A FORMULA FOR THE GROMOV–WITTEN POTENTIAL OF AN ELLIPTIC CURVE

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ABSTRACT. An algorithm to determine all the Gromov–Witten invariants of any smooth projective curve was obtained by Okounkov and Pandharipande in 2006. They identified stationary invariants with certain Hurwitz numbers and then presented Virasoro type constraints that allow to determine all the other Gromov–Witten invariants in terms of the stationary ones. In the case of an elliptic curve, we show that these Virasoro type constraints can be explicitly solved leading to a very explicit formula for the full Gromov–Witten potential in terms of the stationary invariants.

1. INTRODUCTION

The *Gromov–Witten invariants* of a smooth projective variety X are integrals over the moduli space of maps from algebraic curves to X. These invariants are a very rich object of research, where various branches of mathematics, including algebraic geometry, mathematical physics, topology, and combinatorics, interact in a beautiful manner. If X is a point, then the moduli spaces of maps specialize to the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable algebraic curves of genus g with n marked points, and the Gromov–Witten invariants, often called the *intersection numbers* in this case, are the integrals over $\overline{\mathcal{M}}_{g,n}$ of monomials in the first Chern classes of tautological line bundles over $\overline{\mathcal{M}}_{g,n}$. The exponent of the generatings series of intersection numbers, called now the *Kontsevich–Witten (KW) tau-function*, was first described by Kontsevich [Kon92] as a certain tau-function of the KdV hierarchy (this was conjectured before by Witten [Wit91]), and Kontsevich also derived a beautiful matrix model for it. A more detailed description of the associated tau-function of the KdV hierarchy using the Sato Grassmannian can be found in [KS91] (see also [BJP15]). Virasoro constraints for the KW tau-function were derived in [DVV91] (see also [Wit92]). As a result of a huge amount of research during more than 30 years, there exist now a lot of ways to describe the KW tau-function, see, e.g., [Oko02, Ale11, BDY16, Bur17, MM20].

In the case when the target variety is a smooth projective curve, the Gromov–Witten invariants are also well studied. The Gromov–Witten invariants of the complex projective line $\mathbb{P}^1$ were first descibed in [GP99] using localization. Virasoro constraints were derived in [Giv01], a relation with Hurwitz numbers was obtained in [OP06c], and integrable systems controlling the Gromov–Witten invariants were presented in [DZ04, OP06b]. Regarding other descriptions, see, e.g., [DMNPS17, DYZ20, BR21].

The Gromov–Witten invariants of target curves of genus $h \geq 1$ were computed in [OP06a]. The stationary invariants are identified with Hurwitz numbers counting ramified coverings of a Riemann surface of genus h that are branched over some number of fixed points with the ramification profiles given by the so-called *completed cycles*. All the other Gromov–Witten invariants are determined starting from the stationary ones using Virasoro type constraints. For further results on the Gromov–Witten invariants of higher genus target curves, we refer a reader, e.g., to [Ros08, Zho20].

However, there is a certain gap in the understanding of the Gromov–Witten invariants of higher genus target curves. In [OP06b, Section 0.1.6] the authors say the following: "*We do not know whether the Gromov–Witten theories of higher genus target curves are governed by*

integrable hierarchies". As far as we know, this aspect was never clarified in the literature, although it is strongly believed that the Gromov–Witten invariants of any target variety are controlled by an appropriate integrable system, and this is confirmed in a large class of cases (see, e.g., [BPS12]).

In this note, we focus on the case of an elliptic curve as a target variety. We solve the system of Virasoro type constraints given in [OP06a] and obtain an explicit formula (Theorem 2.2) for the full Gromov–Witten potential of the elliptic curve in terms of the stationary Gromov–Witten invariants. As a corollary, we show that the Gromov–Witten invariants of the elliptic curve are controlled by an integrable hierarchy, which is related to a simple dispersionless hierarchy by an explicit Miura transformation (Theorem 3.1). This clarifies the aspect of the Gromov–Witten theory of the elliptic curve pointed out by the authors of [OP06b].

Notation and conventions.

