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ON THE TOPOLOGY OF PLANAR REAL DECOMPOSABLE CURVES OF DEGREE 8

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Abstract. We consider the problem of topological classification of arrangements in the real projective plane of the union of nonsingular curves of degrees 2 and 6 under certain conditions of maximality and general position. We list admissible topological models of such arrangements by the Orevkov method based on the theory of braids and links and prove that most of these models cannot be realized by curves of degree 8.

***Keywords and phrases*:** M-decomposable curve, topological classification, Orevkov method.

***AMS Subject Classification*:** 14P25, 14H99

1. Introduction and statement of the problem. The problem of a systematic study of the topology of real algebraic curves decomposing into the product of two nonsingular curves was first posed by D. A. Gudkov in 1969 in the preface to the monograph [5] for the case of decomposing curves of degree 6, i.e., in the first (lowest-degree) nontrivial case. This problem, directly related to the topology of nonsingular curves of degree 6 (16th Hilbert's problem) was solved by G. M. Polotovsky in [25, 26] under natural conditions of maximality and general position of factor curves. Subsequently, a similar problem was stated for curves of degree 7. However, this problem turned out to be more difficult; its solution required the use of new methods (see [1–3, 6–12, 14–24, 27–31]). Apparently, the solution is close to completion.

In this paper, we begin a similar study of decomposing curves of degree 8. Namely, we discuss the topological classification of curves of degree 8 decomposing into the product of a nonsingular curve of degree 2 (conic) and a nonsingular curve of degree 6 (sextics) under some additional conditions.

We recall the basic definitions and facts from the theory of planar algebraic curves.

Definition 1. A *planar real projective algebraic curve C_m of degree m* is a homogeneous polynomial $C_m(x_0, x_1, x_2)$ with real coefficients of degree m in three variables x_0, x_1, x_2 considered up to a nonzero constant factor.

Definition 2. The set $\mathbb{R}C_m$ ($\mathbb{C}C_m$) of points $(x_0 : x_1 : x_2)$ ($(z_0 : z_1 : z_2)$) of the real (complex) projective plane $\mathbb{R}P^2$ ($\mathbb{C}P^2$) satisfying the equation $C_m(x_0, x_1, x_2) = 0$ is called the *set of real* (respectively, *complex*) *points of the curve C_m*.

Definition 3. A curve C_m is said to be *nonsingular* if the first partial derivatives of the polynomial $C_m(x_0, x_1, x_2)$ with respect to the variables x_0, x_1, and x_2 do not vanish simultaneously (in $\mathbb{C}P^2$).

Each connected component of the set $\mathbb{R}C_m$ of real points of a nonsingular curve C_m (briefly, the real branch of the curve) is homeomorphic to a circle. If the degree of the curve is even, then each such circle is called an *oval*; each oval divides $\mathbb{R}P^2$ into two domains: one domain is homeomorphic to a disk and the other domain is homeomorphic to the Möbius strip. For a given oval, the domain of the first type is considered to be internal, and the area of the second type is considered to be external. If the degree of the curve C_m is odd, then among connected components of the set $\mathbb{R}C_m$, there is exactly one branch embedded in $\mathbb{R}P^2$ one-sidedly; it is called the *odd branch*.

The following classical theorem yields an estimate for the possible number of real branches of a nonsingular curve.

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Theorem (Harnack's theorem, 1876)**.** *Let N be the number of connected components of the set of real points of a planar real projective curve of degree m. Then*

$$N \leq \frac{1}{2}(m-1)(m-2)+1;$$

this estimate is exact for all m.

The set $\mathbb{R}C_m$ considered up to an isotopy in $\mathbb{R}P^2$ is called the *real scheme of the curve C_m.*

A set of s pairwise disjoint topological circles in $\mathbb{R}P^2$, where $0 \leq s \leq (m-1)(m-2)/2+1$, is called a *formal scheme of degree m.*

A real scheme of a nonsingular curve (and, similarly, a formal scheme) can be described by a graph whose vertices correspond to the ovals of the curve and two vertices are connected by an edge if and only if the corresponding ovals are located one inside the other and are not separated by any third oval. The odd branch does not bound a domain in $\mathbb{R}P^2$ and hence it is not necessary to include it in the description; we only must remember that in the case of an odd degree, there is exactly one such branch. It is easy to see that the real scheme of a nonsingular curve is a forest graph in which each outer oval (i.e., an oval that does not lie in the inner domain of any other oval) corresponds to its own tree.

Curves with the maximal possible number of branches (admitted by Harnack's theorem) are called *M-curves.* Schemes with such a number of ovals (both M-curves and formal schemes) are called *M-schemes.* It is well known that the real scheme of a nonsingular conic is either empty or consists of one vertex. Harnack's theorem implies that the numbers of vertices of M-schemes of degrees 6 and 8 are equal to 11 and 22, respectively.

The problem of topological classification of nonsingular planar real algebraic curves stated by D. Hilbert in the first part of his 16th problem, can now be formulated as follows: *For each natural m, find a list of formal schemes of degree m that can be realized as real schemes of some curves of degree m.*

At present, the complete answer to this problem is known for $m \leq 7$. In the first nontrivial case $m = 6$, specially indicated by Hilbert, the answer was obtained by D. A. Gudkov in [5]. A fragment of Gudkov's classification concerning the case of M-curves can be formulated as follows.

Theorem (Gudkov's theorem, 1969)**.** *Nonsingular curves of degree 6 realize only the M-schemes $\frac{1}{1}9$, $\frac{5}{1}5$, and $\frac{9}{1}1$.*

Following Gudkov, we use the following encoding of schemes: $\frac{\alpha}{1}\beta$ means a scheme consisting of $\beta+1$ ovals outside each other, inside one of which lie other α ovals outside each other. This encoding can be easily generalized to forests with taller trees (see, e.g., [22]).

Thus, the set of real points of each M-curve of degree 6 contains exactly one nonempty oval (i.e., an oval with another oval in its inner domain); it is denoted by 1 in the "denominator" of the code; in what follows, a nonempty oval of a degree-6 curve is said to *principal* and ovals located inside and outside the principal oval are said to be *internal* and *external*, respectively.

