i\in\partial\Omega$, $i=1,2,...,n$ can be chosen so that the form t generates an inner product.*

Consider a pre Hilbert space $\mathbf{L}_2^n(\Omega):=\{f:\ f\in\mathbb{L}_2^n(\Omega)\}$ endowed with the inner product

$$(f,g)_{\mathbf{L}_2^n}:=\sum_{i=1}^n\int_{\Omega}(f,\mathbf{e_i})_{\mathbb{E}^n}\overline{(g,\mathbf{e_i})}_{\mathbb{E}^n}dQ,\quad f,g\in\mathbb{L}_2^n(\Omega),$$

where $\mathbf{e_i}$ corresponds to $P_i \in \partial\Omega$, $i = 1, 2, ..., n$, the following condition holds

$$\Delta = \begin{vmatrix} P_{11} & P_{12} & ... & P_{1n} \\ P_{21} & P_{22} & ... & P_{2n} \\ ... & ... & ... & ... \\ P_{n1} & P_{n2} & ... & P_{nn} \end{vmatrix} \neq 0,$$

where $P_i = (P_{i1}, P_{i2}, ..., P_{in})$. The following theorem establishes a norm equivalence.

Theorem 6. *The norms* $||\cdot||_{\mathbb{L}_2^n}$ *and* $||\cdot||_{\mathbf{L}_2^n}$ *are equivalent.*

Consider a pre Hilbert space

$$\widetilde{\mathfrak{H}}_A^n := \left\{ f, g \in C_0^\infty(\Omega), \quad (f, g)_{\widetilde{\mathfrak{H}}_A^n} = \sum_{i=1}^n (A_i f, A_i g)_{L_2} \right\},$$

where $-A_i$ is the infinitesimal generator corresponding to the point P_i. Here, we should point out that the form $(\cdot, \cdot)_{\widetilde{\mathfrak{H}}_A^n}$ generates an inner product due to the fact $\mathrm{N}(A_i) = 0$, $i = 1, 2, ..., n$ proved in Lemma 4. Let us denote a corresponding Hilbert space by $\mathfrak{H}_A^n$.

Corollary 1. *The norms* $||\cdot||_{\mathfrak{H}_A^n}$ *and* $||\cdot||_{H_0^1}$ *are equivalent, we have a bounded compact embedding*

$$\mathfrak{H}_A^n \subset\subset L_2(\Omega).$$

Below, we aim to represent an operator in terms of the infinitesimal generator of the shift semigroup in a direction with the purpose to apply the results [22, 38, 47] to the established representation. In this way we come to natural conditions in terms of the infinitesimal generator of the shift semigroup in a direction what gives us the desired result represented in [5]. The following theorem allows us to express the construction of the partial differential operator in terms of the semigroup theory (having chosen the shift semigroup in the direction) what reveals a mathematical nature of the operator $-\mathcal{T}$.

Theorem 7. *We claim that* $-\mathcal{T} = \frac{1}{n}\sum\limits_{i=1}^n A_i^* G_i A_i$, the following relations hold

$$-\mathrm{Re}(\mathcal{T} f, f)_{L_2} \geq C||f||_{\mathfrak{H}_A^n}; \quad |(\mathcal{T} f, g)_{L_2}| \leq C||f||_{\mathfrak{H}_A^n}||g||_{\mathfrak{H}_A^n}, \quad f, g \in C_0^\infty(\Omega),$$

where G_i *are some operators corresponding to the operators* A_i.

Thus, by virtue of Corollary 1 and Theorem 7, we are able to claim that hypotheses H1, H2 [5] hold for the operator $-\mathcal{T}$. It is rather reasonable to represent analog of Theorem 5 which reflects connection between the operator $-\mathcal{T}$ and its perturbation by the Kipriyanov operator.