- We use the Einstein summation convention for repeated upper and lower Greek indices.
- When it does not lead to a confusion, we use the symbol $*$ to indicate any value, in the appropriate range, of a sub- or superscript.
- For a topological space X, we denote by $H_*(X)$ and $H^*(X)$, respectively, the homology and cohomology groups of X with coefficients in $\mathbb{C}$.

2. The Gromov–Witten potential of an elliptic curve

Here, in Theorem 2.2 we present an explicit formula for the full Gromov–Witten potential of an elliptic curve. Before that, let us recall very briefly basic facts about the Gromov–Witten invariants of an elliptic curve, referring a reader to [OP06a] for further details.

Consider an elliptic curve E. Let us choose a basis $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in H^*(E)$, where the class $\gamma_1 \in H^0(E)$ is the unit, the classes $\gamma_2 \in H^{1,0}(E)$ and $\gamma_3 \in H^{0,1}(E)$ satisfy $\int_E \gamma_2 \gamma_3 = 1$, and the class $\gamma_4 \in H^2(E)$ is the Poincaré dual to a point.

The moduli space $\overline{\mathcal{M}}_{g,n}(E, d)$ parameterizes connected, genus g, n-pointed stable maps $f\colon (C; x_1, \ldots, x_n) \to E$ with $f_*[C] = d[E] \in H_2(E, \mathbb{Z})$. For each $1 \le i \le n$, there is a complex line bundle L_i over $\overline{\mathcal{M}}_{g,n}(E, d)$ given by the cotangent spaces to the i-th marked point in C, and we denote $\psi_i := c_1(L_i) \in H^2(\overline{\mathcal{M}}_{g,n}(E, d))$. For $1 \le i \le n$, there is a map $\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,n}(E, d) \to E$ sending a stable map $f\colon (C; x_1, \ldots, x_n) \to E$ to the point $f(x_i) \in E$. The moduli space $\overline{\mathcal{M}}_{g,n}(E, d)$ is endowed with a *virtual fundamental class* $[\overline{\mathcal{M}}_{g,n}(E, d)]^{\mathrm{vir}} \in H_{2(2g-2+n)}(\overline{\mathcal{M}}_{g,n}(E, d), \mathbb{Z})$.

The *Gromov–Witten invariants* of E are the following integrals:

$$(2.1) \qquad \langle \tau_{k_1}(\gamma_{\alpha_1}) \ldots \tau_{k_n}(\gamma_{\alpha_n}) \rangle_{g,d}^E := \int_{\left[\overline{\mathcal{M}}_{g,n}(E,d)\right]^{\mathrm{vir}}} \psi_1^{k_1} \mathrm{ev}_1^*(\gamma_{\alpha_1}) \ldots \psi_n^{k_n} \mathrm{ev}_n^*(\gamma_{\alpha_n}),$$

where $k_1, \ldots, k_n, d \ge 0$ and $1 \le \alpha_1, \ldots, \alpha_n \le 4$. The Gromov–Witten invariants with $\alpha_1 = \ldots = \alpha_n = 4$ are called *stationary*. The Gromov–Witten invariant (2.1) is zero unless

$$(2.2) \qquad \sum_{i=1}^{n} (k_i + q_{\alpha_i} - 1) = 2g - 2,$$

where $q_\alpha := \frac{1}{2} \deg \gamma_\alpha$. If the subscript g is omitted in the bracket notation $\langle \prod_i \tau_{k_1}(\gamma_{\alpha_i}) \rangle_d^E$, the genus is specified by the constraint (2.2). If the resulting genus is not an integer, the Gromov–Witten invariant is defined as vanishing.

Note that in genus 0 the Gromov–Witten invariant (2.1) is zero unless $d = 0$. In the case $g = d = 0$, the only nontrivial Gromov–Witten invariants (up to simultaneous permutations of the numbers $\alpha_1, \ldots, \alpha_n$ and the numbers $k_1, \ldots, k_n$) are

(2.3)
$$\left\langle \tau_{k_1}(\gamma_4) \prod_{i=2}^{n} \tau_{k_i}(\gamma_1) \right\rangle_{0,0}^{E} = \left\langle \tau_{k_1}(\gamma_2) \tau_{k_2}(\gamma_3) \prod_{i=3}^{n} \tau_{k_i}(\gamma_1) \right\rangle_{0,0}^{E} = \frac{(n-3)!}{k_1! \ldots k_n!}, \quad k_1 + \ldots + k_n = n - 3.$$