We say that an oval containing s ovals in its inner domain, which sequentially surround each other, has weight $s+1$. A scheme consisting of an outer oval (i.e., not lying inside other ovals) of weight $s+1$ and all ovals inside is called a *nest of weight $s+1$.* Thus, an empty outer oval is considered to be a nest of weight 1, and one can say that an M-curve of degree 6 is the union of one nest of weight 2 and nests of weight 1.

Based on his results, D. A. Gudkov formulated (as a hypothesis) the following theorem, which was proved by V. I. Arnold for the special case (modulo 4) and later by V. A. Rokhlin in the general case.

Theorem (Gudkov's congruence, 1969)**.** *The Euler characteristics $\chi(B_+)$ of the oriented part B_+ of the complement in $\mathbb{R}P^2$ to the set of real parts of an M-curve of degree $2k$ satisfies the congruence relation $\chi(B_+) \equiv k^2 \pmod 8$.*

It is easy to see that an M-curve is determined by an irreducible polynomial. Consider decomposable curves.

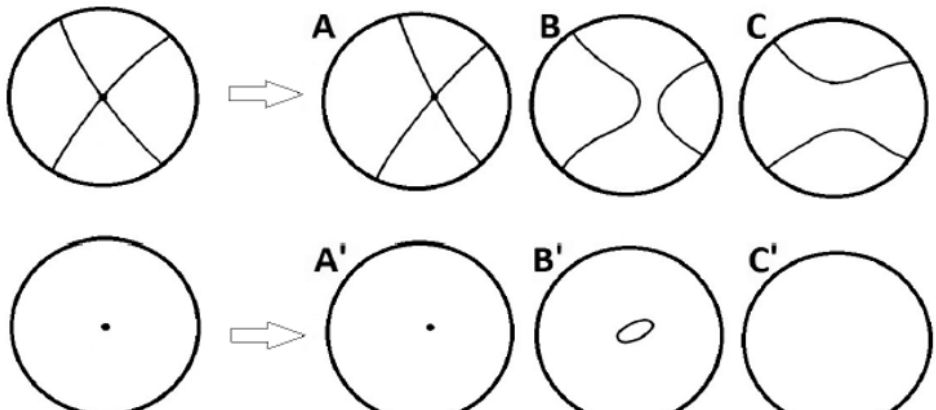


Fig. 1.

Definition 4. A curve C_m is called an *M-decomposable curve of degree m* if the following conditions are fulfilled:

(i) $C_m(x_0, x_1, x_2) = C_k(x_0, x_1, x_2) \cdot C_{m-k}(x_0, x_1, x_2)$, where $k \in \{1, 2, \dots, [m/2]\}$;
(ii) C_k and C_{m-k} are M-curves;
(iii) the set $\mathbb{R}C_k \cap \mathbb{R}C_{m-k}$ consists of $k \cdot (m-k)$ pairwise distinct points;
(iv) all points of the set $\mathbb{R}C_k \cap \mathbb{R}C_{m-k}$ lie on the same branch of the curve C_k and on the same branch of the curve C_{m-k};
(v) for a certain choice of directions of bypassing the intersecting branches, the intersection points (i.e., the points of the set $\mathbb{R}C_k \cap \mathbb{R}C_{m-k}$) lie on them in the same order.

Ovals of the curves C_{m-k} and C_k that do not have intersection points are called *free* ovals.

The problem of topological classification of decomposable curves of degree m is formulated as follows: Find a topological classification of triples $(\mathbb{R}P^2, \mathbb{R}C_m, \mathbb{R}C_k)$, where

$$C_m(x_0, x_1, x_2) = C_k(x_0, x_1, x_2) \cdot C_{m-k}(x_0, x_1, x_2).$$

In this paper, we consider this problem for the case where $m = 8$ and $k = 2$ under the condition that the curve C_8 is an M-decomposable curve, i.e., the conditions (i)–(v) of Definition 4 are fulfilled.

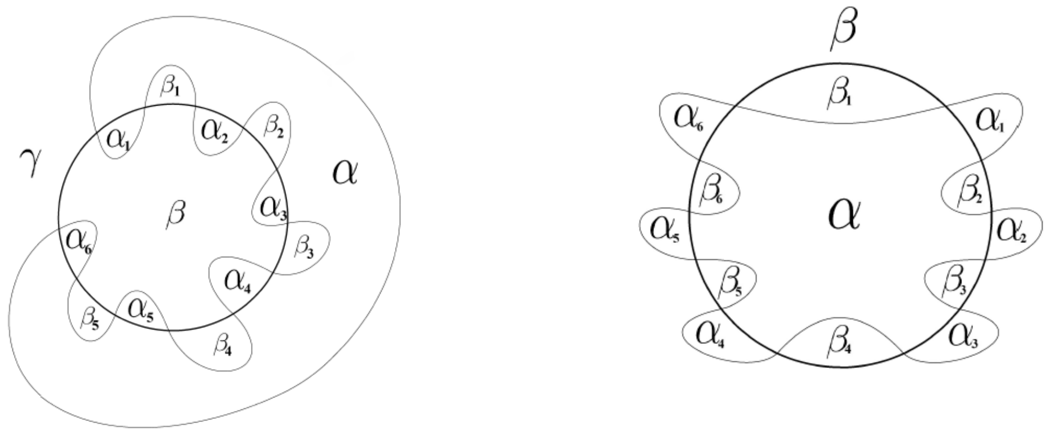
Usually, to solve a classification problem for algebraic curves of a certain class, one first lists admissible (i.e., not prohibited by currently known restrictions) topological models of curves of this class. Then, one tries to realize each admissible model by an algebraic curve of the class under consideration ("to construct") or prove that such a construction is impossible ("to prohibit"). Below, for the problem formulated above, we present admissible models and prohibitions found by using the Orevkov method based on the theory of braids and links.

2. Admissible models. We begin with the formulation of the classical theorem on the independence of perturbations of singular points of a *simple* curve.

Theorem (Brusotti's theorem, 1921)**.** *Assume that a curve C_m has no multiple components and all its singular points are simple double points. Then using sufficiently small changes in the coefficients of the curve C_m, one can obtain a real curve of degree m, which in a neighborhood of each cross-type singular point* (*i.e., a transversal intersection of branches*) *has one of the types A, B, and C, and in a neighborhood of each solitary singular point has one of the types A', B', C'* (*see Fig.* 1).

Now we recall the well-known facts about M-schemes of curves of degree 8.