Theorem 8. *We claim that* $L = \frac{1}{n}\sum\limits_{i=1}^n A_i^* G_i A_i + F A_1^\alpha$, *where* F *is a bounded operator,* $P_1 := P$, *and* G_i *are the same as in Theorem 7. Moreover, if* γ_a *is sufficiently large in comparison with* $||\rho||_{L_\infty}$, *then the following relations hold*

$$\mathrm{Re}(Lf, f)_{L_2} \geq C||f||_{\mathfrak{H}_A^n}; \quad |(Lf, g)_{L_2}| \leq C||f||_{\mathfrak{H}_A^n}||g||_{\mathfrak{H}_A^n}, \quad f, g \in C_0^\infty(\Omega).$$

The theorem reveals a remarkable fact the perturbation preserves the property being in the class satisfying hypotheses H1 and H2 what makes the perturbed operator interesting itself from the theoretical point of view let alone a prospective applications determined by convenience, from the technical point of view, in dealing with the invented operator construction in the multidimensional space.

3. INTEGRO-DIFFERENTIAL CONSTRUCTIONS

3.1. Abel–Lidskii Root Vectors Series Expansion

In this section, we represent a theorem valuable from theoretical and applied points. It is based upon the modification of the Lidskii method, this is why following the the classical approach we divided it into three statements that can be claimed separately. The first statement (Theorem 3 [38]) establishes a character of the series convergence having a principal meaning within the whole concept. The second statement (Theorem 3 [38]) reflects the name of convergence—Abel–Lidskii since the latter can be connected with the definition of the series convergence in the Abel sense, more detailed information can be found in the monograph by Hardy [56]. The third statement (Theorem 4 [38]) is a valuable application of the first one, it is based upon suitable algebraic reasonings having been noticed by the author and allowing to involve a fractional derivative in the first term. We should note that previously, a concept of an operator function represented in the second term was realized in the paper [33], where a case corresponding to a function represented by a Laurent series with a polynomial regular part was considered. Bellow, we consider a comparatively more difficult case obviously related to the infinite regular part of the Laurent series and therefore requiring a principally different method of study.

It is a well-known fact that each eigenvalue μ_q, $q \in \mathbb{N}$ of the compact operator B generates a set of Jordan chains containing eigenvectors and root vectors. Denote by $m(q)$ a geometrical multiplicity of the corresponding eigenvalue and consider a Jordan chain corresponding to an eigenvector e_{q_ξ}, $\xi = 1, 2, ..., m(q)$, we have

$$e_{q_\xi}, e_{q_\xi+1}, ..., e_{q_\xi+k(q_\xi)}, \tag{17}$$

where $k(q_\xi)$ indicates a number of elements in the Jordan chain, the symbols except for the first one denote root vectors of the operator B. Note that combining the Jordan chains corresponding to an eigenvalue, we obtain a Jordan basis in the invariant subspace generated by the eigenvalue, moreover, we can arrange a so-called system of major vectors $\{e_i\}_1^\infty$ (see [34]) of the operator B having combined Jordan chains. It is remarkable that the eigenvalue $\bar{\mu}_q$ of the operator B^* generates the Jordan chains of the operator B^* corresponding to (17). In accordance with [47], we have

$$g_{q_\xi+k(q_\xi)}, \quad g_{q_\xi+k(q_\xi)-1}, ..., g_{q_\xi},$$

where the symbols except for the first one denote root vectors of the operator B^*. Combining Jordan chains of the operator B^*, we can construct a biorthogonal system $\{g_n\}_1^\infty$ with respect to the system of the major vectors of the operator B. This fact is given in detail in the paper [47]. The following construction plays a significant role in the theory created in the papers [33, 47, 57] and, therefore, deserves to be considered separately, denote

$$\mathcal{A}_\nu(\varphi, t)f := \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i}(t),$$

where $\{N_\nu\}_1^\infty$ is a sequence of natural numbers,

$$c_{q_\xi+i}(t) = e^{-\varphi(\lambda_q)t} \sum_{j=0}^{k(q_\xi)-i} H_j(\varphi, \lambda_q, t) c_{q_\xi+i+j}, \quad i = 0, 1, 2, ..., k(q_\xi),$$

$c_{q_\xi+i} = (f, g_{q_\xi+k-i})/(e_{q_\xi+i}, g_{q_\xi+k-i})$, $\lambda_q = 1/\mu_q$ is a characteristic number corresponding to e_{q_ξ},

$$H_j(\varphi, z, t) := \frac{e^{\varphi(z)t}}{j!} \lim_{\zeta \to 1/z} \frac{d^j}{d\zeta^j} \left\{ e^{-\varphi(\zeta^{-1})t} \right\}, \quad j = 0, 1, 2, ...$$

More detailed information on the considered above Jordan chains can be found in [47].

