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GRADIENT-LIKE DIFFEOMORPHISMS AND PERIODIC VECTOR FIELDS

V.Z. GRINES AND L.M. LERMAN

To Yulij on the occasion of his jubilee.
We value and remember well the Yulij's role in the clear and detailed explanation of the orbit method, used in this paper, at the MSU seminar in the beginning of this century.

Abstract. A class of gradient-like nonautonomous vector fields (NVFs) on a smooth closed manifold M is studied and the following problems are solved: 1) can a gradient-like NVF be constructed by means of the nonautonomous suspension over a diffeomorphism of this manifold, and if so, under what conditions on the diffeomorphism? 2) let a diffeomorphism f be gradient-like (see the definition in the text) and diffeotopic to the identity map id_M, when the NVF obtained by means of the nonautonomous suspension over f be gradient-like? Necessary and sufficient conditions to this have been found in the paper. All these questions arise, when studying NVFs on M admitting the uniform classification and a description via combinatorial type invariants.

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1. Introduction

The aim of the paper is a construction of nonautonomous vector fields (NVFs, for a brevity) on a smooth closed manifold M by means of the nonautonmous suspension [20], [21] over gradient-like diffeomorphisms diffeotopic to the identity. One of the problem here is the clarification of those conditions, when the periodic vector field obtained be gradient-like one (see [12], [10], [11] and Section 3 below).

A nonautonomous vector field on M is an uniformly continuous map $v\colon \mathbb{R} \to V^k(M)$, where $V^k(M)$ is the Banach space of C^k-smooth vector fields on M, endowed with C^k-norm. Such NVF has solutions whose graphs in the extended phase space $M \times \mathbb{R}$, i.e., integral curves (ICs), define an orientable foliation $\mathcal{L}_v$. One may to transform the NVF into the autonomous vector field on $M \times \mathbb{R}$, adding

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the equation $\dot{t} = 1$, but we lose here all standard methods of studying dynamics, since all orbits of the vector field obtained are noncompact with empty limit sets. Moreover, the orbit structure of such vector field is topologically equivalent to that of the trivial vector field $\dot{x} = 0,\ x \in M$.

Nonetheless, the study of the foliation into ICs from the standpoint of the theory of dynamical systems is possible, but then one needs to include solutions of the NVF into a greater autonomous system, called usually the extension or the skew product, that is defined on a rather complicated phase space. This approach was introduced by Bebutov [1] and now it is being developed mainly for almost periodic vector fields [17], [12], [3].

Here we follow to the approach to the theory of NVFs proposed in [12] and developed further in [10], [11], where the foliation into ICs is examined by itself on the base of the notion of the uniform equivalence of such foliations. This allows for some classes of of NVFs to get a classification using some invariants of a combinatorial type [10], [11].

In the Section 2 we remind [20], [21] the construction of nonautonomous suspension over a diffeomorphism of a smooth closed manifold and present the proof of a theorem, when such construction gives a periodic in time vector field on the manifold. In Section 3 the conditions are formulated, when an NVF on a smooth closed manifold is gradient-like. Here we specify the conditions formulated first in [10] and used later in [11], [5], [6] at the study of different classes of NVFs. In the Section 4 the main theorem is proved, which gives the necessary and sufficient conditions when a gradient-like diffeomorphism on M generates a gradient-like NVF on M.

2. Nonautonomous Suspension

Let M be a smooth (C^∞) closed manifold and $f\colon M \to M$ be some C^∞-diffeomorphism. Following [20], [21], the nonautonomous suspension over f is a pair $(M \times \mathbb{R},\ \mathcal{L}_f)$, where the metric on $M \times \mathbb{R}$ and the foliation $\mathcal{L}_f$ on $M \times \mathbb{R}$ are defined by the diffeomorphism f using the autonomous suspension over f. According to [19], the autonomous suspension is a smooth closed manifold M_f of the dimension $\dim M + 1$ with a flow on it defined as follows. Consider the manifold $M \times \mathbb{R}$ with the action F of the group $\mathbb{Z}$ due to the rule: for any $m \in \mathbb{Z}$ the iteration F^m acts as $F^m(x,\, s) = (f^m(x),\, s - m)$. This action is free and discrete (any its action orbit has not accumulation points). Hence the factor-manifold $M_f = (M \times \mathbb{R})/F$ is C^∞-smooth manifold being a smooth bundle over the circle $S^1 = \mathbb{R}/\mathbb{Z}$, $p\colon M_f \to S^1$, with the leaf M. On M_f a vector field is generated by the constant vector field $V = (0,\, 1)$ on $M \times \mathbb{R}$ (its orbits are lines $(x,\, t)$, $t \in \mathbb{R}$). After factorization one gets a smooth vector field v_f on M_f with a global cross-section. As such cross-section any submanifold $M_s = p^{-1}(s),\ s \in S^1$ can be taken. The Poincaré map, generated by the flow of the vector field on this cross-section, is conjugated to the diffeomorphism f. This construction allows one to build vector field with a dynamics similar to that of the iterations of the diffeomorphism [19].