1. Fue to Harnack's theorem, an M-curve of degree 8 consists of 22 ovals.



Type I Type II

Fig. 2. Types of arrangement of intersecting ovals and notation of domains

2. Due to topological consequences of Bésout's theorem applied to the intersection of a degree-8 curve with a straight line and a conic, an M-curve of degree 8 can have only one of the following real schemes:
   (a) the union of nests of weight 1 and mo more than three nests of weight 2;
   (b) the union of nests of weight 1 and one nest of weight 3.
3. An M-curve of degree 8 satisfies Gudkov's congruence relation for $k = 4$; in particular, this implies that all 22 ovals of an M-curve of degree 8 cannot be located outside of each other.

The above conditions allow 89 pairwise distinct formal M-schemes of degree 8; 83 of them have been implemented by M-curves of degree 8 by now. The realizability problem for the remaining six schemes remains open. A list of all 89 schemes can be found in [19].

The location of two simple closed curves that intersect at 12 points lying on these curves in the same order can belong to one of the nonequivalent types I and II shown in Fig. 2. In this figure, the bold line shows the conic and the thin line shows the nonfree oval of the scheme of degree 6; by Greek letters, we mark domains in which free ovals of a degree-6 scheme can a priori lie.

Now for each of the cases I and II, it is necessary to enumerate admissible distributions of 10 free ovals in the domains marked with Greek letters for each of the M-schemes $\frac{1}{1}9$, $\frac{5}{1}5$, and $\frac{9}{1}1$ of degree 6; we assume that the nonfree oval is either outer, or principal, or inner. The idea of this enumeration is as follows.

Let us add to an arrangement of type I or II (see Fig. 2) 10 free ovals of a scheme of degree 6 distributed over the domain in some way. Then, bearing in mind Brusotti's theorem, eliminate all double points on the arrangement and obtain a scheme with the maximal possible number of ovals (i.e., we turn each digon into an oval; in the case I, intersecting ovals form the scheme $\frac{11}{1}$ and in the case II we obtain 12 ovals outside each other). As a result, we get an M-scheme of degree 8. If this scheme is not included in the above list of 89 M-schemes of degree 8, then the corresponding arrangement of free ovals cannot be realized. Otherwise, the question on the realizability of the scheme of the decomposable curve must be examined further.

A. Arrangements of type I.

A1. *Nonfree oval is outer.* The domains α and α_i do not contain ovals since the nonfree oval is outer. The domains β_i also cannot contain ovals, otherwise we get a scheme, which does not satisfy the condition 2 on the types of M-schemes of degree 8.

Consider a scheme of degree 6 of the form $\frac{1}{1}9$. Then the following possibilities remain: in one of the domains β or γ, free ovals form the scheme $\frac{1}{1}t$ and in the other the form the scheme "$8 - t$ ovals

Table 1. Type I, outer nonfree oval

No.	Scheme of degree 6	β	γ	M-T	F-M
1	$\frac{1}{1}9$	$\frac{1}{1}3$	5	Hilbert's construction	
2	—"—	$\frac{\star 1}{1}7$	1	✓	×
3	—"—	7	$\frac{1}{1}1$	no M-pencils	
4	—"—	3	$\frac{1}{1}5$	no M-pencils	
5	$\frac{5}{1}5$	$\frac{\star 5}{1}3$	1	✓	✓
6	—"—	3	$\frac{5}{1}1$	no M-pencils	

outside each other," where $0 \leq t \leq 8$. It is easy to verify that Gudkov's congruence relation is valid only for $t \in \{3, 7\}$, where the nest $\frac{1}{1}$ is located in the domain β, and for $t \in \{1, 5\}$, where the nest $\frac{1}{1}$ is located in the domain γ .

These results are contained in the lines 1–4 of Table 1; the meaning of the symbol $\star$ and the last columns in this and the following tables will be explained below in Sec. 4.

Schemes of degree 6 of the form $\frac{5}{1}5$ and $\frac{9}{1}1$ are examined similarly. The lines 5 and 6 of Table 1 correspond to the first of them, and there are no admissible possibilities for the second.

A2. *Non-free oval is principal.* In this case, inner ovals of a scheme of degree 6 must be distributed between the domains α and α_1–α_6. Note that the domains α_i and α_{7-i}, $i \in \{1, 2, 3\}$, have the same properties due to symmetry. Moreover, due to the condition 2 on the types of M-schemes of degree 8, only one of the domains α_i can be nonempty.

A3. *Nonfree oval is outer.* In this case, the principal oval encircles the whole configuration of two intersecting ovals shown in Fig. 2 on the left (Type I). The domains β_1–β_5 cannot contain ovals due to the condition 2: otherwise, a nest of weight 4 is formed.

The unique possibility for the scheme $\frac{1}{1}9$ is as follows: nine free ovals outside each other lie in the domain γ; this contradicts the condition 3 for M-schemes of degree 8.

For the other two schemes, $\frac{5}{1}5$ and $\frac{9}{1}1$, we denote by γ_1 and γ_2 the parts of the domain γ located, respectively, inside and outside the principal oval (in the case considered, the principal oval is not shown in Fig. 2). As above, we obtain the results shown in Table 3.

B. Arrangements of type II.

B1. *Non-free oval is outer.* Since in this case the nonfree oval is empty, free ovals can only lie in the domains β_i, $1 \leq i \leq 6$, and β. At the same time, due to the condition 2, if the principal oval lies in β_i then the other domains β_i must be empty, and if the main oval lies in the domain β, then at most two of the domains β_i may be nonempty. Recall that all domains β_i have the same properties. Also note that in the case where two of the domains β_i are nonempty, there are three pairwise nonequivalent possibilities in the order of bypassing the oval of the conic: nonempty domains are either adjacent, or separated by one empty domain, or separated by two empty domains. We get a list of valid arrangements prsented given in table 4.