3.2. Decomposition Theorem

Denote by $\mathfrak{H}$ the abstract separable Hilbert space and assume that the hypotheses H1 and H2 hold for the operator W acting in $\mathfrak{H}$. We should point out that such chose of the operator class justified by both abstract theoretical relevance related to the spectral properties of non-selfadjoint operators and the concrete applications including ones involving the Kipriyanov operator. Denote by

$$\varphi(W) := \sum_{n=l}^{k} c_n W^n, \quad -\infty \leq l, k \leq \infty \tag{18}$$

a formal construction called by an operator function, where c_n are the coefficients corresponding to the function of the complex variable φ. Here, we ought to make a bibliographic digression and remind that the case $l = -\infty$, $k < \infty$ was considered in [57]. In this case, the complex function φ was supposed to have a decomposition into the Laurent series about the point zero with the coefficients c_n satisfying the additional assumption

$$\max_{n=0,1,\ldots,k} (|\arg c_n| + n\theta) < \pi/2, \tag{19}$$

where θ is the semi-angle of the sector containing the numerical range of values of the operator W. We should note that the problem connected with the representation (18) can be divided on two parts $l \geq -\infty$, $k = 0$ and $l = 0$, $k \leq \infty$, thus the first one was properly studied in [57], the second one was studied in [38], with the following assumptions (we represent a technical variant, the expended variant can be found in [38]): the complex function φ of the order less than a half maps the ray $\arg z = \theta_0$ within a sector $\mathfrak{L}_0(\zeta)$, $0 < \zeta < \pi/2$, the condition holds

$$\mathrm{Re}\varphi(z) > e^{aH(\theta_0)r^{\varrho}}, \quad \arg z = \theta_0, \quad 0 < a < 1, \tag{20}$$

where $H(\theta_0)$ is a positive number in accordance with the Lemma 1 [38]. Taking into account the above, we can consider a function represented by a Laurent series with the arbitrary principal part and the regular part satisfying (19), (20) respectively to the finite, infinite cases. This statement can be proved by repetition of the reasonings represented in Lemma 5 [57], thus we leave the proof to the reader.

Below, we consider a Hilbert space consists of element-functions $u : \mathbb{R}_+ \to \mathfrak{H}$, $u := u(t)$, $t \geq 0$, we understand the differentiation and integration operations in the generalized sense, i.e., the derivative is defined as a limit in the sense of the norm etc. (see [47, 50]). Combining the operations, we can define a generalized fractional derivative in the Riemann–Liouville sense (see [32, 33]), in the formal form, we have

$$\mathfrak{D}_{-}^{1/\alpha} f(t) := -\frac{1}{\Gamma(1-1/\alpha)} \frac{d}{dt} \int_{0}^{\infty} f(t+x) x^{-1/\alpha} dx, \quad \alpha \geq 1,$$

here we should note that facts $\mathfrak{D}_{-}^{1} f(t) = -du/dt, \quad \mathfrak{D}_{-}^{0} f(t) = f(t)$ can be obtained due to the definition of the operator (see [32]). In terms of the expression (18), consider a Cauchy problem

$$\mathfrak{D}_{-}^{1/\alpha} u = \varphi(W) u, \quad u(0) = f \in \mathrm{D}(W^n), \quad n = 1, 2, ..., k. \tag{21}$$

Taking into account the above, combining results [38, 57], we can formulate the following theorem.

