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Article

Cauchy Problem for an Abstract Evolution Equation of Fractional Order

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Abstract: In this paper, we define an operator function as a series of operators corresponding to the Taylor series representing the function of the complex variable. In previous papers, we considered the case when a function has a decomposition in the Laurent series with the infinite principal part and finite regular part. Our central challenge is to improve this result having considered as a regular part an entire function satisfying the special condition of the growth regularity. As an application, we consider an opportunity to broaden the conditions imposed upon the second term not containing the time variable of the evolution equation in the abstract Hilbert space.

Keywords: evolution equations; operator function; fractional differential equations; Abel–Lidskii basis property; Schatten–von Neumann class

MSC: 47B28; 47A10; 47B12; 47B10; 34K30; 58D25

1. Introduction

The urbanization of the sea coast and the active use of shelf resources have led to an increase in accidents of man-made origin and their negative impact on the environment. In these conditions, it becomes critically important to control natural catastrophic phenomena and assess their consequences in order to minimize possible human and material losses. This was the reason for the development in recent years of acoustic measurement methods and long-term monitoring of the parameters of the aquatic environment in shallow waters. Hydroacoustic complexes operating on the basis of the proposed methods provide remote measurement of the parameters of the aquatic environment in bays, straits and inland reservoirs, allowing the early detection of sources of natural and man-made threats. Ferroelectrics are a promising class of polar dielectrics and the study of their nonequilibrium dynamics, phase transitions and domain kinetics is of key importance in acoustics. In the paper [1], the description of the process of switching the polarization of ferroelectrics is implemented by modeling a fractal system. Since the polarization switching process is the result of the formation of self-similar structures, the domain configurations of many ferroelectrics are characterized by a self-similar structure, and electrical responses are characterized by fractal patterns. The manifestation of fractal properties is due to the complex mechanisms of domain boundary movement, the anisotropy properties of real crystals, the stochastic nature of the nucleation process, and the presence of memory effects. The field of application of the results of fractal system modeling is focused on the description of the process of switching the polarization of ferroelectrics. The main mathematical object of research is the Cauchy problem for the evolution equation with a fractional Riemann–Liouville derivative in the first term. The proposed methods are numerical, based on a finite difference method.

At the same time, the method invented in this paper allows us to solve such problems analytically, which is undoubtedly a great advantage. Having created a direction of the spectral theory of non-self-adjoint operators, we can consider abstract theoretical results as a base for further research studying such mathematical objects as a Cauchy problem

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for the evolution equation of fractional order in the abstract Hilbert space. We consider in the second term an operator function defined on a special operator class covering a generator transform considered in [2], where a corresponding semigroup is supposed to be a C_0 semigroup of contractions. In its own turn, the transform reduces to a linear composition of differential operators of real order in various senses, such as the Riemann–Liouville fractional differential operator, the Kipriyanov operator, the Riesz potential, and the difference operator [2–5]. Moreover, in the paper [6], we broadened the class of differential operators having considered the artificially constructed normal operator that cannot be covered by the Lidskii results [7]. The application part of the theory appeals to the results and problems which can be considered particular cases of the abstract ones; the following papers are worth noting within this context [8–11]. At the same time, we should admit that abstract methods can be "clumsy", as some peculiarities can be considered only by a unique technique, which forms the main contribution of the specialists dealing with concrete differential equations. The significance of the problem for physics and engineering sciences is based upon the wide field of applications; here, we refer to the valuable example considered above [1], which gives us an opportunity to demonstrate the significance in a plain way. The main idea of the results connected with the basis property in a more refined sense—the Abel–Lidskii sense [12]—allows to consider many problems [6] in the theory of evolution equations and in this way obtain remarkable applications.

As a main objective of the paper, we consider a method by virtue of which we can principally weaken conditions imposed upon the second term of the abstract evolution equation. The concept of an operator function, realized in terms of the involved contour integral, is an effective technical tool giving an advantage in solving the applied problems let alone abstract generalizations. In this regard, it is rather reasonable to develop a theory analogous to the spectral theorem for self-adjoint operators having defined a family of projectors. At the same time, from an applied point of view, we intend to realize the idea considering a notion of operator function applicably to a Cauchy problem for an abstract fractional evolution equation with an operator function in the second term not containing the time variable, where the derivative in the first term is supposed to be of a fractional order. Here, we should note that regarding functional spaces, we have that an operator function generates a variety of operators acting in a corresponding space. In this regard, even a power function gives us an interesting result [6]. In the context of existence and uniqueness theorems, a significant refinement that is worth highlighting is the obtained formula for the solution represented by a series on the root vectors. In the absence of the norm convergence of the root vector series, we need to consider a notion of convergence in weaker Bari, Riesz, and Abel–Lidskii senses [7,13,14].

In spite of the claimed rather applied objectives, we admit that the problem of the root vectors expansion for a non-self-adjoint unbounded operator still remains relevant in the context of the paper. It is remarkable that the problem originates nearly from the first half of the last century [2,7,12,13,15–22]. However, we have a particular interest when an operator is represented by a linear combination of operators, where a so-called senior term is non-self-adjoint for a case corresponding to a self-adjoint operator; this was thoroughly studied in the papers [16,18–22]. Thus, the results [2,17] covering the very case corresponding to a non-self-adjoint senior term are worth highlighting; moreover, they have a natural mathematical origin that appears brightly while we are considering abstract constructions expressed in terms of the semigroup theory [2].

2. Preliminaries

2.1. Convergence Exponent

Below, for the reader's convenience, we introduce some basic notions and facts of the entire function theory. To characterize the growth of an entire function f, we introduce the functions

$$M_f(r) = \max_{|z|=r} |f(z)|,\ m_f(r) = \min_{|z|=r} |f(z)|.$$

An entire function $f(z)$ is said to be a function of finite order if there exists a positive constant k such that the inequality

$$M_f(r) < e^{r^k}$$

is valid for all sufficiently large values of r. The greatest lower bound of such numbers k is called the *order* of the entire function $f(z)$. We need the following obvious fact that follows from the definition. If ϱ is the order of the entire function $f(z)$, and if ε is an arbitrary positive number, then

$$e^{r^{\varrho-\varepsilon}} < M_f(r) < e^{r^{\varrho+\varepsilon}}, \tag{1}$$

where the inequality on the right-hand side is satisfied for all sufficiently large values of r, and the inequality on the left-hand side holds for some sequence $\{r_n\}$ of values of r, tending to infinity. Define a *type* σ of the entire function f having the order ϱ as the greatest lower bound of positive numbers C such that for a sufficiently large value r, the following relation holds:

$$M_f(r) < e^{Cr^{\varrho}}.$$

The following relation can be obtained easily by virtue of the definition given above:

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\ln M_f(r)}{r^{\varrho}}.$$

We use the following notations:

$$G(z,p) := (1-z)e^{z+\frac{z^2}{2}+\ldots+\frac{z^p}{p}},\ p \in \mathbb{N},\ G(z,0) := (1-z).$$

Assume that an entire function has zeros for which the following relation holds:

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\xi}} < \infty, \tag{2}$$

where $\xi > 0$. In this case, we denote by p the smallest integer number for which the following condition holds:

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty. \tag{3}$$

It is clear that $0 \leq p < \xi$. Note that in accordance with [23], relation (2) guarantees the uniform convergence of the following infinite product:

$$\prod_{n=1}^{\infty} G\left(\frac{z}{a_n}, p\right). \tag{4}$$

This infinite product is called a *canonical product*, and the value p is called the *genus* of the canonical product. Let us define a *convergence exponent* ρ of the sequence $\{a_n\}_1^{\infty} \subset \mathbb{C}$, $a_n \neq 0$, $a_n \to \infty$ as the greatest lower bound for such numbers ξ that the series (2) converges. However, a more precise characteristic of the sequence $\{a_n\}_1^{\infty}$ density than the convergence exponent can be considered (see [23]). For this purpose, let us define a counting function $n(r)$ as a function that equals a number of points of the sequence in the circle $|z| < r$. The upper density of the sequence is defined as follows:

$$\Delta = \overline{\lim_{r\to\infty}}\, n(r)/r^{\rho}.$$

Note that in the case when the limit exists in the ordinary sense, the upper density is called the density. The following fact is proved in Lemma 1 [23], where we have

$$\lim_{r\to\infty} n(r)/r^{\rho+\varepsilon} \to 0,\ \varepsilon > 0.$$

We need Theorem 13 [23] (Chapter I, § 10) presented below, which gives us a representation of the entire function of the finite order. To avoid any sort of inconvenient form of writing, we will also call a root a zero of the entire function.

Theorem 1. *The entire function f of the finite order ϱ has the following representation:*

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{\omega} G\left(\frac{z}{a_n}; p\right), \ \omega \leq \infty,$$

where a_n are non-zero roots of the entire function, $p \leq \varrho$, $P(z)$ is a polynomial, and $\deg P(z) \leq \varrho$, *m is a multiplicity of the zero root.*

The infinite product represented in Theorem 1 is called a canonical product of the entire function.

2.2. Proximate Order and Angular Density of Zeros

The scale of the growth admits further clarifications. As a simplest generalization, E.L. Lindelöf made a comparison $M_f(r)$ with the functions of the type

$$r^\varrho \ln^{\alpha_1} r \ln_2^{\alpha_2} r ... \ln_n^{\alpha_n} r,$$

where $\ln_j r = \ln \ln_{j-1} r$, $\alpha_j \in \mathbb{R}$, $j = 1, 2, ..., n$. In order to make the further generalization, it is natural (see [23]) to define a class of the functions $L(r)$ having *low growth* and compare $\ln M_f(r)$ with $r^\varrho L(r)$. Following this idea, G. Valiron introduced a notion of proximate order of the growth of the entire function f, in accordance with which a function $\varrho(r)$, satisfying the following conditions

$$\lim_{r\to\infty} \varrho(r) = \varrho; \ \lim_{r\to\infty} r\varrho'(r) \ln r = 0,$$

is said to be of proximate order if the following relation holds:

$$\sigma_f = \overline{\lim_{r\to\infty}} \frac{\ln M_f(r)}{r^{\varrho(r)}}, \ 0 < \sigma_f < \infty.$$

In this case, the value σ_f is said to be a type of function f under the proximate order $\varrho(r)$.