We introduce formal variables ε, q and a two-parameter family of formal variables $t_d^\alpha, 1 \leq \alpha \leq 4$, $d \geq 0$, where the variables t_d^2, t_d^3 are odd, and the variables e, q, t_d^1, t_d^4 are even, and consider the Gromov–Witten potential of E:

$$\mathcal{F}(t_*^*, q, \varepsilon) = \sum_{g \geq 0} \varepsilon^{2g} \mathcal{F}_g(t_*^*, q) := \sum \frac{\varepsilon^{2g}}{n!} q^d t_{d_1}^{\alpha_1} \ldots t_{d_n}^{\alpha_n} \langle \tau_{d_1}(\gamma_{\alpha_1}) \ldots \tau_{d_n}(\gamma_{\alpha_n}) \rangle_{g,d}^{E}.$$

Remark 2.1. The Gromov–Witten potential of E (and, more generally, of any target variety with nonvanishing odd cohomology) is often defined a little bit differently, where the monomial in front of the Gromov–Witten invariant $\langle \tau_{d_1}(\gamma_{\alpha_1}) \ldots \tau_{d_n}(\gamma_{\alpha_n}) \rangle_{g,d}^{E}$ is replaced by $t_{d_n}^{\alpha_n} \ldots t_{d_1}^{\alpha_1}$. We follow the convention from the paper [OP06a].

We introduce a 4×4 matrix $\eta = (\eta_{\alpha\beta})$ by $\eta_{\alpha\beta} := \int_E \gamma_\alpha \gamma_\beta$, and denote by $\eta^{\alpha\beta}$ the entries of the matrix η^{-1}, $\eta^{\alpha\mu}\eta_{\mu\beta} = \delta_\beta^\alpha$. We also introduce formal power series

$$v^\alpha := \eta^{\alpha\mu} \frac{\partial^2 \mathcal{F}_0}{\partial t_0^\mu \partial t_0^1}, \qquad v_k^\alpha := \frac{\partial^k v^\alpha}{(\partial t_0^1)^k}, \qquad 1 \leq \alpha \leq 4, \quad k \geq 0.$$

For any $n \geq 0$ and an n-tuple $\overline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n$, denote

$$C_{\overline{d}}(q) := \sum_{d \geq 0} \left\langle \prod_{i=1}^{n} \tau_{d_i}(\gamma_4) \right\rangle_d^E q^d \in \mathbb{C}[[q]].$$

For example [OP06c, Section 5],

$$C_{()}(q) = \sum_{n \geq 1} \frac{\sigma(n)}{n} q^n, \qquad C_{(2)}(q) = \frac{E_2^2}{2} + \frac{E_4}{12}, \qquad C_{(1,1)}(q) = -\frac{8}{3}E_2^3 + \frac{2}{3}E_2 E_4 + \frac{7}{180}E_6,$$

where $\sigma(n) := \sum_{d|n} d$ and

$$E_k(q) := \frac{\zeta(1-k)}{2} + \sum_{n \geq 1} \left(\sum_{d|n} d^{k-1} \right) q^n, \quad k = 2, 4, 6,$$

is the standard notation for the *Eisenstein series*. Note that if $d_i = 0$ for some i, then

$$C_{\overline{d}}(q) = q \frac{d}{dq} C_{(d_1, \ldots, \widehat{d_i}, \ldots, d_n)}(q) - \frac{\delta_{n,1}}{24}.$$

In [OP06c], the authors derived a formula for the series $C_{\overline{d}}(q)$, which we recall in the appendix to our paper.

For an integer $d \geq 0$, denote by $\mathcal{P}_d$ the set of all partitions of d. We denote by $l(\lambda)$ the length of $\lambda \in \mathcal{P}_d$ and denote $m_j(\lambda) := |\{1 \leq i \leq l(\lambda) | \lambda_i = j\}|$, $j \geq 1$.