B2. *Nonfree oval is principal.* As in the previous case, the domains α_i have the same properties and the domains β_i also have the same properties. Due to the condition 2, at most three of these 12 domains can contain ovals at the same time. If exactly two of them are nonempty (i.e., contain free ovals), then, as in the previous case, up to cyclic order of domains (i.e., along the oval of the conic), there are three pairwise distinct cases for numbers i and j of these domains: $(i, j) \in M_1 = \{(1, 2), (1, 3), (1, 4)\}$. If three domains of the same type are nonempty (i.e., all three domains from α_i or all three from β_i),

Table 2. Type I, principal nonfree oval

No.	Scheme of degree 6	α	$\alpha_1 \veebar \alpha_2 \veebar \alpha_3$	$\beta_1 \veebar \beta_2 \veebar \beta_3$	β	γ	M-T	F-M
1	$\frac{1}{1}9$	1		$\star$ 5	3	1	$\times$	—
2	—"—	1		$\star$ 4	3	2	$\times$	—
3	—"—	1		$\star$ 3	3	3	$\times$	—
4	—"—	1		$\star$ 2	3	4	$\times$	—
5	—"—	1		$\star$ 1	3	5	$\times$	—
6	—"—	1		$\star$ 1	7	1	$\times$	—
7	—"—	1			7	2	no M-pencils	
8	—"—	1			3	6	no M-pencils	
9	—"—		$\star$ 1		8	1	$\times$	—
10	—"—		$\star$ 1		4	5	$\checkmark$	$\checkmark$
11	—"—		$\star$ 1			9	$\times$	—
12	$\frac{5}{1}5$	5		$\star$ 1	3	1	$\times$	—
13	—"—	5			3	2	no M-pencils	
14	—"—	4	$\star$ 1		4	1	$\times$	—
15	—"—	4	$\star$ 1			5	$\times$	—
16	—"—	3	$\star$ 2		1	4	$\times$	—
17	—"—	2	$\star$ 3		2	3	$\checkmark$	$\times$
18	—"—	1	$\star$ 4		3	2	$\times$	—
19	—"—		$\star$ 5		4	1	$\times$	—
20	—"—		$\star$ 5			5	$\times$	—
21	$\frac{9}{1}1$	8	$\star$ 1			1	$\checkmark$	$\times$
22	—"—	4	$\star$ 5			1	$\times$	—
23	—"—	0	$\star$ 9			1	$\times$	—

Table 3. Type I, inner nonfree oval

No.	Scheme of degree 6	β	γ_1	γ_2	M-T	F-M
1	$\frac{5}{1}5$	$\star$ 2	2	5	$\checkmark$	$\checkmark$
2	$\frac{9}{1}1$	$\star$ 2	6	1	$\times$	—
3	$\frac{9}{1}1$	6	2	1	Hilbert's construction	

Table 4. Type II, outer nonfree oval

No.	Scheme of degree 6	β	β_1	β_j, $j = 2 \veebar 3 \veebar 4$	M-T	F-M
1	$\frac{1}{1}9$	$\frac{1}{1}6$	2	0	no M-pencils	
2	—"—	—"—	1	1	—"—	
3	—"—	$\frac{1}{1}2$	6	0	—"—	
4	—"—	—"—	5	1	—"—	
5	—"—	—"—	3	3	—"—	
6	—"—	6	$\frac{1}{1}2$		Hilbert's construction	
7	—"—	2	$\frac{\star 1}{1}6$		✓	✓
8	$\frac{5}{1}5$	$\frac{5}{1}2$	2	0	no M-pencils	
9	—"—	—"—	1	1	—"—	
10	—"—	2	$\frac{\star 5}{1}2$		✓	✓

then there are also three possibilities: $(i, j, k) \in M_2 = \{(1, 2, 3), (1, 2, 4), (1, 3, 5)\}$. If two domains of one type and one domain of another type are nonempty (for example, α_i, α_j, and β_k), then there are 9 pairwise nonequivalent possibilities:

$$(i, j, k) \in M_3 = \Big\{(1, 2, 2),\ (1, 2, 3),\ (1, 2, 4),\ (1, 2, 5),\ (1, 3, 2),\ (1, 3, 4),\ (1, 3, 5),\ (1, 4, 2),\ (1, 4, 3)\Big\}.$$

Admissible possibilities for the case considered are listed in Tables 5 and 6; note that the headers of these tables contain only one element (i.e., a set of index values) from each of the sets M_1, M_2, and M_3 indicated above since calculations in Sec. 4 and their results are independent of the choice of these elements.

Note that in the case of a set of indices from the set M_2, if the numbers of ovals in each of the three nonempty domains corresponding to these indices are the same, then the corresponding lines of the table (for example, the line 3 of Table 5) correspond to three pairwise nonisotopic arrangements. If only two of these three numbers coincide (e.g., as in the line 7 of Table 5), then there are 5 pairwise different possibilities (up to the cyclic order of domains along the oval of the conic). Recall that for the sets M_2 and M_3 the situation where these three numbers are all different does not occur in the tables. Similarly, for the case of a set M_3, if the numbers of ovals in two nonempty domains of the same type are the same, then the corresponding lines of the table (for example, the lines 11 and 14 of Table 5) correspond to nine pairwise nonisotopic arrangements, whereas these numbers do not match (e.g., as in the line 13 of Table 5), then there are 15 pairwise different possibilities.

B3. *Nonfree oval is inner.* As in the case of an inner nonfree oval for an arrangement of type I (the case A3 above), the principal oval encircles the whole configuration of two intersecting ovals, and free ovals may lie only in one of the domains β_i, $1 \le i \le 6$ (recall that they have the same properties), and the parts β' (inside the principal oval) and β'' (outside the principal oval) of the domain β. The list of admissible possibilities is given in Table 7.

3. Orevkov's method. Orevkov's method used below for prohibiting isotopic types of algebraic curves using the theory of braids and links has been repeatedly stated in the literature (except for the fundamental work [17], see, e.g., [9, 19, 21]), so here we give only a summary necessary for further understanding.

