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Symplectic partially hyperbolic automorphisms of 6-torus

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ABSTRACT

We study topological properties of automorphisms of a 6-dimensional torus $\mathbb{T}^6$ generated by integer matrices with simple eigenvalues being symplectic with respect to either the standard symplectic structure in $\mathbb{R}^6$ or a nonstandard symplectic structure given by an integer skew-symmetric non-degenerate matrix. Such a symplectic matrix generates a partially hyperbolic automorphism of the torus, if its eigenvalues lie both outside and on the unit circle. We study transitive and decomposable cases possible here and present a classification in both cases.

1. Introduction

The aim of this paper is the investigation of the topological dynamics of automorphisms on a 6-dimensional torus generated by an integer symplectic transformation of $\mathbb{R}^6$ in the case of partial hyperbolicity. The hyperbolic case is rather well understood already [1,10,26,18].

There are many results about partially hyperbolic diffeomorphisms [8,13–15] starting from the seminal paper [7]. Concerning partially hyperbolic automorphisms on a torus $\mathbb{T}^n$, the most detailed study was done in [15], which solved the question on their stable ergodicity posed in [16]. Recall that stable ergodicity of a C^r-smooth diffeomorphism f of a manifold M, being ergodic with respect to a smooth Lebesgue measure on M, means the existence of a neighborhood U of f in the space of C^r-smooth diffeomorphisms such that each $g \in U$ is ergodic. Our goal here is to classify, with respect to topological conjugacy, possible types of orbit behavior of symplectic automorphisms of $\mathbb{T}^6$. We trust this will be useful as a source of interesting examples.

Consider the standard torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ as the factor group of the abelian group $\mathbb{R}^n$ with respect to its discrete subgroup $\mathbb{Z}^n$ of integer vectors. Denote by $p : \mathbb{R}^n \to \mathbb{T}^n$ the related covering map, which is also a group homomorphism. The standard coordinates in the space $\mathbb{R}^n$ will be denoted by $x = (x_1, \dots, x_n)$. Let A be a unimodular matrix with integer entries. Since the linear mapping $L_A : x \to Ax$ maps the subgroup $\mathbb{Z}^n$ onto itself, such a matrix generates a diffeomorphism f_A of the torus $\mathbb{T}^n$ called an automorphism of the torus [1,2,10]. Topological properties of such maps are the classical object of research (see, for example, [1,10,18,26]). Because this toral automorphism also preserves the standard volume element $dx_1 \wedge \dots \wedge dx_n$ on the torus carried over from $\mathbb{R}^n$, its ergodic properties have also been the subject of research [5,12,28]. The following classical theorem of Halmos holds for automorphisms of a torus [12].

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Theorem 1.1 *(Halmos). A continuous automorphism f of a compact abelian group G is ergodic (and mixing) if and only if the induced automorphism on the character group G^* has no finite orbits.*

In the case of the abelian group $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, this theorem is equivalent to the statement that the automorphism L_A is ergodic if and only if none of the eigenvalues of the matrix A are roots of unity [12]. This implies, in particular, that Anosov automorphisms are ergodic, since all eigenvalues of A are outside of the unit circle.

In fact, the following assertions are equivalent [19]:

- the automorphism f_A is ergodic with respect to Lebesgue measure;
- the set of periodic points of f_A coincides with the set of points in $\mathbb{T}^n$ with rational coordinates;
- none of the eigenvalues of the matrix A is a root of unity;
- the matrix A has at least one eigenvalue of absolute value greater than one and has no eigenvector with all rational coordinates;
- all orbits of the dual map $A^* : \mathbb{Z}^n \to \mathbb{Z}^n$, besides the zero orbit, are infinite.

One more result about toral automorphisms is due to Bowen [5] and allows one to calculate its topological entropy.

Theorem 1.2 *(Bowen). If $L_A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map generated by a unimodular integer matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$, then*

$$h_d(f_A) = \sum_{|\lambda_i|>1} \log|\lambda_i|.$$

When the dimension of the torus is even, $n = 2m$, then we can introduce the standard symplectic structure on $\mathbb{T}^{2m}$ using coordinates in $\mathbb{R}^{2m} : \Omega = dx_1 \wedge dx_{m+1} + \dots + dx_m \wedge dx_{2m}$ and consider symplectic automorphisms of the torus which preserve this symplectic structure. A symplectic automorphism f_A is then defined by a symplectic matrix A with integer entries. Such matrices satisfy the identity $A^\top J_0 A = J_0$, where the skew-symmetric matrix J_0 has the form

$$J_0 = \begin{pmatrix} O & E \\ -E & O \end{pmatrix} \tag{1.1}$$

where E is $m \times m$ identity matrix. The identity $A^\top J_0 A = J_0$ implies the product of two symplectic matrices and the inverse matrix of a symplectic matrix be symplectic, i.e. symplectic matrices form a subgroup of $\mathrm{GL}(n,\mathbb{R})$, which is denoted $\mathrm{Sp}(n,\mathbb{R})$. This is one of the standard matrix Lie groups [9].

Another (non-standard) symplectic structure on $\mathbb{T}^{2m}$ can also be defined as follows. Choose a real non-degenerate skew-symmetric $2m \times 2m$ matrix J: $J^\top = -J$. Then we have a bilinear 2-form $[x, y] = (Jx, y)$ on $\mathbb{R}^{2m}$, where $(\cdot,\cdot)$ is the standard coordinate inner product. This form is called sometimes the skew inner product [3]. Then a linear map $S : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is called symplectic, if for all $x, y \in \mathbb{R}^{2m}$ the equality $[Sx, Sy] = [x, y]$ holds. Using the representation of the skew inner product via J and properties of the inner product, we obtain the following identity for the matrix S of the symplectic map: $S^\top J S = J$. This construction allows one to define some symplectic structures on the torus.

Remark 1.3. This way of constructing symplectic structures on the torus $\mathbb{T}^{2m}$ by means of its covering space $\mathbb{R}^{2m}$ is not unique. One may define a symplectic structure on $\mathbb{T}^{2m}$ choosing any symplectic 2-form on $\mathbb{R}^{2m}$ being invariant w.r.t. the translations by the group $\mathbb{Z}^{2m}$ and projecting this form via $p : \mathbb{R}^{2m} \to \mathbb{Z}^{2m}$. Symplectic structures on $\mathbb{T}^{2m}$ defined by means of the constant non-degenerate skew-symmetric matrices J can be called homogeneous. Due to the standard linear Darboux theorem [3], all homogeneous symplectic structures on $\mathbb{R}^{2m}$ are equivalent. However, these forms become different when lowered them onto the torus $\mathbb{T}^{2m}$. Simplest discriminating invariant is the volume of the torus w.r.t. the cube of the symplectic 2-form. Also, invariants are integrals over 2-dimensional sub-tori being basic in the homology group $H_2(\mathbb{T}^4,\mathbb{Z})$.[1]

The unimodularity of S follows from its symplecticity. Then a symplectic 2-form on the torus is given and the map S defines a symplectic automorphism of the torus with respect to this symplectic form. For example, let B be any non-degenerate integer matrix. Having the standard skew inner product $(J_0 x, y)$ in $\mathbb{R}^{2m}$ with J_0 as (1.1), we can define the new skew inner product $[x, y] = (JBx, By) = (B^\top JBx, y)$. Since the matrix $B^\top JB$ has integer entries and is non-degenerate and skew-symmetric, this new skew inner product generates a symplectic 2-form on the torus. Later on, we study the case $n = 6$, i.e. $\mathbb{T}^6$.

Let $P(x) = \lambda^6 + a\lambda^5 + b\lambda^4 + c\lambda^3 + b\lambda^2 + a\lambda + 1$ be an integer self-reciprocal polynomial being irreducible over the field $\mathbb{Q}$. The companion matrix [17] of this polynomial P is of the form

[1] We are thankful to Prof. A. Bolsinov for the explanation us this point.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -a & -b & -c & -b & -a \end{pmatrix}$$

This matrix is unimodular ($\det A = 1$) but not always symplectic with respect to the standard symplectic 2-form $[x, y] = (J_0 x, y)$ for $x, y \in \mathbb{R}^6$ with (1.1) as J_0. Let us show, however, that A is symplectic with respect to the nonstandard symplectic structure on $\mathbb{R}^6$ defined by $[x, y] = (Jx, y)$ for $x, y \in \mathbb{R}^6$ with some integer non-degenerate skew-symmetric matrix J. Rewrite the matrix identity $A^\top JA = J$ (with an unknown J) as $A^\top J - JA^{-1} = 0$. A solution J of this matrix equation is

$$J = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & a & a+1 & a+1 & 1 \\ -1 & -a & 0 & a+b-c & a+1 & 1 \\ -1 & -a-1 & -a-b+c & 0 & a & 1 \\ -1 & -a-1 & -a-1 & -a & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \end{pmatrix}. \tag{1.2}$$

Its determinant $\det J = (a + b - c - 2)^2$ is non-degenerate, if $a + b - c \neq 2$, then A is symplectic with respect to this nonstandard symplectic structure. This will be used later on.

Recall the notion of partially hyperbolic diffeomorphisms of a smooth manifold M, they were first introduced and studied in [7]. Here we use a modification of the definition like in [15]. Let L be a linear transformation between two normed linear spaces. The norm, respectively co-norm, of L are defined as

$$||L|| := \sup_{||v||=1} ||Lv||, \quad m(L) := \inf_{||v||=1} ||Lv||.$$

Definition 1.4. A diffeomorphism $f : M \to M$ is *partially hyperbolic*, if there is a continuous Df-invariant splitting $TM = E^u \oplus E^c \oplus E^s$ in which E^u and E^s are nontrivial sub-bundles and

$$m(D^u f) > ||D^c f|| \geq m(D^c f) > ||d^s f||, \quad m(D^u f) > 1 > ||D^s f||,$$

where $D^\sigma f$ is the restriction of Df to E^σ for $\sigma = s, c$ or u.

A well known class of partially hyperbolic diffeomorphisms are hyperbolic or Anosov diffeomorphisms introduced and studied by Anosov [1]. For such diffeomorphism $E^c = \{0\}$ for any $x \in M$ but both E^s, E^u are not empty. Hyperbolic automorphisms on the torus $\mathbb{T}^n$ are defined by an integer unimodular hyperbolic matrix A, its eigenvalues are all outside of the unit circle in $\mathbb{C}$. A particular example of such automorphism on $\mathbb{T}^2$ was proposed earlier by Thom, as an example of a (presumably) structurally stable diffeomorphism with countable set of periodic points. More details on Anosov diffeomorphisms can be found in [10].

In [24], we presented a classification of automorphisms f_A of the four-dimensional torus $\mathbb{T}^4$ generated by symplectic integer matrices $A \in \mathrm{Sp}(4, \mathbb{Z})$. These automorphisms can possess either a transitive unstable one-dimensional foliation or they are decomposable. In the first case two such automorphisms are (topologically) conjugate, if their matrices are integrally similar (conjugate in $\mathrm{M}_n(\mathbb{Z})$).[2] In the second case, a decomposable f_A is conjugate to the direct product of two 2-dimensional automorphisms acting on $\mathbb{T}^2 \times \mathbb{T}^2$, one of which is an Anosov automorphism of the 2-torus and another one is periodic on a 2-torus.

It is natural to extend these results to the higher-dimensional case. This is more involved problem because several different possibilities can occur: the dimensions of a center sub-bundle and stable/unstable sub-bundles can vary even for a torus of a fixed dimension. For instance, for symplectic automorphisms on $\mathbb{T}^6$, which we study in this paper, there are symplectic partially hyperbolic integer matrices with

- $\dim W^c = 4, \quad \dim W^s,\ W^u = 1;$
- $\dim W^c = 2, \quad \dim W^s,\ W^u = 2.$

A symplectic integer matrix generates a partially hyperbolic automorphism of the torus, if its eigenvalues are both outside the unit circle and on the unit circle. The topological classification of such automorphisms is determined in the first turn by the topology of a foliation generated by unstable (stable) leaves of the automorphism, since this foliation is invariant with respect to the action of f, and also by the action of f on the local center submanifold and its extension. Hence, the structure

[2] In [24] it was erroneously asserted that two such automorphisms are topologically conjugate, if they have the same characteristic polynomial.

of stable and unstable foliations has to be investigated in the first turn for the classification problem. Recall how stable, unstable and center foliation are generated for the case of symplectic automorphisms of $\mathbb{T}^6$. A symplectic 6×6 matrix A has a decomposition of its spectrum into three parts: $Sp(A)=\sigma_s\cup\sigma_c\cup\sigma_u$ where eigenvalues in σ_s lie within the unit circle, of σ_c on the unit circle and those in σ_u are out of the unit circle. We assume below that all eigenvalues are simple. Therefore in the first case above the projection of the subspace $W^u\subset\mathbb{R}^6$ onto the torus $\mathbb{T}^6$ gives an embedded line. The subspace W^u is also a one-parametric subgroup of $\mathbb{R}^6$, hence its shifts by vectors of $\mathbb{R}^6$ gives other affine lines representing other classes of the factor-group $\mathbb{R}^6/W^u$. Their projections on the torus give unstable foliation of the automorphism f_A, its leaves are embedded infinite lines. Similar construction gives the stable foliation and the center foliation. More details on the structure of these foliations will be presented in Sec. 2.