To construct the nonautonomous suspension we consider a covering manifold $\tilde{M}_f$ for M_f, generated by the standard covering $\mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$, which gives a commutative diagram

$$\begin{array}{ccc} \tilde{M}_f & \xrightarrow{\widetilde{\exp}} & M_f \\ \Big\downarrow \tilde{p} & & \Big\downarrow p \\ \mathbb{R} & \xrightarrow{\exp} & S^1 \end{array} \qquad (1)$$

The manifold $\tilde{M}_f$ is homeomorphic to $M \times \mathbb{R}$, since $\mathbb{R}$ is a contractible space.

Let us endow M_f with a smooth Riemannian metric and lift this metric to $\tilde{M}_f$ by the covering map $\widetilde{\exp}$. Since $\widetilde{\exp}$ is a local diffeomorphism, we get a Riemannian metric on $\tilde{M}_f$ for which $\widetilde{\exp}$ is the local isometry. With this covering the foliation in M_f on the orbits of the vector field v_f is lifted into $\tilde{M}_f$ as a smooth foliation $\mathcal{L}_f$ consisting of infinite curves[1].

Thus the nonautonomous suspension $(\tilde{M}_f, \mathcal{L}_f)$ is generated by the diffeomorphism f. Let us emphasize, the foliation $\mathcal{L}_f$ be homeomorphic to the foliation (x, t), $t \in \mathbb{R}$ onto straight-lines in the manifold $M \times \mathbb{R}$ but, generally speaking, it is not equimorphic to this foliation[2]. For instance, this is the case, when $M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and f is an Anosov diffeomorphism (see details in [21]).

The nonautonomous suspension was introduced [20] in order to construct nontrivial examples of NVFs. But it was discovered that the construction does not always give a foliation generated by a NVF on M. Therefore both necessary and also sufficient conditions were derived when two diffeomorphisms f, g on M lead to equimorphic suspensions [21], [11]. In particular, the following assertion was proved there

Proposition 1. *Let f, $g\colon M \to M$ be diffeomorphisms. Then*

- *Nonautonomous suspensions over f and f^n are equimorphic.*
- *If f and g are conjugated, then their nonautonomous suspensions are equimorphic.*

Also the following question turned out important: whether a nonautonomous vector field v on M exists such that in $M \times \mathbb{R}$ endowed with the metric of the direct product its foliation $\mathcal{L}_v$ into ICs is equimorphic to the foliation $\mathcal{L}_f$ into infinite curves in the nonautonomous suspension $(\tilde{M}_f, \mathcal{L}_f)$? This gives a meaning to the definition introduced in [21]

Definition 1. A nonautonomous vector field v reproduces the structure of a diffeomorphism f, if its foliation $\mathcal{L}_v$ in $M \times \mathbb{R}$ is equimorphic to the foliation $\mathcal{L}_f$ in the nonautonomous suspension $(\tilde{M}_f, \mathcal{L}_f)$.

To keep the exposition self-contained, we present the statement and the proof of a theorem from [5] that provides sufficient conditions on a diffeomorphism f

[1]In fact, to construct a nonautonomous suspension it is sufficient to endow $M \times \mathbb{R}$ with the structure of a uniform space [9] instead of a Riemannian metric. The compact manifold M_f has the unique uniform structure compatible with the topology [9]. The uniform structure in $\tilde{M}_f$ is given by lifting the uniform structure from M_f by means of the map $\widetilde{\exp}$.

[2]An equimorhism of two metric spaces (respectively, uniform spaces) X, Y is a uniformly continuous homeomorphism $h\colon X \to Y$ that has a uniformly continuous inverse mapping.

implying the existence of an NVF that reproduces the structure of f. First we formulate an evident lemma.

Lemma 1. *Let f be diffeotopic to* id_M. *Then there is a diffeotopy $F_t\colon M \to M$, $t \in [0, 1]$, joining* id_M *and f such that diffeomorphisms F_t depend smoothly on t and for some $\varepsilon > 0$ small enough the relations hold*: $F_t \equiv \mathrm{id}_M$ *for* $t \in [0, \varepsilon]$, *and* $f_t \equiv f$ *for* $t \in [1 - \varepsilon, 1]$.