To guarantee some technical results, we need to consider a class of entire functions whose zero distributions have a certain type of regularity. We follow the monograph [23], where the regularity of the distribution of the zeros is characterized by a certain type of density of the set of zeros.

We will say that the set Ω of the complex plane has an *angular density of index*

$$\xi(r) \to \xi, \ r \to \infty,$$

if for an arbitrary set of values ϕ and ψ $(0 < \phi < \psi \leq 2\pi)$, maybe except for denumerable sets, there exists the limit

$$\Delta(\phi, \psi) = \lim_{r\to\infty} \frac{n(r, \phi, \psi)}{r^{\xi(r)}}, \tag{5}$$

where $n(r, \phi, \psi)$ is the number of points of the set Ω within the sector $|z| \leq r$, $\phi < \arg z < \psi$. The quantity $\Delta(\phi, \psi)$ will be called the angular density of the set Ω within the sector $\phi < \arg z < \psi$. For a fixed ϕ, the relation

$$\Delta(\psi) - \Delta(\phi) = \Delta(\phi, \psi)$$

determines, within the additive constant, a nondecreasing function $\Delta(\psi)$. This function is defined for all values of ψ, maybe except for a denumerable set of values. It is shown in the monograph [23] (p. 89), that the exceptional values of ϕ and ψ for which there does not exist an angular density must be the points of discontinuity of the function $\Delta(\psi)$. A set will be said to be *regularly distributed* relative to $\xi(r)$ if it has an angular density $\xi(r)$ with a ξ non-integer.

The asymptotic equalities which we will establish are related to the order of growth. By the asymptotic equation

$$f(r) \approx \varphi(r)$$

we will mean the fulfillment of the following condition:

$$[f(r) - \varphi(r)]/r^{\varrho(r)} \to 0, \, r \to \infty.$$

Consider the following conditions allowing us to solve technical problems related to the estimation of contour integrals.

(I) There exists a value $d > 0$ such that circles of radii

$$r_n = d|a_n|^{1-\frac{\varrho(|a_n|)}{2}}$$

with the centers situated at the points a_n do not intersect each other, where a_n.

(II) The points a_n lie inside angles with a common vertex at the origin but with no other points in common, which are such that if one arranges the points of the set $\{a_n\}$ within any one of these angles in the order of increasing moduli, then for all points which lie inside the same angle, the following relation holds:

$$|a_{n+1}| - |a_n| > d|a_n|^{1-\varrho(|a_n|)}, \, d > 0.$$

The circles $|z - a_n| \le r_n$ in the first case, and $|z - a_n| \le d|a_n|^{1-\varrho(|a_n|)}$ in the second case, will be called the exceptional circles.

The following theorem is a central point of the study. Below, for the reader's convenience, we present Theorem 5 [23] (Section II, § 1) in a slightly changed form.

Theorem 2. *Assume that the entire function f of the proximate order $\varrho(r)$, where ϱ is not an integer, is represented by its canonical product, i.e.,*

$$f(z) = \prod_{n=1}^{\infty} G\left(\frac{z}{a_n}; p\right),$$

the set of zeros is regularly distributed relative to the proximate order and satisfies one of the conditions (I) *or* (II)*. Then, outside of the exceptional set of circulus, the entire function satisfies the following asymptotical inequality:*

$$\ln|f(re^{i\psi})| \approx H(\psi) r^{\varrho(r)},$$

where

$$H(\psi) := \frac{\pi}{\sin \pi\varrho} \int\limits_{\psi-2\pi}^{\psi} \cos \varrho(\psi - \varphi - \pi) d\Delta(\varphi).$$

The following lemma gives us a key for the technical part of being constructed theory. Although it does not contain implications of any subtle sort, it is worth being presented in the expanded form for the reader's convenience.

Lemma 1. *Assume that $\varrho \in (0, 1/2]$ then the function $H(\psi)$ is positive if $\psi \in (-\pi, \pi)$.*

Proof. Taking into account the facts $\cos\varrho(\psi-\varphi-\pi)=\cos\varrho(|\psi-\varphi|-\pi)$, $\psi-2\pi<\varphi<\psi$, $\cos\varrho(|\psi-\varphi|-\pi)=\cos\varrho(|\psi-(\varphi+2\pi)|-\pi)$, we obtain the following form:

$$H(\psi):=\frac{\pi}{\sin\pi\varrho}\int\limits_{0}^{2\pi}\cos\varrho(|\psi-\varphi|-\pi)d\Delta(\varphi).$$

Having noticed the following correspondence between sets $\varphi\in[0,\psi]\Rightarrow\xi\in[\varrho(\psi-\pi),-\varrho\pi]$, $\varphi\in[\psi,\psi+\pi]\Rightarrow\xi\in[-\varrho\pi,0]$, $\varphi\in[\psi+\pi,2\pi]\Rightarrow\xi\in[0,\varrho(\pi-\psi)]$, where $\xi:=\varrho(|\psi-\varphi|-\pi)$, we conclude that $\cos\varrho(|\psi-\varphi|-\pi)\geq 0$, $\varphi\in[0,2\pi]$. Taking into account the fact that the function $\Delta(\varphi)$ is non-decreasing, we obtain the desired result. □

3. Main Results

This section is devoted to a method allowing us to consider an entire function as an operator function. In this regard, we involve a special technique providing a proof of convergence of contour integrals, a similar scheme of reasonings was implemented in the papers [6,24]. At the same time, the behavior of the entire function in the neighborhood of the point at infinity is the main obstacle to realize the scheme of reasonings. Thus, to overcome difficulties related to the evaluation of improper contour integrals, we need to study more comprehensive innate properties of the entire function. The property of the growth regularity is a key for the desired estimates for the involved integral constructions. However, the lack of the latter approach is that the condition of the growth regularity is supposed to be satisfied within the complex plane, except for the exceptional set of circles, the location of which in general cannot be described. On the other hand, we need not use the subtle estimates for the Fredholm determinant established in [7], as we can be completely satisfied by the application of the Wieman theorem in accordance with which we can obtain the required estimate on the boundary of circle. Finally, we represent a suitable algebraic reasoning, allowing to involve a fractional derivative in the first term. The idea gives an opportunity to reformulate in abstract form many results in the framework of the theory of fractional differential equations, to say nothing for previously unsolved problems.

3.1. Estimate of a Real Component from Below

In this subsection, we aim to produce estimates of the real component from below for the technical purposes formulated in the further paragraphs. We should admit that it is formulated in rather a rough manner, but its principal value is the discovered way of constructing entire functions fallen in the scope of the theory of fractional evolution equations with the operator function in the second term. Apparently, having put a base, we can weaken conditions imposed upon the entire functions class afterwards, and in this way, come to the natural theory. We need to involve some technicalities related to the estimates of the entire unction from below; we remind that this matter is very important in the constructed theory. Below, we consider a sector $\mathfrak{L}_0(\theta_0,\theta_1):=\{z\in\mathbb{C},\ \theta_0\leq argz\leq\theta_1\}$ and use a short-hand notation $\mathfrak{L}_0(\theta):=\mathfrak{L}_0(-\theta,\theta)$.

Lemma 2. *Assume that the entire function f is of the proximate order $\varrho(r)$, $\varrho\in(0,1/2]$, maps the ray $\arg z=\theta_0$ within a sector $\mathfrak{L}_0(\zeta)$, $0<\zeta<\pi/2$, the set of zeros is regularly distributed relative to the proximate order and satisfies one of the conditions* (I) *or* (II)*, there exists $\varepsilon>0$ such that the angle $\theta_0-\varepsilon<arg\,z<\theta_0+\varepsilon$ does not contain the zeros with a sufficiently large absolute value. Then, for a sufficiently large value r, the following relation holds:*

$$\mathrm{Re}f(z)>Ce^{H(\theta_0)r^{\varrho(r)}},\arg z=\theta_0.$$

Proof. Using Theorem 1, we obtain the following representation:

$$f(z)=Cz^m\prod_{n=1}^{\infty}G\left(\frac{z}{a_n};p\right),$$

here, we should remark that $\deg P(z) = 0$. Let us show that the proximate order of the canonical product of the entire function is the same, and we have

$$M_f(r) = Cr^m M_F(r),\ F(z) = \prod_{n=1}^{\infty} G\left(\frac{z}{a_n}; p\right).$$

Therefore, in accordance with the definition of proximate order, we have

$$\overline{\lim_{r\to\infty}} \left\{ \frac{m \ln r + \ln C}{r^{\varrho(r)}} + \frac{\ln M_F(r)}{r^{\varrho(r)}} \right\} = \sigma_f,\ 0 < \sigma_f < \infty,$$

from which follows easily the fact that $0 < \sigma_F < \infty$, and further, $\sigma_F = \sigma_f$. Note that due to the condition that guarantees that the image of the ray $\arg z = \theta_0$ belongs to a sector in the right half-plane, we obtain

$$\mathrm{Re} f(z) \geq (1 + \tan \zeta)^{-1/2} |f(z)|,\ r = |z|,\ \arg z = \theta_0.$$

