



Automorphism Groups of ind-Varieties of Generalized Flags

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Abstract

We compute the group of automorphisms of an arbitrary ind-variety of (possibly isotropic) generalized flags. Such an ind-variety is a homogeneous ind-space for one of the ind-groups $SL(\infty)$, $O(\infty)$ or $Sp(\infty)$. We show that the respective automorphism groups are much larger than $SL(\infty)$, $O(\infty)$ or $Sp(\infty)$, and present the answer in terms of Mackey groups. The latter are groups of automorphisms of non-degenerate pairings of (in general infinite-dimensional) vector spaces. An explicit matrix form of the automorphism group of an arbitrary ind-variety of generalized flags is also given. The case of the Sato grassmannian is considered in detail, and its automorphism group is the projectivization of the connected component of unity in the group known as Japanese $GL(\infty)$.

Keywords ind-variety · ind-group · Generalized flag · Homogeneous space · Automorphism group · Makey group · Sato grassmannian · Japanese $GL(\infty)$

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1 Introduction

Given a homogenous space X , it is a natural problem to compute its automorphism group $\text{Aut } X$. In the case when X is a complex flag variety, that is, $X = G/P$ for a connected reductive complex algebraic group and a parabolic subgroup $P \subset G$, the automorphism group of X is well known. Moreover, it is a classical result that here

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the connected component $\text{Aut}^0 X$ of the identity equals the projectivized group PG , except in some special cases as described in [12].

In this paper, we would like to pose and solve the problem of computing $\text{Aut } X$ for a class of homogeneous ind-varieties X . This is the class of ind-varieties of generalized flags introduced by that name in [3] but also considered earlier in several works, see for instance [4, 11]. These ind-varieties can be defined simply as G/P where G is one of the ind-groups $SL(\infty) = \varinjlim SL(n)$, $SO(\infty) = \varinjlim SO(n)$, $Sp(\infty) = \varinjlim Sp(2n)$ and P is a splitting parabolic subgroup, i.e., a subgroup for which the intersections $P \cap SL(n)$, $P \cap SO(2n)$, $P \cap SO(2n + 1)$, $P \cap Sp(2n)$ are parabolic subgroups of $SL(n)$, $SO(2n)$, $SO(2n + 1)$, $Sp(2n)$ for all n , respectively. The definition from [3] can be considered as a flag realization of the ind-varieties G/P as above, and is recalled in Section 2 below. The main idea of that approach is that one designates certain chains of subspaces in the natural representation V of $SL(\infty)$ as generalized flags, and then defines an ind-variety of generalized flags as the ind-variety of generalized flags which differ only “slightly” from a fixed generalized flag W in V . For the exact definition see Section 2. One then shows that the so obtained ind-variety is isomorphic to G/P for $G = SL(\infty)$ and some splitting parabolic subgroup $P \subset G$.

An ind-grassmannian is an ind-variety of generalized flags for which the fixed generalized flag consists of a single proper subspace $W \subset V$. For $\dim W = \text{codim}_V W = \infty$ the ind-grassmannian is isomorphic to the Sato grassmannian. This has been pointed out for instance in [5].

In the cases of the groups $SO(\infty)$ and $Sp(\infty)$ we consider ind-varieties of isotropic generalized flags, as stated in Section 4.

Our main result is the explicit determination of the group $\text{Aut } X$ for an arbitrary ind-variety of, possibly isotropic, generalized flags. A notable feature is that the answer is very different from the ind-groups $PGL(\infty)$, $PO(\infty)$, or $PSp(\infty)$, and we present it in the language of Mackey groups. Such a group is defined in terms of a non-degenerate pairing of vector spaces $T \times R \rightarrow \mathbb{C}$, and is a subgroup of the group of all linear operators $\varphi: T \rightarrow T$ for which the dual operator φ^* determines a well-defined automorphism $\bar{\varphi}: R \rightarrow R$. This definition of Mackey group is inspired by G. Mackey’s dissertation [10]. If T and R are finite dimensional, then the Mackey group is nothing but $GL(T) \simeq GL(R)$. The group known as Japanese $GL(\infty)$ is a Mackey group and plays a crucial role in our work. In the Appendix we discuss the structure of this group in detail.

The precise statement of our main result, Theorem 2.1, is presented in Section 2. The consideration of the isotropic case is postponed to Section 8. The proof of Theorem 2.1 is divided into two parts: the case of an ind-grassmannian and the case of an arbitrary generalized flag. For the Sato grassmannian (which is the most interesting ind-grassmannian) our result implies that its automorphism group is isomorphic to the projectivization of the connected component of the identity in the group Japanese $GL(\infty)$. In Section 7 we give a matrix realization of the group of automorphisms of an arbitrary ind-variety of generalized flags. In the isotropic case such a realization is given in Corollary 8.1.

We would like to point out that $\text{Aut } X$ depends essentially on the ind-variety X , despite the fact that all X are homogeneous spaces for the same group $SL(\infty)$ (or,

respectively, $SO(\infty)$, $Sp(\infty)$). This is in contrast with the finite-dimensional case in which the connected component of the identity in the automorphism group of a variety $SL(n)/P$ (respectively, $SO(2n)/P$, $SO(2n + 1)/P$ or $Sp(2n)/P$) depends only on n and not on the choice of P . Further research should be carried out to compare the isomorphism classes of ind-varieties of generalized flags with the isomorphism classes of their automorphism groups.

Our possible application of the results of the present paper would be the study of locally reductive ind-groups \tilde{G} different from $G = SL(\infty)$, $SO(\infty)$, $Sp(\infty)$ for which G/P is a homogeneous \tilde{G} -space.

In conclusion of this short introduction, we should mention that some particular cases of the automorphism groups of ind-varieties of generalized flags have been considered in [13] and [17].

2 Brief Background and Statement of the Main Result

The ground field is \mathbb{C} . If R is a vector space, we set $GL(R) = \{\varphi \in \text{Hom}_{\mathbb{C}}(R, R) \mid \varphi \text{ is invertible}\}$ and $R^* = \text{Hom}_{\mathbb{C}}(R, \mathbb{C})$. We also use the superscript \cdot^* to denote the dual of a vector bundle, as well as the pullback functor for vector bundles along a morphism of varieties. In what follows we consider infinite matrices, in particular, infinite rows and columns. We call such matrices, rows or columns *finitary* if they have at most finitely many nonzero entries. By $\langle \cdot \rangle_{\mathbb{C}}$ we denote the linear span over \mathbb{C} .

We fix a countable-dimensional vector space V . A *chain of subspaces* $W = \{W_{\alpha}\}$ is a set of subspaces $W_{\alpha} \subseteq V$, parameterized by some index set with elements α such that for $\alpha \neq \alpha'$ we have $W_{\alpha} \not\subseteq W_{\alpha'}$ or $W_{\alpha'} \not\subseteq W_{\alpha}$. The relation of inclusion induces a total order on the set of indices of a chain. A chain of subspaces $W = \{W_{\alpha}\}$ is a *generalized flag in V* if every index α has either an immediate predecessor or an immediate successor, and every nonzero vector v of V is contained in some difference $W_{\alpha''} \setminus W_{\alpha'}$, where α'' is the immediate successor of α' . For a more detailed discussion of generalized flags, and for an introduction to ind-varieties of generalized flags, see, e.g., [3, 8] and [15].

We say that a generalized flag W is *compatible with a basis \tilde{E}* of V (or that W is *\tilde{E} -compatible*) if any space W_{α} of W is spanned by elements of \tilde{E} , i.e., $W_{\alpha} = \langle W_{\alpha} \cap \tilde{E} \rangle_{\mathbb{C}}$ for any α . We set $E_{\alpha} = \tilde{E} \cap W_{\alpha}$. Then E_{α} is a basis of W_{α} . By $(W_{\alpha})_*$ we denote the span of the system of linear functions \tilde{E}^* dual to the basis E_{α} . We have $(W_{\alpha})_* \subseteq W_{\alpha}^*$. We also let V_* equal the span of the system of linear functions dual to the basis \tilde{E} . The group $GL(\tilde{E}, V)$ is the subgroup of $GL(V)$ consisting of all invertible operators $\varphi: V \rightarrow V$ each of which acts as the identity on all but finitely elements of \tilde{E} .

In what follows, we fix a basis \tilde{E} and a generalized flag W in V compatible with \tilde{E} . The set $Fl(W, \tilde{E}, V)$ is the set of all generalized flags $W' = \{W'_{\alpha}\}$ which are *\tilde{E} -commensurable* with W . This latter requirement spells out as the following three conditions for each element W' of $Fl(W, \tilde{E}, V)$:

- the index set which parameterizes the generalized flag W' is the same as the index set of W ;

- there exists a finite-dimensional subspace $Z \subsetneq V$ depending on W' , such that for any α we have $W'_\alpha + Z = W_\alpha + Z$ and $\dim(W_\alpha \cap Z) = \dim(W'_\alpha \cap Z)$;
- the generalized flag W' is compatible with a basis \tilde{E}' of V , depending on W' , such that \tilde{E}' differs from \tilde{E} by finitely many vectors.

The set $Fl(W, \tilde{E}, V)$ has a natural structure of ind-variety. This is explained in detail in [3] (and in [8] and [15]). Briefly, the ind-variety structure on $Fl(W, \tilde{E}, V)$ arises as follows. Enumerate the basis \tilde{E} by the set $\mathbb{Z}_{>0}$ and put $V_i := \langle e_1, \dots, e_i \rangle_{\mathbb{C}}$ for $i \in \mathbb{Z}_{>0}$. Each intersection $W \cap V_i$ is a flag in V_i of certain type $\underline{d}_i = (d_i^1, \dots, d_i^{k_i})$, and this ordering of the basis \tilde{E} induces embeddings

$$Fl(\underline{d}_i, V_i) \hookrightarrow Fl(\underline{d}_{i+1}, V_{i+1}), \tag{1}$$

called *strict standard extensions*, such that $Fl(W, \tilde{E}, V) = \varinjlim Fl(\underline{d}_i, V_i)$. The embeddings (1) endow $Fl(W, \tilde{E}, V)$ with an ind-variety structure. In Section 4 below we recall the definition of a strict standard extension.

Next we recall that if T and R are two (in general, infinite-dimensional) vector spaces endowed with a non-degenerate pairing $\mathbf{p}: T \times R \rightarrow \mathbb{C}$, then the Mackey group $G(T, R)$ is defined as

$$G(T, R) = \{ \varphi \in GL(T) \mid \varphi^*(R) = R \}, \tag{2}$$

see [10]. Here $\varphi^*: T^* \rightarrow T^*$ is the operator dual (adjoint) to the operator $\varphi: T \rightarrow T$, and R is considered as a subspace of T^* via the embedding $R \hookrightarrow T^*$ induced by the pairing \mathbf{p} . Equivalently, $G(T, R)$ can be defined as the group

$$\{ \psi \in GL(R) \mid \psi^*(T) = T \} \tag{3}$$

where T is considered as a subspace of R^* via \mathbf{p} . The correspondence

$$\varphi \mapsto (\varphi^*|_R)^{-1}$$

is a canonical isomorphism between the groups (2) and (3). In what follows, when writing $\varphi \in G(T, R)$ we will assume that $\varphi \in GL(T)$, and will denote the operator $\varphi^*|_R$ by $\bar{\varphi}$. Note that, given a subspace A of T , one has $\varphi(A) = \bar{\varphi}^{-1}(A^\perp)$, where A^\perp is the annihilator of A in R .

Consider again the ind-variety of generalized flags $Fl(W, \tilde{E}, V)$. Define the spaces $V_{\tilde{E}}^W$ and $V_{*\tilde{E}}^W$ as

$$\begin{aligned} V_{\tilde{E}}^W &:= \bigcap_{\alpha} ((W_\alpha)_*)^* \oplus U_\alpha, \\ V_{*\tilde{E}}^W &:= \bigcap_{\alpha} (W_\alpha)_* \oplus U_\alpha^*, \end{aligned}$$

where α runs over the indices parameterizing the generalized flag W , and the spaces U_α are direct complements of the spaces W_α , i.e., $V = W_\alpha \oplus U_\alpha$, with the assumption that $U_\alpha \cap \tilde{E}$ is a basis of U_α . Note that the spaces $V_{\tilde{E}}^W, V_{*\tilde{E}}^W$ are necessarily infinite-dimensional and there is a canonical non-degenerate pairing

$$V_{\tilde{E}}^W \times V_{*\tilde{E}}^W \rightarrow \mathbb{C},$$

therefore the group $G(V_{\tilde{E}}^W, V_{*\tilde{E}}^W)$ is well defined.

An essential observation is that the spaces $V_{\tilde{E}}^W$ and $V_{*\tilde{E}}^W$ depend only on the ind-variety $Fl(W, \tilde{E}, V)$ and not on the specific point $W \in Fl(W, \tilde{E}, V)$. This follows from the fact that, for each α , the spaces $((W_\alpha)_*)^* \oplus U_\alpha$ and $(W_\alpha)_* \oplus U_\alpha^*$ do not change when W_α is replaced by a subspace $W'_\alpha \subset W$ which is \tilde{E} -commensurable with W_α , and U_α is replaced by a direct complement U'_α of W'_α containing all but finitely many vectors from $\tilde{E} \cap U_\alpha$.