Now we pass to the theorem.

Theorem 1. *Suppose for some $n \in \mathbb{N}$ diffeomorphism $f^n\colon M \to M$ is diffeotopic to* id_M. *Then*

(1) *the manifold M_{f^n} is leafwise*[3] *diffeomorphic to $M \times S^1$;*
(2) *there exists a periodic vector field v on M which reproduces the structure of f.*

Proof. To simplify the exposition, we assume that f itself is diffeotopic to id_M. For $|n| \geqslant 2$ we prove the assertion for $g = f^n$, and then a fact is used that a periodic vector field w on M, existing due to the item 2 of the theorem for g, has its foliation into ICs in $M \times \mathbb{R}$ being equimorphic to the nonautonomous suspension over $g = f^n$. But, by the Proposition 1 above the nonautonomous suspensions over f and f^n are equimorphic, i.e., w reproduces the structure of f.

Let us show first that the manifold M_f is diffeomorphic to the direct product. To this purpose we construct on M_f two transverse foliations. The first is given by leaves of the bundle $p\colon M_f \to S^1$, its leaves are diffeomorphic to M. Closed curves, as leaves of the second foliation, will be defined as follows. Let $F_s\colon M \to M$, $s \in [0, 1] = I$, is the diffeotopy joining id_M and f, therefore $F_0 = \mathrm{id}_M$ and $F_1 = f$. According to Lemma 1, we can assume that diffeomorphisms F_s smoothly depend on s, $F_s = \mathrm{id}_M$ for $s \in [0, \varepsilon]$ and $F_t = f$ for $s \in [1-\varepsilon, 1]$, where $\varepsilon \in (0, 1/3)$. For any point $x \in p^{-1}(0) = M_0$ we define a smooth curve through the point $(x, 0) \in M \times I$, it is given as $(F_s^{-1}(x), s)$ for $s \in [0, 1]$, and then we employ the factor map using the identification $(x, s) = (f(x), s - 1)$.

The extreme point for the curve in $M \times I$ with the initial point $(x, 0)$ is

$$(F_1^{-1}(x), 1) = (f^{-1}(x), 1)$$

at $s = 1$. Under the identification this point becomes $(f \circ f^{-1}(x), 0) = (x, 0)$. Thus, all constructed curves in M_f are closed, 1-periodic and provide a smooth foliation in M_f. Diffeomorphism $h\colon M_f \to M \times S^1$ is defined as follows. For any point $a \in M_f$ denote as l_a a closed curve of the second foliation passing through the point a. Define the map $p_1\colon M_f \to M_0$ as $p_1(a) = l_a \cap M_0$ and the map h is given by the formula $h(a) = (p_1(a), p(a))$. Lemma 1 above guarantees that if $f\colon M \to M$ is diffeotopic to id_M, then a diffeotopy F_t, joining id_M and f, exists such that the curves constructed above give a smooth foliation, that is, curves are smooth and their dependence on a point is also smooth, so that the map p_1 is smooth. Indeed, it is enough for proving this to verify this property locally near arbitrary point

[3]The term “leafwise” means the existence of a diffeomorphism $\Psi\colon M_{f^n} \to M \times S^1$ acting as $(x, s) \to (\psi(x, s), s)$.

$a \in M_f$. Since M_f is a smooth locally trivial bundle over S^1 and the coordinate s is introduced on S^1, then near the point a coordinates (u, s) can be introduced, where u are coordinates in the leaf $p(a) = s_0$. In these coordinates a smooth family of diffeomorphisms F_s is written as $u_s = R(u, s)$, where u_s are u-coordinates in the leaf M_s, and $R(u, s)$ is smooth in s family of smooth vector-functions of u which have derivatives up to the order $r \geqslant 1$ and all Jacobians $\partial R/\partial u$ do not vanish. Differentiation in s at each point (u, s) gives a smooth vector field $(R_s, 1)$ on M_f, whose orbits provide smooth foliation on the curves due to the standard theorems of the theory of ordinary differential equations. This proves the first part of the theorem.