For the second case the subspace W^u is two-dimensional and this commutative subgroup has two generators. Here there are two different cases: 1) σ_u consists of two simple real eigenvalues $\lambda_{1,2}$, $|\lambda_i|>1$, with the related independent real eigenlines; 2) σ_u consists of two complex conjugate eigenvalues λ,λ^* $|\lambda|>1$, and W^u is the invariant subspace w.r.t. the action of L_A, as generators of this subgroup one can choose real and imaginary parts of the complex eigenvector of the matrix A.

The projection of W^u onto the torus gives an embedded plane (it is the unstable manifold of the fixed point $\hat{O}$), other classes of $\mathbb{R}^6/W^u$ are affine 2-planes, their projections give the unstable foliation of f_A. Analogously we get the stable foliation and the center foliation. More details on the structure of these foliations will be presented in Sec. 3.

The automorphism f_A is an isomorphism of the abelian group $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$. Every isomorphism of the topological group $\mathbb{T}^n$ in standard angular coordinates is given by an integer unimodular matrix. The topological classification of automorphisms of the group $\mathbb{T}^n$ is determined by the similarity of the related matrices, due to Arov's theorem [4].

Theorem 1.5 *(Arov). Two automorphisms T and P of a compact abelian group G are topologically conjugate if and only if they are isomorphic, that is, there is an isomorphism $Q:G\to G$ such that $Q\circ T=P\circ Q$.*

To check that two integer matrices are similar via an unimodular integer matrix is not an easy problem, the discussion is postponed till the last section. For the case of the group $\mathbb{T}^n$ the necessity to be unimodular for a conjugating matrix leads to the possibility that two integer matrices can be similar over $\mathbb{Q}$ but not over $\mathbb{Z}$. This holds true even for the case of Anosov automorphisms on the 2-torus. For instance, consider two automorphisms of $\mathbb{T}^2$ generated by matrices

$$A=\begin{pmatrix}4&3\\5&4\end{pmatrix}\quad\text{and}\quad C=\begin{pmatrix}0&1\\-1&8\end{pmatrix},$$

where C is the companion matrix of the characteristic polynomial $\chi(\lambda)=\lambda^2-8\lambda+1$ of A. Two matrices A,C are rationally similar $TA=CT$ (since their minimal and characteristic polynomials are equal) but not integrally similar. Clearing a common denominator of the entries of a conjugating rational matrix allows us to get the set of integer matrices

$$P=\begin{pmatrix}m&n\\4m+5n&3m+4n\end{pmatrix},\ m,n\in\mathbb{Z},$$

for which $\det P=3m^2-5n^2\neq 0$. Since determinant of this matrix is not equal to ±1, it generates not an isomorphism but a monomorphism of the group $h:\mathbb{Z}^2\to\mathbb{Z}^2$ with the image being some subgroup G of $\mathbb{Z}^2$ with two integer vectors as generators. The parallelogram formed by these generators is the fundamental domain with the area $|\det P|$ for the factor-group $\mathbb{R}^2/G$ being the torus. The number of integer points in the fundamental domain is the same $|\det P|$. Topologically, this means that topological semi-conjugacy holds: the diagram

$$\begin{array}{ccc}\mathbb{T}^2 & \xrightarrow{f_A} & \mathbb{T}^2\\ {\scriptstyle P}\downarrow & & {\scriptstyle P}\downarrow\\ \mathbb{T}^2 & \xrightarrow{f_C} & \mathbb{T}^2\end{array},$$

is commutative but P is not an isomorphism of the torus $\mathbb{T}^2$: it is a $|\det P|$-fold covering of $\mathbb{T}^2$. A general assertion in this case is as follows

Proposition 1.6. *Suppose A be an integer real unimodular matrix with simple eigenvalues. Then this matrix is rationally similar to the companion matrix C of its characteristic polynomial. Automorphisms f_A, f_C of the torus $\mathbb{T}^n$ generated by these matrices are semi-conjugate, i.e. there is an integer non-degenerate matrix T such that the relation $TA=CT$ holds and $f_T f_A=f_C f_T$. The mapping f_T is the k-fold covering map $f_T:\mathbb{T}^n\to\mathbb{T}^n$ with k equal to the determinant $|\det T|$.*

Below we present automorphisms which have a transitive unstable (and stable) foliation, i.e. all its leaves are dense in $\mathbb{T}^6$. Then a natural question arises: suppose the unstable foliation of an automorphism f_A is transitive, whether the automorphism f_A itself is transitive, that is, is there a dense orbit for f_A? In this direction the following assertion is valid.

Proposition 1.7. *An automorphism f_A with the transitive unstable foliation is transitive as a diffeomorphism of $\mathbb{T}^6$.*

Proof. Because a leaf through the fixed point $\hat{O}$ is transitive, then all leaves are transitive, since they are obtained by the shifts on the group $\mathbb{T}^6$. In order to prove the transitivity of f_A, we need for any two open sets U, V in $\mathbb{T}^6$ to find some $m \in \mathbb{Z}$ such that $f_A^m(U) \cap V \neq \emptyset$. Since A has no eigenvalues being roots of unity, the periodic orbits of f_A coincides with $\mathbb{Q}^n$ (the set of points in $\mathbb{T}^6$ with rational coordinates) and therefore they form a dense set. Hence there is a periodic point $s \in U$, let k be its period. The unstable leaf $W^u(s)$ is dense in $\mathbb{T}^6$ and so intersects V. Let $q \in V \cap W^u(s)$, then the sequence $f_A^{-kn}(q)$, $n \in \mathbb{N}$, tends to s, as $n \to \infty$, and $f_A^{-kN}(q) \in U$ for N large enough, so $f_A^{kN}(U) \cap V \neq \emptyset$, we can set $m = kN$. $\square$

Remark 1.8. It worth remarking that the proof works for any partially hyperbolic automorphism of $\mathbb{T}^n$, $n \geq 2$, with transitive unstable foliation.

The structure of the paper is as follows. In Section 2 we describe invariant foliations related to automorphisms of the first class, both transitive and decomposable. In Section 2.1 we show the existence of partially hyperbolic symplectic automorphisms with transitive unstable foliations. In Section 3 we describe the case of automorphisms with two-dimensional unstable (and stable) foliations. Classification problems are discussed in Section 4. The Addendum is devoted to the explanations of the interrelations between integral conjugacy in $\mathrm{M}_n(\mathbb{Z})$ and ideal classes.

2. Partially hyperbolic automorphisms with one-dimensional unstable foliations

Here we study symplectic partially hyperbolic automorphisms defined by symplectic matrices having two (simple) real eigenvalues λ, λ^{-1}, $|\lambda| > 1$, outside of the unit circle and two pairs of complex conjugate eigenvalues on the unit circle. The corresponding eigenspaces $l^s, l^u \subset \mathbb{R}^6$ and the center subspace W^c, when projecting on the torus and shifting them to any point on $\mathbb{T}^6$, give the required decomposition of the partial hyperbolicity definition.

Let us investigate a possible behavior of projections onto the torus of eigenspaces l^s, l^u. We assume later on that λ is positive, otherwise we may consider A^2. In the space $\mathbb{R}^6$ we have the following orbit structure for the linear map L_A generated by the matrix A. Recall that in the case of a symplectic linear map with a 2-elliptic fixed point O (having two pairs of simple eigenvalues on the unit circle and a pair of reals), there is a four-dimensional center invariant subspace W^c corresponding to two pairs of eigenvalues on the unit circle $\nu_1, \nu_2, \bar{\nu_1}, \bar{\nu_2}$, $\nu_1 = \exp[i\alpha_1]$, $\nu_2 = \exp[i\alpha_2]$. If $\alpha_i/2\pi \neq \mathbb{Q}, i = 1, 2$ (this is equivalent to $\nu_i^n \neq 1$ for any $n \in \mathbb{Z}$), the subspace W^c is foliated into invariant 2-tori everywhere except two invariant 2-planes corresponding to pairs $\nu_1, \bar{\nu_1}$ and $\nu_2, \bar{\nu_2}$ which are foliated into closed invariant curves. This follows from the fact that the restriction of the map L_A on W^c has two quadratic positive definite invariant functions (integrals) whose joint levels are invariant tori. The restriction of the map L_A to any such torus is conjugated to the shift $\theta_1 = \theta + \alpha_1 \pmod{2\pi}$, $\theta_2 = \theta + \alpha_2 \pmod{2\pi}$. If $m\alpha_1 + n\alpha_2 \neq k$ for any integer (m, n, k), then the shift on the torus is transitive, but if there is the only integer triple (up to a factor) that gives the equality, then the torus itself is foliated into closed invariant curves. For the case if none ν_i is a root of the unity, these invariant curves are defined correctly and the shift on each such curve is transitive.

In addition to the indicated invariant stable/unstable lines and the center subspace, there are more two five-dimensional invariant subspaces in $\mathbb{R}^6$ spanned, respectively, by vectors from subspaces W^c and l^s (the center-stable 5-plane W^{cs}) and vectors from W^c and l^u (the center-unstable 5-plane W^{cu}). The factor-classes $\mathbb{R}^6/W^{cu}$ and $\mathbb{R}^6/W^{cs}$ define two invariant foliations in $\mathbb{R}^6$ into five-dimensional affine planes. Here, the invariance is understood in the following sense: the image with respect to L_A of a leaf of the foliation coincides with some (possibly another one) leaf of the same foliation. All orbits of L_A, not lying in the union $W^{cs} \cup W^{cu}$, go to infinity for both positive and negative iterations of L_A. In particular, the mapping L_A has no other five-dimensional invariant subspaces except those two W^{cs} and W^{cu}. If $\exp[i\alpha_1], \exp[i\alpha_2]$ are not roots of unity, then subspaces W^{cs}, W^{cu} are foliated into three-dimensional submanifolds $T^2 \times \mathbb{R}$ being stable (respectively, unstable) invariant manifolds of invariant tori on the center plane W^c.

The projection of the eigenlines l^s, l^u onto the torus can lead to different situations. To understand this, it is necessary to clarify what is a projection of a subspace in $\mathbb{R}^6$ onto the torus $\mathbb{T}^6 = \mathbb{R}^6/\mathbb{Z}^6$. Since $\mathbb{T}^6$ is a commutative Lie group, its tangent space at zero possesses the structure of the commutative Lie algebra, this is identified with $\mathbb{R}^6$ in the standard way. Then the projection $p : \mathbb{R}^6 \to \mathbb{T}^6$ is the exponential map of the Lie algebra onto the Lie group. One-dimensional subspace l^u (or l^s) coincides with the one-parameter subgroup $t\gamma^u$ generated by the vector γ^u, and its projection is the image under the exponential mapping of the algebra into the group. This subgroup is included into orbits of the constant vector field on $\mathbb{T}^6$ invariant under group shifts. In angular coordinates θ on $\mathbb{T}^6$ induced by coordinates in $\mathbb{R}^6$ we get the vector field $\dot{\theta} = \gamma^u$. Any trajectory of this vector field coincides with a leaf of the unstable (respectively, stable) invariant foliation on the torus which consists of projections of all straight lines corresponding to the factor-classes $\mathbb{R}^6/l^u$ (respectively, $\mathbb{R}^6/l^s$). Its orbit structure depends on the number of rationally independent integer solutions of the equation $(m, \gamma^u) = 0$, $m \in \mathbb{Z}^6$. This number can take values $0, 1, 2, 3, 4, 5$. In the first case, as is known, the related orbits of the constant vector field are transitive in $\mathbb{T}^6$ (it is a partial case of the Kronecker theorem, see, for instance, [6] and Section 3).

The one-parameter subgroup in $\mathbb{T}^6$ generated by γ^u is an invariant subset with respect to the automorphism f_A (this is simultaneously the strong unstable curve of the fixed point $\hat{O} = p(O)$). Therefore, its closure is also an invariant subset, here it is a smooth invariant torus of some dimension in $\mathbb{T}^6$ [6]. As was said above, the dimension of this torus depends on the number of integer linear independent relations of the form $(m, \gamma^u) = 0$. If there are such relations, vector γ^u is called resonant.

In the case if the only integer relation $(m, \gamma^u) = 0$ exists, the linear 5-dimensional subspace in $\mathbb{R}^6$, defined by the equation $(m, x) = 0$, is projected onto a 5-torus in $\mathbb{T}^6$, and the straight-line spanned by vector γ^u does not contain points of its integer sub-lattice of this subspace. Therefore, this line is projected into a transitively immersed line on this 5-torus.

Similarly, for the case of two integer relations, the corresponding subspace is four-dimensional, it is projected onto a 4-torus in $\mathbb{T}^6$. The straight-line, spanned by vector γ^u, is projected into a transitively immersed line in this 4-torus, etc. In the case, when five independent integer relations exist, the related straight line necessarily passes through an integer point in $\mathbb{R}^6$ and therefore it is projected onto a simple closed curve in $\mathbb{T}^6$. Obviously, the eigen-line l^u corresponding to $\lambda > 1$ (l^s for $\lambda < 1$) can not intersect the integer lattice $\mathbb{Z}^6$. If, on the contrary, this is the case, the projection on the torus of this straight-line is a closed invariant curve for the map f_A and the restriction of this map to this curve gives a diffeomorphism on the circle with the only unstable (respectively, stable) fixed point that is impossible.