Moreover, $GL(\tilde{E}, V)$ is a subgroup of $G(V_{\tilde{E}}^W, V_{*\tilde{E}}^W)$. To see this, consider a linear operator $\varkappa : V \rightarrow V$, $\varkappa \in GL(\tilde{E}, V)$, and fix α . There exist subspaces $W'_\alpha \subset W_\alpha$, $U'_\alpha \subset U_\alpha$, such that $\varkappa|_{W'_\alpha} = \text{id}_{W'_\alpha}$, $\varkappa|_{U'_\alpha} = \text{id}_{U'_\alpha}$, and $V = W'_\alpha \oplus K \oplus U'_\alpha$ for some \varkappa -invariant finite-dimensional subspace $K \subset V$. Then $(W'_\alpha)_*$, $(U'_\alpha)^*$ and K^* are \varkappa^* -invariant subspaces of V^* , and $(W_\alpha)_* \oplus U_\alpha^* = (W'_\alpha)_* \oplus K^* \oplus (U'_\alpha)^*$ is a \varkappa^* -invariant subspace of V^* . This shows that $V_{*\tilde{E}}^W = \bigcap_\alpha ((W_\alpha)_* \oplus U_\alpha^*)$ is a \varkappa^* -invariant subspace of V^* . The same argument applied to the \varkappa^* -invariant subspace V_* of V^* implies that $V_{\tilde{E}}^W$ is a $(\varkappa^*|_{V_*})^*$ -invariant subspace of $(V_*)^*$. This allows to consider \varkappa as an element of $G(V_{\tilde{E}}^W, V_{*\tilde{E}}^W)$.

Next, if W' is any chain of subspaces in V and $\varphi : V_{\tilde{E}}^W \rightarrow V_{\tilde{E}}^{W'}$ is any linear operator from the group $G(V_{\tilde{E}}^W, V_{*\tilde{E}}^W)$, then

$$\bar{\varphi}^{-1} \left(W'^{\perp} \right)^\perp \cap V = \varphi \left((W'^{\perp})^\perp \right) \cap V \tag{4}$$

is a chain of subspaces in V . Here W'^{\perp} is the chain in $V_{*\tilde{E}}^{W'}$ consisting of the annihilators in $V_{*\tilde{E}}^W \subseteq V^*$ of the spaces W'_α , and similarly $\bar{\varphi}^{-1} \left(W'^{\perp} \right)^\perp$, $W'^{\perp\perp}$ are chains in $V_{\tilde{E}}^W \subseteq V^{**}$.

In what follows, we use the notation W'^{\perp} for chains perpendicular to W' also in appropriate subspaces of V^* different from $V_{*\tilde{E}}^W$, and indicate the respective subspace as necessary. A similar convention applies to the notation $W'^{\perp\perp}$. Moreover, we call the generalized flag W symmetric if the chain $W^\perp \subset V_*$ is the image of W under a linear isomorphism $V_* \simeq V$ sending \tilde{E}^* to \tilde{E} .

- Theorem 2.1** a) *If W is not symmetric, then the group $\text{Aut } Fl(W, \tilde{E}, V)$ is isomorphic to $P(GL(\tilde{E}, V) \cdot St_W)$, where St_W is the stabilizer of the generalized flag W in the group $G(V_{\tilde{E}}^W, V_{*\tilde{E}}^W)$ under the action (4). Here the product \cdot is taken inside $G(V_{\tilde{E}}^W, V_{*\tilde{E}}^W)$, and $P \cdot$ indicates passage to the quotient modulo scalar operators.*
 b) *If W is symmetric, then the group $\text{Aut } Fl(W, \tilde{E}, V)$ is isomorphic to $P(GL(\tilde{E}, V) \cdot St_W) \rtimes \mathbb{Z}_2$.*

In Section 7 we present an explicit matrix realization of the group $GL(\tilde{E}, V) \cdot St_W$. Let's also point out that, since Theorem 2.1 implies that $GL(\tilde{E}, V) \cdot St_W$ is a group, we have $GL(\tilde{E}, V) \cdot St_W = St_W \cdot GL(\tilde{E}, V)$.

Remark 2.1 In the case of a finite-dimensional flag variety X , every automorphism of X belonging to the connected component of unity in the automorphism group has a fixed point on X . This no longer holds in the generality of Theorem 2.1. Indeed, if

X is the projective ind-space $Fl(W, E, V)$ for $\dim W = 1$, then $\text{Aut}X = PGL(V)$ and it is well known that not every invertible linear automorphism of V has an eigenvector.

3 Examples

Before we embark on proving Theorem 2.1, we present five examples in which we compute the respective group $GL(\tilde{E}, V) \cdot St_W$ from Theorem 2.1. In all five cases our claims follow from Theorem 7.1 below, which provides a matrix form of the group $GL(\tilde{E}, V) \cdot St_W$ in the general case.

3.1 The Case of an ind-Grassmannian

Let's consider the case where the generalized flag W has the form $0 \subsetneq W \subsetneq V$, where W is a single proper subspace of V (we slightly abuse notation by using the same letter W for a flag and a subspace). There are three cases: $\dim W < \infty$, or $\dim W = \text{codim}_V W = \infty$, or $\text{codim}_V W < \infty$. If $\dim W < \infty$, then $Gr(W, \tilde{E}, V)$ does not depend on the basis \tilde{E} , and the points of $Gr(W, \tilde{E}, V)$ are all subspaces of V of the same dimension as W . In this case we may write $Gr(W, \tilde{E}, V) = Gr(\dim W, V)$. If $\text{codim}_V W < \infty$ then $Gr(W, \tilde{E}, V)$ depends as a set on the choice of the basis \tilde{E} , but up to isomorphism $Gr(W, \tilde{E}, V)$ depends only on $\text{codim}_V W$. Moreover, as an ind-variety $Gr(W, \tilde{E}, V)$ is isomorphic to $Gr(\text{codim}_V W, V)$. If $\dim W = \text{codim}_V W = \infty$, the ind-variety $Gr(W, \tilde{E}, V)$ does not depend up to isomorphism on the choice of both W and \tilde{E} . It is known, see for instance [5], that in this case $Gr(W, \tilde{E}, V)$ is isomorphic to the Sato grassmannian introduced in [16].

If $\dim W < \infty$, then $V_{\tilde{E}}^W = V$, $V_{*\tilde{E}}^W = V^*$ and $G(V, V^*) = GL(V)$. Since W is not symmetric, Theorem 2.1 asserts that $\text{Aut} Gr(W, \tilde{E}, V) \cong P(GL(\tilde{E}, V) \cdot St_W)$, and we note that here

$$P(GL(\tilde{E}, V) \cdot St_W) \cong PGL(V). \tag{5}$$

Indeed, the action of $G(V, V^*) = GL(V)$ on $Gr(W, \tilde{E}, V)$ via the formula (4) is easily checked to coincide with the obvious action of $GL(V)$ on subspaces of V , and the isomorphism (5) is a consequence of the transitive action of the group $GL(\tilde{E}, V)$ on finite-dimensional subspaces of fixed dimension in V .

If $\text{codim}_V W < \infty$, then $V_{\tilde{E}}^W = (V_*)^*$, $V_{*\tilde{E}}^W = V_*$ and $P(GL(\tilde{E}, V) \cdot St_W) \cong PGL(V_*)$, i.e.,

$$\text{Aut} Gr(W, \tilde{E}, V) \cong PGL(V_*).$$

In the case where $\dim W = \text{codim}_V W = \infty$, we prove in the Appendix that the group $G(V_{\tilde{E}}^W, V_{*\tilde{E}}^W)$ can be represented as invertible $(\mathbb{Z} \setminus \{0\}) \times (\mathbb{Z} \setminus \{0\})$ -matrices which together with their inverses satisfy the condition: in the block structure

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \tag{6}$$

induced by the equality $\mathbb{Z} \setminus \{0\} = \mathbb{Z}_{<0} \sqcup \mathbb{Z}_{>0}$, the matrix A has finitary rows (no restriction on the columns), the matrix D has finitary columns (no restriction on the rows) and the matrix C is finitary. The group $GL(\tilde{E}, V) \cdot St_W$ consists of matrices M such that M and M^{-1} have the form (6) and satisfy the additional condition $\text{rk } C = \text{rk } C'$ where

$$M^{-1} = \left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right).$$

Moreover, in this case W is symmetric.

3.2 The Case of $FI(W, \tilde{E}, V)$, Where $W = \{W_n\}$, $\dim W_n = n$ for $n \in \mathbb{Z}_{>0}$, $\bigcup_n W_n = V$

In this case $V_{\tilde{E}}^W = V$, $V_{*\tilde{E}}^W = V^*$, and the group $GL(\tilde{E}, V) \cdot St_W$ can be identified with all invertible $\mathbb{Z}_{>0} \times {}^* \mathbb{Z}_{>0}$ -matrices with finitely many nonzero entries below the main diagonal, cf. [13].

3.3 The Case of $FI(W, \tilde{E}, V)$, Where $\tilde{E} = \{e_i\}_{i \in \mathbb{Z}}$ and $W = \{W_n = \langle e_i, i \leq n \rangle_{\mathbb{C}}, n \in \mathbb{Z}\}$

Here

$$V_{\tilde{E}}^W = ((W_0)_*)^* \oplus \langle e_i, i > 0 \rangle_{\mathbb{C}}, \quad V_{*\tilde{E}}^W = (W_0)_* \oplus (\langle e_i, i > 0 \rangle_{\mathbb{C}})^*.$$

In coordinate form, the vectors from $V_{\tilde{E}}^W$ are columns $(a_j)_{j \in \mathbb{Z}}$ with $a_j = 0$ for $j \gg 0$, and $GL(\tilde{E}, V) \cdot St_W$ consists of all invertible $\mathbb{Z} \times \mathbb{Z}$ -matrices M which, together with their inverses, have finitely many nonzero entries below the main diagonal and satisfy the condition $\text{rk } C = \text{rk } C'$, where C and C' are respectively the strictly lower-triangular parts of M and M^{-1} .

3.4 The Case of $FI(W, \tilde{E}, V)$, Where $W = \{0 \subset W_1 \subset W_{-1} \subset V\}$, $\dim W_1 = 1$, $\text{codim}_V W_{-1} = 1$

Here \tilde{E} can be ordered by any countable ordered set I with a minimal and a maximal element. We have $V_{\tilde{E}}^W = V$, $V_{*\tilde{E}}^W = V_*$, and $GL(\tilde{E}, V) \cdot St_W$ consists of all invertible $I \times I$ -matrices which, together with their inverses, satisfy the condition that each row and column is finitary.

3.5 The Case of $FI(W, \tilde{E}, V)$ for $\tilde{E} = \{e_j\}_{j \in \mathbb{Z}_{>0} \sqcup \mathbb{Z}_{<0}}$ and $W = \{W_n\}$, Where $W_n = \langle e_1, \dots, e_n \rangle_{\mathbb{C}}$ if $n \in \mathbb{Z}_{>0}$ and $W_n = \langle \dots, e_{n-2}, e_{n-1}, e_1, e_2, \dots \rangle_{\mathbb{C}}$ if $n \in \mathbb{Z}_{<0}$

Then $V_{\tilde{E}}^W = V$, $V_{*\tilde{E}}^W = V_*$, $I = \mathbb{Z}_{>0} \sqcup \mathbb{Z}_{<0}$ with $k < l$ if $k \in \mathbb{Z}_{>0}$, $l \in \mathbb{Z}_{<0}$, and $GL(\tilde{E}, V) \cdot St_W$ consists of all invertible $I \times I$ -matrices which, together with their inverses, satisfy the condition: each row and column is finitary, and there are at most finitely many nonzero entries below the main diagonal.

Note that in the cases 3.3, 3.4, and 3.5 the generalized flag W is symmetric, while in the case 3.2 W is not symmetric.

4 More Background

We need to recall some facts about linear embeddings of finite-dimensional grassmannians and flag varieties. If T is a finite-dimensional space and $\underline{d} = \{d^1, \dots, d^l\}$ is a vector of positive integers satisfying $d^k < d^l < \dim T$ for $k < l$, then $Fl(\underline{d}, T)$ denotes the variety of all flags of subspaces $T_1 \subsetneq \dots \subsetneq T_l \subsetneq T$, where $\dim T_j = d^j$. If \underline{d} consists of one integer d , we write simply $Gr(d, T)$. If T is endowed with a non-degenerate symmetric or antisymmetric (symplectic) form, we write respectively $FIO(\underline{d}, T)$ and $FIS(\underline{d}, T)$ for the varieties of isotropic flags in T with respect to the fixed form. We also write $GrO(d, T)$ and $GrS(d, T)$. An isotropic flag has always length less or equal $\frac{\dim T}{2}$ but, for convenience, in this paper by an *isotropic flag* we will mean a flag of the form

$$W_1 \subset W_2 \subset \dots \subset W_k \subset W_k^\perp \subset \dots \subset W_1^\perp,$$

where the spaces W_1, \dots, W_k are isotropic and the spaces $W_k^\perp, \dots, W_1^\perp$ are coisotropic. All flag varieties $FIO(\underline{d}, T)$ and $FIS(\underline{d}, T)$ are connected, except $GrO(d, T)$ for $\dim T = 2d$. In what follows, by $FIO(\underline{d}, T)$ or $GrO(\underline{d}, T)$ we always denote a connected component.