Theorem 2.2. *For any $g \geq 1$ we have*

(2.4)
$$\mathcal{F}_g = \sum_{\lambda \in \mathcal{P}_{2g-2}} \frac{\prod_{i=1}^{l(\lambda)} v_{\lambda_i}^4}{\prod_{j \geq 1} m_j(\lambda)!} C_\lambda(qe^{v^4}) - \frac{v^4}{24} \delta_{g,1}.$$

Example 2.3. We have

$$\mathcal{F}_1 = C_{()}(qe^{v^4}) - \frac{v^4}{24}, \qquad \mathcal{F}_2 = v_2^4 C_{(2)}(qe^{v^4}) + \frac{(v_1^4)^2}{2} C_{(1,1)}(qe^{v^4}).$$

Remark 2.4. By (2.3), $v_k^4|_{t_*^1=t_*^2=t_*^3=0} = t_k^4$, so formula (2.4) is obviously true if we substitute $t_*^1 = t_*^2 = t_*^3 = t_0^4 = 0$, because after this substitution the right-hand side of (2.4) becomes

$$\sum_{n\geq 0} \frac{1}{n!} \sum_{\substack{k_1,\dots,k_n\geq 1 \\ \sum k_i = 2g-2}} \sum_{d\geq 0} q^d \left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_4) \right\rangle_{g,d}^E \prod_{i=1}^n t_{k_i}^4.$$

So formula (2.4) doesn't tell anything about the Gromov–Witten invariants

$$\left\langle \prod_{i=1}^n \tau_{k_i}(\gamma_4) \right\rangle_{g,d}^E, \quad k_1,\dots,k_n \geq 1,$$

and should be understood as a way to reconstruct all the other Gromov–Witten invariants starting from them.

Proof of Theorem 2.2. The divisor equation in Gromov–Witten theory [KM94] implies that

$$(2.5) \qquad \frac{\partial \mathcal{F}_g}{\partial t_0^4} = q \frac{\partial \mathcal{F}_g}{\partial q} + \sum_{n\geq 0} t_{n+1}^1 \frac{\partial \mathcal{F}_g}{\partial t_n^4} + \delta_{g,0} \frac{(t_0^1)^2}{2} - \delta_{g,1} \frac{1}{24}.$$

For $a \in \mathbb{C}$ and $b \in \mathbb{Z}_{\geq 0}$ let

$$(a)_b := \begin{cases} 1, & \text{if } b = 0, \\ a(a+1)\dots(a+b-1), & \text{if } b \geq 1. \end{cases}$$

Denote also $b_1 = b_3 := 0$ and $b_2 = b_4 := 1$. In [OP06a] the authors derived the following family of constraints for the potential $\mathcal{F}$:

$$(2.6) \qquad L_k \exp(\mathcal{F}) = D_k \exp(\mathcal{F}) = \overline{D}_k \exp(\mathcal{F}) = 0, \quad k \geq -1,$$

where

$$L_k := -(k+1)! \frac{\partial}{\partial t_{k+1}^1} + \sum_{m\geq 0} (b_\alpha + m)_{k+1} t_m^\alpha \frac{\partial}{\partial t_{m+k}^\alpha} + \delta_{k,-1} \eta_{\alpha\beta} \frac{t_0^\alpha t_0^\beta}{2}, \qquad k \geq -1,$$

$$D_k := -(k+1)! \frac{\partial}{\partial t_{k+1}^2} + \sum_{m\geq 0} \left((m)_{k+1} t_m^1 \frac{\partial}{\partial t_{m+k}^2} + (m+1)_{k+1} t_m^3 \frac{\partial}{\partial t_{m+k}^4} \right), \qquad k \geq -1,$$

$$\overline{D}_k := -(k+1)! \frac{\partial}{\partial t_{k+1}^3} + \sum_{m\geq 0} \left((m)_{k+1} t_m^1 \frac{\partial}{\partial t_{m+k}^3} - (m+1)_{k+1} t_m^2 \frac{\partial}{\partial t_{m+k}^4} \right), \qquad k \geq -1.$$