Thus, the closure of any trajectory of the constant vector field on the torus is a smooth torus of some dimension, its dimension is equal to $6 - r$ where r is the number of rationally independent linear relations for the components of the vector $\gamma^u : (m, \gamma^u) = 0, m \in \mathbb{Z}^6$. In fact, there are more restrictions.

Proposition 2.1. *The closure of the unstable leaf of the fixed point $\hat{O}$ for f_A is either the whole $\mathbb{T}^6$, or a four-dimensional torus, or a two-dimensional torus, in the first case the leaf is transitive.*

Proof. We should exclude the possible cases of 5- and 3-dimensional tori as the closure. Assume the closure of the unstable manifold of a fixed point $\hat{O}$ on $\mathbb{T}^6$ to form a 5-dimensional torus T^5. This means the eigenvector γ^u to satisfy the only integer relation $(m, \gamma^u) = 0, \quad m \in \mathbb{Z}^6$. Consider 5-dimensional hyperplane in $\mathbb{R}^6$ defined by co-vector m: $(m, x) = 0$, T^5 is the projection of this hyperplane.

Let us show in this case the stable one-dimensional manifold of the fixed point $\hat{O}$ be transversal to T^5. Torus T^5 contains the fixed point $\hat{O}$ of f_A. Since the torus is smooth invariant manifold with respect to the map f_A (as the closure of its invariant set), the tangent space to the torus at the fixed point is invariant with respect to differential $Df_A = L_A$. So, T^5 is the projection of the hyperplane in $\mathbb{R}^6$ passing through the fixed point O of the map L_A. This plane is invariant with respect to L_A, but linear partially hyperbolic map L_A has no other invariant 5-planes through O except for W^{cs}, W^{cu}. Only the second of them contains the eigen-line spanned by the vector γ^u. So, the torus T^5 is a projection of the center-unstable plane W^{cu}. But then the stable eigenvector γ_s is transversal to this plane, therefore the projection of the stable eigen-line onto $\mathbb{T}^6$ is a smooth curve being transversal to the torus T^5 at the point $\hat{O}$. Since T^5 is a smooth embedded submanifold in $\mathbb{T}^6$, there is a neighborhood V of the point $\hat{O}$ such that all points in V, lying on the stable curve $W^s(\hat{O})$, do not belong to T^5.

Due to transversality of T^5 and $W^s(\hat{O})$ at $\hat{O}$, they must intersect each other at more than one point in $\mathbb{T}^6$ (in fact, at infinitely many points), since the eigen-line, being the covering of this curve, intersect in $\mathbb{R}^6$ infinitely many 5-dimensional planes being integer shifts of W^{cu}. Stable invariant curve is given as $t\gamma^s$ (*modd* 1), it contains a point z other than $\hat{O}$ which belongs to T^5. Since z belongs to $W^s(\hat{O})$, its forward iterations $f_A^n(z)$ must lie on the stable curve near the point $\hat{O}$ for positive n large enough. As a consequence, these points do not belong to T^5 for such iterations. On the other hand, the torus T^5 is invariant with respect to f_A, therefore, all iterations of z should lie on it. This contradiction proves that the closure of an unstable curve cannot be a 5-torus.

Now suppose the closure of the unstable manifold of a fixed point $\hat{O}$ on $\mathbb{T}^6$ be a 3-dimensional torus T^3. This means the eigenvector γ^u to satisfy three rationally independent relations $(n, \gamma^u) = 0, (m, \gamma^u) = 0, (l, \gamma^u) = 0$, where integer vectors $n, m, l \in \mathbb{Z}^6$ are linearly independent over $\mathbb{Q}$.

The torus T^3 is a smooth invariant manifold for f_A containing a fixed point $\hat{O}$. The tangent plane to T^3 at $\hat{O}$ is the invariant 3-plane w.r.t. differential Df_A. In the covering space $\mathbb{R}^6$ the pre-image of this 3-plane w.r.t. the projection p, passing through the origin O, is an invariant 3-plane for L_A. There are only four such invariant 3-planes: 1) W_1^*, that is spanned by subspaces l^u (for λ) and 2-plane corresponding to eigenvalues $\exp[\pm i\alpha_1]$; 2) W_2^*, generated by l^u and 2-planes corresponding to eigenvalues $\exp[\pm i\alpha_2]$; 3) W_3^*, generated by l^s (for λ^{-1}) and 2-plane corresponding to eigenvalues $\exp[\pm i\alpha_1]$; 4) W_4^*, generated by l^s and 2-plane corresponding to eigenvalues $\exp[\pm i\alpha_2]$. Only the first two of them contain γ^u, so W_1^*, W_2^* can be these planes. Thus we get two possible cases of splitting the space $\mathbb{R}^6$ into two 3-dimensional invariant subspaces: 1) $W_1^* \oplus W_4^*$; 2) $W_2^* \oplus W_3^*$. Since these cases are similar, it is sufficient to examine the first case only. The p-projection of the plane W_1^* to $\mathbb{T}^6$ is the 3-dimensional torus T^3, therefore W_1^* contains three linear independent integer vectors. The lattice generated by these three integer vectors does not intersect the eigen-line l^u, because its projection is everywhere dense on T^3. This implies the restriction of f_A to this invariant torus be a map which is generated by the restriction of L_A onto invariant 3-plane W_1^*.

In the space $\mathbb{R}^6$ the linear map L_A is generated by the integer unimodular matrix A. Since W_1^* is its invariant subspace, we consider the linear transformation L_B, induced by L_A on this invariant subspace. It is given by 3×3 matrix B in the basis made up of three integer vectors. Thus, B has rational coefficients and the characteristic polynomial Q of the matrix B will also have rational coefficients. The space $\mathbb{R}^6$ splits into the direct sum of subspaces $\mathbb{R}^6 = W_1^* \oplus W_4^*$. Therefore the following statement holds [25]: *If the space splits into a direct sum of subspaces invariant with respect to a linear transformation L_A, then the characteristic polynomial of L_A is equal to the product of the characteristic polynomials induced by the transformation L_A in invariant subspaces*. Thus, we obtain the characteristic polynomial P of the transformation L_A be equal to the product of two polynomials of the third degree $P = QS$, where Q corresponds to the induced transformation in the invariant subspace W_1^*, and S, respectively, to W_4^*. Since P has integer coefficients and Q has rational coefficients, then, by Euclidean algorithm, we get S also to have rational coefficients.

On the other hand, the eigenvalues of the matrix B are $\lambda, \exp[\pm i\alpha_1]$. Then the characteristic polynomial Q is represented in the form

$$Q = (x-\lambda)(x-\exp[i\alpha_1])(x-\exp[-i\alpha_1]) = (x-\lambda)(x^2+2x\cos\alpha_1+1) =$$
$$x^3+(2\cos\alpha_1-\lambda)x^2+(1-2\lambda\cos\alpha_1)x-\lambda.$$

We conclude λ be a rational number, therefore the eigenvector γ^u can be chosen as an integer vector, but the straight-line l^u cannot pass through an integer point. □

The cases of two and four rationally independent relations for partially hyperbolic matrix are indeed possible. The related examples are presented at the end of this Section. Notice that in these cases the characteristic polynomial of such a matrix is the product of two monic polynomials, one of the second degree and the other of the fourth degree with integer coefficients.

2.1. Examples of automorphisms on $\mathbb{T}^6$ with transitive unstable 1-dimensional foliations

In order to construct examples of symplectic partially hyperbolic automorphisms of a torus with one-dimensional stable foliation having different dynamic properties, we need to find a matrix in $\mathrm{Sp}(6,\mathbb{Z})$ which has two pairs complex conjugate eigenvalues and two real eigenvalues λ and λ^{-1}, $|\lambda| > 1$. Moreover, we would like to obtain an automorphism of the torus whose one-dimensional unstable foliation is transitive. To find such a matrix, we start, following [20], with an irreducible over rational numbers cubic polynomial $P(z)$ with integer coefficients having three roots, one greater than 2 and two others being less than 2 in modulus. We make a change of variable $z = x + x^{-1}$ in this polynomial and obtain a self-reciprocal polynomial of the sixth degree that serves as the characteristic polynomial of the matrix with the desired properties. This polynomial is also irreducible over $\mathbb{Q}$.

For example, let us start with the cubic polynomial $P(z) = z^3 - 2z^2 - z + 1$ that gives the polynomial $Q(x) = x^6 - 2x^5 + 2x^4 - 3x^3 + 2x^2 - 2x + 1$ which is irreducible over the field $\mathbb{Q}$. Its companion matrix of $Q(x)$ is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & -2 & 3 & -2 & 2 \end{pmatrix}.$$

This matrix possesses the required properties for its eigenvalues. However, A is not symplectic with respect to the standard symplectic 2-form $[x, y] = (Ix, y)$ on $\mathbb{R}^6$ (see (1.1)): $A^\top IA \neq I$. Acting as before, we construct a skew-symmetric integer non-degenerate matrix J which provides this structure defined as $[x, y] = (Jx, y)$ for $x, y \in \mathbb{R}^6$. The matrix J is as follows

$$J = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \\ -1 & 2 & 0 & 3 & -1 & 1 \\ -1 & 1 & -3 & 0 & -2 & 1 \\ -1 & 1 & 1 & 2 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 \end{pmatrix}.$$

Real eigenvalues of matrix A are

$$\lambda \approx 1.6355, \quad \lambda^{-1} \approx 0.6114.$$

The eigenvector corresponding to eigenvalue λ is $\gamma^u = (1, \lambda, \lambda^2, \lambda^3, \lambda^4, \lambda^5)^\top$. Factor-classes $\mathbb{R}^6/l^u$, generated by this eigenspace l^u, form a foliation into affine lines being invariant under L_A. These lines are projected onto $\mathbb{T}^6$ as trajectories of the vector field

$$\dot{x}_1 = 1,\ \dot{x}_2 = \lambda,\ \dot{x}_3 = \lambda^2,\ \dot{y}_1 = \lambda^3,\ \dot{y}_2 = \lambda^4\ \dot{y}_3 = \lambda^5.$$

To prove the orbit of this vector field be transitive in $\mathbb{T}^6$, we need to verify the vector γ^u be incommensurate, i.e. $(m, \gamma^u) \neq 0$ for any nonzero integer vector $m \in \mathbb{Z}^6$. The number λ is an algebraic number of the degree six being the root of the irreducible sixth degree polynomial with integers coefficients.

Recall a number $\xi \in \mathbb{C}$ is called an *algebraic number*, if it is a root of a nonzero polynomial with rational coefficients, and it is called *an algebraic integer*, if it is the root of a polynomial with integral coefficients and unitary leading coefficient (a monic polynomial). The *degree* of an algebraic integer ξ is the degree of the *minimal* polynomial of ξ (a monic polynomial of the lowest degree with integer coefficients and ξ as a root).

The polynomial $Q(x)$ above has degree six due to the following theorem.

Theorem 2.2. *Let ξ be an algebraic number with minimal polynomial $p(x)$. Then*

1. *the polynomial $p(x)$ is irreducible over $\mathbb{Q}$;*
2. *the polynomial $p(x)$ is unique;*
3. *if ξ is a root of a polynomial f over $\mathbb{Q}$, then f is divided by p.*

The last statement of the theorem implies that the number λ, being a root of the irreducible polynomial $Q(x)$ of degree six, is not a root of a polynomial of smaller degree with rational (integer) coefficients.

Now we can prove

Lemma 2.3. *The vector γ is incommensurate.*

Proof. Suppose, on the contrary, the vector γ be commensurate. Then an integer vector $(m_1, m_2, m_3, m_4, m_5, m_6)$ exists such that the equality $m_1 + m_2\lambda + m_3\lambda^2 + m_4\lambda^3 + m_5\lambda^4 + m_6\lambda^5 = 0$ holds, i.e. λ is a root of the polynomial P of the fifth (or lesser) degree with integer coefficients. So Q divides P and Q is reducible. $\square$

2.2. Decomposable automorphisms with one-dimensional unstable foliations

Here we present automorphisms with one-dimensional unstable and stable foliations which we call decomposable. They are characterized by the property: the closure of their unstable (stable) leaves are tori of a dimension lesser than six. It follows from Proposition 2.1 that in the decomposable case the closure of the unstable (stable) leaf is either a two-dimensional or four-dimensional torus.