The Picard group of any grassmannian or ind-grassmannian Z is isomorphic to \mathbb{Z} except in the case of $GrO\left(\frac{\dim T}{2} - 1, T\right)$ for $2 < \dim T \in 2\mathbb{Z}_{>0}$, and $\mathcal{O}_Z(1)$ always denotes the ample generator of $\text{Pic } Z$. In the case of $GrO\left(\frac{\dim T}{2} - 1, T\right)$ for $2 < \dim T \in 2\mathbb{Z}_{>0}$ we have $\text{Pic } GrO\left(\frac{\dim T}{2} - 1, T\right) \simeq \mathbb{Z} \times \mathbb{Z}$.

The automorphism groups of the flag varieties $Fl(\underline{d}, T)$, $FIO(\underline{d}, T)$, $FIS(\underline{d}, T)$ have been known for long time. The fact that the automorphism group of the projective space \mathbb{P}^n is $PGL(n+1)$ goes back to the nineteenth century. Wei-Liang Chow [2] extended this result to grassmannians in 1949. For a general flag variety $X = Fl(\underline{d}, T)$, $FIO(\underline{d}, T)$, $FIS(\underline{d}, T)$ the connected component of the identity in the automorphism group $\text{Aut } X$ is the respective group $PGL(T)$, $SO(T)$, or $Sp(T)$, except in several cases listed by A.L. Onishchik in [12]. These special cases are $GrS(1, T) \simeq \mathbb{P}(T)$, the five dimensional quadric $GrO(1, T)$ for $\dim T = 7$, and $GrO\left(\frac{\dim T - 1}{2}, T\right)$ for $\dim T \in 2\mathbb{Z}_{>0} + 1$.

In all cases, see for instance [1, Section 3.3], the full automorphism group G is always a semidirect product of its connected component of unity G^0 and a finite group of automorphisms of the Dynkin diagram of the Lie algebra $\mathfrak{g} = \text{Lie } G^0$ which keep fixed the simple roots of the Lie algebra of the isotropy subgroup of a point on the respective flag variety. In the present paper we only consider classical groups of large enough rank, hence we can summarize the relevant part of this result as follows:

- for $\dim T \geq 3$,

$$\text{Aut } Fl(\underline{d}, T) \simeq \begin{cases} PGL(T) \times \mathbb{Z}_2 \text{ if } \underline{d} = (d^0 = 0, d^1, d^2, \dots, d^s, d_n^{s+1} = \dim T) \text{ satisfies} \\ \text{the condition } d^i - d^{i-1} = d^{s+2-i} - d^{s+1-i} \text{ for all } 1 \leq s \leq n + 1 \\ PGL(T) \text{ in all other cases.} \end{cases}$$

- for $\dim T \geq 8$, $\text{Aut } FlO(\underline{d}, T) \simeq O(T)$, except for $GrO(\frac{\dim T}{2}, T)$ where $\text{Aut } GrO(\frac{\dim T}{2}, T) \simeq SO(T)$, and for $GrO(\frac{\dim T - 1}{2}, T)$ where $\text{Aut } GrO(\frac{\dim T - 1}{2}, T) \simeq SO(T')$ for $\dim T' = \dim T + 1$.
- for $\dim T \geq 4$, $\text{Aut } FlS(\underline{d}, T) \simeq Sp(T)$, except for $GrS(1, T)$ where $\text{Aut } GrS(1, T) \simeq PGL(T)$.

Based on the above exception concerning $GrO(\frac{\dim T - 1}{2}, T)$ for $\dim T \in 2\mathbb{Z}_{>0} + 1$, in what follows we will automatically assume that this case is excluded from consideration. This leads to no loss of generality as $GrO(\frac{\dim T - 1}{2}, T)$ is isomorphic to $GrO(\frac{\dim T'}{2}, T')$ where T' is an orthogonal space of dimension $\dim T + 1$.

A nice class of embeddings of flag varieties $Fl(\underline{d}_1, T_1) \hookrightarrow Fl(\underline{d}_2, T_2)$ for $\dim T_1 < \dim T_2$ is the class of standard extensions. Embeddings of ind-grassmannians are discussed in detail in [14], and of arbitrary flag varieties in [15]. Here we just recall a definition and a basic fact needed to understand our arguments in Sections 5, 6, and 8.

Definition 4.1 a) An embedding

$$\eta: Fl(d_1^1, \dots, d_1^k, T) \hookrightarrow Fl(d_2^1, \dots, d_2^l, T'),$$

respectively,

$$\eta: FlO(d_1^1, \dots, d_1^k, T) \hookrightarrow FlO(d_2^1, \dots, d_2^l, T'),$$

respectively,

$$\eta: FlS(d_1^1, \dots, d_1^k, T) \hookrightarrow FlS(d_2^1, \dots, d_2^l, T')$$

is a *strict standard extension* if there exists a surjection

$$p: \{0, 1, \dots, l, l + 1\} \rightarrow \{0, 1, \dots, k, k + 1\}$$

satisfying $p(i) \leq p(j)$ for $i < j$, together with an isomorphism

$$V' = V \oplus \widehat{W}, \tag{7}$$

satisfying $\widehat{W} = V^\perp$ in the orthogonal and symplectic case, and subspaces $W_i \in \widehat{W}$ for $1 \leq i \leq l$ with $W_i \subset W_j$ for $i < j$ such that η has the form

$$\begin{aligned} \eta(\{0\}) &= V_{p(0)} \subset V_{p(1)} \subset \dots \subset V_{p(l)} \subset V \\ &= (\{0\} = V_{p(0)} \subset V_{p(1)} \oplus W_1 \subset \dots \subset V_{p(l)} \oplus W_l \subset V'). \end{aligned} \tag{8}$$

Here the spaces W_1, \dots, W_l are not required to be pairwise distinct, while the spaces in the right-hand side of (8) are pairwise distinct by definition. In the orthogonal or symplectic case we require that for each i there is j such that $W_j = W_i^\perp$.

b) An embedding

$$\eta: Fl(d_1^1, \dots, d_1^k, T) \hookrightarrow Fl(d_2^1, \dots, d_2^l, T')$$

is a *standard extension* if after composing with one of the duality isomorphisms

$$Fl(d_1^1, \dots, d_1^k, T) \simeq Fl(\dim T - d_1^k, \dots, \dim T - d_1^1, T^*),$$

$$Fl(d_2^1, \dots, d_2^l, T') \simeq Fl(\dim T' - d_2^l, \dots, \dim T' - d_2^1, T'^*)$$

η becomes a strict standard extension. For varieties of isotropic flags, standard extension and strict standard extension are synonyms.

The following theorem follows directly from Corollary 4.4 in [15].

Theorem 4.1 *Let $X_1 \hookrightarrow X_2$ and $Y_1 \hookrightarrow Y_2$ be embeddings of flag varieties or of isotropic flag varieties, such that Y_1 is the image of X_1 under some isomorphism $\varphi: X_2 \rightarrow Y_2$. Then the embedding $Y_1 \hookrightarrow Y_2$ is a standard extension whenever the embedding $X_1 \hookrightarrow X_2$ is a standard extension.*

We need to recall also the notion of an ind-variety of isotropic generalized flags. There are several cases. If a symmetric non-degenerate form (\cdot, \cdot) on V is given, then there are two types of relevant bases \tilde{E} we consider:

$$\{e_i\}_{i \in \mathbb{Z}} \text{ with } (e_i, e_{-i}) = 1 \text{ for all } i, (e_i, e_k) = 0 \text{ for } k \neq -i,$$

or

$$\{e_i\}_{i \in \mathbb{Z} \setminus \{0\}} \text{ with } (e_i, e_{-i}) = 1 \text{ for all } i \in \mathbb{Z}_{>0}, (e_i, e_k) = 0 \text{ for } k \neq -i. \quad (9)$$

In the case of a symplectic non-degenerate form on V , we consider bases satisfying (9) (here $(e_{-i}, e_i) = -1$ for $i \in \mathbb{Z}_{>0}$). We refer to bases as above as *isotropic bases* of V . An *isotropic generalized flag* in V is by definition a generalized flag $W = \{W_\alpha\}$ in V such that each space W_α is either isotropic or coisotropic and W_α belongs to W if and only if $W_\alpha^\perp = \{w' \in V \mid (w', w) = 0 \text{ for all } w \in W_\alpha\}$ belongs to W . If W is an E -compatible isotropic generalized flag in V , then by definition, $FLO(W, \tilde{E}, V)$ in the case of a symmetric form, or $FIS(W, \tilde{E}, V)$ in the case of a symplectic form, consists of all isotropic generalized flags in V which are \tilde{E} -commensurable with W . In all cases, $FLO(W, \tilde{E}, V)$ or, respectively, $FIS(W, \tilde{E}, V)$ is a direct limit of finite-dimensional varieties of isotropic flags under standard extensions.

Finally, we should point out that in order to follow the proof of our main results in Sections 5–8 the readers should first familiarize themselves with the results of the [Appendix](#).

5 Proof of Theorem 2.1 for ind-Grassmannians

We start by proving Theorem 2.1 under the assumption that the generalized flag W has exactly one proper subspace, which we also denote by W . In what follows, we write most of the time $Gr(W, E, V)$ instead of $Gr(W, \tilde{E}, V)$, where $E = \tilde{E} \cap W$. The set E is a basis of W , and the ind-variety $Gr(W, \tilde{E}, V)$ depends only on E

and not on the entire basis \tilde{E} . We feel that this notation makes the argument more transparent. Also, the space W is fixed and we write V_E and V_{*E} instead of $V_{\tilde{E}}^W$ and $V_{*\tilde{E}}^W$, respectively.

Recall that $Gr(W, E, V)$ is defined as the direct limit of strict standard extensions

$$Gr(d_n, V_n) \hookrightarrow Gr(d_{n+1}, V_{n+1})$$

for some d_n and some subspaces $V_n \subset V$, $\dim V_n \geq n$, $\varinjlim V_n = V$. If d_n stabilizes at $k \in \mathbb{Z}_{>0}$ for large n , then $\varinjlim Gr(d_n, V_n) = Gr(k, V)$ is the ind-grassmannian of all k -dimensional subspaces in \tilde{V} . If $\dim V_n - d_n$ stabilizes at $k > 0$, then $Gr(W, E, V)$ is isomorphic to $Gr(k, V)$ as an ind-variety via the map

$$\delta: Gr(W, E, V) \rightarrow Gr(k, V_*),$$

$$W' \mapsto W'^{\perp} \subset V_*,$$

where W' denotes a variable point of $Gr(W, E, V)$ and $W'^{\perp} := \{\alpha \in V_* \mid \alpha(w') = 0 \forall w' \in W'\}$.

As we already mentioned, the automorphism groups of (finite-dimensional) grassmannians have been described in the classical paper [2]. This description implies that if

$$\alpha: Gr(d_n, T_n) \xrightarrow{\sim} Gr(d_n, T'_n) \tag{10}$$

is any isomorphism of grassmannians, where $\dim T_n = \dim T'_n$, then the pullback $\alpha^*S'_n$ of the tautological bundle S'_n on $Gr(d_n, T'_n)$ is isomorphic to the tautological bundle S_n on $Gr(d_n, T_n)$, or to the bundle $(\tilde{T}_n/S_n)^*$ in case $2d_n = n$, where \tilde{T}_n is the trivial bundle on $Gr(d_n, T_n)$ with fiber T_n . Moreover, if $\alpha^*S'_n \simeq S_n$, then the isomorphism (10) is determined by the linear operator $\eta: (T'_n)^* \rightarrow T_n^*$ which it induces via pullback: we have $\alpha(T_{d_n}) = \eta^*(T_{d_n})$ where $T_{d_n} \in Gr(d_n, T_n)$ and the operator η^* is dual to η . Recall also that any global endomorphism of the bundle S_n or $(\tilde{T}/S_n)^*$ is scalar.

Set $X_n = Gr(d_n, V_n)$. Then $Gr(W, E, V) = \varinjlim X_n$. In the rest of the argument we assume in addition that $\dim W = \text{codim}_V W = \infty$ and that $d_n = n, \dim V_n = 2n$. This is the case of the Sato grassmannian. The remaining cases where $\dim W < \infty$ or $\text{codim}_V W < \infty$ have been considered in [13], and it has been proved there that $\text{Aut } Gr(W, E, V) \cong PGL(V)$ for $\dim W < \infty$, and $\text{Aut } Gr(W, E, V) \cong PGL(V_*)$ for $\text{codim } W < \infty$. This is in agreement with Theorem 2.1, as in these two cases W is not symmetric and there is an isomorphism

$$GL(\tilde{E}, V) \cdot St_W \cong \begin{cases} GL(V) & \text{for } \dim W < \infty \\ GL(V_*) & \text{for } \text{codim}_V W < \infty. \end{cases} \tag{11}$$

We leave the proof of (11) as an exercise to the reader.

Our first step will be to prove that the group of automorphisms of the ind-variety $Gr(W, E, V)$ is a subgroup of $P(GL(\tilde{E}, V) \cdot St_W) \rtimes \mathbb{Z}_2$. Let $\tilde{\varphi}: Gr(W, E, V) \rightarrow Gr(W, E, V)$ be an arbitrary automorphism. Denote by Y_n the image of X_n , that is, $\tilde{\varphi}(X_n) = Y_n$. Clearly, Y_n is a grassmannian isomorphic to X_n . Moreover, the embeddings $Y_n \hookrightarrow Y_{n+1}$ can be assumed to be strict standard extensions by Theorem 4.1.