To prove the second statement of the theorem, we construct a periodic vector field v on M such that its foliation $\mathcal{L}_v$ onto ICs is equimorphic to the foliation $\mathcal{L}_f$ in $\tilde{M}_f$ onto the infinite curves. The defined above diffeomorphism $h\colon M_f \to M \times S^1$ allows us to identify M_f and $M \times S^1$. Therefore the vector field of the autonomous suspension is given as $(V(x, s), r(x, s))$, where $r \neq 0$, it has a global cross-section $s = 0$. The Poincaré map $g\colon M_0 \to M_0$ generated on this section is evidently conjugated to f. Hence, one can consider diffeomorphism g instead of f on M. According to the item 2 of the Proposition 1, the nonautonomous suspensions over f, g are equimorphic, since f, g are topologically conjugated. Let us define a periodic vector field on M, as $v(x, t) = V(x, t)/r(x, t)$. Integral curves in $M \times \mathbb{R}$ of the periodic vector field are unfoldings of orbits of the vector field $(V(x, t), r(x, t))$ in M_f, since they are obtained by the change of variables being bounded from above and below. So, the theorem has been proved completely. □

3. Gradient-Like NVFs: Definitions

Nonautonomous gradient-like vector fields were introduced first on two-dimensional closed surfaces in [12], and later in more details and in a more convenient form in [10]. The structure of such NVFs and invariants providing the classification were studied in the two-dimensional case in [10, Chapter 3], and for a one-dimensional case (NVFs on S^1) in [11]. In the concentrated form these conditions were formulated in the overview [6]. The main goal that is pursued when distinguishing this class of NVFs is a possibility of their description and classification using invariants of a combinatorial type as well as an opportunity to prove their structural stability. Below we recall the assumptions which single out the gradient-like NVFs on a smooth closed manifold M [6].

Assumption 1. *Any integral curve of a NVF v possesses an exponential dichotomy on both semi-axes $\mathbb{R}_+$ and $\mathbb{R}_-$, types of these dichotomies can be different.*

The notion of exponential dichotomy goes back to Perron [16] and in the explicit form was introduced by Maizel [14], later on it was studied thoroughly in [15], [4]. Main definitions and corollaries can be found in [5].

Condition 1 implies the existence of global stable and unstable manifolds which are defined for any IC with this property [8], [2]. Such manifold is a smooth immersion J of the space $\mathbb{R}^k \times \mathbb{R}$ such that the restriction of J on the section $\mathbb{R}^k_t = \mathbb{R}^k \times \{t\}$ is a smooth immersion of $\mathbb{R}^k$ into $M_t = M \times \{t\}$, uniformly continuously depending on t on compact sets. By the type $(p,\, q)$ of an exponential dichotomy on $\mathbb{R}_+$ for an IC we understand the dimension p of the trace of its stable manifold on M_0, then $q = \dim\, M - p$. Similarly, by the type $(p,\, q)$ of an exponential dichotomy on $\mathbb{R}_-$ for an IC, we understand the dimension q of the trace of its unstable manifold on M_0, then $p = \dim\, M - q$. Recall [15], [4] that for an IC possessing an exponential dichotomy on $\mathbb{R}_+$ its stable manifold is defined uniquely, the same is true for an IC possessing an exponential dichotomy on $\mathbb{R}_-$: its unstable manifold is defined uniquely.

Any two ICs, lying on the same stable (respectively, unstable) manifold, asymptotically approach to each other as $t \to \infty$ (respectively, as $t \to -\infty$). Therefore the whole extended phase manifold $M \times \mathbb{R}$ is partitioned into stable manifolds: $M \times \mathbb{R} = \bigcup_\alpha W^s_\alpha$, here index α belongs to some set of indexes. The same holds true for the partition into global unstable manifolds W^u_β: $M \times \mathbb{R} = \bigcup_\beta W^u_\beta$ with the set of indexes of other cardinality. It is worth remarking that no relations exist between partitions into stable and unstable manifolds, if the NVF v has not any recurrence in time.

Assumption 2. *Both partitions are finite, i.e., the sets of indexes α and β are finite.*

The experience of studying autonomous and periodic vector fields suggests that a necessary condition of their structural stability is the transversality of intersection for stable and unstable manifolds. Let M_0 be the section $t = 0$.

Assumption 3. *For any pair $\{W^s_\alpha,\, W^u_\beta\}$ the intersection $W^s_\alpha \cap W^u_\beta$ is transversal. Moreover, if the sum of dimensions $\dim(W^s_\alpha \cap M_0) + \dim(W^u_\beta \cap M_0)$ is equal to $\dim M_0$, then $W^s_\alpha \cap W^u_\beta$ consists of finitely many ICs, but if this sum is greater than $\dim M_0$, then $W^s_\alpha \cap W^u_\beta \cap M_0$ consists of finitely many compact connected submanifolds (possibly, this number is zero) and a finite number of noncompact submanifolds, whose closures are compact manifolds with boundaries (possibly, this number is zero).*

As an illustration we remark that if $\dim M = 2$, then any one-dimensional unstable manifold W^u consists of alone IC γ with the type of dichotomy $(2,\, 0)$ on R_-. Then transversality of W^u with any stable manifold W^s is possible only if $\dim W^s = 3$, and hence the trace $W^s \cap M_0$ is an open two-dimensional disk containing the point $\gamma \cap M_0$. Also, here the transversality condition prohibits the intersection of one-dimensional stable manifold with one-dimensional or two-dimensional unstable manifolds.