The case, when the closure of the unstable (stable) leaf is a four-dimensional torus, is indeed possible. It is sufficient to choose, for example, a block-diagonal integer matrix S_1 composed of two integer blocks, one of which (4×4)-block has eigenvalues $\lambda, \lambda^{-1}, \exp[i\alpha_1], \exp[-i\alpha_1]$, $\lambda > 1$, that generates a partially hyperbolic transitive automorphism of $\mathbb{T}^4$, and the second (2×2)-block has two complex conjugate eigenvalues $\exp[i\alpha_2], \exp[-i\alpha_2]$ (then $\alpha_2/2\pi = 1/3,\ 1/4, 1/6$) [24]. As an example one can take the matrix

$$S_1 = \begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Its reducible characteristic polynomial $\chi_1(\lambda) = (\lambda^4 - 4\lambda^3 + \lambda^2 - 4\lambda + 1)(\lambda^2 - \lambda + 1)$ of the related symplectic matrix S_1 has the roots:

$$\lambda_{1,2} = 1 + \frac{\sqrt{5}}{2} \pm \frac{\sqrt{5+4\sqrt{5}}}{2},\ \lambda_{3,4} = 1 - \frac{\sqrt{5}}{2} \pm i\frac{\sqrt{4\sqrt{5}-5}}{2},\ \lambda_{5,6} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

The case, when the closure of the unstable (stable) leaf is a two-dimensional torus, can also be realized. Two different cases are possible here. In the first case, it is sufficient to choose, for example, a block-diagonal integer matrix S_2 composed of two integer blocks, one of which (2×2)-block generates an Anosov automorphism, and the second (4×4)-block has two pairs complex conjugate eigenvalues $\exp[i\alpha_1], \exp[-i\alpha_1], \exp[i\alpha_2], \exp[-i\alpha_2]$ on the unit circle. Since the characteristic polynomial of this (4×4)-matrix has integer coefficients and monic, its eigenvalues are in fact roots of unity [23]. An example is given by the matrix

$$S_2 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & 2 & 1 \\ 0 & 0 & -1 & -2 & 1 & 1 \\ 0 & 0 & -3 & -2 & 1 & 2 \\ 0 & 0 & 0 & -7 & 4 & 1 \end{pmatrix}. \tag{2.1}$$

The characteristic polynomial $\chi_2(\lambda) = (\lambda^2 - 3\lambda + 1)(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1)$ of the related symplectic matrix S_2 is reducible with roots:

$$\lambda_{1,2} = -\frac{1}{4} - \frac{\sqrt{5}}{4} \pm i\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}},\ \lambda_{3,4} = -\frac{1}{4} + \frac{\sqrt{5}}{4} \pm i\sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}},\ \lambda_{5,6} = \frac{3 \pm \sqrt{5}}{2},$$

the eigenvalues on the unit circle are roots of unity of degree 10.

In the second case, we choose a block-diagonal integer matrix S_3 composed of three integer blocks, one of which (2×2)-block generates an Anosov automorphism, and the second and third (2×2)-blocks have pairs complex conjugate eigenvalues $\exp[i\alpha_1], \exp[-i\alpha_1]$ and $\exp[i\alpha_2], \exp[-i\alpha_2]$ on the unit circle, respectively (then $\alpha_i/2\pi = 1/3,\ 1/4, 1/6, \quad i = 1,2$ [24]) provided that $\alpha_1/2\pi \neq \alpha_2/2\pi$. This case is almost the same as previous one, the difference is that the roots of unity here are only of degrees $\{3, 4, 5\}$. This leads to other structures of a periodic transformation on the center manifold $\mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2$ of the fixed point $\hat{O}$.

An example is provided by the matrix

$$S_3 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The characteristic polynomial $\chi_3(\lambda) = (\lambda^2 - 3\lambda + 1)(\lambda^2 + \lambda + 1)(\lambda^2 - \lambda + 1)$ of the related symplectic matrix S_3 has roots:

$$\lambda_{1,2} = \frac{3}{2} \pm \frac{\sqrt{5}}{2},\ \lambda_{3,4} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2},\ \lambda_{5,6} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

3. Automorphisms with 2-dimensional unstable foliation

In this section we study the case, when the dimension of W^u (and W^s) equals two. This corresponds to those symplectic (6×6)-matrices which have two eigenvalues λ_1, λ_2 outside of the unit disk in $\mathbb{C}$, two eigenvalues $\lambda_1^{-1}, \lambda_2^{-1}$ inside of this disk and two complex conjugate eigenvalues on the unit circle. In its turn, these λ_1, λ_2 can be either two different real ones or a pair of complex conjugate numbers. Respectively, W^u is spanned either by two independent real eigenvectors γ_1^u, γ_2^u of real eigenvalues, or it is an invariant 2-dimensional real subspace corresponding to complex conjugate eigenvalues. In the latter case the invariant subspace is generated by real and complex parts γ_r^u, γ_i^u of the complex eigenvector $\gamma_r^u + i\gamma_i^u$, these are two independent real vectors.

Projection by p of W^u on $\mathbb{T}^6$ generates related unstable foliation in $\mathbb{T}^6$. In order to determine how leaves of this foliation behave in $\mathbb{T}^6$, we observe that this foliation is formed by orbits of the action of the group $\mathbb{R}^2$ on $\mathbb{T}^6$ generated by two commuting constant vector fields on $\mathbb{T}^6$ given as $\gamma_1^u \cdot \partial/\partial\theta$ and $\gamma_2^u \cdot \partial/\partial\theta$, $\theta = (\theta_1, \ldots, \theta_6)$. The related orbits are given as

$$\theta(t_1, t_2) = t_1\gamma_1^u + t_2\gamma_2^u + \theta_0 \text{ (modd 1)}.$$

In particular, for $\theta_0 = \hat{O}$ we have the orbit through the fixed point $\hat{O}$ of the automorphism. The action and the shift on the torus commute.

Recall the Kronecker theorem [22] (see also [6], chapter VII-1 and [11]) and its corollary. In their formulations we use the common notations: $(\cdot,\cdot)$ for the standard inner product in $\mathbb{R}^n$ and $||\cdot||$ for the maximum norm of a vector in $\mathbb{R}^n$.

Theorem 3.1 *(Kronecker). Let m vectors $\mathbf{a_i}$, $1 \le i \le m$, and a vector $\mathbf{b}$ in $\mathbb{R}^n$ be given. In order for any $\varepsilon > 0$ there exist m numbers $t_i \in \mathbb{R}$ and an integer vector $\mathbf{p} \in \mathbb{Z}^\mathbf{n}$ such that*

$$||\sum_{i=1}^{m} t_i\mathbf{a_i} - \mathbf{p} - \mathbf{b}|| \le \varepsilon,$$

it is necessary and sufficient that for any integer vector $\mathbf{r} \in \mathbb{Z}^\mathbf{n}$, such that all m equalities $(\mathbf{r}, \mathbf{a_i}) = \mathbf{0}$ hold, the relation $(\mathbf{r}, \mathbf{b}) = \mathbf{0}$ also holds.

Let us briefly discuss the geometric sense of this theorem. To be closer to the case under consideration, we assume vectors $\mathbf{a_i}$ be independent, in particular, $m \le n$. The linear combination $\sum_i t_i\mathbf{a_i} \in \mathbb{R}^\mathbf{n}$ with real t_i, $1 \le i \le m$, belongs to the linear subspace of $\mathbb{R}^n$ generated by the vectors $\{\mathbf{a_1}, \ldots, \mathbf{a_m}\}$. When m-tuple $t = (t_1, \ldots, t_m)$ runs all $\mathbb{R}^m$, we get the whole subspace. Adding (or subtracting) vectors $\mathbf{p} \in \mathbb{Z}^\mathbf{n}$ shifts this subspace to any point of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. When $\mathbf{p}$ run all

$\mathbb{Z}^n$, this gives some set $\mathcal{L}_a$ in $\mathbb{R}^n$ and the question is: if vectors of $\mathcal{L}_a$ approximates a given vector $\mathbf{b} \in \mathbb{R}^{\mathbf{n}}$ with an arbitrary preciseness? Or, in other words, when $\mathbf{b}$ belongs to the closure $\overline{\mathcal{L}_a}$? The Kronecker's theorem gives the exhausting answer to this question: if there is an integer vector $\mathbf{r} \in \mathbb{Z}^{\mathbf{n}}$ such that all $(\mathbf{r}, \mathbf{a_i}) = \mathbf{0}$, $1 \le i \le m$, then $\mathbf{b}$ has to belong to the plane $(\mathbf{r}, \mathbf{x}) = \mathbf{0}$. If there are several independent such vectors $\mathbf{r}$, then $\mathbf{b}$ has to belong to all of them.

Observe that the equality $(\mathbf{r}, \mathbf{x}) = \mathbf{0}$, $\mathbf{x} \in \mathbb{R}^{\mathbf{6}}$, with integer vector $\mathbf{r}$ defines a torus $T^5 \subset \mathbb{T}^6$. So, any such equality defines a torus. If there are two such equalities with independent $\mathbf{r_1}$ and $\mathbf{r_2}$ over $\mathbb{Q}$, then two 5-dimensional planes in $\mathbb{R}^6$ intersect each other transversely along a 4-dimensional plane, therefore related 5-dimensional tori in $\mathbb{T}^6$ intersect each other transversely and provide a 4-dimensional torus. Orbits of the action of $\mathbb{R}^2$ lie in this 4-torus.

If such integer vector $\mathbf{r}$ exists, we can call the equality as a *resonance relation* for the collection $\{\mathbf{a_1}, \dots, \mathbf{a_m}\}$. The number of linearly independent (over $\mathbb{Q}$) such relations can be called the degree of the degeneration for the resonance. So, if none such relations exist, one can take any $\mathbf{b} \in \mathbb{R}^{\mathbf{n}}$ and the following statement is valid

Corollary 3.2. *In order for any $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$ and any $\varepsilon > 0$ there exist m real numbers $t_1, \dots, t_m$ and an integer vector $\mathbf{p}$ such that*

$$\left|\left|\sum_{i=1}^{m} t_i \mathbf{a_i} - \mathbf{p} - \mathbf{x}\right|\right| \le \varepsilon,$$

it is necessary and sufficient that there is no nonzero integer vector $\mathbf{r} \in \mathbb{Z}^{\mathbf{n}}$ such that $(\mathbf{r}, \mathbf{a_i}) = \mathbf{0}$ for all i.

This corollary simultaneously gives the criterion when the leaves of the unstable foliation are transitive in $\mathbb{T}^6$. Indeed, suppose the conditions of the corollary to hold for two independent vectors $\mathbf{a_1}, \mathbf{a_2}$ in W^u. Then the subspace in $\mathbb{R}^6$ generated by these two vectors, along with all its shifts by $\mathbb{Z}^6$, is dense in $\mathbb{R}^6$ and hence $p(W^u)$ is dense when projecting onto $\mathbb{T}^6$.

For the classification problem we have to prove an analog of the Proposition 2.1. As we know, the leaves of the unstable foliation are simultaneously orbits of the action of the group $\mathbb{R}^2$ generated by two commuting vector fields corresponding to either eigenvectors of two real eigenvalues greater than one or vectors being the real and imaginary parts of the complex eigenvector of the eigenvalue greater in modulus than one.

Proposition 3.3. *The closure of the 2-dimensional unstable leaf of the fixed point $\hat{O}$ for f_A is either the whole $\mathbb{T}^6$, or a four-dimensional torus, in the first case the leaf is transitive.*

Proof. Let us consider first the case of complex conjugate complex greater in modulus than one. In this case, the closure of the leaf of the fixed point $\hat{O}$ for f_A cannot be either a three-dimensional torus or a five-dimensional torus, since the linear map L_A in $\mathbb{R}^6$ has neither five-dimensional nor three-dimensional invariant subspaces.

In the case of two real eigenvalues greater than one, the proof of the fact that the closure of the fixed point $\hat{O}$ for f_A cannot be either a three-dimensional torus or a five-dimensional torus is similar to the proof of Proposition 2.1. □

3.1. Automorphisms on $\mathbb{T}^6$ with transitive two-dimensional unstable foliations

These two subsections are devoted to constructing examples of both transitive and decomposable automorphisms. We start again with a degree three monic real polynomial with integer coefficients a, b, c being irreducible over field $\mathbb{Q}$: $P = z^3 + az^2 + bz + c$. Two different cases are considered. In the first case this polynomial should have one real root z_1 lesser than two in absolute value and a pair of different real roots z_2, z_3 being greater than two in absolute values. In the second case this polynomial should have one real root z_1 lesser than two in absolute value and a pair of complex conjugate roots z_2, z_2^* with $|z_2| > 2$.

Having such degree three polynomial, we make the change $z = x + x^{-1}$ and multiply the function obtained at x^3, therefore we get the sixth degree integer polynomial

$$Q = x^6 + ax^5 + (3+b)x^4 + (2a+c)x^3 + (3+b)x^2 + ax + 1.$$

Its roots are two complex conjugate numbers $x_{1,2}$ being the roots of the quadratic polynomial $x^2 - z_1 x + 1$, they belong to the unit circle in $\mathbb{C}$. Four other roots for the first case are four real roots $x_3, x_3^{-1}, x_4, x_4^{-1}$ being roots of the quadratic equations $x^2 - z_2 x + 1 = 0$ and $x^2 - z_3 x + 1 = 0$. Since $|z_i| > 2$, the absolute values of x_j, $j = 3, 4$, are greater than one. As an example, we take the irreducible third order polynomial $P = z^3 - 2z^2 - 8z + 1$ [20]. It has two real roots of absolute value larger than two, and a real root of absolute value lesser than two. Its self-reciprocal polynomial is

$$Q(x) = x^6 - 2x^5 - 5x^4 - 3x^3 - 5x^2 - 2x + 1, \tag{3.1}$$

which is irreducible over $\mathbb{Q}$, has four real roots λ_1, λ_2, $\lambda_1^{-1}, \lambda_2^{-1}$, $|\lambda_{1,2}| > 1$, and a pair of complex conjugate roots of absolute value 1. All these numbers are algebraic and Q is their minimal polynomial. Its companion matrix A has two-dimensional subspaces W^u, W^s and generates a partially hyperbolic action on $\mathbb{T}^6$ with two-dimensional unstable and stable foliations, and two-dimensional center foliation. Matrix A can be made symplectic w.r.t. the nonstandard symplectic structure generated skew symmetric nondegenerate matrix J in (1.2) with $a=-2, b=-5, c=-3$, then $\det J = 36$.