Applying Theorem 2, we conclude that excluding the intersection of the exceptional set of circulus with the ray $\arg z = \theta_0$, the following relation holds for sufficiently large values r:

$$|f(z)| = Cr^m \left| \prod_{n=1}^{\infty} G\left(\frac{z}{a_n}; p\right) \right| \geq Cr^m e^{H(\theta_0) r^{\varrho(r)}},$$

where $H(\theta_0) > 0$ in accordance with Lemma 1. It is clear that if we show that the intersection of the ray $\arg z = \theta_0$ with the exceptional set of circulus is empty, then we complete the proof. Note that the character of the zeros distribution allows us to claim that that is true. In accordance with the lemma conditions, it suffices to consider the neighborhoods of the zeros defined as follows $|z - a_n| < d|a_n|^{1-\varrho(|a_n|)}$, $|z - a_n| < d|a_n|^{1-\varrho(|a_n|)/2}$ and note that $0 < \varrho(|a_n|) < 1$ for a sufficiently large number $n \in \mathbb{N}$, since $\varrho(|a_n|) \to \varrho$, $n \to \infty$. Here, we ought to remind that the zeros are arranged in order with their absolute value growth. Thus, using simple properties of the power function with the positive exponent less than one, we obtain the fact that the intersection of the exceptional set of circulus with the ray $\arg z = \theta_0$ is empty for a sufficiently large $n \in \mathbb{N}$. $\square$

3.2. Classical Lemmas in the Refined Form

Denote by $\mathfrak{H}$ the abstract separable Hilbert space and consider an invertible operator $B : \mathfrak{H} \to \mathfrak{H}$ with a dense range. We use notation $W := B^{-1}$. Note that such agreements are justified by the significance of the operator with a compact resolvent, of which the detailed information of spectral properties can be found in the papers cited in the introduction section. Consider an entire function φ; due to the Taylor series expansion, we can write formally

$$\varphi(W) := \sum_{j=0}^{\infty} c_j W^j. \tag{6}$$

The latter construction is called an operator function, where c_n are the Taylor coefficients. Below, we consider conditions that guarantee the convergence of series (6) on some elements of the Hilbert space $\mathfrak{H}$; here, we ought to note that in this case, the operator $\varphi(W)$ is defined.

Assume that a compact operator T acts in the Hilbert space $\mathfrak{H}$, $\overline{\Theta(T)} \subset \mathfrak{L}_0(\theta_0, \theta_1)$, here we used notations accepted in [25], define the following contour

$$\Upsilon(T) := \{\lambda : \ |\lambda| = r > 0,\ \theta_0 \leq \arg\lambda \leq \theta_1\} \cup \{\lambda : \ |\lambda| > r,\ \arg\lambda = \theta_0,\ \arg\lambda = \theta_1\},$$

where the number r is chosen so that the operator $(I - \lambda T)^{-1}$ is regular within the corresponding closed circle. Consider the following hypotheses separately written for the convenience of the reader.

(HI) The operator B is compact, $\overline{\Theta(B)} \subset \mathfrak{L}_0(\theta_0, \theta_1)$, the entire function φ of the order of less than a half maps the sector $\mathfrak{L}_0(\theta_0, \theta_1)$ into the sector $\mathfrak{L}_0(\varpi)$, $\varpi < \pi/2\alpha$, $\alpha > 0$, its zeros with a sufficiently large absolute value do not belong to the sector $\mathfrak{L}_0(\theta_0, \theta_1)$.

Lemma 3. *Assume that the condition* (HI) *holds, then we have the following relation:*

$$\int\limits_{\Upsilon(B)} \varphi(\lambda) e^{-\varphi^{\alpha}(\lambda)t} B(I-\lambda B)^{-1} f d\lambda = \varphi(W) \int\limits_{\Upsilon(B)} e^{-\varphi^{\alpha}(\lambda)t} B(I-\lambda B)^{-1} f d\lambda,\ f \in \mathrm{D}(W^n),\ \forall n \in \mathbb{N},$$

and moreover,

$$\lim_{t\to+0} \frac{1}{2\pi i} \int\limits_{\Upsilon(B)} e^{-\varphi^{\alpha}(\lambda)t} B(I-\lambda B)^{-1} f d\lambda = f,\ f \in \mathrm{D}(W).$$

Proof. Firstly, we should note that the made assumptions regarding the order allow us to claim that the latter integral converges for a concrete value of the parameter t. Let us establish the formula

$$\int\limits_{\Upsilon(B)} \varphi(\lambda) e^{-\varphi^{\alpha}(\lambda)t} B(I-\lambda B)^{-1} f d\lambda = \sum_{n=0}^{\infty} c_n \int\limits_{\Upsilon(B)} e^{-\varphi^{\alpha}(\lambda)t} \lambda^n B(I-\lambda B)^{-1} f d\lambda. \tag{7}$$

To prove this fact, we should show that

$$\int\limits_{\Upsilon_j(B)} \varphi(\lambda) e^{-\varphi^{\alpha}(\lambda)t} B(I-\lambda B)^{-1} f d\lambda = \sum_{n=0}^{\infty} c_n \int\limits_{\Upsilon_j(B)} e^{-\varphi^{\alpha}(\lambda)t} \lambda^n B(I-\lambda B)^{-1} f d\lambda, \tag{8}$$

where

$$\Upsilon_j(B) := \{\lambda : |\lambda| = r > 0,\ \theta_0 \le \arg\lambda \le \theta_1\} \cup \{\lambda : r < |\lambda| < r_j,\ \arg\lambda = \theta_0,\ \arg\lambda = \theta_1\},$$

$r_j \uparrow \infty$. Note that in accordance with Lemma 6 [12], we obtain

$$\|(I-\lambda B)^{-1}\| \le C,\ r < |\lambda| < r_j,\ \arg\lambda = \theta_0,\ \arg\lambda = \theta_1.$$

Using this estimate, we can easily obtain the fact

$$\sum_{n=0}^{\infty} |c_n| |e^{-\varphi^{\alpha}(\lambda)t}| |\lambda^n| \cdot \|B(I-\lambda B)^{-1} f\| \le C\|B\| \cdot \|f\| \sum_{n=0}^{\infty} |c_n| |\lambda|^n e^{-\mathrm{Re}\varphi^{\alpha}(\lambda)t},\ \lambda \in \Upsilon_j(B),$$

where the latter series is convergent. Therefore, reformulating the well-known theorem of calculus on the absolutely convergent series in terms of the norm, we obtain (8). Now, let us show that the series

$$\sum_{n=0}^{\infty} c_n \int\limits_{\Upsilon_j(B)} e^{-\varphi^{\alpha}(\lambda)t} \lambda^n B(I-\lambda B)^{-1} f d\lambda \tag{9}$$

is uniformly convergent with respect to $j \in \mathbb{N}$. Using Lemma 1 [24], we obtain a trivial inequality

$$\left\| \int\limits_{\Upsilon_j(B)} e^{-\varphi^{\alpha}(\lambda)t} \lambda^n B(I-\lambda B)^{-1} f d\lambda \right\|_{\mathfrak{H}} \le C\|f\|_{\mathfrak{H}} \int\limits_{\Upsilon_j(B)} e^{-\mathrm{Re}\varphi^{\alpha}(\lambda)t} |\lambda|^n |d\lambda| \le$$

$$\le C\|f\|_{\mathfrak{H}} \int\limits_{\Upsilon_j(B)} e^{-C|\varphi(\lambda)|^{\alpha}t} |\lambda|^n |d\lambda|.$$

Here, we should note that to obtain the desired result, one is satisfied with a rather rough estimate dictated by the estimate obtained in Lemma 2. We obtain

$$\int\limits_{\Upsilon_j(B)} e^{-|\varphi(\lambda)|^\alpha t}|\lambda|^n|d\lambda| \leq C\int\limits_r^{r_j} e^{-xt}x^n dx \leq Ct^{-n}\Gamma(n+1).$$

Thus, we obtain

$$\left\|\int\limits_{\Upsilon_j(B)} e^{-\varphi^\alpha(\lambda)t}\lambda^n B(I-\lambda B)^{-1}fd\lambda\right\|_{\mathfrak{H}} \leq Ct^{-n}n!\,.$$

Using the standard formula establishing the estimate for the Taylor coefficients of the entire function, then applying the Stirling formula, we obtain

$$|c_n| < (e\sigma\varrho)^{n/\varrho}n^{-n/\varrho} < (2\pi)^{1/2\varrho}(\sigma\varrho)^{n/\varrho}\left(\frac{\sqrt{n}}{n!}\right)^{1/\varrho},$$

where $0<\sigma<\infty$ is a type of the function φ. Thus, we obtain

$$\sum_{n=1}^{\infty}|c_n|\left\|\int\limits_{\Upsilon_j(B)} e^{-\varphi^\alpha(\lambda)t}\lambda^n B(I-\lambda B)^{-1}fd\lambda\right\| \leq C\sum_{n=1}^{\infty}(\sigma\varrho)^{n/\varrho}t^{-n}(n!)^{1-1/\varrho}n^{1/2\varrho}.$$