Different types of intersections of traces of stable and unstable manifolds on the section M_0 for NVFs given on a three-dimensional manifold are shown on Fig. 1 and Fig. 2. On Fig. 1 a case of noncompact intersection is shown and on Fig. 2 the case of compact intersection is presented.

The next condition concerns with a mutual disposition of stable (unstable) manifolds of different dimensions. The clue here is the notion of the Smale boundary for a given global stable (respectively, unstable) manifold W^s [19]. We reformulate it for the case of NVFs.

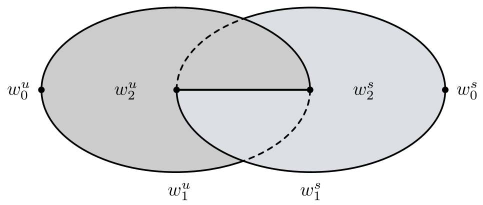

FIGURE 1. Traces of stable and unstable manifolds for a noncompact intersection. Letters w_i^s, w_j^u, $'w_i^s$, $'w_j^u$, i, $j \in \{0, 1, 2, 3\}$, denote traces on M_0 of the related stable or unstable manifolds of the dimension $i+1$, $j+1$, respectively.

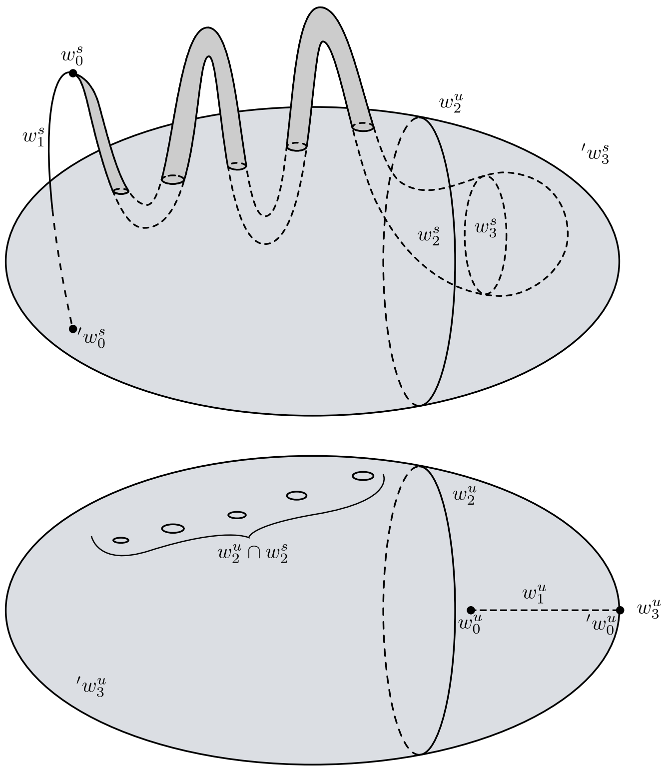

FIGURE 2. Traces of stable and unstable manifolds for a compact intersection. Upper picture shows the traces of stable manifolds on M_0 denoting as w_i^s, w_j^u, $'w_i^s$, $'w_j^u$, i, $j \in \{0, 1, 2, 3\}$. Also this picture presents the trace w_2^u of 3-dimensional unstable manifold shaded by grey. The lower picture shows traces of only unstable manifolds of different dimensions. Five circles denote intersections of traces stable and unstable 3-dimensional manifolds.

As follows from the stable manifold theorem, the intersection of W^s with any section $M_t = M \times \{t\}$ is the image of the immersion $J_t\colon \mathbb{R}^k \to M_t$, $k = \dim W^s - 1$. These immersions depend uniformly continuously in t in C^r-topology on compact sets in $\mathbb{R}^k$. The Smale boundary for the trace $w^s_t = W^s \cap M_t$ is the set $\partial w^s_t = \mathrm{clos}(w^s_t) \setminus w^s_t$, here $\mathrm{clos}(A)$ denotes the closure of the set A. The Smale boundary ∂W^s for the manifold W^s itself we call the union of all ICs passing through the set $\partial w^s_0 = \mathrm{clos}(w^s_0) \setminus w^s_0$.