Next, we have two possibilities: for some n , the isomorphism

$$\tilde{\varphi}_n := \tilde{\varphi}|_{X_n} : X_n \longrightarrow Y_n$$

has the property

$$\tilde{\varphi}_n^* S_{Y_n} \cong (\tilde{V}_n/S_n)^*, \tag{12}$$

where S_n is the tautological bundle on X_n and S_{Y_n} is the tautological bundle on Y_n , or the property

$$\tilde{\varphi}_n^* S_{Y_n} \cong S_n \tag{13}$$

for all n . Since our chains of embeddings $X_n \hookrightarrow X_{n+1}$ and $Y_n \hookrightarrow Y_{n+1}$ are strict standard extensions, if the isomorphism (12) holds for some n , it must hold for all n . However, if this happens we can compose our automorphism $\tilde{\varphi}$ with the following automorphism which represents an element of \mathbb{Z}_2 in the semidirect product from the statement of the theorem:

$$\delta : Gr(W, \tilde{E}, V) \longmapsto Gr(W^\perp, \tilde{E}^*, V_*) \longrightarrow Gr(W, \tilde{E}, V),$$

where the left arrow sends W to $W^\perp \subset V_*$ and the right arrow is induced by an appropriate linear isomorphism

$$V_* \longrightarrow V$$

which maps W^\perp to W . Option (13) certainly holds for the composition $\delta \circ \tilde{\varphi}$, so without loss of generality we can assume in the rest of the argument that (13) holds.

The latter assumption implies

$$\tilde{\varphi}_n^* S_{Y_n}^* = S_n^* \tag{14}$$

for all n . We write equality, as such an isomorphism is determined up to a scalar c_n , and we assume that the scalars c_n are chosen in a way compatible with the restriction maps

$$\begin{array}{ccc} \tilde{\varphi}_n^* S_{Y_n}^*|_{X_{n-1}} & \longrightarrow & \tilde{\varphi}_{n-1}^* S_{Y_{n-1}}^* \\ \parallel & & \parallel \\ S_n^*|_{X_{n-1}} & \longrightarrow & S_{n-1}^*. \end{array}$$

Then, by our above remark that isomorphisms of grassmannians are encoded by linear operators, the isomorphisms $\tilde{\varphi}_n : X_n \xrightarrow{\sim} Y_n$ are recovered by a choice of compatible invertible linear operators

$$\varphi_n^* : (V_n')^* = H^0(Y_n, S_{Y_n}^*) \xrightarrow{\sim} H^0(X_n, S_n^*) = V_n^*.$$

The operators φ_n^* are dual to unique operators $\varphi_n = (\varphi_n^*)^* : V_n \rightarrow V_n'$ which we will also consider.

Since both chains of embeddings $X_n \hookrightarrow X_{n+1}$ and $Y_n \hookrightarrow Y_{n+1}$ are strict standard extensions, we have $\varinjlim V_n' = V = \varinjlim V_n$, and consequently, $\varprojlim (V_n')^* = V^* = \varprojlim V_n^*$. Therefore, the operators φ_n^* induce a linear operator

$$\Phi : V^* \rightarrow V^*,$$

and more precisely, a commutative diagram

$$\begin{array}{ccc}
 V^* & \xleftarrow{\Phi} & V^* \\
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 V_n^* & \xleftarrow{\varphi_n^*} & (V'_n)^* \\
 \downarrow & & \downarrow \\
 V_{n-1}^* & \xleftarrow{\varphi_{n-1}^*} & (V'_{n-1})^* \\
 \vdots & & \vdots
 \end{array} \tag{15}$$

The diagram (15) encodes the automorphism $\tilde{\varphi}$ in the following way. Let $W' = \varinjlim (W' \cap V_n)$ be a point of $Gr(W, E, V)$. Then

$$\tilde{\varphi}(W') = \varinjlim \varphi_n(W' \cap V_n) = \varinjlim \left((\varphi_n^*)^{-1} \left((W' \cap V_n)^\perp \right)^\perp \right), \tag{16}$$

where the orthogonal to $W' \cap V_n$ is taken in V_n^* and the orthogonal to $(\varphi_n^*)^{-1} \left((W' \cap V_n)^\perp \right)^\perp$ is taken in V'_n .

Next, it is essential to observe that the subspace $V_{*E} \subset V^*$ is nothing but the subspace of global sections μ of the sheaf $\varprojlim S_n^*$ satisfying the condition: the value of μ at any point $W' \in Gr(W, E, V)$ is a linear function on W' which belongs to the subspace $W'_* \subset W'^*$. Here W'_* is defined in terms of a basis of W' which differs from E by finitely many vectors (W'_* is the span of the system of linear functionals dual to such a basis). Note that, for each $W' \in Gr(W, E, V)$, the subspace $W'_* \subset W'^*$ is determined solely by the set $Gr(W, E, V)$, and hence the above subspace of global sections μ , i.e., the space V_{*E} , must be invariant under the operator Φ . Next, the subspace V_E is the counterpart of the space V_{*E} for the ind-grassmannian $Gr(W^\perp, \tilde{E}^*, V_*)$, where V_* is defined by the fixed extension \tilde{E} of E to a basis of V , and E^* are the linear functions in \tilde{E}^* which do not vanish on W . Since $Gr(W^\perp, \tilde{E}^*, V_*)$ is isomorphic to $Gr(W, E, V)$, the space V_E is also invariant under the linear map $\Phi': (V_*)^* \rightarrow (V_*)^*$ induced by the automorphism $\tilde{\varphi}$ of $Gr(W, E, V)$.

We have shown that any automorphism $\tilde{\varphi}: Gr(W, E, V) \rightarrow Gr(W, E, V)$ satisfying (13) induces a pair of invertible operators $\varphi := \Phi'|_{V_E}: V_E \rightarrow V_E$ and $\bar{\varphi} := \Phi|_{V_{*E}}: V_{*E} \rightarrow V_{*E}$ which determine an element of the Mackey group $G(V_E, V_{*E})$. Moreover, if $\text{Aut}^0 Gr(W, E, V)$ stands for the group of automorphisms of the ind-variety $Gr(W, E, V)$ satisfying (13), then the assignment $\tilde{\varphi} \mapsto \varphi$, or equivalently $\tilde{\varphi} \mapsto \bar{\varphi}^{-1}$, defines an injective group homomorphism

$$\varepsilon: \text{Aut}^0 Gr(W, E, V) \hookrightarrow PG(V_E, V_{*E}).$$

We now check that the action of the image in $PG(V_E, V_{*E})$ of $\text{Aut}^0 Gr(W, E, V)$ is given by the formula

$$\tilde{\varphi}(W') = \bar{\varphi}^{-1} \left((W'^{\perp})^{\perp} \right) \cap V,$$

where $W'^{\perp} \subset V_{*E}$ and $\bar{\varphi}^{-1} \left((W'^{\perp})^{\perp} \right) \subset (V_{*E})^*$ (clearly, $V \subset (V_{*E})^*$). To do this, recall that $\tilde{\varphi}(W')$ is given by formula (16). Therefore, we need to verify that

$$\lim_{\rightarrow} \left((\varphi_n^*)^{-1} \left((W' \cap V_n)^{\perp} \right)^{\perp} \right) = \bar{\varphi}^{-1} \left(\left(\lim_{\leftarrow} \left((W' \cap V_n)^{\perp} \right) \right)^{\perp} \right) \cap V. \tag{17}$$

However, formula (17) follows from the observation that both its left-hand and right-hand sides coincide with the subspace of vectors in V which vanish on $(\varphi_n^*)^{-1} \left((W' \cap V_n)^{\perp} \right)^{\perp}$ whenever they belong to V_n ; we consider here vectors in V_n as linear functions on V_n^* .

In conclusion, the image of $\text{Aut}^0 Gr(W, E, V)$ in $PG(V_E, V_{*E})$ is a subgroup of $PG(V_E, V_{*E})$ which acts on $Gr(W, E, V)$ via formula (17). Next, we note that since $GL(\tilde{E}, V)$ acts transitively on $Gr(W, E, V)$, for any $\tilde{\varphi} \in \text{Aut}^0 Gr(W, E, V)$ there are $\varkappa \in GL(\tilde{E}, V)$ and $\tilde{\varphi}_W \in \text{Aut}^0 Gr(W, E, V)$ such that $\tilde{\varphi}_W(W) = W$ and $\tilde{\varphi} = \varkappa \circ \tilde{\varphi}_W$. Indeed if $\tilde{\varphi}(W) = W'$ for $W' \in Gr(W, E, V)$, then $\tilde{\varphi} = \varkappa^{-1} \circ \varkappa \circ \tilde{\varphi}$ where $\varkappa \in GL(\tilde{E}, V)$ satisfies $\varkappa(W') = W$. Consequently, $\varphi = \varkappa^{-1} \varphi_W$ for $\varphi_W \in St_W$, in other words, the image of ε lies in $P(GL(\tilde{E}, V) \cdot St_W)$.

To complete the proof, we need to show that any operator $\varphi \in GL(\tilde{E}, V) \cdot St_W$ determines a well-defined automorphism of $Gr(W, E, V)$. In the Appendix we introduce the degree $d(\varphi)$ of an operator φ and show that the space $\varphi \cdot W' = \bar{\varphi}^{-1} (W'^{\perp})^{\perp} \cap V$ is \tilde{E} -commensurable with W for any $\varphi \in G(V_E, V_{*E})$ such that $d(\varphi) = 0$. Denote by $G^0(V_E, V_{*E})$ the group of all operators $\varphi \in G(V_E, V_{*E})$ with $d(\varphi) = 0$.

We have to convince ourselves that the action of $G^0(V_E, V_{*E})$ on $Gr(W, E, V)$ is by automorphisms of ind-varieties, and not merely by bijections of the set $Gr(W, E, V)$. Let $V = W \oplus U$ where $\tilde{E} \cap U$ spans U . Recall that our fixed nested finite-dimensional spaces $V_1 \subset \dots \subset V_n \subset V_{n+1} \subset \dots$ are spanned by elements of \tilde{E} and $Gr(W, E, V) = \lim Gr(d_n, V_n)$ where $d_n = \dim V_n \cap W$. The embeddings $X_n = Gr(d_n, V_n) \hookrightarrow Gr(\overrightarrow{d}_{n+1}, V_{n+1}) = X_{n+1}$ are strict standard extensions

$$F_{d_n} \longmapsto F_{d_n} \oplus W_{n|n+1},$$

where $W_{n+1} = W \cap V_{n+1} = W_n \oplus W_{n|n+1}$ for $W_n = W \cap V_n$, and $W_{n|n+1} \cap \tilde{E}$ spans $W_{n|n+1}$. Therefore, we have a decomposition $V_{*E} = \overline{W}_{n*} \oplus V_n^* \oplus U_n^*$, where \overline{W}_{n*} is the \tilde{E} -compatible direct complement of V_n^* within $W_* + V_n^*$ and U_n^* is the dual of the \tilde{E} -compatible direct complement U_n of $W + V_n$ in V . Any invertible linear operator $\zeta : V_{*E} \rightarrow V_{*E}$ induces a decomposition

$$V_{*E} = \zeta(\overline{W}_{n*}) \oplus \zeta(V_n^*) \oplus \zeta(U_n^*),$$

and hence an operator

$$\zeta_n : V_n^* := V_{*E}/(\overline{W}_{n*} \oplus U_n^*) \longrightarrow V_{*E}/(\zeta(\overline{W}_{n*}) \oplus \zeta(U_n^*)) = \zeta_n(V_n^*).$$

Moreover, we have $\zeta = \left(\lim_{\leftarrow} \zeta_n \right) |_{V_{*E}}$.

Each linear operator ζ_n induces an isomorphism of grassmannians

$$Gr(d_n, \zeta_n(V_n^*)^*) \longrightarrow Gr(d_n, V_n),$$

and the varieties $Gr(d_n, \zeta_n(V_n^*)^*)$ form an ind-variety isomorphic to $Gr(W, E, V)$. We conclude that if $\varinjlim Gr(d_n, \zeta_n(V_n^*)^*) = Gr(W, E, V)$, then ζ induces an automorphism of the ind-variety $Gr(W, E, V)$. Set now $\zeta := \bar{\varphi}^{-1}$ for $\varphi \in G^0(V_E, V_{*E})$. Then $\zeta_n = (\varphi_n^*)^{-1}$ and $\zeta_n(V_n^*)^* = \varphi_n(V_n)$. By the above mentioned result from the [Appendix](#), we know that $\varphi \cdot W' \in Gr(W, E, V)$ whenever $\varphi \in G^0(V_E, V_{*E})$ and $W' \in Gr(W, E, V)$. Therefore $\varinjlim Gr(d_n, \varphi_n(V_n)) = Gr(W, E, V)$, and we have shown that $PG^0(V_E, V_{*E}) \subset \text{im } \varepsilon$.

Finally, Theorem A.1 (i) from the [Appendix](#) implies that $St_W \subset G^0(V_E, V_{*E})$, and hence that also $GL(\tilde{E}, V) \cdot St_W \subset G^0(V_E, V_{*E})$. Consequently, $GL(\tilde{E}, V) \cdot St_W = G^0(V_E, V_{*E})$ and $GL(\tilde{E}, V) \cdot St_W$ is a group. In particular, $GL(\tilde{E}, V) \cdot St_W = St_W \cdot GL(\tilde{E}, V)$. The proof is now complete as we have shown that $\text{im } \varepsilon = P(GL(\tilde{E}, V) \cdot St_W)$.