Introducing the operator $O := \frac{\partial}{\partial t_0^4} - q \frac{\partial}{\partial q} - \sum_{n\geq 0} t_{n+1}^1 \frac{\partial}{\partial t_n^4}$, we rewrite the constraint (2.5) as

$$(2.7) \qquad O\mathcal{F}_g = \delta_{g,0} \frac{(t_0^1)^2}{2} - \frac{\delta_{g,1}}{24}.$$

Note also that the constraints (2.6) can be equivalently written as

$$(2.8) \qquad \widetilde{L}_k \mathcal{F}_g = -\delta_{k,-1} \delta_{g,0} \eta_{\alpha\beta} \frac{t_0^\alpha t_0^\beta}{2}, \qquad D_k \mathcal{F}_g = \overline{D}_k \mathcal{F}_g = 0, \quad k \geq -1,$$

where $\widetilde{L}_k := L_k - \delta_{k,-1} \eta_{\alpha\beta} \frac{t_0^\alpha t_0^\beta}{2}$.

Clearly, the constraints (2.7) and (2.8) uniquely determine the potential $\mathcal{F}_g$ starting from the part $\mathcal{F}_g|_{t^1_*=t^2_*=t^3_*=t^4_0=0}$. Since in Remark 2.4 we explained that equation (2.4) is true if we substitute $t^1_* = t^2_* = t^3_* = t^4_0 = 0$, it remains to prove that following lemma.

Lemma 2.5.
1. *The operators $\widetilde{L}_k$, D_k, and $\overline{D}_k$, $k \geq -1$, annihilate the right-hand side of* (2.4).
2. *Applying the operator O to the right-hand side of* (2.4) *gives $-\frac{\delta_{g,1}}{24}$.*

Proof. 1. Since the operators $\widetilde{L}_k$, D_k, and $\overline{D}_k$ are linear combinations of the operators $\frac{\partial}{\partial t^\alpha_a}$, it is sufficient to check that they annihilate the formal power series v^4_d for all $d \geq 0$. This is obtained by applying the operator $\frac{\partial^{d+2}}{(\partial t^1_0)^{d+2}}$ to the equations in (2.8) with $g = 0$ and using that the operator $\frac{\partial}{\partial t^1_0}$ commutes with the operators $\widetilde{L}_k$, D_k, and $\overline{D}_k$.

2. Since the operator O is a linear combination of the operators $\frac{\partial}{\partial t^\alpha_a}$ and $\frac{\partial}{\partial q}$, and $Oq = -q$, it is sufficient to check that $Ov^4_d = \delta_{d,0}$. For this, as in Part 1, we apply $\frac{\partial^{d+2}}{(\partial t^1_0)^{d+2}}$ to both sides of equation (2.7) with $g = 0$ and use that $[\frac{\partial}{\partial t^1_0}, O] = 0$. $\qquad\square$

$\square$

3. INTEGRABLE SYSTEMS ASSOCIATED TO AN ELLIPTIC CURVE

In this section, we determine an integrable system controlling the Gromov–Witten invariants of an elliptic curve.

The *topological recursion relations* in genus 0 (see, e.g., [KM98]) are the following PDEs for the generating series of genus 0 Gromov–Witten invariants of an elliptic curve E:

$$\frac{\partial^3 \mathcal{F}_0}{\partial t^{\alpha_1}_{d_1+1} \partial t^{\alpha_2}_{d_2} \partial t^{\alpha_3}_{d_3}} = \eta^{\nu\mu} \frac{\partial^2 \mathcal{F}_0}{\partial t^{\alpha_1}_{d_1} \partial t^\mu_0} \frac{\partial^3 \mathcal{F}_0}{\partial t^\nu_0 \partial t^{\alpha_2}_{d_2} \partial t^{\alpha_3}_{d_3}}, \quad 1 \leq \alpha_1, \alpha_2, \alpha_3 \leq 4, \quad d_1, d_2, d_3 \geq 0.$$

It implies that (see, e.g., [BPS12, Proposition 3])

$$\frac{\partial^2 \mathcal{F}_0}{\partial t^\alpha_a \partial t^\beta_b} = \Omega_{\alpha,a;\beta,b}|_{t^\gamma_0=v^\gamma}, \quad \text{where} \quad \Omega_{\alpha,a;\beta,b} = \Omega_{\alpha,a;\beta,b}(t^*_0) := \left.\frac{\partial^2 \mathcal{F}_0}{\partial t^\alpha_a \partial t^\beta_b}\right|_{t^*_{\geq 1}=0}.$$