Theorem 9. *Assume that conditions (19), (20) hold respectively to the cases corresponding to the finite, infinite regular part of the series (18), then there exists a solution of the Cauchy problem (21) in the form*

$$u(t) = \sum_{\nu=0}^{\infty} \mathcal{A}_{\nu}(\varphi^{\alpha}, t) f, \quad \sum_{\nu=0}^{\infty} \|\mathcal{A}_{\nu}(\varphi^{\alpha}, t) f\| < \infty.$$

Moreover, the existing solution is unique if the operator $\mathfrak{D}_{-}^{1-1/\alpha}\varphi(W)$ *is accretive.*

Proof. To avoid any kind of repetition, we do not represent the complete proof having restricted reasonings by the scheme appealing to Lemma 5, Theorem 1 [57], Lemma 3, Theorem 4 [38]. Thus, the detailed calculation is left to the reader. □

Further, considering an operator function, we will assume that conditions (19), (20) hold respectively to the case. Now, consider transform (16)

$$Z_{G,F}^{\alpha}(J) = J^{*}GJ + FJ^{\alpha}, \quad \alpha \in [0,1),$$

assuming that the conditions of Theorem 6 hold, we can consider a Cauchy problem involving the transform which represents an integro-differential construction in the generalized sense. The latter problem appeals to a plenty of concrete evolution equations which form a base for the modern engineering sciences. Let us contemplate then a magnificent representative of the generators creating the transform—the directional derivative which fractional power is the Kipriyanov operator. Consider a Cauchy problem

$$\mathfrak{D}_{-}^{1/\alpha}u = \sum_{n=l}^{k} c_n L^n u, \quad u(0) = f \in C_0^{\infty}(\Omega), \quad n = 1,2,...,k, \tag{22}$$

where we are dealing with the following integro-differential construction with the sufficiently smooth coefficients for which the conditions (13) hold

$$L := -\mathcal{T} + \mathfrak{I}_{0+}^{\sigma}\rho\,\mathfrak{D}_{d-}^{\gamma}, \quad \sigma,\gamma \in [0,1).$$

Thus, we can easily see that the case $\sigma = 0,\ \gamma = 0,\ -\infty < l,\ k < \infty$ leads to the class of integro-differential equations of the integer order and the obtained results give us a method to solve the corresponding Cauchy problems (22). Certainly, we can consider a closure of the defined operator function on the set $C_0^{\infty}(\Omega)$, the fact that it admits closure is proved in [58]. The case corresponding to arbitrary values of σ,γ within the range, requires more peculiar technique that was considered in paragraph 3.1 [33]. However, Theorem 9 becomes relevant in solving evolution equations with an integro-differential operator in the second term to say nothing on the far reaching generalizations corresponding to an operator function with the infinite principal or regular part of its Laurent series.

4. CONCLUSIONS

In the paper, there was represented a historical survey devoted to the achievements of Kipriyanov, where the exclusively constructed fractional calculus theory was discussed convexly. We produced a comparison analysis in the framework of ways and means related to further prospective generalizations of fractional derivative as a notion. The qualitative properties of the Kipriyanov fractional differential operator were studied by the methods of the classical fractional calculus theory in contrast to the exclusive approach invented by Kipriyanov. Having taken as a basis the concept of multidimensional generalization of the fractional differential operator in the sense of Marchaud, we adapted the previously known technique of the proofs related to the theory of fractional calculus of one variable. Along with the previously known definition of a fractional derivative introduced by Kipriyanov we used a new definition of a multidimensional fractional integral in the direction what allows to describe the range of the Kipriyanov adjoint operator. A number of statements having analogues in the theory of fractional calculus of one variable and previously proved by the author were discussed. In particular the classical result—a sufficient condition for representability by a fractional directional integral in the direction was observed. The strictly accretive property of the Kipriyanov operator, being an outstanding author's result, was observed. On the base of the given technique, there were developed methods of the semigroup theory and the spectral theory of non-selfadjoint operators what leads us to significant applications to the abstract evolution equations in the Hilbert space. The latter gives us an opportunity to solve a whole class of problems related to integro-differential equations of the real order wherein the one related to the semigroup connected with the Kipriyanov operator was studied properly.

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