Corollary 5.1 *If $\dim W = \text{codim}_V W = \infty$, the group $\text{Aut}Gr(W, E, V)$ is isomorphic to the projectivization of the connected component of unity in the group Japanese $GL(\infty)$.*

Proof We proved that $\text{Aut}Gr(W, E, V) \cong PG^0(V_E, V_{*E})$. As pointed out in the [Appendix](#), the group $G^0(V_E, V_{*E})$ is isomorphic to the connected component of unity in the group Japanese $GL(\infty)$. □

6 Proof of Theorem 2.1 in the General Case

Step 1. Reduction to the case of an automorphism which preserves all inverse limits of dual tautological bundles. Consider the ind-variety $Fl(W, \tilde{E}, V)$ for our fixed generalized flag $W = \{W_\alpha\}$ compatible with the fixed basis \tilde{E} of V . Fix an exhaustion of $Fl(W, \tilde{E}, V)$ as a direct limit $\varinjlim Fl(\underline{d}_n, V_n)$ of strict standard extensions of finite-dimensional flag varieties $Fl(\underline{d}_n, V_n)$. If the generalized flag W is symmetric, then the exhaustion can be chosen so that each flag variety $Fl(\underline{d}_n, V_n)$ is symmetric, i.e., the vector $(d_n^0 = 0, d_n^1, d_n^2, \dots, d_n^s, d_n^{s+1} = \dim V_n)$ satisfies the condition

$$d_n^i - d_n^{i-1} = d_n^{s+2-i} - d_n^{s+1-i}$$

for all $1 \leq s \leq n + 1$. If W is not symmetric, then infinitely many vectors \underline{d}_n are not symmetric, so (by passing to a subsequence of the sequence $\{n\}$) we can assume that all vectors \underline{d}_n are not symmetric.

Let $\tilde{\varphi}: Fl(W, \tilde{E}, V) \rightarrow Fl(W, \tilde{E}, V)$ be an automorphism. Set $X_n := Fl(\underline{d}_n, V_n)$ and let $Y_n = \tilde{\varphi}(X_n)$. The varieties are finite-dimensional flag varieties, and there are fixed isomorphisms $\tilde{\varphi}_n := \tilde{\varphi}|_{X_n}: X_n \xrightarrow{\sim} Y_n$. Moreover, Theorem 4.1 implies that the embeddings $Y_n \hookrightarrow Y_{n+1}$ are standard extensions. By replacing Y_n or Y_{n+1} by its dual flag variety we can further assume that the embeddings $Y_n \hookrightarrow Y_{n+1}$ are strict standard extensions.

Denote by S_n^j and $S_{Y_n}^j$ the tautological bundles of rank d_n^j on X_n and Y_n , respectively. There are two possibilities: either

$$\tilde{\varphi}_n^* S_{Y_n}^j \simeq S_n^j \tag{18}$$

for all n , or

$$\tilde{\varphi}_n^* S_{Y_n}^j \simeq (\tilde{V}_n / S_n^{s+1-j})^* \tag{19}$$

for some $n = n_0$ and all $j, 1 < j \leq s$. Case (19) can occur only if the vector \underline{d}_{n_0} is symmetric. Moreover, then (19) will necessarily hold for all $n > n_0$ due to the assumption that all embeddings $X_n \hookrightarrow X_{n+1}$ and $Y_n \hookrightarrow Y_{n+1}$ are strict standard extensions. In that case, we may as well assume that (19) holds for all n .

Similarly to the case of an ind-grassmannian, if (19) holds for all n we can compose $\tilde{\varphi}$ with an automorphism

$$\delta : Fl(W, \tilde{E}, V) \xrightarrow{\sim} Fl(W^\perp, \tilde{E}^*, V_*) \xrightarrow{\sim} Fl(W, \tilde{E}, V)$$

which maps W first to W^\perp and then maps W^\perp to a point of $Fl(W, \tilde{E}, V)$ under the isomorphism $Fl(W^\perp, \tilde{E}^*, V_*) \xrightarrow{\sim} Fl(W, \tilde{E}, V)$ induced by an appropriate linear isomorphism $V_* \rightarrow V$ mapping \tilde{E}^* to \tilde{E} . Then the composition $\delta \circ \tilde{\varphi}$ satisfies the condition (18) for all n . Therefore, in order to prove Theorem 2.1, it suffices to prove that

$$\text{Aut}^0 Fl(W, \tilde{E}, V) \simeq P(GL(\tilde{E}, V) \cdot St_W)$$

where $\text{Aut}^0 Fl(W, \tilde{E}, V)$ denotes the group of automorphisms of $Fl(W, \tilde{E}, V)$ satisfying (18) for all n .

Step 2. From automorphisms to linear operators. Since (18) holds, the automorphism

$$\tilde{\varphi} : Fl(W, \tilde{E}, V) \rightarrow Fl(W, \tilde{E}, V)$$

induces automorphisms

$$\tilde{\varphi}_\alpha : Gr(W_\alpha, E_\alpha, V) \rightarrow Gr(W_\alpha, E_\alpha, V) \tag{20}$$

for each subspace W_α in W . In turn, the automorphisms (20) induce linear operators, defined up to scalar multiples,

$$\Phi_\alpha : V^* \rightarrow V^*$$

as explained in Section 5.

We now point out that the operators Φ_α can be chosen to coincide, i.e., to define a single operator

$$\Phi : V^* \rightarrow V^*$$

not depending on α . This observation is justified as follows. Denote by S_α^* the pullback to $Fl(W, \tilde{E}, V)$ of the inverse limit of the tautological bundles S_n^* on $Gr(W_\alpha, E_\alpha, V)$. Let $\alpha' < \alpha$ be two indices in the chain $W = \{W_\alpha\}$. Then the morphism of inverse limits $S_\alpha^* \rightarrow S_{\alpha'}^*$, arising from the respective morphism of inverse systems, induces a commutative diagram

$$\begin{CD} H^0(Fl(W, \tilde{E}, V), S_\alpha^*) @>>> H^0(Fl(W, \tilde{E}, V), S_{\alpha'}^*) \\ @| @| \\ V^* @= id = V^* \end{CD} \tag{21}$$

the vertical equalities being the identifications $V^* = \varprojlim V_n^* = \varprojlim H^0(Gr(d_n^\gamma, V_n), S_n^*)$ where $Gr(W_\gamma, E_\gamma, V) = \varprojlim Gr(d_n^\gamma, V_n)$ for $\gamma = \alpha$ and $\gamma = \alpha'$, respectively. Therefore, for a fixed α and all $\alpha' < \alpha$, the spaces of the form $H^0(S_{\alpha'}^*)$ are identified with V^* in a way compatible with the upper horizontal arrows of the diagrams (21). Next, the following diagram is commutative

$$\begin{array}{ccc}
 H^0(Fl(W, \tilde{E}, V), S_\alpha^*) & \longrightarrow & H^0(Fl(W, \tilde{E}, V), S_{\alpha'}^*) \\
 \downarrow \Phi_\alpha & \begin{array}{c} \parallel \\ \xrightarrow{id} \\ \parallel \end{array} & \downarrow \Phi_{\alpha'} \\
 V^* & \xrightarrow{id} & V^* \\
 \downarrow \Phi_\alpha & \begin{array}{c} \parallel \\ \xrightarrow{id} \\ \parallel \end{array} & \downarrow \Phi_{\alpha'} \\
 V^* & \xrightarrow{id} & V^* \\
 \downarrow \Phi_\alpha & \begin{array}{c} \parallel \\ \xrightarrow{id} \\ \parallel \end{array} & \downarrow \Phi_{\alpha'} \\
 H^0(Fl(W, \tilde{E}, V), \tilde{\varphi}^* S_\alpha^*) & \longrightarrow & H^0(Fl(W, \tilde{E}, V), \tilde{\varphi}^* S_{\alpha'}^*)
 \end{array}$$

and this implies $\Phi_\alpha = \Phi_{\alpha'}$.

Step 3. *Injective homomorphism* $\text{Aut}^0 Fl(W, \tilde{E}, V) \rightarrow PG(V_{\tilde{E}}^W, V_{*\tilde{E}}^W)$. Note first that each homomorphism

$$\begin{aligned}
 \text{Aut}^0 Fl(W, \tilde{E}, V) &\longrightarrow \text{Aut}^0 Gr(W_\alpha, \tilde{E}, V) \\
 \tilde{\varphi} &\longmapsto \tilde{\varphi}_\alpha
 \end{aligned}$$

is injective, since $\tilde{\varphi}_\alpha$ recovers $\tilde{\varphi}$ through the formula

$$\tilde{\varphi}(W')_{\alpha'} = \Phi^{-1}(W_{\alpha'}^\perp)^\perp \cap V$$

for any α' , where $W_{\alpha'}^\perp \subset V_{*E_{\alpha'}}^{W_{\alpha'}}$, $\Phi^{-1}(W_{\alpha'}^\perp)^\perp \subset V_{E_{\alpha'}}^{W_{\alpha'}}$ and, as explained above, $\Phi = \Phi_\alpha$.

Moreover, the automorphism $\Phi : V^* \rightarrow V^*$ induces pairs of automorphisms

$$\varphi_\alpha = \Phi'_\alpha|_{V_{E_\alpha}^{W_\alpha}} : V_{E_\alpha}^{W_\alpha} \longrightarrow V_{E_\alpha}^{W_\alpha}, \quad \bar{\varphi}_\alpha : V_{*E_\alpha}^{W_\alpha} \longrightarrow V_{*E_\alpha}^{W_\alpha}, \tag{22}$$

compatible with all inclusions of the form $V_{E_{\alpha'}}^{W_{\alpha'}} \subset V_{E_\alpha}^{W_\alpha}$, $V_{*E_\alpha}^{W_\alpha} \subset V_{*E_{\alpha'}}^{W_{\alpha'}}$ for $\alpha' < \alpha$, and such that $\varphi_\alpha \in G(V_{E_\alpha}^{W_\alpha}, V_{*E_\alpha}^{W_\alpha})$. The compatibility of φ_α and $\varphi_{\alpha'}$ is clear, while the compatibility of $\bar{\varphi}_\alpha$ and $\bar{\varphi}_{\alpha'}$ follows from the inclusion

$$\varphi_{\alpha'}^{-1}(W_{\alpha'}^\perp)^\perp \subset \varphi_\alpha^{-1}(W_\alpha^\perp)^\perp,$$

where $W_\alpha^\perp \subset V_{*E_\alpha}^{W_\alpha}$, $W_{\alpha'}^\perp \subset V_{*E_{\alpha'}}^{W_{\alpha'}}$, $\varphi_\alpha^{-1}(W_\alpha^\perp)^\perp \subset V_{E_\alpha}^{W_\alpha}$, $\varphi_{\alpha'}^{-1}(W_{\alpha'}^\perp)^\perp \subset V_{E_{\alpha'}}^{W_{\alpha'}}$. Therefore, we conclude that the system of linear operators (22) determines a unique element in $G(V_{\tilde{E}}^W, V_{*\tilde{E}}^W)$ for $V_{\tilde{E}}^W = \bigcap_\alpha V_{E_\alpha}^{W_\alpha}$ and $V_{*\tilde{E}}^W = \bigcap_\alpha V_{*E_\alpha}^{W_\alpha}$, and we obtain a homomorphism

$$\varepsilon : \text{Aut}^0 Fl(W, \tilde{E}, V) \longrightarrow PG(V_{\tilde{E}}^W, V_{*\tilde{E}}^W).$$

The fact that $GL(\tilde{E}, V)$ acts transitively on $Fl(W, \tilde{E}, V)$ shows, by the same argument as in Section 5, that the image of ε lies in the subset $P(GL(\tilde{E}, V) \cdot St_W)$ of $PG(V_{\tilde{E}}^W, V_{*\tilde{E}}^W)$.

We have to prove that ε is injective, i.e., that the image of ε determines all operators φ_α and $\bar{\varphi}_\alpha$ as above. We will do this by recalling that each operator $\varphi_\alpha : V_{E_\alpha}^{W_\alpha} \rightarrow V_{E_\alpha}^{W_\alpha}$ admits a matrix as described in the Appendix. The key point is that if an invertible operator on $V_{E_\alpha}^{W_\alpha}$ admits such a matrix, then this matrix is unique. Now the compatibility of the operators φ_α under all inclusions $V_{E_{\alpha'}}^{W_{\alpha'}} \hookrightarrow V_{E_\alpha}^{W_\alpha}$ for $\alpha' < \alpha$ implies that the matrices of all operators φ_α coincide. Since each homomorphism

$$\tilde{\varphi}_\alpha \longmapsto \varphi_\alpha$$

is injective according to Section 5, we conclude that ε is injective.

Step 4. The image of ε . As a final step of the proof, we need to show that the image of ε coincides with the set $P(GL(\tilde{E}, V) \cdot St_W)$. For this it suffices to prove that $P(GL(\tilde{E}, V) \cdot St_W)$ belongs to the image of ε , i.e., that $GL(\tilde{E}, V) \cdot St_W$ acts on the ind-variety $Fl(W, \tilde{E}, V)$ via the formula (4).