We then denote

$$P^\alpha_{\beta,b} := \eta^{\alpha\mu} \frac{\partial^2 \mathcal{F}_0}{\partial t^\beta_b \partial t^\mu_0}$$

and using (2.3) compute

$$P^1_{1,b} = \frac{(v^1)^{b+1}}{(b+1)!}, \qquad\qquad P^1_{2,b} = 0, \qquad P^1_{3,b} = 0, \qquad P^1_{4,b} = 0,$$

$$P^2_{1,b} = v^2 \frac{(v^1)^b}{b!}, \qquad\qquad P^2_{2,b} = \frac{(v^1)^{b+1}}{(b+1)!}, \qquad P^2_{3,b} = 0, \qquad P^2_{4,b} = 0,$$

$$P^3_{1,b} = v^3 \frac{(v^1)^b}{b!}, \qquad\qquad P^3_{2,b} = 0, \qquad P^3_{3,b} = \frac{(v^1)^{b+1}}{(b+1)!}, \qquad P^3_{4,b} = 0,$$

$$P^4_{1,b} = v^4 \frac{(v^1)^b}{b!} + v^2 v^3 \frac{(v^1)^{b-1}}{(b-1)!}, \quad P^4_{2,b} = v^3 \frac{(v^1)^b}{b!}, \quad P^4_{3,b} = -v^2 \frac{(v^1)^b}{b!}, \quad P^4_{4,b} = \frac{(v^1)^{b+1}}{(b+1)!}.$$

We see that the functions v^α satisfy the system of evolutionary PDEs with one spatial variable:

(3.1) $$\frac{\partial v^\alpha}{\partial t^\beta_b} = \partial_x P^\alpha_{\beta,p}, \quad 1 \leq \alpha, \beta \leq 4, \quad b \geq 0,$$

where we identify ∂_x with $\frac{\partial}{\partial t_0^1}$.

Let
$$w^\alpha := \eta^{\alpha\mu} \frac{\partial^2 \mathcal{F}}{\partial t_0^\mu \partial t_0^1}.$$
The above considerations together with Theorem 2.2 imply the following result.

Theorem 3.1. *The functions w^α satisfy the system of evolutionary PDEs with one spatial variable that is obtained from the system (3.1) by the Miura transformation*

$$v^\alpha \mapsto w^\alpha = v^\alpha + \sum_{g\geq 1} \varepsilon^{2g} \eta^{\alpha\mu} \frac{\partial^2}{\partial t_0^\mu \partial t_0^1} \left(\sum_{\lambda \in \mathcal{P}_{2g-2}} \frac{\prod_{i=1}^{l(\lambda)} v_{\lambda_i}^4}{\prod_{j\geq 1} m_j(\lambda)!} C_\lambda(qe^{v^4}) - \frac{v^4}{24} \delta_{g,1} \right).$$

Remark 3.2. The part of the system (3.1) given by the flows $\frac{\partial}{\partial t_b^\beta}$ with $\beta = 1$ or $\beta = 4$ can be restricted to the submanifold given by $v^2 = v^3 = 0$. By [BDGR18, Proposition 10.1], the resulting system coincides with the DR hierarchy associated to the even part of the cohomological field theory corresponding to the elliptic curve. Therefore, Theorem 3.1 proves the DR/DZ equivalence conjecture for the elliptic curve.

Appendix A. Stationary Gromov–Witten invariants of an elliptic curve

In this section, in order to make the paper more self-contained, we recall the formula for the stationary Gromov–Witten invariants of an elliptic curve given in [OP06c].