The latter series is convergent for an arbitrary fixed $t>0$, which proves the uniform convergence of the series (9) with respect to j. Therefore, reformulating the well-known theorem of calculus applicably to the norm of the Hilbert space, taking into account the facts

$$\int\limits_{\Upsilon_j(B)} \varphi(\lambda)e^{-\varphi^\alpha(\lambda)t}B(I-\lambda B)^{-1}fd\lambda \xrightarrow{\mathfrak{H}} \int\limits_{\Upsilon(B)} \varphi(\lambda)e^{-\varphi^\alpha(\lambda)t}B(I-\lambda B)^{-1}fd\lambda,$$

$$\int\limits_{\Upsilon_j(B)} e^{-\varphi^\alpha(\lambda)t}\lambda^n B(I-\lambda B)^{-1}fd\lambda \xrightarrow{\mathfrak{H}} \int\limits_{\Upsilon(B)} e^{-\varphi^\alpha(\lambda)t}\lambda^n B(I-\lambda B)^{-1}fd\lambda,\ j\to\infty,$$

we obtain formula (7). Further, using the formula

$$\lambda^k B^k(I-\lambda B)^{-1} = (I-\lambda B)^{-1} - (I+\lambda B+...+\lambda^{k-1}B^{k-1}),\ k\in\mathbb{N},$$

taking into account the facts that the operators B^k and $(I-\lambda B)^{-1}$ commute, we obtain

$$\int\limits_{\Upsilon(B)} e^{-\varphi^\alpha(\lambda)t}\lambda^n B(I-\lambda B)^{-1}fd\lambda =$$

$$= \int\limits_{\Upsilon(B)} e^{-\varphi^\alpha(\lambda)t}B(I-\lambda B)^{-1}W^n fd\lambda - \int\limits_{\Upsilon(B)} e^{-\varphi^\alpha(\lambda)t}\sum_{k=0}^{n-1}\lambda^k B^{k+1}W^n fd\lambda = I_1(t)+I_2(t).$$

Since the operators W^n and $B(I-\lambda B)^{-1}$ commute, this fact can be obtained by direct calculation, and we obtain

$$I_1(t) = W^n\int\limits_{\Upsilon(B)} e^{-\varphi^\alpha(\lambda)t}B(I-\lambda B)^{-1}fd\lambda.$$

Consider $I_2(t)$, using the technique applied in Lemma 5 [24]. It is rather reasonable to consider the following representation:

$$I_2(t) := -\sum_{k=0}^{n-1} \beta_k(t) B^{k-n+1} f,\ \beta_k(t) := \int\limits_{\Upsilon(B)} e^{-\varphi^{\alpha}(\lambda)t} \lambda^k d\lambda.$$

Analogous to the scheme of reasonings of Lemma 5 [24], we can show that $\beta_k(t) = 0$ under the imposed condition of the entire function growth regularity. Below, we produce a complete reasoning to avoid any kind of misunderstanding. Since the function under the integral is analytic inside the contour, then

$$\oint\limits_{\Upsilon_R(B)} e^{-\varphi^{\alpha}(\lambda)t} \lambda^k d\lambda = 0,$$

where $\Upsilon_R(B) := \mathrm{Fr}\{\mathrm{int}\,\Upsilon(B) \cap \{\lambda : r < |\lambda| < R\}\}$. Hence, it suffices to show that there exists such a sequence $\{R_n\}_1^{\infty}$, $R_n \uparrow \infty$ that

$$\oint\limits_{\Upsilon_{R_n}(B)} e^{-\varphi^{\alpha}(\lambda)t} \lambda^k d\lambda \to \beta_k(t),\ n \to \infty. \tag{10}$$

Consider $\tilde{\Upsilon}_R := \{\lambda : |\lambda| = R, \theta_0 \leq \arg\lambda \leq \theta_1\}$, $\hat{\Upsilon}_R := \{\lambda : |\lambda| = r, \theta_0 \leq \arg\lambda \leq \theta_1\} \cup \{\lambda : r < |\lambda| < R, \arg\lambda = \theta_0, \arg\lambda = \theta_1\}$. We have obviously

$$\oint\limits_{\Upsilon_R(B)} e^{-\varphi^{\alpha}(\lambda)t} \lambda^k d\lambda = \int\limits_{\tilde{\Upsilon}_R} e^{-\varphi^{\alpha}(\lambda)t} \lambda^k d\lambda + \int\limits_{\hat{\Upsilon}_R} e^{-\varphi^{\alpha}(\lambda)t} \lambda^k d\lambda.$$

Therefore, it suffices to prove that

$$\int\limits_{\tilde{\Upsilon}_{R_n}} e^{-\varphi^{\alpha}(\lambda)t} \lambda^k d\lambda \to 0, \tag{11}$$

where $\{R_n\}_1^{\infty}$, $R_n \uparrow \infty$. Observe that the latter claim is not so trivial and requires to involve some subtle estimates on the boundary of a circle. However, the following approach gives us what we need, and we have

$$\left| \int\limits_{\tilde{\Upsilon}_R} e^{-\varphi^{\alpha}(\lambda)t} \lambda^k d\lambda \right| \leq R^k \int\limits_{\tilde{\Upsilon}_R} |e^{-\varphi^{\alpha}(\lambda)t}||d\lambda| \leq R^{k+1} \int\limits_{\theta_0}^{\theta_1} e^{-\mathrm{Re}\varphi^{\alpha}(\lambda)t} d\arg\lambda.$$

Using Theorem 30, §18, Chapter I [23], we can establish the fact that there exists a sequence $R_n \uparrow \infty$ such that for arbitrary positive ε the following estimate holds for sufficiently large numbers

$$e^{-C|\varphi(\lambda)|^{\alpha} t} \leq e^{-C m_{\varphi}^{\alpha}(R_n) t} \leq e^{-Ct[M_{\varphi}(R_n)]^{(\cos \pi\varrho - \varepsilon)\alpha}},\ \lambda \in \tilde{\Upsilon}_{R_n},$$

where ϱ is the order. Applying this result, taking into account condition (HI), we obtain

$$\int\limits_{\theta_0}^{\theta_1} e^{-\mathrm{Re}\varphi^{\alpha}(\lambda)t} d\arg\lambda \leq \int\limits_{\theta_0}^{\theta_1} e^{-Ct|\varphi(\lambda)|^{\alpha}} d\arg\lambda \leq e^{-Ct[M_{\varphi}(R_n)]^{(\cos \pi\varrho - \varepsilon)\alpha}} \int\limits_{\theta_0}^{\theta_1} d\arg\lambda,$$

which gives us (11). Having recollected the previously made implications, we obtain the fact $\beta_k(t) = 0$, hence $I_2(t) = 0$ and we get

$$\int\limits_{\Upsilon(B)} e^{-\varphi^{\alpha}(\lambda)t}\lambda^n B(I-\lambda B)^{-1} f d\lambda = W^n \int\limits_{\Upsilon(B)} e^{-\varphi^{\alpha}(\lambda)t} B(I-\lambda B)^{-1} f d\lambda.$$

Substituting the latter relation into the formula (7), we obtain the first statement of the lemma.

The scheme of the proof corresponding to the second statement is absolutely analogous to the one presented in Lemma 4 [24]. We should just use Lemma 2 providing the estimates along the sides of the contour. Thus, the completion of the reasonings is due to the technical repetition of the Lemma 4 [24] reasonings, which we leave to the reader. □

3.3. Series Expansion and Its Application to the Existence and Uniqueness Theorems

In this paragraph, we represent two theorems valuable from theoretical and applied points of view, respectively. The first one is a generalization of the Lidskii method, which is why following the classical approach, we divide it into two statements that can be claimed separately. The first statement establishes a character of the series convergence having a principal meaning within the whole concept. The second statement reflects the name of convergence, Abel–Lidskii; since the latter can be connected with the definition of the series convergence in the Abel sense, more detailed information can be found in the monograph by Hardy G.H. [26]. The second theorem is a valuable application of the first one, and it is based upon suitable algebraic reasonings noticed by the author, allowing us to involve a fractional derivative in the first term. We should note that previously, a concept of an operator function represented in the second term was realized in the paper [6], where a case corresponding to a function represented by a Laurent series with a polynomial regular part was considered. Below, we consider a comparatively more difficult case obviously related to the infinite regular part of the Laurent series and therefore requiring a principally different method of study.

It is a well-known fact that each eigenvalue $\mu_q,\ q \in \mathbb{N}$ of the compact operator B generates a set of Jordan chains containing eigenvectors and root vectors. Denote by $m(q)$ a geometrical multiplicity of the corresponding eigenvalue and consider a Jordan chain corresponding to an eigenvector $e_{q_\xi},\ \xi = 1, 2, ..., m(q)$. We have

$$e_{q_\xi}, e_{q_\xi+1}, ..., e_{q_\xi+k(q_\xi)}, \tag{12}$$

where $k(q_\xi)$ indicates a number of elements in the Jordan chain, the symbols except for the first one denote root vectors of the operator B. Note that combining the Jordan chains corresponding to an eigenvalue, we obtain a Jordan basis in the invariant subspace generated by the eigenvalue; moreover, we can arrange a so-called system of major vectors $\{e_i\}_1^\infty$ (see [7]) of the operator B having combined Jordan chains. It is remarkable that the eigenvalue $\bar{\mu}_q$ of the operator B^* generates the Jordan chains of the operator B^* corresponding to (12). In accordance with [12], we have

$$g_{q_\xi+k(q_\xi)},\ g_{q_\xi+k(q_\xi)-1}, ..., g_{q_\xi},$$

where the symbols, except for the first one, denote root vectors of the operator B^*. Combining Jordan chains of the operator B^*, we can construct a biorthogonal system $\{g_n\}_1^\infty$ with respect to the system of the major vectors of the operator B. This fact is given in detail in the paper [12]. The following construction plays a significant role in the theory created in the papers [6,12,24] and therefore deserves to be considered separately. Denote

$$\mathcal{A}_\nu(\varphi, t) f := \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i}(t), \tag{13}$$

where $\{N_\nu\}_1^\infty$ is a sequence of natural numbers,

$$c_{q_\xi+i}(t) = e^{-\varphi(\lambda_q)t} \sum_{j=0}^{k(q_\xi)-i} H_j(\varphi,\lambda_q,t)c_{q_\xi+i+j},\ i = 0,1,2,...,k(q_\xi), \tag{14}$$

$c_{q_\xi+i} = (f, g_{q_\xi+k-i})/(e_{q_\xi+i}, g_{q_\xi+k-i})$, $\lambda_q = 1/\mu_q$ is a characteristic number corresponding to e_{q_ξ},

$$H_j(\varphi,z,t) := \frac{e^{\varphi(z)t}}{j!} \cdot \lim_{\zeta\to 1/z} \frac{d^j}{d\zeta^j}\left\{e^{-\varphi(\zeta^{-1})t}\right\},\ j = 0,1,2,\ldots .$$

More detailed information on the considered above Jordan chains can be found in [12].