For the second case complex root $z_2 = u+iv$, $uv \neq 0$, can be represented as $u=(\rho+\rho^{-1})\cos\alpha$, $u=(\rho-\rho^{-1})\sin\alpha$, where $x_{3,4}=\rho\exp[\pm i\alpha]$, $\rho>1$, $x_{3,4}^{-1}=\rho^{-1}\exp[\mp i\alpha]$. From these equalities we get the system for finding ρ,α at the given u,v:

$$\frac{u^2}{\rho^2+\rho^{-2}+2}+\frac{u^2}{\rho^2+\rho^{-2}-2}=1,\ \tan\alpha=\frac{u(\rho^2+1)}{v(\rho^2-1)}.$$

Introducing variable $s=\rho^2+\rho^{-2}>2$, we come to the quadratic equation for s

$$s^2-(u^2+v^2)s-4+2(u^2-v^2)=0.$$

The monic polynomial $P=z^3+az^2+bz+c$ has a unique real root of absolute value less than 2, two other roots have to be complex conjugate. As an example, one can take the polynomial $P=z^3-3z^2+6z-1$ with $z_1\approx 0.182$, $z_{2,3}\approx 1.409\pm 1.871i$ or $P=z^3-z-1$ with $z_1\approx 1.325, z_{2,3}\approx -0.662\pm 0.562i$. This gives two reciprocal sixth degree integer irreducible polynomial

$$x^6-3x^5+9x^4-7x^3+9x^2-3x+1 \text{ and } x^6+2x^4-x^3+2x^2+1 \tag{3.2}$$

Consider now the automorphism in $\mathbb{R}^6$ generated by the companion matrix of the polynomial (3.1). Its eigenvectors of real roots have the form

$$\mathbf{f}_\lambda=(1,\lambda,\lambda^2,\lambda^3,\lambda^4,\lambda^5)^\top,\ \lambda=\lambda_{1,2}.$$

Thus, no integers vectors $\mathbf{n}$ can exist such that $(\mathbf{n},\mathbf{f}_{\lambda_\mathbf{k}})=\mathbf{0}$, $k=1,2$, otherwise, at least one of λ_k is a root of a monic integer polynomial of degree five or lesser.

For the case of the polynomial (3.2) we have for its companion matrix complex conjugate eigenvalues λ,λ^* with $|\lambda|>1$ with two complex conjugate eigenvectors

$$\mathbf{f}_\lambda=(1,\lambda,\lambda^2,\lambda^3,\lambda^4,\lambda^5)^\top,\ \mathbf{f}^*_\lambda.$$

Their real and imaginary parts provide two real independent vectors $\mathbf{g_r},\mathbf{g_i}$, $\mathbf{f}_\lambda=\mathbf{g_r}+\mathbf{i}\mathbf{g_i}$. Suppose there is an integer nonzero vector $\mathbf{n}\in\mathbb{Z}^\mathbf{6}$ such that $(\mathbf{n},\mathbf{g_r})=\mathbf{0}$ and $(\mathbf{n},\mathbf{g_i})=\mathbf{0}$. This implies the equality

$$(\mathbf{n},\mathbf{g_r}+\mathbf{i}\mathbf{g_i})=\mathbf{n_1}+\mathbf{n_2}\lambda+\mathbf{n_3}\lambda^\mathbf{2}+\mathbf{n_4}\lambda^\mathbf{3}+\mathbf{n_5}\lambda^\mathbf{4}+\mathbf{n_6}\lambda^\mathbf{5}=\mathbf{0},$$

i.e. λ is the algebraic number of the degree five or lesser. We come to the contradiction. So, in both cases the leaves of the unstable foliation are dense in $\mathbb{T}^6$.

One more example of an automorphism on $\mathbb{T}^6$ with transitive unstable two-dimensional foliation is given as follows. Take a block-diagonal integer matrix S_4 composed of two integer blocks, one of which (4×4)-block has eigenvalues $\lambda,\lambda^{-1},\exp[i\alpha_1]$, $\exp[-i\alpha_1]$, $\lambda>1$, that generates a partially hyperbolic transitive automorphism on $\mathbb{T}^4$, and the second (2×2)-blocks that generates an Anosov automorphism on $\mathbb{T}^2$

$$S_4=\begin{pmatrix} 2 & 1 & 2 & 1 & 0 & 0\\ 1 & 1 & 1 & 1 & 0 & 0\\ 0 & 2 & 1 & 1 & 0 & 0\\ 2 & -2 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 0 & 1 & 1\end{pmatrix}.$$

Characteristic polynomial $\chi_4(\lambda)=(\lambda^4-4\lambda^3+\lambda^2-4\lambda+1)(\lambda^2-3\lambda+1)$ of the related symplectic matrix S_4 is reducible with roots

$$\lambda_{1,2}=1+\frac{\sqrt{5}}{2}\pm\frac{\sqrt{5+4\sqrt{5}}}{2},\ \lambda_{3,4}=1-\frac{\sqrt{5}}{2}\pm i\frac{\sqrt{4\sqrt{5}-5}}{2},\ \lambda_{5,6}=\frac{3}{2}\pm\frac{\sqrt{5}}{2}.$$

Here the two-dimensional unstable foliation is generated by real eigenvectors of real eigenvalues λ_1 and λ_5. Let us verify that the Corollary 3.2 holds here. Indeed, eigenvectors of these eigenvalues have the form: $\mathbf{e_1}=(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{0},\mathbf{0})^\top$ and $\mathbf{e_2}=(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{f},\mathbf{g})$. Assume, on the contrary, there is an integer vector $\mathbf{r}$ such that both equalities $(\mathbf{r},\mathbf{e_1})=\mathbf{0}$ and $(\mathbf{r},\mathbf{e_2})=\mathbf{0}$ hold. Then the second of them gives $r_5 f+r_6 g=0$, that says the ratio f/g is rational. But it is impossible for the Anosov automorphism on $\mathbb{T}^2$ [20]. Therefore, the unstable foliation is transitive.

3.2. Decomposable automorphisms with 2-dimensional unstable foliations

Here we present automorphisms with 2-dimensional unstable and stable foliations which we call decomposable. They are characterized by the property the closure of their unstable (stable) leaves be tori of a dimension lesser than six. As was proved in the Proposition 3.3, the closure of a leaf of the unstable foliation can be a four-dimensional torus. Let us construct corresponding examples.

Three different cases are possible here. In the first case, we select a block-diagonal integer matrix S_5 composed of two integer blocks, one of which (4×4)-block with eigenvalues $\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}$, $\lambda_i > 1$, $\quad i = 1, 2$, generates an Anosov automorphism on $\mathbb{T}^4$ with transitive two-dimensional unstable foliation, and the second (2×2)-block has two complex conjugate eigenvalues $\exp[i\alpha], \exp[-i\alpha]$ (then $\alpha/2\pi = 1/3,\ 1/4, 1/6$ [24]). For instance, the following matrix suits

$$S_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -4 & 12 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The characteristic polynomial $\chi_5(\lambda) = (\lambda^4 + 4\lambda^3 - 12\lambda^2 + 4\lambda + 1)(\lambda^2 - \lambda + 1)$ of the related symplectic matrix S_5 is reducible with roots

$$\lambda_1 = -1 + \tfrac{3\sqrt{2}}{2} + \tfrac{\sqrt{6}\sqrt{3-2\sqrt{2}}}{2} \approx 1.629,\ \lambda_2 = -1 + \tfrac{3\sqrt{2}}{2} - \tfrac{\sqrt{6}\sqrt{3-2\sqrt{2}}}{2} \approx 0.614,$$

$$\lambda_3 = -1 - \tfrac{3\sqrt{2}}{2} + \tfrac{\sqrt{6}\sqrt{3+2\sqrt{2}}}{2} \approx -0.165,\ \lambda_4 = -1 - \tfrac{3\sqrt{2}}{2} - \tfrac{\sqrt{6}\sqrt{3+2\sqrt{2}}}{2} \approx -6.078,$$

$$\lambda_{5,6} = \tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2}.$$

This matrix has a pair of negative eigenvalues $\lambda_{3,4}$. If one wish to get the case when all eigenvalues outside of the unit circle are positive, one needs to take the square of this (4×4)-block.

In the second case, it is sufficient to choose, for example, a block-diagonal integer matrix S_6 composed of two integer blocks, one of which (4×4)-block has two pairs complex conjugate eigenvalues $\rho\exp[i\alpha], \rho\exp[-i\alpha]$, $\rho^{-1}\exp[i\alpha]$, $\rho^{-1}\exp[-i\alpha]$ outside the unit circle (it is also an Anosov automorphism on $\mathbb{T}^4$ with two-dimensional unstable and stable foliations), and another (2×2)-block has a pair complex conjugate eigenvalues $\exp[i\alpha], \exp[-i\alpha]$ on the unit circle (then $\alpha_/2\pi = 1/3, 1/4, 1/6$ [24]).

$$S_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 5 & -9 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The characteristic polynomial $\chi_6(\lambda) = (\lambda^4 - 5\lambda^3 + 9\lambda^2 - 5\lambda + 1)(\lambda^2 - \lambda + 1)$ of the related symplectic matrix S_6 is reducible with roots

$$\lambda_{1,2} = \tfrac{5}{4} - \tfrac{i\sqrt{3}}{4} \pm \tfrac{\sqrt{2}\sqrt{3-i5\sqrt{3}}}{4} \approx 0.378 \pm i0.188,$$

$$\lambda_{3,4} = \tfrac{5}{4} + \tfrac{i\sqrt{3}}{4} \pm \tfrac{\sqrt{2}\sqrt{3+i5\sqrt{3}}}{4} \approx 2.122 \pm i1.054,$$

$$\lambda_{5,6} = \tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2}.$$

The first multiplier is irreducible polynomial that is deduced from the second order polynomial $z^2 - 5z + 7$ by means of the change $z = \lambda + 1/\lambda$. Both roots $(5 \pm i\sqrt{3})/2$ are greater than two in modulus.

The third case is generated by a block-diagonal integer matrix S_7 composed of three integer blocks, two of which (2×2)-blocks generate Anosov automorphisms on $\mathbb{T}^2$, and the third block has a pair complex conjugate eigenvalues $\exp[i\alpha], \exp[-i\alpha]$ on the unit circle (then $\alpha/2\pi = 1/3, 1/4, 1/6$, [24]):

$$S_7 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 3 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The characteristic polynomial $\chi_7(\lambda) = (\lambda^2 - 3\lambda + 1)(\lambda^2 - 7\lambda + 1)(\lambda^2 - \lambda + 1)$ of the related symplectic matrix S_7 is reducible with roots

$$\lambda_{1,2} = \frac{3}{2} \pm \frac{\sqrt{5}}{2},\ \lambda_{3,4} = \frac{7}{2} \pm \frac{3\sqrt{5}}{2},\ \lambda_{5,6} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

This case is similar, in a sense, to the first and second ones, since the related unstable foliation for f_{S_7} is two-dimensional and transitive on $\mathbb{T}^2 \times \mathbb{T}^2$. Indeed, related eigenvectors of the real eigenvalues, that are greater than one, obey the conditions of the Corollary 3.2. Nevertheless, we distinguish this case since such automorphisms do not topologically conjugate to any automorphism of the first or second cases.

4. On classification of partially hyperbolic automorphisms

At the study of partially hyperbolic symplectic automorphisms a natural question arises: when two such automorphisms are topologically conjugate. Recall that two homeomorphisms f_1, f_2 of a metric space M are called topologically conjugate, if there is a homeomorphism $h : M \to M$ such that $h \circ f_1 = f_2 \circ h$.

Note, that classification of ergodic automorphisms of the torus from the measure theory point of view is given by their entropy [21], which is equal to the sum of logarithms of the absolute values for eigenvalues greater than the unity of the matrix A. This follows from the Ornstein isomorphism theorem [27] and the fact that every ergodic automorphism of a torus is Bernoulli one with respect to the Lebesgue measure [21].