A characteristic feature of gradient-like systems is the decrease of the dimensions, when passing to the Smale boundary of stable (respectively, unstable) manifolds. Therefore, we require

Assumption 4. *For any stable manifold W^s its Smale boundary ∂W^s consists of the whole stable manifolds of the lesser dimensions. The same holds for any unstable manifold.*

On the Fig. 3 possible violations of this Assumption are shown, when a NVF on a two-dimensional manifold is considered. On the left picture the boundary contains wholly another manifold of the same dimension, on the right picture the dimension preserves, when passing to the Smale boundary, but the limiting set of a stable manifold do not consists of the whole stable manifold.

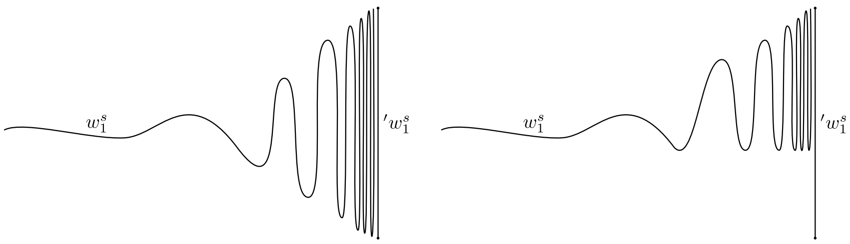

Figure 3. Possible types of limiting sets for the trace of a stable manifold

Finally, to formulate the last condition 5, we consider the partition of $M \times \mathbb{R}$ into stable manifolds of a NVF v, assuming all previous Assumptions be fulfilled. As was said above, we endow the manifold $M \times \mathbb{R}$ with the Riemannian metric of the direct product of some metric in M and the standard metric in $\mathbb{R}$. By a cylindric ε-neighborhood of the integral curve Γ one understands its neighborhood $U(\varepsilon)$ in the manifold $M \times \mathbb{R}$ such that for the pair $(\Gamma, U(\varepsilon))$ an equimorphism $h\colon U(\varepsilon) \to D(\varepsilon) \times \mathbb{R}$ exists with $D(\varepsilon)$ being the n-dimensional disk in $\mathbb{R}^n$: $x_1^2 + \cdots + x_n^2 < \varepsilon^2$, $h(\Gamma) = \{0\} \times \mathbb{R}$, and for any $t \in \mathbb{R}$ the distance between $\partial U(\varepsilon) \cap M_t$ and $\Gamma \cap M_t$ is uniformly in t bounded by positive constants $c(\varepsilon) < C(\varepsilon)$ from below and above, both functions $c(\varepsilon)$, $C(\varepsilon)$ are of the order ε at $\varepsilon = 0$.

Let us choose one IC from each stable manifold and denote them by $\Gamma_1^+, \ldots, \Gamma_k^+$. They are separated in $M \times \mathbb{R}_+$ in the sense that an $\varepsilon > 0$ exists such that the cylindric ε-neighborhoods $U_i(\varepsilon)$ of these ICs do not intersect each other in $M \times \mathbb{R}_+$. Such neighborhoods can be constructed using Lyapunov functions.

For any i the boundary $\partial U_i(\varepsilon)$ possesses the following properties. It consists of three parts: the entrance set S_e, where ICs intersect transversely the boundary

$\partial U_i(\varepsilon)$, entering inside of $U_i(\varepsilon)$ as time increases, the exit set S_l, where ICs intersect transversely the boundary $\partial U_i(\varepsilon)$, going out $U_i(\varepsilon)$, as time increases, and the set $S_\tau = S_e \cap S_l$, containing a finite number of connected components of the dimension lesser than n and possessing the property: any IC, passing through a point z of the set S_τ, does not intersect more this set at the times from a sufficiently small neighborhood of the time t_z, corresponding to the point z.

Consider an IC passing through a point m from the set $(M \times \mathbb{R}_+) \setminus \bigcup_{i=1}^k U_i(\varepsilon)$. The connected piece of this IC between neighboring points of intersection with either the boundary of the set $\cup_{i=1}^k U_i(\varepsilon)$, or with M_0, has the temporal length, which we denote as $T_m(\varepsilon)$.

Assumption 5. *For any ε small enough there is $T(\varepsilon) > 0$ such that for any integral curve the temporal interval $T_m(\varepsilon)$ is bounded from above by the constant $T(\varepsilon)$. The same holds true for the union of ICs $\Gamma_1^-, \ldots, \Gamma_r^-$, taken by one from the partition into unstable manifolds for $M \times \mathbb{R}_-$.*

Those NVFs, which satisfy to Assumptions 1–5, are called *gradient-like.*

4. Gradient-Like Diffeomorphisms and Periodic Vector Fields

Recall the notion of gradient-like diffeomorphism on a smooth closed manifold M (see details in [7]).