Pick an operator $\varphi \in GL(\tilde{E}, V) \cdot St_W$ and a flag $W' \in Fl(W, \tilde{E}, V)$. Since W and W' are \tilde{E} -commensurable, one has $GL(\tilde{E}, V) \cdot St_W = GL(\tilde{E}, V) \cdot St_{W'}$. Therefore $\varphi = \varkappa^{-1}\varphi_{W'}$ for some $\varphi_{W'} \in St_{W'}$ and some $\varkappa \in GL(\tilde{E}, V)$. Consequently, $\varphi(W') = \varkappa^{-1}(W')$, i.e., $\varphi^{-1}(W')$ is \tilde{E} -commensurable with W . In conclusion, $\text{im } \varepsilon = P(GL(\tilde{E}, V) \cdot St_W)$, and since $GL(\tilde{E}, V) \cdot St_W$ is a group we have also $\text{im } \varepsilon = P(St_W \cdot GL(\tilde{E}, V))$. The proof is complete.

7 An Explicit Matrix Form of the Group $GL(\tilde{E}, V) \cdot St_W$

Now we would like to characterize the product $GL(\tilde{E}, V) \cdot St_W$ in terms of matrices. We start by describing a matrix form of the group St_W .

Choose a linear order on \tilde{E} such that $e_j \in W_\alpha \setminus W_{\alpha'}$, $e_k \in W_{\alpha'}$ for $\alpha' < \alpha$ implies $k < j$. It follows from the Appendix that for each space W_α the stabilizer of W_α in $G(V_{E_\alpha}^{W_\alpha}, V_{*\tilde{E}_\alpha}^{W_\alpha})$ under the action (4) can be represented by infinite matrices (with rows and columns ordered by the fixed order on \tilde{E}) which, together with their inverses, have the form

$$\left(\begin{array}{c|c} A & B \\ \hline 0 & D \end{array} \right) \tag{23}$$

where A has finitary rows, D has finitary columns, and there are no restrictions on the rows and columns of B . Certainly, the splitting (23) depends on the space W_α .

Next, the fact that the operators φ_α and $\bar{\varphi}_\alpha$ form a system compatible with the inclusions $V_{E_{\alpha'}}^{W_{\alpha'}} \subset V_{E_\alpha}^{W_\alpha}$, $V_{*\tilde{E}_{\alpha'}}^{W_{\alpha'}} \subset V_{*\tilde{E}_\alpha}^{W_\alpha}$ for $\alpha' < \alpha$, implies that all operators φ_α , respectively $\bar{\varphi}_\alpha$, are represented by the same matrix. This means that all matrices (23) are just one matrix which satisfies the above conditions for all spaces W_α .

Consequently, St_W consists of matrices M which, together with their inverses, have the form

$$M = \begin{pmatrix} \ddots & & & & & & \\ & 0 & \boxed{A_{\alpha'}} & & & & \\ & & & \ddots & & & \\ & 0 & 0 & & \boxed{A_{\alpha}} & & \\ & 0 & 0 & & 0 & \ddots & \end{pmatrix}, \tag{24}$$

the rows and columns of M being ordered by the ordered set which orders the elements of \tilde{E} as above, and the diagonal blocks being of size $\dim(W_{\alpha_2}/W_{\alpha_1}) \times \dim(W_{\alpha_2}/W_{\alpha_1})$ where α is one of the indices α_1 and α_2 , and α_1 is the immediate predecessor of α_2 . This datum induces a block structure on the entire matrix M , and all strictly lower-triangular blocks are set to equal zero. Furthermore, since M and M^{-1} are subject to the above additional conditions for all splittings arising from spaces W_{α} of W , the matrices M and M^{-1} satisfy the following:

- if there exists a rightmost highest block L (this depends on the order on \tilde{E} , i.e., ultimately on the structure of the generalized flag W), then there are no conditions on the rows and columns of L , all columns of M which intersect L have at most finitely many nonzero entries outside of L , all rows of L which intersect L have at most finitely many nonzero entries outside of L , and all other rows and columns of M have at most finitely many nonzero entries in the region in which the first index is greater than the first index of any entry of L and the second index is smaller than the second index of any entry of L ;
- if M has no rightmost highest block, then all rows have at most finitely many nonzero entries in direction to the left (from any point on) and all columns have at most finitely many nonzero entries in the downward direction (from any point on).

In order to pass to the full group $GL(\tilde{E}, V) \cdot St_W$, consider matrices of the form (24) and replace the zeros in the lower-triangular part by finitely many nonzero entries. Let's refer to such matrices as W -aligned. If M is a W -aligned matrix, then every space W_{α} induces a splitting of M into four blocks

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \tag{25}$$

where C is a finitary matrix. A W -aligned matrix M is *eligible* if M^{-1} is also W -aligned, and for any α the splittings

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \text{ and } \left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array} \right)$$

of M and M^{-1} respectively, satisfy the condition

$$\operatorname{rk} C = \operatorname{rk} C'.$$

We leave it as an exercise to the reader to check that the condition of eligibility is empty (i.e., it is automatically satisfied) if the ordered set parameterizing the rows and columns of A or D is finite.

Our result in this section states as follows.

Theorem 7.1 *The group $GL(\tilde{E}, V) \cdot St_W$ is isomorphic to the group of all eligible W -aligned matrices.*

Proof Any matrix in $GL(\tilde{E}, V) \cdot St_W$ has the form $M_f M$ where M_f is an element of $GL(\tilde{E}, V)$ and M is a matrix from St_W having the form (24). The necessary and sufficient condition for $M_f M$ to lie in $GL(\tilde{E}, V) \cdot St_W$ is that the splitting (25) of $M_f M$ for any space W_α in W satisfies the condition of Theorem A.1. For all α these conditions amount precisely to the requirement that the matrix $M_f M$ be eligible. \square

8 The Isotropic Case

Theorem 8.1 *Let $X = FLO(W, \tilde{E}, V)$ or $FIS(W, \tilde{E}, V)$ for some isotropic generalized flag W compatible with an isotropic basis \tilde{E} of V . We assume in addition that W does not have the form $0 \subset W_1 \subset W_1^\perp \subset V$ where $\dim W_1 = 1$ in the symplectic case, and that $\dim W_1^\perp / W_1 \neq 2$ in the orthogonal case if the basis \tilde{E} has the form (9). Then*

$$\operatorname{Aut} X \simeq P(O(\tilde{E}, V) \cdot St_W^O)$$

or

$$\operatorname{Aut} X \simeq P(Sp(\tilde{E}, V) \cdot St_W^{Sp}),$$

where St_W^O and St_W^{Sp} denote respectively the stabilizer of W in $O(V)$ and $Sp(V)$. The action of $O(V)$ on $FLO(W, \tilde{E}, V)$, or respectively of $Sp(V)$ on $FIS(W, \tilde{E}, V)$, is induced by the linear action of $O(V)$ or $Sp(V)$ on V .

Proof Step 1. The case of an isotropic ind-grassmannian. The argument starts in the same way as for ordinary ind-grassmannians. The embeddings $X_n \hookrightarrow X_{n+1}$ are assumed to be standard extensions of isotropic grassmannians, and by Theorem 4.1 the embeddings $Y_n := \varphi(X_n) \xrightarrow{\eta_n} Y_{n+1} := \varphi(X_{n+1})$ are also standard extensions.

Next, in the isotropic case the isomorphisms (14) must hold, so we arrive to diagram (15) or, equivalently, to a commutative diagram

$$\begin{array}{ccc}
 V = \varinjlim V_n & \xrightarrow{\varinjlim \varphi_n} & V = \varinjlim V'_n \\
 \vdots & & \vdots \\
 \uparrow & & \uparrow \\
 V_n & \xrightarrow{\varphi_n} & V'_n \\
 \uparrow & & \uparrow \\
 V_{n-1} & \xrightarrow{\varphi_{n-1}} & V'_{n-1} \\
 \uparrow & & \uparrow \\
 \vdots & & \vdots
 \end{array}$$

The isomorphism $V = \varinjlim V'_n$ holds since the embeddings $Y_n \hookrightarrow Y_{n+1}$ are standard extensions, and hence any V'_n is a subspace of V_N for some $N > n$. Furthermore, the restriction of the fixed (symmetric or antisymmetric) form on V to V'_n is a form defining the corresponding isotropic grassmannian in V'_n . (Such a form is unique up to a scalar if W satisfies the conditions of the theorem.) Recall (from Section 5) that the maps φ_n are defined up to compatible scalars c_n . The key observation is that there is a unique choice of these scalars such that the maps φ_n are isomorphisms of orthogonal or, respectively, symplectic vector spaces. This follows from the fact that the automorphism groups of our finite-dimensional isotropic grassmannians X_n and Y_n are the respective orthogonal or symplectic groups.

Set

$$\varphi := \varinjlim \varphi_n : V \rightarrow V.$$

Then by construction φ is an orthogonal or, respectively, symplectic operator, and φ determines our automorphism $\tilde{\varphi}$ which acts on a point W' by formula (16). However, in the case considered we have

$$\varinjlim \left(\varphi_n \left((W' \cap V_n)^\perp \right)^\perp \right) = \varinjlim (\varphi_n (W' \cap V_n)) = \varphi(W'), \tag{26}$$

hence the action of $\tilde{\varphi}$ on X is simply induced by the action of φ as a linear automorphism of V . This implies that there is an injective homomorphism

$$\varepsilon : \text{AutGr}O(W, \tilde{E}, V) \hookrightarrow PO(V)$$

or, respectively,

$$\varepsilon : \text{AutGr}S(W, \tilde{E}, V) \hookrightarrow PSp(V),$$

$$\tilde{\varphi} \mapsto \varphi$$

and that the action of the image of ε on $GrO(W, \tilde{E}, V)$ or $GrS(W, \tilde{E}, V)$ is induced by the linear action of $O(V)$ or, respectively, $Sp(V)$ on V .

In the case of a general ind-variety of isotropic generalized flags, an injective homomorphism ε is constructed exactly as in Step 3 of the proof of Theorem

2.1. Here the images of all homomorphisms ε_α lie in $PO(V)$ or, respectively, $PSp(V)$, therefore ε is just the homomorphism into the intersection of all images of ε_α . Moreover, the image of ε coincides respectively with $P(O(\tilde{E}, V) \cdot St_W^O)$ or $P(Sp(\tilde{E}, V) \cdot St_W^{Sp})$. Indeed, since $O(\tilde{E}, V)$ or, respectively, $Sp(\tilde{E}, V)$, acts transitively on $FLO(W, \tilde{E}, V)$ or $FIS(W, \tilde{E}, V)$, the image of ε must be a subgroup of $P(O(\tilde{E}, V) \cdot St_W^O)$ or, respectively, $P(Sp(\tilde{E}, V) \cdot St_W^{Sp})$. On the other hand, by the same argument as in Section 6, $O(\tilde{E}, V) \cdot St_W^O$ or, respectively, $Sp(\tilde{E}, V) \cdot St_W^{Sp}$ acts on the ind-variety $FLO(W, \tilde{E}, V)$ or $FIS(W, \tilde{E}, V)$ via the formula (16). Since in our case the equality (26) holds, the proof is complete. \square

Corollary 8.1 *The group $O(\tilde{E}, V) \cdot St_W^O$, or $Sp(\tilde{E}, V) \cdot St_W^{Sp}$, is isomorphic to the group of all invertible W -aligned matrices M with finitary rows and columns satisfying $\overline{M} = \pm M^{-1}$, where $\overline{}$ denotes reflection along the antidiagonal, plus corresponds to the case of O and minus corresponds to the case of Sp .*

Proof The group $O(\tilde{E}, V) \cdot St_W^O$ or $Sp(\tilde{E}, V) \cdot St_W^{Sp}$ is clearly the intersection of the group $GL(\tilde{E}, V) \cdot St_W$ with $O(V)$ or, respectively, with $Sp(V)$, and this implies the claim. Note that the condition $\overline{M} = \pm M^{-1}$ makes M eligible automatically. \square

Appendix. On the Structure of the Group $G(V_E, V_{*E})$

In this Appendix we have collected some basic statements about the Mackey group $G(V_E, V_{*E})$. The isomorphism of this group to the group sometimes referred to as Japanese $GL(\infty)$ is stated at the end.

A.1 A Lemma from Linear Algebra

Lemma A.1 *A be a countable-dimensional vector space, and $\mu: A^* \rightarrow A$, $\nu: A^* \rightarrow A$ be linear maps such that $\alpha(\mu(\beta)) = \beta(\nu(\alpha))$ for all $\alpha, \beta \in A^*$. Then the dimensions of the images of μ and ν are finite and equal.*

Proof We first check that $\dim \operatorname{im} \mu < \infty$. This will imply that $\dim \operatorname{im} \nu$ is also finite because the conditions on μ and ν are symmetric. Pick a basis $\{f_1, f_2, \dots\}$ of A , and let $\{f_1^*, f_2^*, \dots\}$ be the dual system in A^* . Assume that $\operatorname{im} \mu$ is infinite dimensional. Then there exist linear functions $\lambda_n \in A^*$ for $n \in \mathbb{Z}_{>0}$, such that $\dim \langle \mu(\lambda_1), \dots, \mu(\lambda_n) \rangle_{\mathbb{C}} = n$. It is easy to see that these linear functions can be chosen so that $\lambda_n(f_i) = 0$ for $i < n$ and $\lambda_n(f_n) \neq 0$. Let $B := \langle \lambda_n | n \in \mathbb{Z}_{>0} \rangle_{\mathbb{C}}$. The formula $(a, b) \mapsto b(a)$ for $a \in A, b \in B$ defines a non-degenerate pairing between A and B . By a result of G. Mackey [10, Lemma on p. 171], we may assume without loss of generality that $\lambda_n = f_n^*$ for all n .