The Arov's theorem (see above) provides a topological conjugation of two automorphisms f_A, f_B on $\mathbb{T}^n$. Any isomorphism of the group $\mathbb{T}^n$ in cyclic coordinates θ is given by an integer unimodular matrix. The existence of a conjugating homeomorphism $h : \mathbb{T}^6 \to \mathbb{T}^6$ for two automorphisms f_A, f_B, i.e. $h \circ f_A = f_B \circ h$, implies the relation $H \circ A = B \circ H$ in the fundamental group $\mathbb{Z}^6$ of the torus, where H is the linear isomorphism in $\mathbb{Z}^6$ generated by h. Thus, matrices A, B are integrally similar by H. Conversely, if two integer unimodular matrices A, B are integrally similar, i.e. there is an integer unimodular matrix H such that $HA = A'H$, then H induces the automorphism $h = f_H$ of the torus $\mathbb{T}^6$. The covering map for $h \circ f_A$ is $L_H L_A = L_{HA}$ and for $f_{A'} \circ h$ is $L_{A'} L_H = L_{A'H}$. Hence, $L_H L_A = L_{A'} L_H$ and the relation $h \circ f_A = f_{A'} \circ h$ holds. So, for classification of partially hyperbolic automorphisms on $\mathbb{T}^6$ the following theorem, being reformulation of the Arov's theorem, holds

Theorem 4.1. *Two symplectic partially hyperbolic automorphisms f_A, f_B on $\mathbb{T}^6$ are topologically conjugate if and only if their matrices A, B are integrally similar.*

Similar matrices A, B have the same integer characteristic polynomial. As we have seen, the constructions of proper examples of automorphisms are easier using companion matrices of the characteristic polynomials. Suppose now that there are two integer symplectic partially hyperbolic matrices A, B having the same characteristic polynomial. When two automorphisms f_A, f_B are topologically conjugate? We have seen on examples above that these two automorphisms can be not conjugated, since their matrices are not integrally similar, though rationally similar. So, we need to determine whether these two integer matrices are integrally similar, the related discussion we postpone till the Section 5.

In particular, as was shown above, there are two cases of automorphisms on $\mathbb{T}^6$ with a transitive unstable two-dimensional foliation. In the first case the characteristic polynomial is irreducible over $\mathbb{Z}$, but it is reducible over $\mathbb{Z}$ into the product of monic integer polynomials of a fourth and second order in the second case. Automorphisms of the first and second case lead to automorphisms being non conjugated, since they have different characteristic polynomials.

To distinguish automorphisms being not topologically conjugate, it is useful to have some simple features that guarantee this. The following statement is one of them.

Proposition 4.2. *Let f_A be a symplectic partially hyperbolic automorphism of a 6-dimensional torus $\mathbb{T}^6$ with 1-dimensional unstable/stable foliations and matrix A has simple eigenvalues. Then f_A has transitive one-dimensional unstable (stable) foliation if and only if the characteristic polynomial of the matrix A is irreducible over $\mathbb{Q}$.*

Proof. Let A have irreducible characteristic polynomial and assume the unstable foliation is not transitive. Then by Proposition 2.1 the closure of the unstable leaf of the fixed point $\hat{O}$ is a torus of the dimension four or two. In both case, as was proved in this Proposition, the characteristic polynomial is reducible, that is a contradiction.

Suppose now, this polynomial is reducible over $\mathbb{Q}$ into two real polynomials with rational coefficients. Then four cases are possible.

- Characteristic polynomial is decomposed into the product of two polynomials, the first degree one corresponding to the eigenvalue λ, and the fifth degree, corresponding to four eigenvalues lying on the unit circle and λ^{-1}. Then we conclude $\lambda \in \mathbb{Q}$. In this case there is an eigenvector γ for λ with rational coordinates that is impossible: multiplication at the least common denominator makes this vector integer and the projection $p(t\gamma)$ is a closed invariant curve with the unique unstable point. The same holds, if the linear multiplier corresponds to λ^{-1}.
- Characteristic polynomial is the product of two ones: of the second degree, corresponding to the eigenvalues of λ, λ^{-1}, and of the fourth degree, corresponding to four eigenvalues on the unit circle. The characteristic polynomial has the form $P = x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_2x^2 + a_1x + 1 = (x^2 + ax + 1)(x^4 + bx^3 + cx^2 + bx + 1)$ with rational a, b, c. Then λ is the root of the second degree polynomial $x^2 + ax + 1$ with rational a. Since eigenvalues of A are all simple, the minimal and characteristic polynomials of matrix A coincide and A is conjugated by means of a rational non-degenerate matrix T to the companion matrix B of the characteristic polynomial of the matrix A, $A = T^{-1}BT$. The eigenvector of the matrix B, corresponding to eigenvalue λ, is $\rho = (1, \lambda, \lambda^2, \lambda^3, \lambda^4, \lambda^5)^\top$. Because λ is the root of the second degree polynomial, four rational linearly independent relations for the corresponding vector ρ exist: $(m_i, \rho) = 0$, where $i = 1, 2, 3, 4$, namely $m_1 = (1, a, 1, 0, 0, 0)^\top$, $m_2 = (0, 1, a, 1, 0, 0)^\top$, $m_3 = (0, 0, 1, a, 1, 0)^\top$, $m_4 = (0, 0, 0, 1, a, 1)^\top$. If $a = p/q$ with co-prime integers p, q, then we get four linear independent integer vectors.
  We have $TA = BT$ with rational non-degenerate matrix T, hence multiplication at the common denominator of entries of T gives the integer non-degenerate matrix T_1 satisfying the relation $T_1A = BT_1$. The vector $T_1\gamma^u$ is the eigen-vector of B with eigenvalues λ: $BT_1\gamma^u = T_1A\gamma^u = \lambda T_1\gamma^u$. Since λ is a simple eigenvalue, we have $T_1\gamma^u = c\rho$. Now we get four independent integer vectors $T_1^*m_i$, $i = 1 - 4$, which satisfy to relations $(T_1^*m_i, \gamma^u) = (m_i, T_1\gamma^u) = c(m_i, \rho) = 0$. Thus, we obtain that the closure of an unstable manifold is a torus of smaller dimension, namely two-dimensional. This contradicts to the transitivity of f_A on $\mathbb{T}^6$.
- Suppose that the characteristic polynomial is the product of two polynomials of the third degree corresponding to the eigenvalues $\lambda, \exp[\pm i\alpha_k]$ and $\lambda^{-1}, \exp[\pm i\alpha_j]$, where $i, j = 1, 2$ and $i \neq j$. Then the polynomial of the third degree with rational coefficients has the form

$$(x-\lambda)(x-\exp[i\alpha_1])(x-\exp[-i\alpha_1]) = (x-\lambda)(x^2+2x\cos\alpha_1+1) =$$
$$x^3 + (2\cos\alpha_1 - \lambda)x^2 + (1 - 2\lambda\cos\alpha_1)x - \lambda.$$

  We conclude λ be a rational number. But then there is again an eigenvector with rational coefficient, hence a contradiction.
- Suppose that the characteristic polynomial is decomposed into the product of two polynomials, one of which is a polynomial of the fourth degree corresponding to the eigenvalues of $\lambda, \lambda^{-1}, \exp[\pm i\alpha_k]$, and of the second degree corresponding to the eigenvalues of $\exp[\pm i\alpha_j]$, $j \neq k$. The characteristic polynomial has the form $P = x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_2x^2 + a_1x + 1 = (x^2 + ax + 1)(x^4 + b_1x^3 + b_2x^2 + b_1x + 1)$. Since $a, b_l \in \mathbb{Q}, \quad l = 1, 2$, then $b_l = \frac{p_l}{q_l}$, where p_l, q_l are relatively prime integers. Denote by N the common denominator for $b_l, l = 1, 2$. Thus λ is a root of a polynomial of the fourth degree with rational coefficients. Reasoning as in the item 2, we conclude that there are two integer relations for the eigenvector $\gamma^u : (m_i, \gamma^u) = 0$, where $i = 1, 2$, namely, $m_1 = N(1, b_1, b_2, b_1, 1, 0)^\top, m_2 = N(0, 0, 0, 0, b_1, 1)^\top$. Thus, we obtain that the closure of an unstable manifold is a torus of smaller dimension, namely four-dimensional. This contradicts transitivity.

So, the proposition has been proved. □

Now we prove

Proposition 4.3. *If f_A is a symplectic partially hyperbolic automorphism of $\mathbb{T}^6$ with the transitive unstable one-dimensional foliation, then its stable and center foliations have also dense leaves.*

Proof. Let γ^u, γ^s be eigenvectors for λ, λ^{-1} and the projection of the line $t\gamma^u$ into $\mathbb{T}^6$ is dense. Then the characteristic polynomial of A is irreducible. Suppose the closure of the projection of $t\gamma^s$ in $\mathbb{T}^6$ is either a two-dimensional torus or four-dimensional torus. It implies the characteristic polynomial of A is reducible, that is a contradiction.

Now we show that the central foliation is also dense in $\mathbb{T}^6$, provided that the unstable foliation is dense. The characteristic polynomial of the matrix A is irreducible. Suppose the closure of the projection of the center subspace W^s is not dense and, being a torus, forms a four-dimensional torus in $\mathbb{T}^6$. This torus is a smooth invariant manifold for f_A (as the closure of an invariant set) containing a fixed point $\hat{O}$. The tangent plane to T^4 at $\hat{O}$ is the invariant 4-plane w.r.t. differential Df_A. In the covering space $\mathbb{R}^6$ the p-pre-image of this 4-plane passing through the origin O, is the invariant 4-plane w.r.t. L_A. There is the only such 4-plane W^c for which the restriction of L_A onto this invariant plane has eigenvalues $\exp[\pm i\alpha_1], \exp[\pm i\alpha_2]$. Because the projection of this plane to $\mathbb{T}^6$ is the invariant 4-dimensional torus T^4, there exist four linear independent integer vectors in W^c.

Since W^c is an invariant subspace, we consider the induced linear transformation L_B of the transformation L_A to this invariant subspace, which is given by 4×4 matrix B in the basis made up of integer vectors. Thus, matrix B has rational coefficients. Hence, the characteristic polynomial Q of the matrix B also has rational coefficients. The space $\mathbb{R}^6$ splits into a direct sum of two invariant w.r.t. linear transformation L_A subspaces. Then the characteristic polynomial is equal to the product of the characteristic polynomials induced by the transformation L_A in invariant subspaces. So, the characteristic polynomial the transformation L_A is reducible, that is a contradiction.

Now suppose the closure of $p(W^c)$ be a five-dimensional torus T^5 in $\mathbb{T}^6$. This torus is also a smooth invariant manifold for f_A containing a fixed point $\hat{O}$. The tangent plane to T^5 at $\hat{O}$ is the invariant 5-plane w.r.t. differential $Df_A = L_A$. In the covering space $\mathbb{R}^6$ the pre-image of this 5-plane w.r.t. the projection p, that passes through the origin O, is the invariant 5-plane w.r.t. L_A. There are only two such invariant 5-planes W^{cs}, W^{cu}, for which the restriction of L_A onto the invariant plane possesses eigenvalues $\exp[\pm i\alpha_1], \exp[\pm i\alpha_2]$. But the plane W^{cu} contains γ^u and we get that the closure of the projection of $t\gamma^u$ belongs to T^5, but this cannot be the case, since it contradicts the condition of transitivity of $t\gamma^u$. Similarly, reasoning about W^{cs}, we get a contradiction, since we showed above that the closure of γ^s is dense on $\mathbb{T}^6$. □

Now we turn to the classification of automorphisms f_A of $\mathbb{T}^6$ which we called decomposable. In fact, the Arov's theorem works in this case as well. Nevertheless, one can say about the conjugating automorphism more. Recall that the automorphism f_A and the automorphism f_C generated by the companion matrix C of its characteristic polynomial are connected by the semi-conjugation 1.6.

We consider first those f_A which have one-dimensional unstable foliation. Let f_A be generated by matrix A (symplectic w.r.t. a standard or nonstandard symplectic structure) with one-dimensional unstable foliation and being decomposable. This means that the closure of any unstable leaf is a torus of a dimension lesser than six. Then the following statement holds.

Proposition 4.4. *If f_A has one-dimensional unstable foliation being decomposable, then the integer characteristic polynomial $\chi(\lambda)$ of A is reducible over $\mathbb{Q}$.*

Proof. Since the closure of the infinite line $p(t\gamma^u)$ forms a torus of a dimension lesser than six (four or two), therefore the eigenvector γ^u of the matrix A with the eigenvalue $\lambda_u > 1$ is resonant, i.e. there are nontrivial integer relations $(m, \gamma^u) = 0$ with nonzero vector $m \in \mathbb{Z}$. Suppose, on the contrary, that the polynomial $\chi(\lambda)$ of A is irreducible over $\mathbb{Q}$. This implies λ_u to be an algebraic number of degree six, not lesser. Since all eigenvalues of A are simple and A is an integral matrix, it is rationally similar to its companion matrix C: $AT = TC$ with non-degenerate rational T. As we know, the eigenvector ρ of matrix C corresponding to real eigenvalue λ_u is of the form $(1, \lambda_u, \dots, \lambda_u^5)^\top$: $C\rho = \lambda_u\rho$. Hence the vector $T\rho$ is an eigenvector of the matrix A: $AT\rho = TC\rho = \lambda_u T\rho$. Since λ_u is the simple eigenvalue, we get $T\rho = c\gamma^u$. Thus, we get for the integer vector $m \in \mathbb{Z}^6$: $(m, T\rho) = c(m, \gamma^u) = 0$. So, for the integer non-degenerate vector T^*m we have $(T^*m, \rho) = 0$. Thus, there is a nonzero polynomial of the degree five or lesser with integer coefficients such that λ_u is its root. This contradicts to that λ_u is an algebraic number of degree six. So, the characteristic polynomial $\chi(\lambda)$ is reducible over $\mathbb{Q}$. □

The polynomial $\chi(\lambda)$ of the integer symplectic matrix A can be decomposed into real polynomials with rational coefficients by only two ways: 1) $\chi(\lambda) = P_4(\lambda)Q_2(\lambda)$ with irreducible P_4 of the degree four or 2) $\chi(\lambda) = P_2(\lambda)Q_2(\lambda)R_2(\lambda)$ with irreducible polynomials of the second degree. Otherwise, if to assume $\chi(\lambda)$ be a product of two rational polynomials of the odd degrees, then, as before, we conclude that there is a rational root and therefore, an integer eigenvector, this leads to the existence of an invariant circle in $\mathbb{T}^6$ with the only unstable fixed point, i.e. to a contradiction.