Table 5. Type II, principal nonfree oval, schemes $\frac{1}{1}9$ and $\frac{5}{1}5$

No.	Scheme of degree 6	α	α_1	α_2	α_3	β	β_1	β_2	β_3	M-T	F-M
1	$\frac{1}{1}9$	1				6	3			Hilbert's construction	
2	—"—	$\star$ 1				6	2	1		✓	×
3	—"—	$\star$ 1				6	1	1	1	✓	×
4	—"—	$\star$ 1				2	7			×	—
5	—"—	$\star$ 1				2	6	1		×	×
6	—"—	$\star$ 1				2	5	2		×	×
7	—"—	$\star$ 1				2	5	1	1	×	—
8	—"—	$\star$ 1				2	4	3		×	×
9	—"—	$\star$ 1				2	3	3	1	×	—
10	—"—		1			7	2			no M-pencils	
11	—"—		1			7	1	1		no M-pencils	
12	—"—		1			3	6			no M-pencils	
13	—"—		1			3	5	1		no M-pencils	
14	—"—		1			3	3	3		no M-pencils	
15	$\frac{5}{1}5$	$\star$ 5				2	3			×	×
16	—"—	$\star$ 5				2	2	1		×	×
17	—"—	$\star$ 5				2	1	1	1	×	×
18	—"—	$\star$ 4	1			3	2			×	—
19	—"—	$\star$ 4	1			3	1	1		✓	×
20	—"—	$\star$ 3	2			4	1			✓	✓
21	—"—	$\star$ 3	2				5			✓	×
22	—"—	$\star$ 3	1	1		4	1			✓	×
23	—"—	$\star$ 3	1	1			5			×	×
24	—"—	$\star$ 2	3			5				✓	×
25	—"—	$\star$ 2	3			1	4			✓	×
24	—"—	$\star$ 2	3			1	3	1		✓	×
25	—"—	$\star$ 2	2	1		5				✓	×
26	—"—	$\star$ 2	1	1	1	5				✓	×
27	—"—	$\star$ 1	4			2	3			✓	×
28	—"—	$\star$ 1	3	1		2	3			✓	×
29	—"—		5			3	2			no M-pencils	
30	—"—		5			3	1	1		no M-pencils	

Table 6. Type II, principal nonfree oval, scheme $\frac{9}{1}1$

No.	Scheme of degree 6	α	α_1	α_2	α_3	β	β_1	β_2	β_3	M-T	F-M
31	$\frac{9}{1}1$	$\star$ 7	2				1			✓	×
32	—"—	$\star$ 7	1	1			1			×	—
33	—"—	6	3			1				Hilbert's construction	
34	—"—	$\star$ 6	2	1		1				✓	✓
35	—"—	$\star$ 6	1	1	1	1				✓	×
36	—"—	$\star$ 3	6				1			✓	×
37	—"—	$\star$ 3	5	1			1			✓	×
38	—"—	$\star$ 3	3	3			1			✓	×
39	—"—	$\star$ 2	7			1				✓	×
40	—"—	$\star$ 2	6	1		1				×	—
41	—"—	$\star$ 2	5	2		1				✓	×
42	—"—	$\star$ 2	5	1	1	1				✓	×
43	—"—	$\star$ 2	4	3		1				×	—
44	—"—	$\star$ 2	3	3	1	1				✓	×

Table 7. Type II, inner nonfree oval

No.	Scheme of degree 6	β_1	β'	β''	M-T	F-M
1	$\frac{5}{1}5$	$\star$ 1	3	5	✓	✓
2	$\frac{9}{1}1$	$\star$ 1	7	1	✓	×
3	$\frac{9}{1}1$	5	3	1	Hilbert's construction	

Let C_m be a curve whose singularities are nondegenerate double points. Assume that there exists a point $p \in \mathbb{R}P^2 \setminus \mathbb{R}C_m$ such that the pencil L_p of straight lines centered at this point has the following properties:

(a) there exists a straight line l_0 in L_p intersecting the curve $\mathbb{R}C_m$ at m distinct points (the *maximal line*);

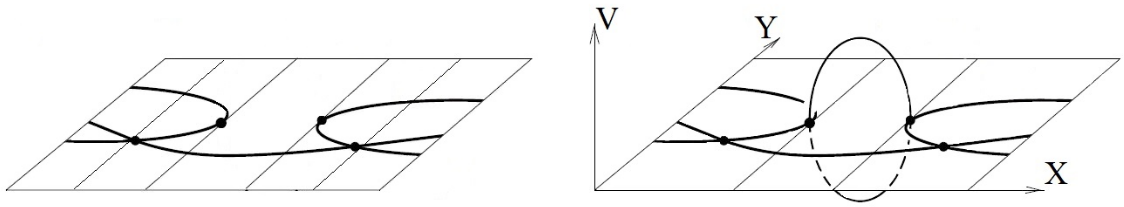
(b) any straight line $l \in L_p$ intersects the curve $\mathbb{R}C_m$ at least at $m-2$ distinct points;

(c) each straight line of the pencil has no more than one point of double intersection with $\mathbb{R}C_m$, i.e., each of such *critical lines* either touches $\mathbb{R}C_m$ or intersects $\mathbb{R}C_m$ at its double point transversally.

A pensil satisfying the conditions (a)–(c)is said to be *maximal*. Note that the condition (c) can always be satisfied by a small perturbation of the center of the pencil.

We choose affine coordinates (x, y) so that the line l_0 (and hence the point p) is located at infinity and the pencil L_p turns into a pencil of parallel lines $\{l_t\}$ (see Fig. 3(a)), where l_t is the straight line given by the equation $x = t$.

Let $\{l_{t_1}, \dots, l_{t_s}\}$ be the set of all critical straight lines ordered so that the numbers t_i increase. The scheme of the arrangement of the curve $\mathbb{R}C_m$ relative to the pencil L_p is encoded by the word $u_1 \dots u_s$,



(a) (b)

Fig. 3.