Since $\dim \operatorname{im} \mu = \infty$, for any $n \in \mathbb{Z}_{>0}$ there exists $i_n \in \mathbb{Z}_{>0}$ such that $\mu(f_{i_n}^*)$ is not contained in $A_n := \langle f_1, \dots, f_n \rangle_{\mathbb{C}}$. In other words, there exist two infinite sequences of integers $1 \leq i_1 < i_2 < \dots$ and $2 \leq k_1 < k_2 < \dots$ so that $\mu(f_{i_n}^*) \in A_{k_n} \setminus A_{k_n-1}$ for all $n \geq 1$.

Now we define a sequence c_1, c_2, \dots of complex numbers inductively as follows. Set c_1 to be an arbitrary nonzero scalar. For $n > 1$ we let c_n to be an arbitrary complex number such that

$$\sum_{j=1}^n c_j f_{k_j}^*(\mu(f_{i_n}^*)) = c_n f_{k_n}^*(\mu(f_{i_n}^*)) + \sum_{j=1}^{n-1} c_j f_{k_j}^*(\mu(f_{i_n}^*)) \neq 0.$$

(Such a number exists because $f_{k_n}^*(\mu(f_{i_n}^*)) \neq 0$.) Finally, we define $\alpha \in A^*$ by the formula

$$\alpha(f_j) = \begin{cases} c_n & \text{if } j = k_n \text{ for some } n \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$f_{i_n}^*(v(\alpha)) = \alpha(\mu(f_{i_n}^*)) = \sum_{j=1}^n c_j f_{k_j}^*(\mu(f_{i_n}^*)) \neq 0$$

for any $n \in \mathbb{Z}_{>0}$. This means that $v(\alpha) \notin A_{i_n-1}$ for all $n \geq 1$, i.e., that $v(\alpha)$ can not be expressed as a finite linear combination of f_1, f_2, \dots , and this is a contradiction.

Hence, $\dim \operatorname{im} \mu < \infty$, and also $\dim \operatorname{im} v < \infty$. It remains to check that $\dim \operatorname{im} \mu = \dim \operatorname{im} v$. Since $\dim \operatorname{im} \mu < \infty$, there exist $n \in \mathbb{Z}_{>0}$ and linear functions $\alpha_1, \dots, \alpha_n \in A^{**}$ such that $\mu(a) = \sum_{i=1}^n \alpha_i(a) f_i$ for all $a \in A^*$. We note that in fact $\alpha_i \in A$ for all i . Indeed,

$$f_j^*(\mu(a)) = f_j^*\left(\sum_{i=1}^n \alpha_i(a) f_i\right) = \alpha_j(a) = a(v(f_j^*)),$$

so $\alpha_j = v(f_j^*) \in A$ for all j . It follows that there exist $m \in \mathbb{Z}_{>0}$ and scalars $\alpha_{ij} \in \mathbb{C}$ so that $\alpha_j = \sum_{i=1}^m \alpha_{ij} f_i$ for all $1 \leq j \leq n$. Clearly, $\dim \operatorname{im} \mu = \operatorname{rk}(\alpha_{ij})_{i,j=1}^{m,n}$. Similarly, there exist $n', m' \in \mathbb{Z}_{>0}$, vectors $\beta_j \in A$, and scalars $\beta_{ij} \in \mathbb{C}$ so that $v(b) = \sum_{j=1}^{n'} \beta_j(b) f_j$ and $\beta_j = \sum_{i=1}^{m'} \beta_{ij} f_i$ for all $1 \leq j \leq n'$. Then $\dim \operatorname{im} v = \operatorname{rk}(\beta_{ij})_{i,j=1}^{m',n'}$. However, $\alpha_{ij} = f_i^*(\alpha_j) = f_i^*(v(f_j^*)) = f_j^*(\mu(f_i^*)) = f_j^*(\beta_i) = \beta_{ji}$ for all i, j . This means that $m' = n, n' = m$, and the matrices $(\alpha_{ij})_{i,j=1}^{m,n}$ and $(\beta_{ij})_{i,j=1}^{n,m}$ are transpose to each other. The result follows. \square

A.2 A Grading on the Group $G(V_E, V_{*E})$

We work in the setup of Section 5. Recall that we consider a generalized flag of the form $\{0\} \subset W \subset V$, where $\dim W = \operatorname{codim}_V W = \infty$, and that $E = \tilde{E} \cap W$ is a basis of W . Let $U = (\tilde{E} \setminus E)_{\mathbb{C}}$. We have $V_E = (W_*)^* \oplus U, V_{*E} = W_* \oplus U^*$ and there is a natural non-degenerate pairing $V_E \times V_{*E} \rightarrow \mathbb{C}$. The group $G(V_E, V_{*E})$ consists of invertible linear operators $\varphi: V_E \rightarrow V_E$ such that $\tilde{\varphi} = \varphi^*|_{V_{*E}}: V_{*E} \rightarrow V_{*E}$ is a well-defined isomorphism.

Choose an ordering of E via $\mathbb{Z}_{<0}$ and an ordering of $\tilde{E} \setminus E$ via $\mathbb{Z}_{>0}$. Set

$$E^k := \begin{cases} E \setminus \{e_k, \dots, e_{-1}\} & \text{for } k < 0 \\ E \cup \{e_1, \dots, e_k\} & \text{for } k > 0 \end{cases}$$

and $W^k := \langle E^k \rangle_{\mathbb{C}}$. The disjoint union $\bigsqcup_{k \in \mathbb{Z}} Gr(W^k, E^k, V)$, where $E = E^0$, $W = W^0$, is an ind-variety which we denote by X .

Proposition A.2 *The formula (4) defines an action of $G(V_E, V_{*E})$ on X by ind-variety automorphisms.*

Proof We first prove that $\varphi \cdot W' \in X$ for any $W' \in X$ and any $\varphi: V_E \rightarrow V_E$, $\varphi \in G(V_E, V_{*E})$. Since in the definition of X , W can be replaced by W' , it suffices to check that $\varphi \cdot W \in X$. Denote by $\pi_U: V_E \rightarrow U$ and $\pi_{W_*}: V_{*E} \rightarrow W_*$ the canonical projections onto the corresponding direct summands. Set also $\mu := \pi_U \circ (\varphi|_{(W_*)^*})$ and $\nu := \pi_{W_*} \circ (\overline{\varphi}|_{U^*})$.

We claim that $\dim \operatorname{im} \mu$ and $\dim \operatorname{im} \nu$ are finite (and equal). Fix an isomorphism of vector spaces $\eta: W_* \rightarrow U$ and the dual isomorphism $\eta^*: U^* \rightarrow (W_*)^*$. Consider the linear maps $\mu' = \mu \circ \eta^*$ and $\nu' = \eta \circ \nu$ from U^* to U . Then, for all $\alpha, \beta \in U^*$, we have

$$\begin{aligned} \alpha(\mu'(\beta)) &= \alpha(\mu(\eta^*(\beta))) = \alpha(\pi_U(\varphi(\eta^*(\beta)))) = \alpha(\varphi(\eta^*(\beta))) = \overline{\varphi}(\alpha)(\eta^*(\beta)) \\ &= \pi_{W_*}(\overline{\varphi}(\alpha))(\eta^*(\beta)) = \nu(\alpha)(\eta^*(\beta)) = \eta(\nu(\alpha))(\beta) = \beta(\nu'(\alpha)). \end{aligned}$$

Hence, $\dim \operatorname{im} \mu' = \dim \operatorname{im} \nu' < \infty$ (and, consequently, $\dim \operatorname{im} \mu = \dim \operatorname{im} \nu < \infty$) by Lemma A.1.

Now, denote by A_0 the preimage of zero under the restriction of π_U to $\varphi((W_*)^*)$. Obviously, $A_0 = (W_*)^* \cap \varphi((W_*)^*)$. According to the proof of Lemma A.1, there exist $n \in \mathbb{Z}_{>0}$ and $\alpha_1, \dots, \alpha_n \in W_*$ such that

$$\mu(\omega) = \pi_U(\varphi|_{(W_*)^*}(\omega)) = \sum_{i=1}^n \omega(\alpha_i)e_i$$

for all $\omega \in (W_*)^*$. A vector ω of $(W_*)^*$ belongs to A_0 if and only if $\mu(\omega) = 0$, i.e., if $\omega(\alpha_i) = 0$ for $1 \leq i \leq n$. It follows immediately that $A_0 \cap W$ has finite codimension in W and, moreover, given $w \in W$, one has $w \in A_0$ if and only if $\alpha_i(w) = 0$ for all $1 \leq i \leq n$. Hence, $A_0 \cap W$ contains all but finitely many vectors of E .

Recall that $\varphi \cdot W = \overline{\varphi}^{-1}(W^\perp)^\perp \cap V$. Clearly, $W^\perp = U^*$, and since $\overline{\varphi}^{-1}(B)^\perp = \varphi(B^\perp)$ for any subspace B of V_{*E} , we conclude that

$$\overline{\varphi}^{-1}(U^*)^\perp = \varphi((U^*)^\perp) = \varphi((W_*)^*).$$

Hence $\varphi \cdot W = \varphi((W_*)^*) \cap V$. Since $A_0 \subset \varphi((W_*)^*)$, we have $A_0 \cap W \subset \varphi \cdot W$, and so $\varphi \cdot W$ contains all but finitely many vectors of E . On the other hand,

$$\varphi((W_*)^*) \subset (W_*)^* \oplus \pi_U(\varphi((W_*)^*)) = (W_*)^* \oplus \operatorname{im} \mu.$$

This implies $\varphi \cdot W \subset W \oplus \operatorname{im} \mu$, and consequently $\varphi \cdot W \in X$.

Next, a trivial modification of the argument at the end of the proof of Theorem 2.1 for ind-grassmannians shows that φ induces an automorphism of ind-varieties

$$\tilde{\varphi}: X \rightarrow X.$$

Finally, the fact that $\widetilde{\varphi' \circ \varphi''} = \tilde{\varphi}' \circ \tilde{\varphi}''$ follows from the realization of $\tilde{\varphi}$ as an inverse limit of morphisms of finite-dimensional grassmannians where the property $\widetilde{\varphi'_n \circ \varphi''_n} = \tilde{\varphi}'_n \circ \tilde{\varphi}''_n$ is obvious. □

The ind-variety X is the disjoint union of the ind-grassmannians $Gr(W^k, E^k, V)$ for $k \in \mathbb{Z}$. These closed ind-subvarieties are not stable with respect to this action.

Example A.3 Let a *reverse sequence* be a set parameterized by $\mathbb{Z}_{<0}$. The vectors in V_E and V_{*E} can be written respectively as $x = \sum_{i>0} \omega_i e_{-i} + \sum_{i>0} u_i e_i \in V_E$, $y = \sum_{i>0} w_i e_{-i}^* + \sum_{i>0} v_i e_i^* \in V_{*E}$ where $\omega = (\dots, \omega_{-2}, \omega_{-1})$ and $w = (\dots, w_{-2}, w_{-1})$ are reverse sequences of complex numbers, and $u = (u_1, u_2, \dots)$, $v = (v_1, v_2, \dots)$ are usual sequences of complex numbers. In addition, u and w are finitary, i.e., have at most finitely many nonzero entries. In what follows, we write simply (ω, u) for vectors in V_E , and (w, v) for vectors in V_{*E} .

Consider the shift (linear) operator $Sh: V_E \rightarrow V_E$

$$Sh((\omega, u)) = ((\dots, \omega_{-3}, \omega_{-2}), (\omega_{-1}, u_1, u_2 \dots)).$$

One checks immediately that Sh is an element of the group $G(V_E, V_{*E})$ with \overline{Sh} having the form

$$\overline{Sh}((w, v)) = ((\dots, w_{-2}, w_{-1}, v_1), (v_2, v_3, \dots)).$$

Moreover, $Sh \cdot W$ belongs to $Gr(W^1, E^1, V)$, and in fact $Sh \cdot Gr(W, E, V) = Gr(W^1, E^1, V)$. In addition, we note that

$$\begin{aligned} & \dim \pi_U(Sh((W_*)^*)) - \dim \pi_U(Sh^{-1}((W_*)^*)) \\ &= \dim \pi_{W_*}(\overline{Sh}(U^*)) - \dim \pi_{W_*}(\overline{Sh}^{-1}(U^*)) = 1. \end{aligned}$$

This example motivates the following.

Definition A.4 Let $\varphi \in G(V_E, V_{*E})$. We define the integer

$$\begin{aligned} d(\varphi) &= d(\overline{\varphi}) := \dim \pi_U(\varphi((W_*)^*)) - \dim \pi_U(\varphi^{-1}((W_*)^*)) \\ &= \dim \pi_{W_*}(\overline{\varphi}(U^*)) - \dim \pi_{W_*}(\overline{\varphi}^{-1}(U^*)) \end{aligned}$$

to be the *degree* of φ . (The latter equality follows immediately from the fact that $\dim \text{im}(\pi_U \circ \varphi|_{(W_*)^*}) = \dim \text{im}(\pi_{W_*} \circ \overline{\varphi}|_{U^*})$ checked in the proof of Proposition A.2.)