A distribution of roots among these polynomials can be as follows: $\lambda, \lambda^{-1}, \exp[\pm i\alpha_1]$ for P_4 and $\exp[\pm i\alpha_2]$ for Q_2, or $\exp[\pm i\alpha_1], \exp[\pm i\alpha_2]$ for P_4 and λ, λ^{-1} for Q_2, for the first case, for the second case the eigenvalues are pair-wisely λ, λ^{-1}, $\exp[\pm i\alpha_1]$, $\exp[\pm i\alpha_2]$ for every polynomial.

Keeping this in mind, we have the following classification theorems.

Theorem 4.5. *Let a symplectic automorphism f_A of $\mathbb{T}^6$ with one-dimensional unstable foliation be decomposable and the closure of the unstable (stable) leaf is a four-dimensional torus. Then the matrix A is rationally similar to a block-diagonal matrix (H, I) with blocks 4×4 for H and 2×2 for I being companion matrices of its reducible characteristic polynomials P_4 and Q_2, respectively. Two such block-diagonal matrices $(H, I), (H', I')$ generate topologically conjugate decomposable automorphisms on $\mathbb{T}^4 \times \mathbb{T}^2$ if and only if partially hyperbolic integer matrices H and H' are integrally similar and periodic matrices I, I' have the same period k from $k \in \{3, 4, 6\}$.*

Theorem 4.6. *Let f_B be a decomposable symplectic automorphism $\mathbb{T}^6$ with one-dimensional unstable leaves such that the closure of the unstable (stable) leaf is a two-dimensional torus. Then B is rationally similar to a block-diagonal matrix (H, I) whose blocks are either companion 2×2 and 4×4 matrices H, I for the related polynomials-multipliers P_2, Q_4, or three 2×2 matrices (H, I_1, I_2) for related polynomials-multipliers P_2, Q_2, R_2.*

In the first case, two such block-diagonal matrices $(H, I), (H', I')$ generate topologically conjugate decomposable automorphisms on $\mathbb{T}^2 \times \mathbb{T}^4$ if and only if their hyperbolic integer matrices H and H' are integrally similar and periodic matrices I, I' have the same period k.

In the second case, two collections of block-diagonal matrices $(H, I_1, I_2), (H', I'_1, I'_2)$ *generate topologically conjugate decomposable automorphisms on* $\mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$ *if and only if hyperbolic integer matrices* H *and* H' *are integrally similar and periodic matrices* I_1, I'_1 *have the same period* k_1 *and* I_2, I'_2 *have the same period* k_2, $k_i = 3, 4, 6, \quad i = 1, 2$ *provided that* $k_1 \neq k_2$.

Remark 4.7. When dealing with the first case of the two-dimensional torus as a result of the closure, we get a companion matrix I for $\mathbb{T}^4$ that should have an integer characteristic self-reciprocal polynomial with two pairs of simple complex roots on the unit circle. As follows from the Kronecker's theorem [22], in this case all four roots are roots of unity. If $\lambda^4 + a\lambda^3 + b\lambda^2 + a\lambda + 1$ be such polynomial with integers a, b, then $a = -2(\cos\alpha_1 + \cos\alpha_2)$, $b = 2 + 4\cos\alpha_1\cos\alpha_2$. Therefore, $|a| \le 4$, $|b - 2| \le 4$, hence there are 81 possible polynomials. Among them there are reducible, for instance, $\lambda^4 + \lambda^3 + 2\lambda^2 + \lambda + 1 = (\lambda^2 + 1)(\lambda^2 + \lambda + 1)$, $\lambda^4 + 2\lambda^3 + 3\lambda^2 + 2\lambda + 1 = (\lambda^2 + \lambda + 1)^2$, but irreducible polynomials with roots on the unit circle also exist. For example, the polynomial $\lambda^4 - \lambda^3 + \lambda^2 - \lambda + 1$ is cyclotomic and has roots on the unit circle

$$\frac{\sqrt{5} + 1 \pm i\sqrt{10 - 2\sqrt{5}}}{4} = \cos(\pi/5) \pm i\sin(\pi/5),$$
$$\frac{1 - \sqrt{5} \pm i\sqrt{10 + 2\sqrt{5}}}{4} = \cos(3\pi/5) \pm i\sin(3\pi/5).$$

All they are roots of unity of degree ten. Of course, not all of 81 polynomials have only roots on the unit circle, some of them have roots out of the unit circle: these inequalities on a, b are only necessary but not sufficient.

But if all eigenvalues of the matrix I are roots of the unity, some degree of the matrix has only unity as its eigenvalues. Because all eigenvalues of I are simple, the unity is the multiple eigenvalue with four 1-dimensional Jordan boxes, so it is the identity matrix and I is periodic matrix. It is a funny problem to investigate the orbit structure of the related automorphisms f_I on $\mathbb{T}^4$.

Similar to decomposable symplectic automorphisms on $\mathbb{T}^6$ with one-dimensional unstable foliation, the classification theorems for decomposable automorphisms on $\mathbb{T}^6$ with two-dimensional unstable foliation hold. Nevertheless, there is a difference here: an analog of the Proposition 4.2 is not valid. Indeed, consider the matrix S_4 in the Section 3. Its unstable two-dimensional foliation is transitive, but its characteristic polynomial is reducible over $\mathbb{Z}$. We classify different classes of the topological conjugation using another approach.

We know from the corollary of the Halmos theorem (see above) that the automorphism f_A on $\mathbb{T}^6$ is ergodic, if and only if, when among eigenvalues of A no roots of unity exist. All eigenvalues of A are algebraic integer numbers of some degree being, due to symplecticity, an even integer, six or lesser. Among them only two complex conjugate ones lie on the unit circle. So, remaining four eigenvalues outside of the unit circle are algebraic numbers of some degree, they form either a complex quadruple or they break into two different real pairs $\lambda_1, \lambda_1^{-1}$ and $\lambda_2, \lambda_2^{-1}$, $\lambda_{1,2} > 1$, $\lambda_1 \neq \lambda_2$. Thus, we come to the following distributions of eigenvalues in dependence of degrees of algebraic eigenvalues for decomposable automorphisms.

- the set of eigenvalues breaks into two subsets: fourth order algebraic numbers $\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}$ and second order numbers $\exp[\pm i\alpha]$ ($\chi(\lambda)$ is reducible over $\mathbb{Z}$ into the product of monic integer polynomials of the fourth and second order);
- the set of eigenvalues breaks into two subsets: fourth order algebraic numbers composing the quadruple $\rho\exp[i\alpha]$, $\rho\exp[-i\alpha]$, $\rho^{-1}\exp[i\alpha]$, $\rho^{-1}\exp[-i\alpha]$, $\rho > 1$, and second order numbers $\exp[\pm i\alpha]$ ($\chi(\lambda)$ is reducible over $\mathbb{Z}$ into the product of monic integer polynomials of the fourth and second order);
- there are three pairs of eigenvalues being algebraic numbers of the second orders: $\lambda_1, \lambda_1^{-1}$, $\lambda_2, \lambda_2^{-1}$, $\exp[\pm i\alpha]$ ($\chi(\lambda)$ is reducible over $\mathbb{Z}$ into the product of three monic quadratic integer polynomials).

In these cases stable and unstable two-dimensional foliations have leaves whose closures are four-dimensional tori (decomposable automorphisms), the center foliation is a two-dimensional torus with a periodic automorphism of the periods $\{3, 4, 6\}$ [24] as the restriction of f_A.

Keeping this in mind, we have the following classification theorems.

Theorem 4.8. *Let* f_A *be a decomposable symplectic automorphism of* $\mathbb{T}^6$ *for which the closure of any two-dimensional unstable (stable) leaf is a four-dimensional torus, and the matrix* A *has either a quadruple of complex conjugate eigenvalues outside the unit circle, or two different pairs of real eigenvalues outside the unit circle. Then* A *is rationally similar to a block-diagonal matrix* S *whose blocks are* 4×4 *and* 2×2 *companion matrices* (H, I) *of factors for the characteristic polynomial. Two such block-diagonal matrices* $(H, I), (H', I')$ *generate topologically conjugate decomposable automorphisms on* $\mathbb{T}^4 \times \mathbb{T}^2$ *if and only if hyperbolic integer matrices* H *and* H' *are integrally similar and periodic matrices* I, I' *have the same period* k *from* $k \in \{3, 4, 6\}$.

Theorem 4.9. *Let* f_B *be a decomposable symplectic automorphism* $\mathbb{T}^6$ *such that the closure of the two-dimensional unstable (stable) leaf is a four-dimensional torus, and its characteristic polynomial is the product of three monic polynomials of the second order. Then* B *is rationally similar to a block-diagonal matrix* S *whose blocks are* 2×2, 2×2 *and* 2×2 *factor matrices* (H_1, H_2, I) *for the characteristic polynomial. Two such block-diagonal matrices* $(H_1, H_2, I), (H'_1, H'_2, I')$ *generate topologically conjugate decomposable* *automorphisms on* $\mathbb{T}^6$*, if and only if hyperbolic integer matrices* H_1 *and* H_1' *are integrally similar, hyperbolic integer matrices* H_2 *and* H_2' *are integrally similar and integer matrices* I, I' *have the same period* k, $k = 3, 4, 6$.

5. Addendum

In this section we construct examples of four- and six-dimensional linear symplectic integer matrices being not integrally conjugate to its companion matrix. This demonstrates that the conditions of integral similarity in the classification theorems are essential. Recall that a symplectic matrix S in the standard linear symplectic space with coordinates $(x, y) = (x_1, x_2, y_1, y_2)$ satisfy the relation $S^\top I S = I$. If S is represented in the form of four (2×2) integer matrices A, B, C, D

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then these matrices obey the relations

$$A^\top C = C^\top A,\ B^\top D = D^\top B,\ A^\top D - C^\top B = E, \tag{5.1}$$

with E being the unitary (2×2) integer matrix. Let us denote entries of these four matrices as follows

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},\ C = \begin{pmatrix} x & y \\ z & w \end{pmatrix},\ D = \begin{pmatrix} \xi & \eta \\ \zeta & \omega \end{pmatrix}.$$

Then relations in (5.1) are reduced to the six equalities

$$\begin{aligned} & bx - ay + dz - cw = 0,\ \beta\xi - \alpha\eta + \delta\zeta - \gamma\omega = 0, \\ & a\xi + c\zeta - \alpha x - \gamma z = 1,\ a\eta + c\omega - \beta x - \delta z = 0, \\ & b\xi + d\zeta - \alpha y - \gamma w = 0,\ b\eta + d\omega - \beta y - \delta w = 1. \end{aligned} \tag{5.2}$$

Thus we have 6 equations for 16 variables, hence 10 free variables exist. The last four equations can be grouped into two sets of two linear equations with respect to those variables which matrix entries compose a nonzero determinant. Suppose, for instance, that $\det B = \Delta \neq 0$. Then variables (x, y, z, w) can be expressed via the remaining variables as follows

$$\begin{aligned} x &= \frac{\delta(a\xi + c\zeta) - \delta - \gamma(a\eta + c\omega)}{\Delta},\ y = \frac{\delta(b\xi + d\zeta) + \gamma - \gamma(b\eta + d\omega)}{\Delta}, \\ z &= \frac{\alpha(a\eta + c\omega) + \beta - \beta(a\xi + c\zeta)}{\Delta},\ w = \frac{\alpha(b\eta + d\omega) - \alpha - \beta(b\xi + d\zeta)}{\Delta}. \end{aligned}$$

If we insert these expressions for (x, y, z, w) into the first relation in (5.2), then we get with the account of the second equality in (5.2) the relation $b\delta + a\gamma - d\beta - c\alpha = 0$. Hence, instead of two first relations in (5.2) we can use two relations

$$\beta\xi - \alpha\eta + \delta\zeta - \gamma\omega = 0,\ b\delta + a\gamma - d\beta - c\alpha = 0,$$

where coefficients are the set $(\alpha, \beta, \gamma, \delta)$.

To get an example, let us set $\alpha = \gamma = \delta = 1, \beta = 0$. Then $\Delta = 1$ and calculations give $c = a + b,\ \zeta = \eta + \omega$. So, we have six arbitrary parameters $a, b, d, \xi, \eta, \omega$.