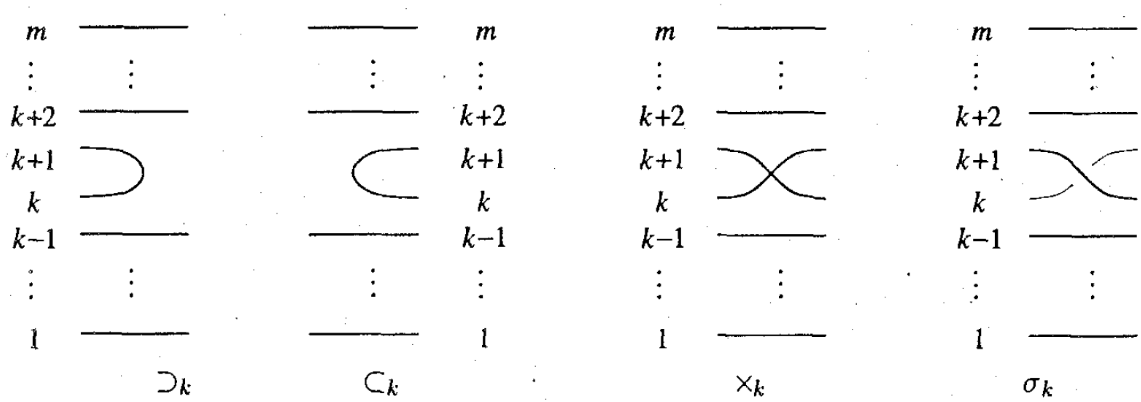


Fig. 4. Symbols of the ×-code and standard generators of the braid group

where the letter u_i characterizes the location of the curve $\mathbb{R}C_m$ in a neighborhood of the line l_{t_i} and takes one of the values $\supset_k$, $\subset_k$, or $\times_k$ ($k \in \{1, \dots, m-1\}$) according to Fig. 4[1]. A pair of consecutive symbols $\subset_k$, $\supset_k$ is replaced by a single symbol o_k ("a free oval in the $(k-1)$th strip counting from below"), the coding word is called the ×-code.

In the complex projective plane $\mathbb{C}P^2$, consider the set $M = \mathbb{C}C_m \cap \mathbb{C}L_p$, where $\mathbb{C}L_p$ is a pencil of complex straight lines $\mathbb{C}l$. The set M is homeomorphic to a set of circles, some of which are pairwise glued together at double points of the curve $\mathbb{R}C_m$ and at the points of tangency of lines of the pencil L_p with this curve (see Fig. 3(b)[2]).

Having eliminated all gluing points in some standard way (see Fig. 5; for details, see [17, 21]), we obtain a link $K(C_m, p)$. Let $b(C_m, p)$ be a braid of m threads whose closure coincides with $K(C_m, p)$. For what follows, it is important that, due to the assumption that the pencil is maximal, the braid $b(C_m, p)$ is uniquely determined (up to conjugacy in the group B_m of braids with m threads) by the mutual position of the curve $\mathbb{R}C_m$ and the pencil L_p in $\mathbb{R}P^2$.

Recall that the group B_m has the following presentation by standard generators σ_k:

$$\left\langle \sigma_1, \dots, \sigma_{m-1} \;\middle|\; \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1,\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1 \right\rangle.$$

It is well known (see [13]) that the braid $b(C_m, p)$ obtained by the method described above is *quasi-positive*, i.e., can be represented in the form

$$\prod_{j=1}^{k} \omega_j \sigma_{i_j} \omega_j^{-1},$$

[1]Figure 4 is taken from [21].

[2]This drawing is conditional: the "imaginary axis" V is two-dimensional.

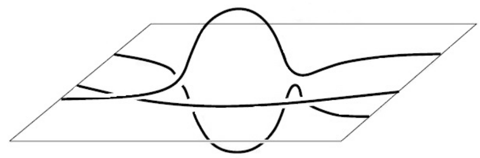


Fig. 5.

where ω_j, $j \in \{1, 2, \dots, k\}$, are some words in the alphabet $\{\sigma_1, \dots, \sigma_{m-1}, \sigma_1^{-1}, \dots, \sigma_{m-1}^{-1}\}$. Therefore, if for the topological model of a hypothetic curve $\mathbb{R}C_m$, the braid $b(C_m, p)$ is not quasi-positive for each possible mutual arrangement of the pencil L_p and this model, then the model cannot be realized by an algebraic curve of degree m.

As a necessary condition for quasi-positivity, S. Yu. Orevkov suggested to use the Murasugi–Tristram inequality in [17] and, later, then the Fox–Milnor condition in [19, 20].

Murasugi–Tristram inequality. *If $b = \prod \sigma_i^{k_i}$ is a quasi-positive braid of m threads, then its closure satisfies the inequality*

$$|\sigma(b)| + m - e(b) - n(b) \leq 0,$$

where $\sigma(b)$ and $n(b)$ are the signature and the closure defect of the braid b and $e(b) = \sum k_j$ is the algebraic degree of the braid b.

Fox–Milnor condition. *Let b be a quasi-positive braid of m threads. If $e(b) = m-1$, then there exists a polynomial $f(t) \in \mathbb{Z}[t]$ such that the Alexander polynomial $\Delta_L(t)$ of the closure L of the braid b is represented in the form*

$$\Delta_L(t) \doteq f(t) \cdot f(t^{-1}),$$

where $\doteq$ means equality up to multiplication by units of the ring $\mathbb{Z}[t, t^{-1}]$; if $e(b) < m - 1$, then the equality $\Delta_L(t) = 0$ holds (for details, see [27]).

Proposition 1. *The Alexander polynomial satisfies the Fox–Milnor condition if its value at $t = -1$ is the square of an integer.*

Proposition 2. *If the decomposition of the Alexander polynomial into irreducible factors contains a symmetric polynomial to an odd degree, then the Alexander polynomial cannot be represented in the form specified in the Fox–Milnor condition.*

4. Prohibitions of mutual positions of a conic and a sextic using the Orevkov method. Let us illustrate the application of the Orevkov method in our problem with examples and describe the results obtained.

Example 1. Consider the hypothetical arrangement of a conic and a sextic indicated in the line 1 of Table 2 (see Fig. 6, where p is the center of the maximum pencil, a is the maximal line in l_0, the outer circle is the boundary of the model of the projective plane, i.e., diametrically opposite points of this circle are considered identified).

Let us choose the center p of the maximal pencil inside one of the ovals lying in the domain β_1. Next, as the straight line l_0, we choose the line a passing through the point p and a point inside one of the ovals lying in the domain β (see Fig. 6).