Theorem 3. *Assume that the condition* (HI) *holds, $B \in \mathfrak{S}_s$, $0 < s < \infty$. Then a sequence of natural numbers $\{N_\nu\}_0^\infty$ can be chosen so that*

$$\sum_{\nu=0}^{\infty} \|\mathcal{A}_\nu(\varphi^\alpha,t)f\|_{\mathfrak{H}} < \infty,\ f \in \mathfrak{H};\quad f = \lim_{t\to+0} \sum_{\nu=0}^{\infty} \mathcal{A}_\nu(\varphi^\alpha,t)f,\ f \in \mathrm{D}(W). \tag{15}$$

Proof. Firstly, we will establish the fact of the series convergence. Let us choose $R > 0$, $0 < \kappa < 1$, so that $R(1-\kappa) = r$, thus we get a sequence $\{R_\nu\}_0^\infty$, $R_\nu = R(1-\kappa)^{-\nu+1}$. Applying Lemma 5 [12], we obtain

$$\|(I-\lambda B)^{-1}\|_{\mathfrak{H}} \le e^{\gamma(\xi_\nu)\xi_\nu^\sigma}\xi_\nu^m,\ \sigma > s,\ m = [\sigma],\ R_\nu < \xi_\nu < R_{\nu+1},$$

where

$$\gamma(\tau) = \beta(\tau^{m+1}) + C_1\beta(C_2\tau^{m+1}),\ C_1, C_2 > 0,$$

$$\beta(\tau) = \tau^{-\frac{\sigma}{m+1}}\left(\int_0^\tau \frac{n_{B^{m+1}}(t)dt}{t} + \tau\int_\tau^\infty \frac{n_{B^{m+1}}(t)dt}{t^2}\right),\ \tau > 0.$$

Applying Lemma 3 [7], we can claim

$$\sum_{i=1}^{\infty} \lambda_i^{\frac{\sigma}{(m+1)}}(\tilde{B}) \le \sum_{i=1}^{\infty} s_i^\sigma(B) < \infty, \tag{16}$$

where $\tilde{B} := (B^{*m+1}A^{m+1})^{1/2}$. Using (16), we obtain easily $\tilde{B} \in \mathfrak{S}_\phi$, $\phi < \sigma/(m+1)$. Consider a contour $\Upsilon_\nu := \mathrm{Fr}\{\mathrm{int}\,\Upsilon(B) \cap \{\lambda : \xi_\nu < |\lambda| < \xi_{\nu+1}\}\}$, denote by N_ν a number of poles of the resolvent contained in the set $\mathrm{int}\,\Upsilon(B) \cap \{\lambda : r < |\lambda| < \xi_\nu\}$. Applying Lemma 3 [24], we have

$$\frac{1}{2\pi i}\oint_{\Upsilon_\nu} e^{-\varphi^\alpha(\lambda)t}B(I-\lambda B)^{-1}f d\lambda = \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i}c_{q_\xi+i}(t),\ f \in \mathfrak{H}. \tag{17}$$

Further reasoning is devoted to estimating the above integral and based on the contour Υ_ν decomposition on terms $\tilde{\Upsilon}_\nu := \{\lambda : |\lambda| = \xi_\nu,\ \theta_0 \le \arg\lambda \le \theta_1\}$, $\tilde{\Upsilon}_{\nu+1}$, $\Upsilon_{\nu_-} := \{\lambda : \xi_\nu < |\lambda| < \xi_{\nu+1},\ \arg\lambda = \theta_0\}$, $\Upsilon_{\nu_+} := \{\lambda : \xi_\nu < |\lambda| < \xi_{\nu+1},\ \arg\lambda = \theta_1\}$. In accordance with Theorem 30, §18, Chapter I [23] (Wieman theorem), we can choose such a sequence $\{x_n\}_1^\infty$, $x_n \uparrow \infty$, $\xi_\nu < x_\nu < \xi_{\nu+1}$ that for an arbitrary positive ε and sufficiently large numbers ν, we have

$$e^{-C|\varphi(\lambda)|^\alpha t} \le e^{-Cm_\varphi^\alpha(x_\nu)t} \le e^{-Ct[M_\varphi(x_\nu)]^{(\cos\pi\varrho-\varepsilon)\alpha}},\ \lambda \in \tilde{\Upsilon}_\nu, \tag{18}$$

where ϱ is the order. We should note that the assumption $\xi_\nu < x_\nu < \xi_{\nu+1}$ is made without loss of generality of the reasonings, as in the context of the proof, we do not care about the accurate arrangement of the contours but need to prove the existence of an arbitrary one. This inconvenience is based upon the uncertainty in the way of choosing the contours in accordance with the Wieman theorem; at the same time at any rate, we can extract a subsequence of the sequence $\{\xi_\nu\}_1^\infty$ in the way we need. Thus, using the given reasonings, Applying Lemma 5 [12], relation (18), we obtain

$$J_\nu := \left\| \int\limits_{\tilde{\Upsilon}_\nu} e^{-\varphi^\alpha(\lambda)t} B(I-\lambda B)^{-1} f d\lambda \right\|_{\mathfrak{H}} \leq \int\limits_{\tilde{\Upsilon}_\nu} e^{-t\mathrm{Re}\,\varphi^\alpha(\lambda)} \left\| B(I-\lambda B)^{-1} f \right\|_{\mathfrak{H}} |d\lambda| \leq$$

$$\leq e^{\gamma(\xi_\nu)\xi_\nu^\sigma} \xi_\nu^{m+1} C e^{-Ct[M_\varphi(x_\nu)]^{(\cos\pi\varrho-\varepsilon)\alpha}} \int\limits_{\theta_0}^{\theta_1} d\arg\lambda.$$

As a result, we obtain

$$J_\nu \leq e^{\gamma(\xi_\nu)\xi_\nu^\sigma} \xi_\nu^{m+1} C e^{-Ct[M_\varphi(x_\nu)]^{(\cos\pi\varrho-\varepsilon)\alpha}},\ m=[\sigma].$$

Using Lemma 2 [12], we have $\gamma(|\lambda|) \to 0,\ |\lambda| \to \infty$. In accordance with the Formula (1) we can extract a subsequence from the sequence $\{x_\nu\}_1^\infty$ and as a result from the sequence $\{\xi_\nu\}_1^\infty$ so that for a fixed t and a sufficiently large ν, we have $\gamma(|\xi_\nu|)|\xi_\nu|^\sigma - Ct[M_\varphi(x_\nu)]^{(\cos\pi\varrho-\varepsilon)\alpha} < 0$. Here, we have not used a subsequence to simplify the form of writing. Therefore, we have

$$\sum_{\nu=0}^{\infty} J_\nu < \infty.$$

Applying Lemma 6 [12], Lemma 2, we obtain

$$J_\nu^+ := \left\| \int\limits_{\Upsilon_{\nu_+}} e^{-\varphi^\alpha(\lambda)t} B(I-\lambda B)^{-1} f d\lambda \right\|_{\mathfrak{H}} \leq C\|f\|_{\mathfrak{H}} \int\limits_{R_\nu}^{R_{\nu+1}} e^{-Ct\mathrm{Re}\,\varphi^\alpha(\lambda)} |d\lambda| \leq$$

$$\leq C \int\limits_{R_\nu}^{R_{\nu+1}} e^{-Ct|\varphi(\lambda)|^\alpha} |d\lambda| \leq C e^{-Ct e^{\alpha H(\theta_1)} R_\nu^{\varrho(R_\nu)}} \int\limits_{R_\nu}^{R_{\nu+1}} |d\lambda| = C e^{-Ct e^{\alpha H(\theta_1)} R_\nu^{\varrho(R_\nu)}} \{R_{\nu+1} - R_\nu\}.$$

$$J_\nu^- := \left\| \int\limits_{\Upsilon_{\nu_-}} e^{-\varphi^\alpha(\lambda)t} B(I-\lambda B)^{-1} f d\lambda \right\|_{\mathfrak{H}} \leq C e^{-Ct e^{\alpha H(\theta_0)} R_\nu^{\varrho(R_\nu)}} \int\limits_{R_\nu}^{R_{\nu+1}} |d\lambda| = C e^{-Ct e^{\alpha H(\theta_0)} R_\nu^{\varrho(R_\nu)}} \{R_{\nu+1} - R_\nu\}.$$

The obtained results allow us to claim (the proof is omitted) that

$$\sum_{\nu=0}^{\infty} J_\nu^+ < \infty,\ \sum_{\nu=0}^{\infty} J_\nu^- < \infty.$$