If we set $\alpha = 2, \beta = \gamma = \delta = 1$, then, as easily verified, one can choose

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix},\ C = \begin{pmatrix} 1 & -1 \\ -1 & 6 \end{pmatrix},\ D = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix},$$

and corresponding symplectic matrix has the characteristic polynomial $\chi(\lambda) = \lambda^4 - 8\lambda^3 + 17\lambda^2 - 8\lambda + 1$ being reducible over $\mathbb{Z}$: $\chi(\lambda) = (\lambda^2 - 5\lambda + 1)(\lambda^2 - 3\lambda + 1)$. Thus we get the Anosov automorphism on $\mathbb{T}^4$ with two-dimensional unstable and stable foliations but it is decomposable, since the closure of a one-dimensional unstable foliations generated by each eigenvector greater than one ($\lambda_1 = (5 + \sqrt{21})/2$, and $\lambda_1 = (3 + \sqrt{5})/2$) gives an invariant two-dimensional torus.

Next choice is when matrices A, B, C, D are as follows

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\ C = \begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix},\ D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomial $\chi(\lambda) = \lambda^4 - 4\lambda^3 + \lambda^2 - 4\lambda + 1$ of the related symplectic matrix S is irreducible with roots

$$\lambda_{1,2} = 1 + \frac{\sqrt{5}}{2} \pm \frac{\sqrt{5 + 4\sqrt{5}}}{2},\ \lambda_{3,4} = 1 - \frac{\sqrt{5}}{2} \pm i\frac{\sqrt{4\sqrt{5} - 5}}{2}$$

It turns out that matrix S is not integrally similar to its companion matrix K

$$K = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{pmatrix}.$$

This means that automorphisms f_S and f_K, being both ergodic and transitive (and even mixing ones) with transitive strong unstable foliations, are not topologically conjugate, since their matrices S, K are not integrally similar, though they are rationally similar.

For a 6-dimensional case the above-used algorithm is not convenient. Here we apply the generating function method [3]. We seek a symplectic matrix for the symplectic transformation in the form

$$X = \frac{\partial S}{\partial Y},\ y = \frac{\partial S}{\partial x},\ X = (X_1, X_2, X_3),\ y = (y_1, y_2, y_3),$$

with a quadratic generating function $S(x, Y) = \frac{1}{2}(Ax, x) + (Bx, Y) + \frac{1}{2}(CY, Y)$, where matrices A, C are symmetric and B is an arbitrary real non-degenerate matrix. Then the transformation casts in the “cross” form

$$X = Bx + CY,\ y = Ax + B^\top Y,$$

or in the direct form

$$x = B^{-1}X - B^{-1}CY,\ y = AB^{-1}X + (B^\top - AB^{-1}C)Y.$$

Let us choose

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ B^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix},\ C = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Then we get a symplectic 6×6 matrix

$$S = \begin{pmatrix} 1 & -1 & 0 & -3 & 1 & -2 \\ 0 & 1 & -1 & 1 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 & -2 \\ 1 & 0 & -2 & 0 & -2 & 3 \\ 1 & 1 & -2 & 0 & -2 & 4 \\ -1 & 1 & 3 & 1 & 3 & -3 \end{pmatrix}$$

with the characteristic polynomial $\chi(\lambda) = \lambda^6 + 2\lambda^5 - 16\lambda^4 + 24\lambda^3 - 16\lambda^2 + 2\lambda + 1$ with approximate roots $\lambda_{1,2} = 0.704 \pm i0.710$, $\lambda_{3,4} = -5.663, 1.907$, $\lambda_{5,6} = -0.177, 0.524$. The last polynomial can be obtained from the cubic polynomial $z^3 + 2z^2 - 19z + 20$ by means of the change $z = \lambda + 1/\lambda$.

5.1. Rational and integral similarity: verification

Results of this subsection were kindly presented to us by Dr. K. Conrad (the University of Connecticut), authors are deeply thank to him for detailed explanations and the help.

Set

$$A = \begin{pmatrix} 1 & -1 & 0 & -3 & 1 & -2 \\ 0 & 1 & -1 & 1 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 & -2 \\ 1 & 0 & -2 & 0 & -2 & 3 \\ 1 & 1 & -2 & 0 & -2 & 4 \\ -1 & 1 & 3 & 1 & 3 & -3 \end{pmatrix},$$

which has characteristic polynomial

$$\chi_A(x) = x^6 + 2x^5 - 16x^4 + 24x^3 - 16x^2 + 2x + 1.$$

This is irreducible in $\mathbf{Q}[x]$ (it is irreducible mod 7, as well as mod 61, 73, 83, 101, and so on). Is A integrally conjugate to its companion matrix? We'll see how to get a computer to tell us the answer is **no**.

Set $K=\mathbf{Q}(\alpha)$ where α is a root of $\chi_A(x)$. Using a computer, $\chi_A(x)$ has discriminant -1639961600 and the ring of integers $\mathcal{O}_K$ has discriminant -102497600, so

$$[\mathcal{O}_K:\mathbf{Z}[\alpha]]^2=\frac{-1639961600}{-102497600}=16.$$

Thus $[\mathcal{O}_K:\mathbf{Z}[\alpha]]=4$.

Remark 5.1. A computer says the splitting field of $K/\mathbf{Q}$ has degree 48 and $\mathcal{O}_K$ has class number 1 (it is a PID), but this does not mean $\mathbf{Z}[\alpha]$ is a PID; in fact it can't be since for every number field F, a proper subring of finite index in $\mathcal{O}_F$ is not a PID since it is not integrally closed.

We want to convert A (or rather its integral conjugacy class) into an ideal (or rather an ideal class) in the ring $\mathbf{Z}[\alpha]$ and then ask a computer if that ideal is principal since the ideal class in $\mathbf{Z}[\alpha]$ corresponding to the companion matrix of A is the class of principal ideals.

A general element of K has the form $g(\alpha)=a_0+a_1\alpha+a_2\alpha^2+a_3\alpha^3+a_4\alpha^4+a_5\alpha^5$ where $\alpha_j\in\mathbf{Q}$. We associate to $g(\alpha)$ in K the 6×6 matrix

$$g(A)=a_0I_6+a_1A+a_2A^2+a_3A^3+a_4A^4+a_5A^5$$

in $\mathrm{M}_6(\mathbf{Q})$. Turn $\mathbf{Q}^6$ into a K-vector space by setting $g(\alpha)\mathbf{v}:=g(A)\mathbf{v}$ for all $\mathbf{v}\in\mathbf{Q}^6$. Using a computer,

$$\mathbf{v}_0:=\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}\Longrightarrow g(\alpha)\mathbf{v}_0=g(A)\mathbf{v}_0=\begin{pmatrix}1&1&1&0&6&-22\\0&0&-4&18&-108&602\\0&0&2&-11&63&-353\\0&1&-4&25&-140&793\\0&1&-5&27&-158&884\\0&-1&6&-36&199&-1136\end{pmatrix}\begin{pmatrix}a_0\\a_1\\a_2\\a_3\\a_4\\a_5\end{pmatrix}.$$

Call that 6×6 integral matrix B. Its determinant is -4. The (fractional) $\mathbf{Z}[\alpha]$-ideal $\mathfrak{a}$ in K that is associated to A is $\mathfrak{a}:=\{\gamma\in K:\gamma\mathbf{v}_0\in\mathbf{Z}^6\}$. What do elements of $\mathfrak{a}$ look like? We want to find a spanning set for $\mathfrak{a}$ as a $\mathbf{Z}$-module and then as a $\mathbf{Z}[\alpha]$-module.

Write γ as

$$g(\alpha)=a_0+a_1\alpha+a_2\alpha^2+a_3\alpha^3+a_4\alpha^4+a_5\alpha^5=\begin{pmatrix}a_0\\a_1\\a_2\\a_3\\a_4\\a_5\end{pmatrix}\cdot\begin{pmatrix}1\\\alpha\\\alpha^2\\\alpha^3\\\alpha^4\\\alpha^5\end{pmatrix}=\mathbf{a}\cdot\boldsymbol{\alpha}.$$

We have

$$\gamma\mathbf{v}_0=g(\alpha)\mathbf{v}_0=g(A)\mathbf{v}_0=B\mathbf{a}\in\mathbf{Z}^6\Longleftrightarrow\mathbf{a}\in B^{-1}(\mathbf{Z}^6)\Longleftrightarrow\mathbf{a}=B^{-1}\mathbf{x}$$

for $\mathbf{x}\in\mathbf{Z}^6$. Then

$$\gamma\in\mathfrak{a}\Longleftrightarrow\gamma=g(\alpha)=\mathbf{a}\cdot\boldsymbol{\alpha}=(B^{-1}\mathbf{x})\cdot\boldsymbol{\alpha}=\mathbf{x}\cdot(B^{-1})^{\top}\boldsymbol{\alpha}\text{ for some }\mathbf{x}\in\mathbf{Z}^6.$$

As $\mathbf{x}$ runs over $\mathbf{Z}^6$, we get $\mathfrak{a}$ is the $\mathbf{Z}$-span of the components of $(B^{-1})^{\top}\boldsymbol{\alpha}$.

Since

$$B^{-1}=\begin{pmatrix}1&-25/2&-23/2&-49/2&15&-17/2\\0&-107/2&-39&-96&62&-35\\0&115&175/2&419/2&-133&153/2\\0&-207/2&-161/2&-371/2&118&-135/2\\0&35/2&14&31&-20&11\\0&7&11/2&25/2&-8&9/2\end{pmatrix},$$

the elements of $\mathfrak{a}$ are in $\frac{1}{2}\mathbf{Z}[\alpha]$. Therefore set $\mathfrak{b}:=2\mathfrak{a}$ is contained in $\mathbf{Z}[\alpha]$ and is the $\mathbf{Z}$-span of the components of $2(B^{-1})^{\top}\boldsymbol{\alpha}$.

We have

$$2(B^{-1})^\top \boldsymbol{\alpha} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ -25 & -107 & 230 & -207 & 35 & 14 \\ -23 & -78 & 175 & -161 & 28 & 11 \\ -49 & -192 & 419 & -371 & 62 & 25 \\ 30 & 124 & -266 & 236 & -40 & -16 \\ -17 & -70 & 153 & -135 & 22 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \\ \alpha^3 \\ \alpha^4 \\ \alpha^5 \end{pmatrix},$$

so $\mathfrak{b}$ is the $\mathbf{Z}$-span of

$$\begin{aligned} &2, \\ &-25 - 107\alpha + 230\alpha^2 - 207\alpha^3 + 35\alpha^4 + 14\alpha^5, \\ &-23 - 78\alpha + 175\alpha^2 - 161\alpha^3 + 28\alpha^4 + 11\alpha^5, \\ &-49 - 192\alpha + 419\alpha^2 - 371\alpha^3 + 62\alpha^4 + 25\alpha^5, \\ &30 + 124\alpha - 266\alpha^2 + 236\alpha^3 - 40\alpha^4 - 16\alpha^5, \\ &-17 - 70\alpha + 153\alpha^2 - 135\alpha^3 + 22\alpha^4 + 9\alpha^5. \end{aligned}$$

Since $\mathfrak{b}$ is an *ideal* in $\mathbf{Z}[\alpha]$, $\mathfrak{b}$ is also the $\mathbf{Z}[\alpha]$-span of the above 5 elements. Because $2 \in \mathfrak{b}$, we can streamline all the generators by replacing each integral coefficient with 0 or 1 depending on whether it is even or odd:

$$\begin{aligned} \mathfrak{b} &= (2, 1 + \alpha + \alpha^3 + \alpha^4, 1 + \alpha^2 + \alpha^3 + \alpha^5, 0, 1 + \alpha^2 + \alpha^3 + \alpha^5) \\ &= (2, 1 + \alpha + \alpha^3 + \alpha^4, 1 + \alpha^2 + \alpha^3 + \alpha^5). \end{aligned}$$

Writing these ideal generators as $2, \beta_1, \beta_2$,

$$\mathfrak{b} = (2, \beta_1, \beta_2) = (2, \beta_1, \beta_2 - \alpha\beta_1) = (2, 1 + \alpha + \alpha^3 + \alpha^4, 1 - \alpha + \alpha^3 - \alpha^4).$$

The last two generators differ by a multiple of 2 in $\mathbf{Z}[\alpha]$, so

$$\boxed{\mathfrak{b} = (2, 1 + \alpha + \alpha^3 + \alpha^4)}$$

as an ideal in $\mathbf{Z}[\alpha]$.

Using this last formula for $\mathfrak{b}$, we now ask a computer algebra system if $\mathfrak{b}$ is a principal ideal in $\mathbf{Z}[\alpha]$. Using Magma's online calculator at

http://magma.maths.usyd.edu.au/calc/ `http://magma.maths.usyd.edu.au/calc/`.

We enter the code

```
P<x>:=PolynomialRing(Integers());
R:=ext<Integers()|x^6+2*x^5-16*x^4+24*x^3-16*x^2+2*x+1>;
R;
k:=R.2;
I:=ideal<R|[2,1+k+k^3+k^4]>;
Norm(I);
IsPrincipal(I);
```

and this returns the result

```
16
false
```

which means (i) $[\mathbf{Z}[\alpha] : \mathfrak{b}] = 16$ and (ii) $\mathfrak{b}$ is **not** a principal ideal in $\mathbf{Z}[\alpha]$.

Data availability

No data was used for the research described in the article.

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