Denote by
$$\langle \tau_{d_1}(\gamma_{\alpha_1}) \dots \tau_{d_n}(\gamma_{\alpha_n}) \rangle_d^{\bullet E}, \quad d_1, \dots, d_n, d \geq 0, \quad 1 \leq \alpha_i \leq 4,$$
the Gromov–Witten invariants of an elliptic curve E obtained by the integration over the moduli space of stable maps with possibly disconnected domains. For $n = 0$ we have
$$\sum_{d\geq 0} \langle \rangle_d^E q^d = \sum_{n\geq 1} \frac{\sigma(n)}{n} q^n, \qquad \sum_{d\geq 0} \langle \rangle_d^{\bullet E} q^d = \exp \left(\sum_{n\geq 1} \frac{\sigma(n)}{n} q^n \right) = \frac{1}{(q)_\infty},$$
where $(q)_\infty := \prod_{j\geq 1}(1 - q^j)$. For $n \geq 1$, a relation between the connected and disconnected stationary Gromov–Witten invariants of E is given by

$$(A.1) \qquad \sum_{d\geq 0} \left\langle \prod_{i=1}^n \tau_{d_i}(\gamma_4) \right\rangle_d^{\bullet E} q^d = \frac{1}{(q)_\infty} \sum_{k=1}^n \frac{1}{k!} \sum_{\substack{I_1 \sqcup \dots \sqcup I_k = \{1,\dots,n\} \\ I_i \neq \emptyset}} \prod_{i=1}^k C_{\overline{d}_{I_i}}(q),$$

where $\overline{d}_{I_i}$ denotes the $|I_i|$-tuple of integers composed of the integers d_j with $j \in I_i$.

A useful convention is to formally set
$$\left\langle \tau_{-2}(\gamma_4)^l \prod_{i=1}^n \tau_{d_i}(\gamma_4) \right\rangle_d^{\bullet E} := \left\langle \prod_{i=1}^n \tau_{d_i}(\gamma_4) \right\rangle_d^{\bullet E}, \quad d_1, \dots, d_n, d, l \geq 0,$$
in the disconnected case; and
$$\langle \tau_{-2}(\gamma_4) \rangle_0^E := 1$$
in the connected case (therefore, $C_{(-2)}(q) = 1$). Then formula (A.1) remains true.

For any $n \geq 0$, consider the following n-point function:

$$F_E(z_1, \dots, z_n) := \sum_{d\geq 0} q^d \sum_{d_1,\dots,d_n \in \{-2\} \cup \mathbb{Z}_{\geq 0}} \left\langle \prod_{i=1}^n \tau_{d_i}(\gamma_4) \right\rangle_d^{\bullet E} \prod_{i=1}^n z_i^{d_i+1} \in (z_1 \cdots z_n)^{-1} \mathbb{C}[[z_1, \dots, z_n, q]],$$

and let

$$\vartheta(z) := \sum_{n \geq 0} (-1)^n q^{\frac{n(n+1)}{2}} \left(e^{\left(n+\frac{1}{2}\right)z} - e^{-\left(n+\frac{1}{2}\right)z} \right).$$

In [OP06c] the authors proved that

$$(A.2) \qquad F_E(z_1, \ldots, z_n) = \frac{1}{(q)_\infty} \sum_{\substack{\text{all } n! \text{ permutations} \\ \text{of } z_1, \ldots, z_n}} \frac{\det \left[\frac{\vartheta^{(j-i+1)}(z_1 + \cdots + z_{n-j})}{(j-i+1)!} \right]_{1 \leq i,j \leq n}}{\vartheta(z_1) \vartheta(z_1 + z_2) \cdots \vartheta(z_1 + \cdots + z_n)},$$

where $\vartheta^{(k)} := \frac{\partial^k \vartheta}{\partial z^k}$ for $k \geq 0$. For $k < 0$, the convention $\frac{1}{k!} := 0$ is followed. So negative derivatives of $\vartheta(z)$ don't appear in formula (A.2).

As it is explained in [OP06c], formula (A.2) implies that for any n-tuple $\overline{d} \in \mathbb{Z}_{\geq 0}^n$ we have

$$C_{\overline{d}}(q) \in \mathbb{Q}[E_2, E_4, E_6]_{\sum(d_i+2)},$$

where $\mathbb{Q}[E_2, E_4, E_6]$ is the ring freely generated by the Eisenstein series E_2, E_4, E_6, and the lower index specifies the homogeneous component of weight $\sum(d_i + 2)$, where the weight of E_k is equal to k.

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