Therefore, applying Formula (17), we obtain the first relation (15). To prove the second relation (15), we should note that in accordance with (17), the properties of the contour integral, we have

$$\frac{1}{2\pi i} \oint\limits_{\Upsilon_{\xi_n}(B)} e^{-\varphi^\alpha(\lambda)t} B(I-\lambda B)^{-1} f\, d\lambda = \sum_{\nu=0}^{n-1} \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i}(t),\ f \in \mathfrak{H},\ n \in \mathbb{N},$$

where $\Upsilon_{\xi_n}(B) := \mathrm{Fr}\{\mathrm{int}\,\Upsilon(B) \cap \{\lambda : r < |\lambda| < \xi_n\}\}$. Using the fact $J_\nu \to 0,\, \nu \to \infty$, we obtain

$$\frac{1}{2\pi i}\oint\limits_{\Upsilon_{\xi_n}(B)} e^{-\varphi^\alpha(\lambda)t}B(I-\lambda B)^{-1}f\,d\lambda \to \frac{1}{2\pi i}\oint\limits_{\Upsilon(B)} e^{-\varphi^\alpha(\lambda)t}B(I-\lambda B)^{-1}f\,d\lambda,\; n\to\infty.$$

The latter relation allows to obtain the formula

$$\frac{1}{2\pi i}\oint\limits_{\Upsilon(B)} e^{-\varphi^\alpha(\lambda)t}B(I-\lambda B)^{-1}f\,d\lambda = \sum_{\nu=0}^{\infty}\mathcal{A}_\nu(\varphi^\alpha,t)f,\; f\in\mathfrak{H}. \tag{19}$$

If $f \in \mathrm{D}(W)$, then applying Lemma 3, we obtain the second relation (15). □

Consider element-functions $u : \mathbb{R}_+ \to \mathfrak{H}$, $u := u(t)$, $t \geq 0$, belonging to the Hilbert space $\mathfrak{H}$; using the approach [6], we understand the differentiation and integration operations in the generalized sense, i.e., the derivative is defined as a limit in the sense of the norm (see [12,27]). Involving a superposition of the operations, we can define a generalized fractional derivative in the Riemann–Liouville sense (see [4,6]). In the formal sense, we have

$$\mathfrak{D}_-^{1/\alpha}f(t) := -\frac{1}{\Gamma(1-1/\alpha)}\frac{d}{dt}\int\limits_0^\infty f(t+x)x^{-1/\alpha}dx,\; \alpha>1.$$

Consider a Cauchy problem

$$\mathfrak{D}_-^{1/\alpha}u = \varphi(W)u,\; u(0) = f \in \mathrm{D}(W^n),\, \forall n\in\mathbb{N}. \tag{20}$$

Theorem 4. *Assume that the conditions of Theorem 3 hold, then there exists a solution of the Cauchy problem (20) in the form*

$$u(t) = \sum_{\nu=0}^{\infty}\mathcal{A}_\nu(\varphi^\alpha,t)f. \tag{21}$$

Moreover, the existing solution is unique if the operator $\mathfrak{D}_-^{1-1/\alpha}\varphi(W)$ is accretive.

Proof. Firstly, we will show that $u(t)$ is a solution of the problem (20), we need prove the following formula

$$\frac{d}{dt}\int\limits_{\Upsilon(B)}\varphi(\lambda)^{1-\alpha}e^{-\varphi^\alpha(\lambda)t}B(I-\lambda B)^{-1}f\,d\lambda = -\int\limits_{\Upsilon(B)}\varphi(\lambda)e^{-\varphi(\lambda)^\alpha t}B(I-\lambda B)^{-1}f\,d\lambda,\; h\in\mathfrak{H}. \tag{22}$$

Using simple reasonings, we obtain the fact that that for an arbitrary

$$\Upsilon_j(B) := \{\lambda:\; |\lambda| = r>0,\, \theta_0 \leq \arg\lambda \leq \theta_1\} \cup \{\lambda:\; r<|\lambda|<r_j,\, \arg\lambda=\theta_0,\, \arg\lambda=\theta_1\},$$

there exists a limit $(e^{-\varphi^\alpha(\lambda)\Delta t}-1)e^{-\varphi^\alpha(\lambda)t}/\Delta t \longrightarrow -\varphi^\alpha(\lambda)e^{-\varphi^\alpha(\lambda)t}$, $\Delta t\to 0$, where convergence is uniform with respect to $\lambda\in\Upsilon_j(B)$. By virtue of the decomposition on the Taylor series, we get

$$\left|\frac{e^{-\varphi^\alpha(\lambda)\Delta t}-1}{\Delta t}e^{-\varphi^\alpha(\lambda)t}\right| \leq |\varphi(\lambda)|^\alpha e^{|\varphi(\lambda)|^\alpha\Delta t}e^{-\mathrm{Re}\,\varphi^\alpha(\lambda)t} \leq |\varphi(\lambda)|^\alpha e^{(\Delta t-Ct)|\varphi(\lambda)|^\alpha},\; \lambda\in\Upsilon(B).$$

Thus, applying the latter estimate, Lemma 6 [12], for a sufficiently small value Δt, we obtain

$$\left\| \int\limits_{\Upsilon(B)} \frac{e^{-\varphi^{\alpha}(\lambda)\Delta t}-1}{\Delta t}\varphi^{1-\alpha}(\lambda)e^{-\varphi^{\alpha}(\lambda)t}B(I-\lambda B)^{-1}fd\lambda \right\|_{\mathfrak{H}} \leq C\|f\|_{\mathfrak{H}} \int\limits_{\Upsilon(B)} e^{-Ct|\varphi(\lambda)|^{\alpha}}|\varphi(\lambda)||d\lambda|. \tag{23}$$

Let us establish the convergence of the last integral. Applying Theorem 2, we obtain

$$\int\limits_{\Upsilon(B)} e^{-Ct|\varphi(\lambda)|^{\alpha}}|\varphi(\lambda)||d\lambda| \leq \int\limits_{\Upsilon(B)} e^{-te^{C|\lambda|^{\varrho(|\lambda|)}}} e^{C|\lambda|^{\varrho}}|d\lambda|.$$

It is clear that the latter integral is convergent for an arbitrary positive value t, which guarantees that the improper integral at the left-hand side of (23) is uniformly convergent with respect to Δt. These facts give us an opportunity to claim that relation (22) holds. Here, we should explain that this conclusion is based on the generalization of the well-known theorem of the calculus; we left a complete investigation of the matter to the reader, having noted that the reasonings are absolutely analogous to the ordinary calculus.

Applying the scheme of the proof corresponding to the ordinary integral calculus, using the contour $\Upsilon_j(B)$, applying Lemma 6 [12] respectively, we can establish a formula

$$\int\limits_{0}^{\infty} x^{-\xi}dx \int\limits_{\Upsilon(B)} e^{-\varphi^{\alpha}(\lambda)(t+x)}B(I-\lambda B)^{-1}f\,d\lambda = \int\limits_{\Upsilon(B)} e^{-\varphi^{\alpha}(\lambda)t}B(I-\lambda B)^{-1}fd\lambda \int\limits_{0}^{\infty} x^{-\xi}e^{-\varphi^{\alpha}(\lambda)x}dx, \tag{24}$$

where $\xi \in (0,1)$. Taking into account the obvious formula

$$\int\limits_{0}^{\infty} x^{-1/\alpha}e^{-\varphi^{\alpha}(\lambda)x}dx = \Gamma(1-1/\alpha)\varphi^{1-\alpha}(\lambda),$$

we get

$$\mathfrak{D}_{-}^{1/\alpha}\int\limits_{\Upsilon(B)} e^{-\varphi^{\alpha}(\lambda)t}B(I-\lambda B)^{-1}f = \int\limits_{\Upsilon(B)} e^{-\varphi^{\alpha}(\lambda)t}\varphi(\lambda)B(I-\lambda B)^{-1}f\,d\lambda. \tag{25}$$

Applying Lemma 3, relation (19), we obtain the fact that u is a solution of the Equation (20). The fact that the initial condition holds, in the sense $u(t) \xrightarrow{\mathfrak{H}} f$, $t \to +0$, follows from the second relation (15) Theorem 3. The scheme of the proof corresponding to the uniqueness part is given in Theorem 6 [6]. We complete the proof. □

3.4. Applications to Concrete Operators and Physical Processes

Note that the method considered above allows to obtain a solution for the evolution equation with the operator function in the second term, where the operator argument belongs to a sufficiently wide class of operators. One can find a lot of examples in [6], where such well-known operators as the Riesz potential, the Riemann–Liouville fractional differential operator, the Kipriyanov operator, and the difference operator are studied. Some interesting examples that cannot be covered by the results established in [22] are represented in the paper [17]. The general approach, applied in the paper [2], creates a theoretical base to produce a more abstract example—a transform of the m-accretive operator. We should point out a significance of the last statement since the class contains the infinitesimal generator of a strongly continuous semigroup of contractions. Here, we recall that fractional differential operators of the real order can be expressed in terms of the infinitesimal generator of the corresponding semigroup [2]. Application of the obtained results appeals to electron-induced kinetics of ferroelectrics polarization switching as the self-similar memory physical systems. The whole point is that the mathematical model of the fractal dynamic system includes a Cauchy problem for the differential equation of the fractional order considered in the paper [1], where computational schemes for solving the problem were constructed using the Adams–Bashforth–Moulton-type predictor–corrector methods. The stochastic algorithm based on the Monte Carlo method was proposed to simulate the domain nucleation process during the restructuring domain structure in ferroelectrics.

At the same time, the results obtained in this paper allow us not only to solve the problem analytically, but consider a whole class of problems for evolution equations of fractional order. As for the mentioned concrete case [1], we just need consider a suitable functional Hilbert space and apply Theorem 4 directly. For instance, it can be the Lebesgue space of square-integrable functions. Here, we should note that in the case corresponding to a functional Hilbert space, we gain more freedom in constructing the theory, and thus, some modifications of the method can appear, but it is an issue for further more detailed study, which is not supposed in the framework of this paper. However, the following example may be of interest to the reader.