Remark. As was noted above, in the presence of a maximum pencil, the braid $b(C_m, p)$ is uniquely determined by the "real picture" (i.e., by the mutual arrangement of the curve $\mathbb{R}C_m$ and the pencil L_p in $\mathbb{R}P^2$). For a nonmaximal beam, it is not seen how the threads intertwine in the imaginary domain. In the cases where it was not possible to find a maximum pencil of lines, investigations were not carried out; these cases are indicated noted in the tables. Although, in principle, Orevkov's method

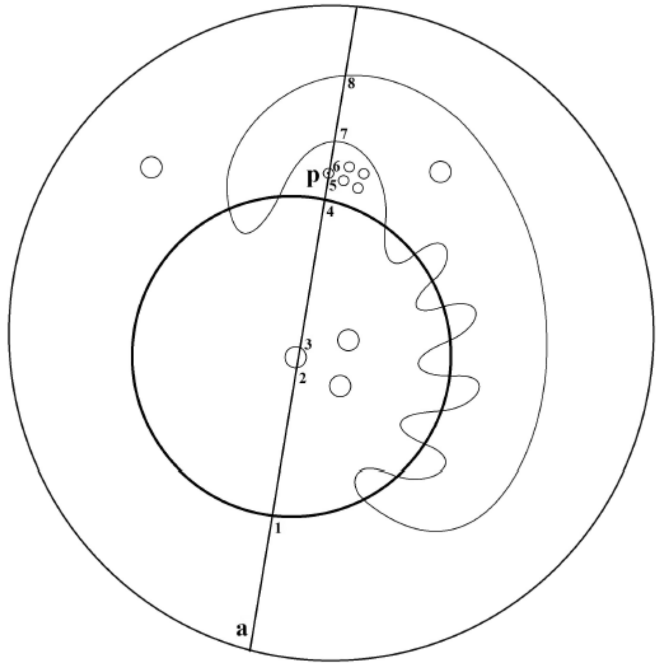


Fig. 6.

is also applicable in these cases (see [17, 19]), consideration of the available possibilities here is rather complicated and requires large calculations.

Now, choosing an affine coordinate system as described in Sec. 3, we obtain an unfolding of the curve shown in Fig. 6; this unfolding is shown in Fig. **??**, where Latin letters with indices mark areas in which free ovals can be located, and in the caption to the figure, the notation $|U|$ means the number of ovals in the area U; moreover, the conditions for these numbers corresponding to the example considered are presented. The $\times$-code for Fig. 7, which does not take into account (undrawn) free ovals, has the form

$$\supset_5 o_5 o_5 \subset_5 \times_6 \times_4 \supset_5 \subset_4 \times_3^{10} \supset_2 \subset_3,$$

where $\times_3^{10}$ means the character $\times_3$ repeated consecutively ten times.

Now we must enumerate logically possible distributions of free ovals over the domains indicated by letters and insert the necessary symbols o_i into the corresponding places of the $\times$-code. Next, for each $\times$-code from the resulting list of $\times$-codes, we verify the Murasugi–Tristram inequality for the corresponding link. If this inequality is not satisfied for each case, then the arrangement considered cannot be realized by any curve of degree 8. The calculation for the example considered lead to this result. The fact that an arrangement is prohibited is noted by $\times$ in the corresponding cells in the columns “M-T” of the tables in Sec. 2. In these cases, we did not verify the Fox–Milnor condition; this fact is indicated by a dash in the corresponding cells of the column “F-M.”

If the Murasugi–Tristram inequality did not prohibit an arrangement (see Example 2 below), i.e., this inequality is satisfied for at least one distribution of ovals, then the corresponding cells of the column “M-T” is filled with the symbol ✓. In such cases, we calculated the Alexander polynomial and verified the Fox–Milnor condition; the results of calculations are marked similarly in the columns “F-M.”

Example 2. Consider the hypothetical arrangement of a conic and a sextic indicated in the line 2 of Table 7 (see Fig. 8, where p is the center of the maximum pencil, a is the maximum line of l_0, and the

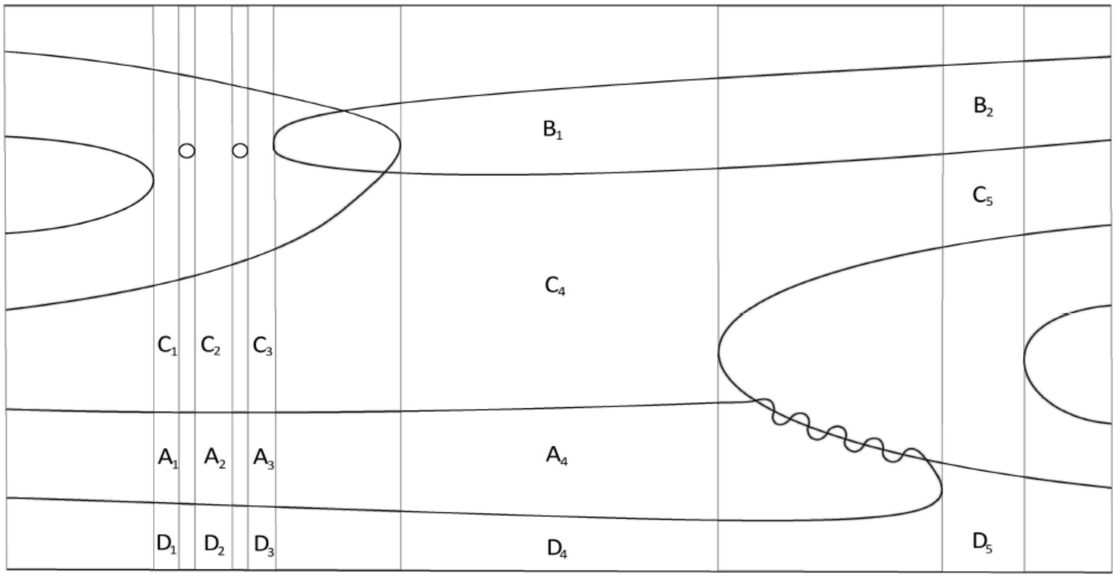


Fig. 7. $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup B_1 \cup B_2| = 1$, $|C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5| = 1$, $|D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5| = 4$; by virtue of Bezout's theorem, ovals cannot be located in vertical strips that do not contain areas indicated by letters

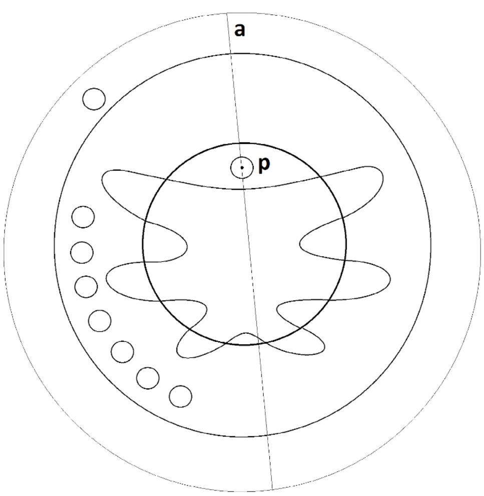


Fig. 8.