Definition 2. A Morse–Smale diffeomorphism on M is called *gradient-like* if for any two different periodic points σ_1, σ_2 with the property $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \varnothing$ the inequality $\dim W^u_{\sigma_1} < \dim W^u_{\sigma_2}$ holds.

For a multidimensional case $\dim M \geqslant 3$ a gradient-like diffeomorphism f can have pairs of saddle periodic points whose stable and unstable manifolds intersect each other both along compact and noncompact connected components. A compact component is iterated along the stable (as well as along unstable one) manifold and the orbit of this component consists of countably many components. In case if the dimension of manifold is three, noncompact connected components are called heteroclinic curves, see Fig. 4.

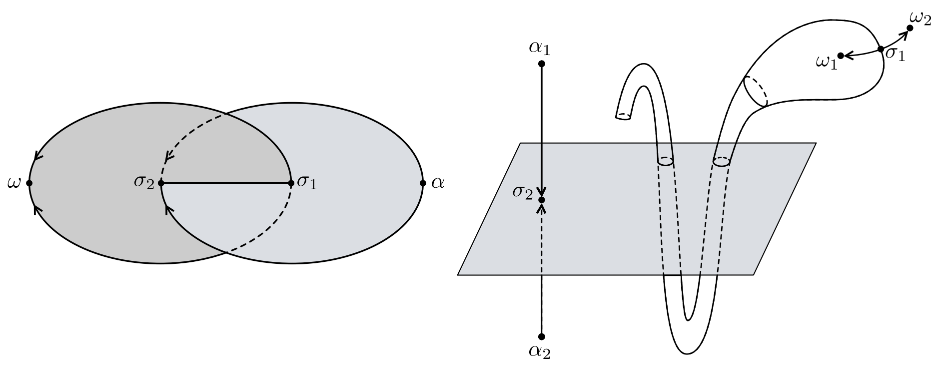

Figure 4. Intersections of invariant manifolds of a gradient-like diffeomorphism

The main result of this paper is the following theorem.

Theorem 2. *Let $f\colon M \to M$ be a diffeotopic to the identity gradient-like diffeomorphism f of a smooth three-dimensional closed manifold M. Then a gradient-like NVF on M, reproducing the structure of f, exists if and only if the intersection of any stable and unstable manifolds of different periodic points for f has no compact connected components.*

Proof. Necessity. Suppose the contrary, i.e., let f be a gradient-like diffeotopic to the identity diffeomorphism of a smooth closed manifold M, dim $M = 3$, and the nonwandering set of f contains a pair of saddle periodic points p, q such that their transversal intersection $W^s_p \cap W^u_q$ has a compact connected component being here a closed curve l_0. Set $l_n = f^n(l_0)$, then we get a collection of disjoint closed curves $\{l_n\}$, whose topological limit, as $n \to +\infty$, is the point p, and as $n \to -\infty$ it is the point q. In the manifold $M_f = M \times S^1$ on the cross-section M_0 we get a curve $\hat{l}_n$ that corresponds to the curve l_n. The curve $\hat{l}_n$ is the directrix of the infinite cylinder consisting of those orbits of the suspension flow which pass at $t = n$ through the closed curve $\hat{l}_n$. The closure of this cylinder contains periodic orbits L_p and L_q, defined by periodic points p, q for f.

When going to the nonautonomous suspension over f, the manifold M_f is cut along the section M_0 and the countable copies of the manifold with the boundary $M_0 \cup M_0$ are glued in accordance with the identification $(x, 1) \sim (f(x), 0)$. There the infinite cylinder in M_f is cut into the countably many pieces C_n with boundaries $\hat{l}_n$, $\hat{l}_{n+1}$. When gluing, C_n is glued to C_{n-1} along the curve $\hat{l}_n$ and to C_{n+1} along the curve $\hat{l}_{n+1}$. Thus, a periodic vector field v_f that is constructed via the nonautonomous suspension, has in the manifold $M \times \mathbb{R}$ infinitely many disjoint invariant cylinders $\mathcal{C}_n$, containing correspondingly the directrix $\hat{l}_n$ and lying in the intersection of $W^u(\mathcal{L}_p)$ and $W^s(\mathcal{L}_q)$. This contradicts to the Assumption 3 of a gradient-like NVF.