Goldstein et al. proved in [28] several new results having replaced the Laplacian by the Kolmogorov operator:
$$L = \Delta + \frac{\nabla \rho}{\rho} \cdot \nabla,$$
here, ρ is a probability density on $\mathbb{R}^N$ satisfying $\rho \in C^{1+\alpha}_{lok}(\mathbb{R}^N)$ for some $\alpha \in (0,1),\ \rho(x) > 0$ for all $x \in \mathbb{R}^N$. A reasonable question can appear: are there possible connections between the developed theory and the operator L? Indeed, the mentioned operator gives us an opportunity to show brightly capacity of the spectral theory methods. First of all, let us note that the relation $L = \rho^{-1}W$ holds, where $W := \mathrm{div}\rho\nabla$. Thus, at first glance, the right direction of the issue investigation should be connected with the operator composition $\rho^{-1}W$ since the operator W is uniformly elliptic and satisfies the following hypotheses (see [2]).

(H1) There exists a Hilbert space $\mathfrak{H}_+ \subset\subset \mathfrak{H}$ and a linear manifold $\mathfrak{M}$ that is dense in $\mathfrak{H}_+$. The operator V is defined on $\mathfrak{M}$.

(H2) $|(Vf,g)_{\mathfrak{H}}| \le C_1\|f\|_{\mathfrak{H}_+}\|g\|_{\mathfrak{H}_+},\ \mathrm{Re}(Vf,f)_{\mathfrak{H}} \ge C_2\|f\|^2_{\mathfrak{H}_+},\ f,g \in \mathfrak{M},\ C_1,C_2 > 0.$

Apparently, the results [2,12,17] can be applied to the operator after an insignificant modification. A couple of words on the difficulties appear while we study the operator composition. Superficially, the problem looks good, but it is not so for the inverse operator (one needs to prove that it is a resolvent) which is a composition of an unbounded operator and a resolvent of the operator W, indeed since $R_W W = I$, then formally, we have $L^{-1}f = R_W \rho f$. Most likely, the general theory created in the papers [2,17] can be adopted to some operator composition, but it is a tremendous work. Instead of that, we may find a suitable pair of Hilbert spaces that is also not so easy. However, we shall see. Below, we consider a space $\mathbb{R}^N$ endowed with the norm
$$|x| = \sqrt{\sum_{k=1}^{n} |x_k|^2},\ x = (x_1, x_2, ..., x_n) \in \mathbb{R}^N.$$

Assume that there exists a constant $\lambda > 2$ such that the following condition holds:
$$\left\|\rho^{1/\lambda - 1}\nabla\rho\right\|_{L_\infty(\mathbb{R}^N)} < \infty,\ \rho^{1/\lambda}(x) = O(1 + |x|).$$

One can verify easily that this condition is not unnatural, as it holds for a function $\rho(x) = (1+|x|)^\lambda,\ x \in \mathbb{R}^N,\ \lambda \ge 1$. Let us define a Hilbert space $\mathfrak{H}_+$ as a completion of the set $C_0^\infty(\mathbb{R}^N)$ with the norm
$$\|f\|^2_{\mathfrak{H}_+} = \|\nabla f\|^2_{L_2(\mathbb{R}^N)} + \|f\|^2_{L_2(\mathbb{R}^N,\varphi^{-2})},\ \varphi(x) = (1+|x|),$$

here, one can easily see that it is generated by the corresponding inner product. The following result can be obtained as a consequence of the Adams theorem (see Theorem 1 [29]). Using the formula

$$\varphi^{\lambda/2}\nabla f=\nabla(f\varphi^{\lambda/2})-f\nabla\varphi^{\lambda/2},\, f=g\varphi^{-\lambda/2},\, g\in C_0^\infty(\mathbb{R}^N),$$

we can easily obtain

$$\left(\int\limits_{\mathbb{R}^N}|\nabla(g\varphi^{-\lambda/2})|^2\varphi^\lambda dx\right)^{1/2}\leq C\|g\|_{\mathfrak{H}_+}.$$

It is clear that the latter relation can be expanded to the elements of the space $\mathfrak{H}_+$. Note that

$$\|g\|_{L_2(\mathbb{R}^N,\varphi^{-\lambda})}\leq\|g\|_{L_2(\mathbb{R}^N,\varphi^{-2})},\, g\in L_2(\mathbb{R}^N,\varphi^{-2}),\,\lambda>2.$$

This relation gives us the inclusion $\mathfrak{H}_+\subset L_2(\mathbb{R}^N,\varphi^{-\lambda})$, thus we conclude that $g\varphi^{-\lambda/2}\in L_2(\mathbb{R}^N)$, $g\in\mathfrak{H}_+$. In accordance with Theorem 1 [29], we conclude that if a set is bounded in the sense of the norm $\mathfrak{H}_+$, then it is compact in the sense of the norm $L_2(\mathbb{R}^N,\varphi^{-\lambda})$.

Thus, we have created a pair of Hilbert spaces $\mathfrak{H}_-:=L_2(\mathbb{R}^N,\varphi^{-\lambda})$ and $\mathfrak{H}_+$, satisfying the condition of compact embedding, i.e., $\mathfrak{H}_+\subset\subset\mathfrak{H}_-$. Let us see how can it help us in studying the operator L. Considering an operator $L':=-L+\eta\rho^{-2/\lambda}I$, $\eta>0$, we ought to remark here that we need involve an additional summand to apply the methods [2]. The crucial point is related to how to estimate the second term of the operator $-L$ from below. Here, we should point out that some peculiar techniques of the theory of functions can be involved. However, along with this, we can consider a simplified case (since we have imposed additional conditions upon the function ρ) in order to show how the invented method works. The following reasonings are made under the assumption that the functions $f,g\in C_0^\infty(\mathbb{R}^N)$. Using simple reasonings based upon the Cauchy–Schwarz inequality, we obtain

$$\left|\int\limits_{\mathbb{R}^N}\frac{\nabla\rho}{\rho}\cdot\nabla f\,\bar{g}dx\right|\leq\int\limits_{\mathbb{R}^N}\left|\frac{\nabla\rho}{\rho}\right||\nabla f|\,|g|dx\leq\left\|\rho^{1/\lambda-1}\nabla\rho\right\|_{L_\infty(\mathbb{R}^N)}\frac{1}{2}\left\{\varepsilon\|\nabla f\|^2_{L_2(\mathbb{R}^N)}+\frac{1}{\varepsilon}\|g\|^2_{L_2(\mathbb{R}^N,\rho^{-2/\lambda})}\right\},$$

where $\varepsilon>0$. Therefore,

$$-\mathrm{Re}\left(\frac{\nabla\rho}{\rho}\cdot\nabla f,f\right)_{L_2(\mathbb{R}^N)}\geq-\left\|\rho^{1/\lambda-1}\nabla\rho\right\|_{L_\infty(\mathbb{R}^N)}\frac{1}{2}\left\{\varepsilon\|\nabla f\|^2_{L_2(\mathbb{R}^N)}+\frac{1}{\varepsilon}\|g\|^2_{L_2(\mathbb{R}^N,\rho^{-2/\lambda})}\right\}.$$

Choosing η,ε, we easily obtain

$$\mathrm{Re}\big(L'f,f\big)_{L_2(\mathbb{R}^N)}\geq C\|f\|^2_{\mathfrak{H}_+},\, C>0.$$

Using the above estimates, we obtain

$$\begin{aligned}\left|(L'f,g)_{L_2(\mathbb{R}^N)}\right|&\leq\|\nabla f\|_{L_2(\mathbb{R}^N)}\|\nabla g\|_{L_2(\mathbb{R}^N)}+\left\|\rho^{1/\lambda-1}\nabla\rho\right\|_{L_\infty(\mathbb{R}^N)}\|\nabla f\|_{L_2(\mathbb{R}^N)}\|g\|_{L_2(\mathbb{R}^N,\rho^{-2/\lambda})}\\&+\eta\|f\|_{L_2(\mathbb{R}^N,\rho^{-2/\lambda})}\|g\|_{L_2(\mathbb{R}^N,\rho^{-2/\lambda})}\leq C\|f\|_{\mathfrak{H}_+}\|g\|_{\mathfrak{H}_+},\, C>0.\end{aligned}$$

Thus, we have a fulfillment of the hypothesis H2 [2]. Taking into account the fact that a negative space $L_2(\mathbb{R}^N,\varphi^{-\lambda})$ is involved, we are forced to involve a modification of the hypothesis H1 [2] expressed as follows. There exist pairs of Hilbert spaces $\mathfrak{H}\subset\mathfrak{H}_-$, $\mathfrak{H}_+\subset\subset\mathfrak{H}_-$, $\mathfrak{H}:=L_2(\mathbb{R}^N)$ and a linear manifold $\mathfrak{M}:=C_0^\infty(\mathbb{R}^N)$ that is dense in $\mathfrak{H}_+$. The operator L' is defined on $\mathfrak{M}$. However, we can go further and modify a norm $\mathfrak{H}_+$ adding a summand; in this case, the considered operator can be changed, and we have

$$\|f\|^2_{\mathfrak{H}_+} := \|\nabla f\|^2_{L_2(\mathbb{R}^N)} + \|f\|^2_{L_2(\mathbb{R}^N,\psi)},\ \psi(x) = (1+|x|)^{-2}+1,\ L' := L+\eta I,\ \eta > 0.$$

Implementing the same reasonings, one can prove that in this case, Hypothesis H2 [2] is fulfilled, and the modified analog of Hypothesis H1 [2] can be formulated as follows.