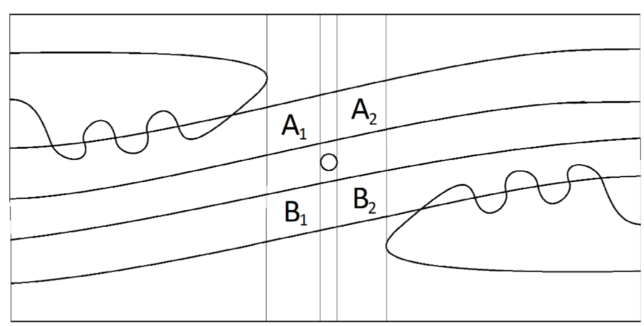


Fig. 9. $|A_1 \cup A_2 \cup B_1 \cup B_2| = 7$

outer circle is the boundary of the model of the projective plane). The unfolding for Fig. 8 is presented in Fig. 9.

Here we have 120 pairwise different distributions of seven ovals in the domains A_1, A_2, B_1, and B_2. Calculations show that for each of them, the corresponding link satisfies the Murasugi–Tristram inequality. Then we apply the Fox–Milnor condition, i.e., we calculate the Alexander polynomial for each of these links. In 68 cases, the value of the Alexander polynomial at the point -1 is not the square of an integer; hence the corresponding arrangements cannot be realized by curves of degree 8 (see Proposition 1). In each of the other 52 cases, calculations show that the factorization of the Alexander polynomial contains a symmetric polynomial to an odd degree, so the Fox–Milnor condition is not

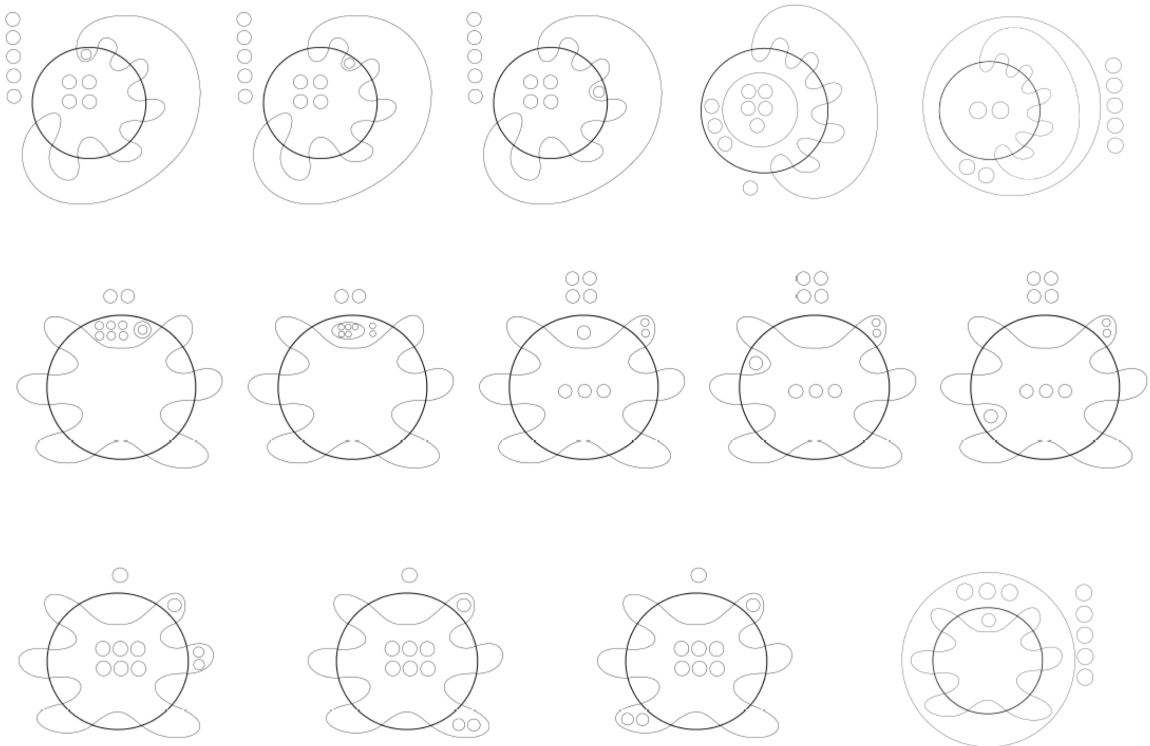


Fig. 10.

satisfied due to Proposition 2. Thus, the arrangement from the line 2 of Table 7 is prohibited; this fact is indicated by the symbol $\times$ in the column "F-M."

The choice of the center of a pencil and the construction of an arrangement for constructing the initial $\times$-code are done manually. The symbol $\star$ in the tables next to the number indicating the number of ovals in the given domain means that the center of the pencil was chosen inside this or one of these ovals. In this case, it does not matter for calculations in which of the ovals (if the number is greater than one) and in which domain (if the domain is indicated ambiguously) the center of the pencil was chosen.

All other calculations are performed using a computer. Namely, the enumeration of admissible arrangements of free ovals for each layered arrangement, the calculation of the corresponding braids from the $\times$-codes, the calculation of all link invariants involved in the Murasugi–Tristram inequality were performed by a software created by M. A. Gushchin and repeatedly used earlier (see, e.g., [9, 10]) for the classification of other classes of decomposable curves.

The calculation of the Alexander polynomial of the link and the verification of its properties from Propositions 1 and 2 were performed by a software created by I. M. Borisov.

5. Statistics of results. In total, there are 323 pairwise nonisotopic schemes of curves of the form considered in the paper. Six of them can be realized by curves of degree 8 in the process of constructing nonsingular curves of even degree by *Hilbert's method*. Since constructions by the Hilbert method have been repeatedly described in the literature (see, e.g., [4, 22, 32]), we restrict ourselves to indicating "Hilbert's construction" in the tables.

Among the remaining 317 schemes, a maximum pencil exists for 245 schemes, of which 231 are prohibited in this paper, and for 14 schemes shown in Fig. 10, the question of realizability is open.

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COMPLIANCE WITH ETHICAL STANDARDS

Conflict of interests. The authors declare no conflict of interest.

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