Sufficiency. Since f is diffeotopic to the identity map, in virtue of the Theorem 1, there is a periodic vector field v_f on M, reproducing the structure of f. Let us show that v_f is gradient-like NVF, i.e., it satisfies to Assumptions 1–5. The nonwandering set of f is finite and hyperbolic, this implies M to partition into the union of stable manifolds of the finite set of hyperbolic periodic points of f. Similar partition of M is provided by unstable manifolds of the same periodic points. The diffeotopy with the identity map for f allows one to regard the manifold M_f to be $M \times S^1$ and hyperbolic periodic orbits of the suspension flow be finite-fold coverings of S^1 in the bundle $M \times S^1 \to S^1$. When going to the nonautonomous suspension, each hyperbolic periodic orbit unfolds in $M \times \mathbb{R}$ into periodic integral curve of the periodic vector field. These periodic ICs have period one, if the related periodic points of f are fixed points, and they have period k, if k was the period of the related periodic point of f. Thus, in view to the hyperbolicity of periodic orbits of the suspension flow, periodic ICs obtained possess an exponential dichotomy of the type corresponding to the type of the periodic point. Because M is partitioned into stable (unstable) manifolds, $M \times \mathbb{R}$ will also be partitioned into a finite number of stable or, respectively, unstable manifolds. In particular, in this case the partition into stable (unstable) manifolds consists of stable (respectively, unstable) manifolds of periodic ICs.

Now let us verify, if the Assumption 3 fulfills. The transversality condition for any pairs of stable and unstable manifolds holds, since it holds for the gradient-like diffeomorphism f. Let us notice first that from the condition of gradient-likeness for f follows that if $W^u(p) \cap W^s(q) \neq \varnothing$ for two periodic points p, q, then no a saddle periodic point r exists (distinct from p, q) such that $W^s(r) \cap W^u(p) \neq \varnothing$, $W^u(r) \cap W^s(q) \neq \varnothing$. Let us delete from M all one-dimensional stable and unstable manifolds of saddle periodic points and their closures, i.e., sinks and sources, as well as isolated sources and sinks, if they exist. Denote the remaining set as V. By construction, V does not contain periodic points, hence all its points are wandering. Then consider the space of orbits $\hat{V}$ for the restriction of f on V, i.e., we introduce the equivalence relation on V: $x \sim y$, if $y = f^n(x)$ and factorize V according to this relation. Denote $\pi\colon V \to \hat{V}$ the natural projection. At the factorization the manifolds $W^u(p)$, $W^s(q)$ become smooth two-dimensional tori or Klein bottles, and if a heteroclinic curve $\gamma \subset W^u(p) \cap W^s(q)$ with limit points p, q exists, it transforms to the closed curve which belongs to the transverse intersection of these smooth tori or Klein bottles, consisting of a finite number connected components. This implies that the intersection $W^u(p) \cap W^s(q)$ consists also of a finite number of heteroclinic curves.

Keeping this in mind, we consider again the autonomous suspension $M_f = M \times S^1$ over f with its suspension flow. If periodic points p, q with the least periods N_p, N_q, have $W^u(p) \cap W^s(q) \neq \varnothing$, in the suspension hyperbolic periodic orbits L_p, L_q correspond to them, being respectively N_p, N_q-covering of the base S^1. Each heteroclinic curve $\gamma_1, \ldots, \gamma_s$ of f, joining points p and q, corresponds in M_f to an open annulus being an invariant set of the suspension flow, its boundaries are closed orbits L_p, L_q, respectively. More precisely, some of these curves γ_j can belong under iteration of f to the same orbit, then the set of heteroclinic curves consists of several orbits and in the autonomous suspension such orbit generates a cylinder which covers several times S^1 when projecting on the base.

When going to the nonautonomous suspension, such cylinder, covering l times the base, is cut into l rectangles and after gluing gives l different infinite strips foliated into ICs. Boundaries of such strip are a pair of periodic ICs with the dichotomy of the type (1, 2) and (2, 1), obtained under unfolding periodic orbits L_p and L_q, respectively. Every such strip belongs to the transverse intersection of three-dimensional unstable and two-dimensional stable manifolds of these periodic ICs. There exists a finite number of these strips. Therefore, the Assumption 3 holds for the NVF.

Assumptions 4 and 5 hold as well, which follows immediately from the gradient-likeness of f and the construction of the nonautonomous suspension. The theorem has been proved. □

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V.G.: National Research University, "Higher School of Economics" (Nizhny Novgorod branch)

L.L.: National Research University, "Higher School of Economics" (Nizhny Novgorod branch)

E-mail address: lermanl@mm.unn.ru