(H1*) There exists a chain of Hilbert spaces $\mathfrak{H}_+ \subset \mathfrak{H} \subset \mathfrak{H}_-$, $\mathfrak{H}_+ \subset\subset \mathfrak{H}_-$ and a linear manifold $\mathfrak{M}$ that is dense in $\mathfrak{H}_+$. The operator L' is defined on $\mathfrak{M}$.

However, we have $\mathfrak{H}_+ \subset\subset \mathfrak{H}_-$ instead of the required inclusion $\mathfrak{H}_+ \subset\subset \mathfrak{H}$. This inconvenience can stress a peculiarity of the chosen method; at the same time, the central point of the theory—Theorem 1 [2]—can be reformulated under newly obtained conditions corresponding to both variants of the operator L'. The further step is how to calculate order of the real component $\mathfrak{Re}L' := (L'+L'^*)/2$ (a more precise definition can be seen see in the paper [2]). Formally, we can avoid the appeared difficulties connected with the fact that the set $\mathbb{R}^N$ is not bounded since we can refer to the Fefferman concept presented in the monograph [30] (p. 47), in accordance with which we can choose such an unbounded subset of $\mathbb{R}^N$ that the relation $\lambda_j(\mathfrak{Re}L') \asymp j^{2/N}$ holds, where the symbol λ_j denotes an eigenvalue. It gives us $\mu(\mathfrak{Re}L') = 2/N$, where the symbol μ denotes the order of the real component of the operator L'(see [2]). Thus, we leave this question to the reader for a more detailed study and reasonably allow ourselves to assume that the operator L' has a finite non-zero order. Having obtained an analog of Theorem 1 [2] and order of the real component of the operator L', we have a key to the theory created in the papers [6,12,24]. Now, we can consider a Cauchy problem for the evolution equation with the operator L' in the second term as well as a function of the operator L' in the second term, which leads us to the integro-differential evolution equation—it corresponds to an operator function having finite principal and major parts of the Laurent series.

One more example of a non-self-adjoint operator that is not completely subordinated in the sense of forms (see [17,22]) is given below. Consider a differential operator acting in the complex Sobolev space

$$\mathcal{L}f := (c_k f^{(k)})^{(k)} + (c_{k-1} f^{(k-1)})^{(k-1)} + \ldots + c_0 f,$$

$$\mathrm{D}(\mathcal{L}) = H^{2k}(I) \cap H^k_0(I),\ k \in \mathbb{N},$$

where $I := (a,b) \subset \mathbb{R}$, the complex-valued coefficients $c_j(x) \in C^{(j)}(\bar{I})$ satisfy the condition $\mathrm{sign}(\mathrm{Re} c_j) = (-1)^j$, $j = 1,2,\ldots,k$. Consider a linear combination of the Riemann–Liouville fractional differential operators (see [4], (p. 44)) with the constant real-valued coefficients

$$\mathcal{D}f := p_n D^{\alpha_n}_{a+} + q_n D^{\beta_n}_{b-} + p_{n-1} D^{\alpha_{n-1}}_{a+} + q_{n-1} D^{\beta_{n-1}}_{b-} + \ldots + p_0 D^{\alpha_0}_{a+} + q_0 D^{\beta_0}_{b-},$$

$$\mathrm{D}(\mathcal{D}) = H^{2k}(I) \cap H^k_0(I),\ n \in \mathbb{N},$$

where $\alpha_j, \beta_j \geq 0$, $0 \leq [\alpha_j], [\beta_j] < k$, $j = 0,1,\ldots,n.$,

$$q_j \geq 0,\ \mathrm{sign}\, p_j = \begin{cases} (-1)^{\frac{[\alpha_j]+1}{2}},\ [\alpha_j] = 2m-1,\ m \in \mathbb{N}, \\ (-1)^{\frac{[\alpha_j]}{2}},\ [\alpha_j] = 2m,\ m \in \mathbb{N}_0. \end{cases}$$

The following result is represented in the paper [17]. Consider the operator

$$G = \mathcal{L} + \mathcal{D},$$

$$\mathrm{D}(G) = H^{2k}(I) \cap H^k_0(I).$$

and suppose $\mathfrak{H} := L_2(I)$, $\mathfrak{H}^+ := H_0^k(I)$, $\mathfrak{M} := C_0^\infty(I)$, then we have that the operator G satisfies the conditions H1, H2. Using the minimax principle for estimating eigenvalues, we can easily see that the operator $\mathfrak{Re}G$ has a non-zero order. Hence, we can successfully apply Theorem 1 [2] to the operator G, in accordance with which the resolvent of the operator G belongs to the Schatten–von Neumann class $\mathfrak{S}_s$ with the value of the index $0 < s < \infty$ defined by the formula given in Theorem 1 [2]. Thus, it gives us an opportunity to apply Theorem 3 to the operator.

A couple of words on condition H1 in the context of operators generating semigroups. Assume that an operator $-A$ acting in a separable Hilbert space $\mathfrak{H}$ is the infinitesimal generator of a C_0 semigroup such that A^{-1} is compact. By virtue of Corollary 2.5 [31] (p. 5), we have that the operator A is densely defined and closed. Let us check the fulfillment of condition H1. Consider a separable Hilbert space $\mathfrak{H}_A := \{f, g \in \mathrm{D}(A), (f,g)_{\mathfrak{H}_A} = (Af, Ag)_{\mathfrak{H}}\}$, where the fact that $\mathfrak{H}_A$ is separable follows from the properties of the energetic space. Note that since A^{-1} is compact, then we conclude that the following relation holds $\|f\|_{\mathfrak{H}} \leq \|A^{-1}\| \cdot \|Af\|_{\mathfrak{H}}$, $f \in \mathrm{D}(A)$ and the embedding provided by this inequality is compact. Thus, we have obtained in the natural way a pair of Hilbert spaces such that $\mathfrak{H}_A \subset\subset \mathfrak{H}$. We may say that this general property of infinitesimal generators is not so valuable, as it requires a rather strong and unnatural condition of compactness of the inverse operator. However, if we additionally deal with the semigroup of contractions, then we can formulate a significant result (see Theorem 2 [2]), allowing us to study the spectral properties of the infinitesimal generator transform

$$T := A^*GA + FA^\alpha,\ \alpha \in [0, 1),$$

where the symbols G, F denote operators acting in $\mathfrak{H}$. Having analyzed the proof of Theorem 2 [2], one can easily see that the condition of contractions can be omitted in the case $\alpha = 0$.

4. Conclusions

In this paper, we invented a method to study a Cauchy problem for the abstract fractional evolution equation with the operator function in the second term. The considered class corresponding to the operator argument is rather wide and includes non-selfadjoint unbounded operators. As a main result, we represent a technique allowing to principally weaken conditions imposed upon the second term not containing the time variable. Obviously, the application section of the paper is devoted to the theory of fractional differential equations.

The invented method allows to solve the Cauchy problem for the abstract fractional evolution equation analytically, which is undoubtedly a great advantage. We used the results of the spectral theory of non-self-adjoint operators as a base for studying the mathematical objects. Characteristically, the operator function is defined on a special operator class covering the infinitesimal generator transform (see [2]), where a corresponding semigroup is assumed to be a strongly continuous semigroup of contractions. The corresponding particular cases lead us to a linear composition of differential operators of real order in various senses listed in the introduction section. In connection with this, various types of fractional integro-differential operators can be considered, which becomes clear if we involve an operator function represented by the Laurent series with finite principal and regular parts. Moreover, the artificially constructed normal operator [6] belonging to the special operator class indicates that the application part is beyond the class of differential operators of real order. Below, we represent a comparison analysis to show brightly the main contribution of the paper, particularly the newly invented method allowing us to consider an entire function as the operator function. First of all, the technique related to the proof of the contour integral convergence is similar to the papers [6,7,12,24]; one can italicize a similar scheme of reasonings, but the last one is nothing without the required properties of the considered entire function. Such theorems as the Wieman theorem, the theorem on the entire function growth regularity and their applications form the main author's creative contribution to the paper. To be honest, it was not so easy to find such a condition that makes the contour integral convergent on the entire function. We should note that the latter idea in its precise statement was not considered previously. The following fact is also worth noting—a suitable algebraic reasonings having noticed by the author and allowing us to involve a fractional derivative in the first term. This idea allows to cover many results in the framework of the theory of fractional differential equations. The latter which is a relevant result. As for other mathematicians, here, the Lidskii name ought to be sounded; however, the peculiarities of the author's own technique were shown and discussed in the papers [2,12,17] and one can study them properly. We may say that the main concept of the root vector series expansion jointly with the method analogous to the Abel's one belongs to Lidskii, which is reflected in the name: Abel–Lidskii sense of the series convergence. As for the author's contribution to this method, it is not so small, as one can observe in the paper [12] since the main result establishes clarification of the results by Lidskii. In the framework of the discussion, the following papers by Markus [20], Matsaev [19], Shkalikov [22] can undergo a comparison analysis. The latter represents in the paper [22] only an idea of the proof of Theorem 5.1 [22] even the statement of which differs from the statement of Theorem 4 [12], which is provided with a detailed proof and clarifies the Lidskii results represented in [12]. Particular attention can be paid to a special class of operators with which, due to the author's results [2], the reader can successfully deal. The latter benefit stresses the relevance of the results for initially the theoretical results in the framework of the developed direction of the spectral theory [2,17] originated from the ones [15] devoted to uniformly elliptic non-selfadjoint operators, which cannot be covered by the results by Markus [20], Matsaev [19], Shkalikov [22] due to the absence of a so-called complete subordination condition imposed upon the operator (a corresponding example is given in the paper [17]).

We hope that the general concept will have a more detailed study, as well as concrete applied problems being solved by virtue of the invented theoretical approach.

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