Proposition A.5 Given $\varphi \in G(V_E, V_{*E})$ and $k \in \mathbb{Z}$, one has $\varphi \cdot Gr(W, E, V) = Gr(W^k, E^k, V)$ if and only if $d(\varphi) = k$.

Proof It suffices to check that the condition $\varphi \cdot W \in Gr(W^k, E^k, V)$ is equivalent to $d(\varphi) = k$. Since $\varphi \cdot W$ belongs to X , the condition $\varphi \cdot W \in Gr(W^k, E^k, V)$ is equivalent to the equality

$$\text{codim}_{\varphi \cdot W}((\varphi \cdot W) \cap W) - \text{codim}_W((\varphi \cdot W) \cap W) = k.$$

We have $\varphi \cdot W = \varphi((W_*)^*) \cap V$ and $W \subset (W_*)^*$, hence

$$(\varphi \cdot W) \cap W \subset (\varphi((W_*)^*) \cap V) \cap (W_*)^* = A_0 \cap V = A_0 \cap W$$

where $A_0 = (W_*)^* \cap \varphi((W_*)^*)$. Moreover, the opposite inclusion is clear, so $(\varphi \cdot W) \cap W = A_0 \cap W$. Therefore, we need to prove that

$$\text{codim}_{\varphi((W_*)^*) \cap V}(A_0 \cap W) - \text{codim}_W(A_0 \cap W) = d(\varphi).$$

As was shown in the proof of Proposition A.2, there exist $n \in \mathbb{Z}_{>0}$ and $\alpha_1, \dots, \alpha_n \in W_*$ such that $\pi_U(\varphi|_{(W_*)^*}(\omega)) = \sum_{i=1}^n \omega(\alpha_i)e_i$ for all $\omega \in (W_*)^*$, and $A_0 = \{\omega \in (W_*)^* \mid \omega(\alpha_i) = 0, 1 \leq i \leq n\}$. Hence, $A_0 \cap W = \{w \in W \mid \alpha_i(w) = 0, 1 \leq i \leq n\}$ and

$$\text{codim}_{(W_*)^*} A_0 = \text{codim}_W(A_0 \cap W) = \dim \pi_U(\varphi((W_*)^*)). \tag{27}$$

It remains to check that $\text{codim}_{\varphi((W_*)^*) \cap V}(A_0 \cap W) = \dim \pi_U(\varphi^{-1}((W_*)^*))$. This is done essentially by the same argument as above. Indeed, the argument in the proof of Proposition A.2 now shows that there exist $m \in \mathbb{Z}_{>0}$ and $\tilde{\alpha}_i \in W_*, 1 \leq i \leq m$, such that $\pi_U(\varphi^{-1}|_{(W_*)^*}(\omega)) = \sum_{i=1}^m \omega(\tilde{\alpha}_i)e_i$ for $\omega \in (W_*)^*$. This, together with fact that $\dim \pi_U(\varphi^{-1}((W_*)^*)) < \infty$, implies that, given $(\omega, u) \in V_E$, the condition $\varphi^{-1}((\omega, u)) \in (W_*)^*$ is equivalent to a finite system of linear equations on finitely many of the coordinates of (ω, u) . Thus,

$$\begin{aligned} \text{codim}_{\varphi((W_*)^*) \cap V}(A_0 \cap W) &= \text{codim}_{\varphi((W_*)^*)} A_0 = \text{codim}_{\varphi((W_*)^*)}(\varphi((W_*)^*) \cap (W_*)^*) \\ &= \text{codim}_{(W_*)^*}((W_*)^* \cap \varphi^{-1}((W_*)^*)) = \dim \pi_U(\varphi^{-1}((W_*)^*)). \end{aligned}$$

The proof is now complete. □

Corollary A.6 *The definition of $d(\varphi)$ does not depend on the choice of W in the following sense: in the definition of $d(\varphi)$ one can replace W by any subspace $W' \subset V$ which is \tilde{E} -commensurable with W , and U — by any direct complement U' of W' in V such that $\text{codim}_{U'}(U' \cap \tilde{E})_{\mathbb{C}} < \infty$.*

Proof The claim follows directly from Proposition A.5 as the ind-varieties $Gr(W^k, E^k, V)$ remain unchanged under a replacement $W \rightsquigarrow W', U \rightsquigarrow U'$. □

Now we are ready to describe the structure of the $G(V_E, V_{*E})$ -action on X in more detail. In particular, we prove that the degree defines a grading on the group $G(V_E, V_{*E})$. In the next theorem we set $X(k) = Gr(W^k, E^k, V)$ for $k \in \mathbb{Z}$. Then $X(0) = Gr(W, E, V)$ and $X = \bigsqcup_{k \in \mathbb{Z}} X(k)$.

Theorem A.7 *Given $d \in \mathbb{Z}$, set $G^d(V_E, V_{*E}) := \{\varphi \in G(V_E, V_{*E}) \mid d(\varphi) = d\}$. Then*

- i) $\varphi \cdot X(k) = X(k + d(\varphi))$ for all $\varphi \in G(V_E, V_{*E}), k \in \mathbb{Z}$;
- ii) $d(\varphi \circ \varphi') = d(\varphi) + d(\varphi')$ for all $\varphi, \varphi' \in G(V_E, V_{*E})$;
- iii) $G^0(V_E, V_{*E})$ is a normal subgroup of $G(V_E, V_{*E})$ whose cosets are $G^d(V_E, V_{*E}), d \in \mathbb{Z}$.

Proof i) We have $d(\varphi^{-1}) = -d(\varphi)$ by the definition of degree. This implies that if $\varphi \cdot X(0) = X(d)$ then $\varphi^{-1} \cdot X(0) = X(-d)$. In particular, since $Sh \cdot X(n) =$

$X(n + 1)$ and consequently $Sh^k \cdot X(n) = X(n + k)$, we obtain $Sh^{-k} \cdot X(n) = X(n - k)$ for all $k, n \in \mathbb{Z}$.

We claim that if $\varphi \cdot X(0) = X(0)$ then $\varphi \cdot X(k) = X(k)$ for all $k \in \mathbb{Z}$. Indeed, assume that $\varphi \cdot X(0) = X(0)$ and pick an integer k . We have $(Sh^{-k} \circ \varphi) \cdot X(0) = Sh^{-k} \cdot (\varphi \cdot X(0)) = Sh^{-k} \cdot X(0) = X(-k)$. By our observation in the first paragraph, we conclude that $(Sh^{-k} \circ \varphi)^{-1} \cdot X(0) = X(k)$, i.e.,

$$(\varphi^{-1} \cdot Sh^{-k}) \cdot X(0) = \varphi^{-1} \cdot (Sh^k \cdot X(0)) = \varphi^{-1} \cdot X(k) = X(k).$$

This clearly implies $\varphi \cdot X(k) = X(k)$.

Now, assume that $\varphi \cdot X(0) = X(d)$, i.e., that $d(\varphi) = d$. Then $(Sh^{-d} \circ \varphi) \cdot X(0) = Sh^{-d} \cdot X(d) = X(0)$. Thus, by the previous paragraph, for all $k \in \mathbb{Z}$ we have $(Sh^{-d} \circ \varphi) \cdot X(k) = X(k)$, i.e.,

$$\varphi \cdot X(k) = Sh^d \cdot X(k) = X(k + d).$$

- ii) Follows immediately from (i).
- iii) Follows immediately from (ii).

□

A.3 Matrix Realization

Consider the spaces $V_E = (W_*)^* \oplus U, V_{*E} = W_* \oplus U^*$, and assume that the isomorphism $\eta : W_* \rightarrow U$ maps e_{-i}^* to e_i for $i \in \mathbb{Z}_{>0}$. Then the pairing $V_E \times V_{*E} \rightarrow \mathbb{C}$ is identified with the pairing

$$\begin{aligned} (U^* \oplus U) \times (U^* \oplus U) &\rightarrow \mathbb{C}: ((\gamma, y), (\varkappa, z)) \mapsto \langle (\gamma, y), (\varkappa, z) \rangle \\ &= \gamma(z) + \varkappa(y), \quad y, z \in U, \gamma, \varkappa \in U^*. \end{aligned}$$

This allows us to denote the group $G(V_E, V_{*E}) = G(U^* \oplus U, U^* \oplus U)$ simply by G_U .

We now present an explicit matrix realization of the group G_U . We have a fixed basis $\{e_1, e_2, \dots\}$ of U , and we identify $U^* \oplus U$ with the space $\{(\gamma, y)\}$ where $\gamma = (\dots, \gamma_{-2}, \gamma_{-1})$ are reverse sequences and $y = (y_1, y_2, \dots)$ are finitary usual sequences. We consider matrices whose rows and columns are parameterized by the ordered set $\mathbb{Z} \setminus \{0\} = \mathbb{Z}_{<0} \sqcup \mathbb{Z}_{>0}$. Such a matrix M naturally splits into four blocks

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right).$$

Let J be the subset of such matrices satisfying the following conditions:

- each row of A and each column of D is finitary;
 - C is a finitary matrix.
- (28)

Matrices from J act on $U^* \oplus U$ via left multiplication: one considers vectors of $U^* \oplus U$ as columns

$$(\gamma, y)^T = \begin{pmatrix} \gamma^T \\ y^T \end{pmatrix},$$

and we have $(\gamma, y)^T \mapsto \psi(\gamma, y)^T$ for $\psi \in J$.

Given a matrix $\psi \in J$, we let $\overline{\psi}$ denote the reflection of ψ with respect to its antidiagonal. Clearly, $\overline{\overline{\psi}} = \psi$. One checks immediately that $\overline{\psi}$ belongs to J , and that $\psi_1\psi_2 = \overline{\psi_2}\overline{\psi_1}$. Furthermore, if $\overline{\gamma} := (\gamma_{-1}, \gamma_{-2}, \gamma_{-3}, \dots)$ for a reverse sequence $\gamma = (\dots, \gamma_{-3}, \gamma_{-2}, \gamma_{-1})$ and $\overline{y} := (\dots, y_3, y_2, y_1)$ for a finitary sequence $y = (y_1, y_2, y_3, \dots)$, then we have $\overline{\psi(\gamma, y)^T} = (\overline{\gamma}, \overline{y})^T \overline{\psi}$ for all $(\gamma, y) \in U^* \oplus U$, where $(\overline{\gamma}, \overline{y})^T = (\overline{y}, \overline{\gamma})$.

Let \widetilde{G}_U be the group of automorphisms of V_E with matrices ψ such that $\psi, \psi^{-1} \in J$.

Theorem A.8 *The group G_U coincides with \widetilde{G}_U .*

Proof First, we prove that G_U is contained in \widetilde{G}_U . Pick an operator $\varphi \in G_U$. Since $\pi_U \circ (\varphi|_U)$ is a well-defined operator on the countable-dimensional space U , it is represented by a (unique) matrix $D = (d_{i,j})_{i,j \in \mathbb{Z}_{>0}}$ with finitary columns.

Next, Lemma A.1 implies that $\dim \text{im}(\pi_U \circ (\varphi|_{U^*})) < \infty$. Moreover, according to the proof of Lemma A.1 there exist $n \in \mathbb{Z}_{\geq 0}$ and vectors $c_1, \dots, c_n \in U$ such that $\pi_U(\varphi|_{U^*}(\gamma)) = \sum_{i=1}^n \gamma(c_i)e_i$ for all $\gamma \in U^*$. This means that $\pi_U \circ (\varphi|_{U^*})$ can be represented as a finitary matrix $C = (c_{i,-j})_{i,j \in \mathbb{Z}_{>0}}$, where $c_i = \sum_{j \geq 1} c_{i,-j}e_j$.

Let π_{U^*} be the projection operator $U^* \oplus U \rightarrow U^*$. The operator $\pi_{U^*} \circ (\varphi|_U)$ is an operator from the space U of finitary sequences to the space U^* of arbitrary reverse sequences, and is therefore given by a matrix $B = (b_{-i,j})_{i,j \in \mathbb{Z}_{>0}}$.

Finally, given $\gamma \in U^*$, we have $\pi_{U^*}(\varphi|_{U^*}(\gamma))(e_i) = \varphi(\gamma)(e_i) = \gamma(\overline{\varphi}(e_i)) = \gamma(\pi_U(\overline{\varphi}(e_i)))$. There exist linear functions $\beta_j \in U^*, j \in \mathbb{Z}_{>0}$, such that $\pi_U(\overline{\varphi}(y)) = (\beta_1(y), \beta_2(y), \dots)$. Since $\pi_U(\overline{\varphi}(e_i))$ belongs to U , there exists $k(i) \in \mathbb{Z}_{>0}$ such that $\beta_j(e_i) = 0$ for $j > k(i)$, i.e., $\pi_U(\overline{\varphi}(e_i)) = \sum_{j=1}^{k(i)} \beta_j(e_i)e_j$. Thus, $\pi_{U^*} \circ (\varphi|_{U^*})$ can be represented as a matrix with finitary rows $A = (a_{-i,-j})_{i,j \in \mathbb{Z}_{>0}}$, where $a_{-i,-j} = \beta_j(e_i)$.

It follows that $\varphi \in G_U$ can be represented by invertible matrices from L . This conclusion applies also to the inverse operator φ^{-1} by the same argument.

In order to verify the inclusion $\widetilde{G}_U \subset G_U$, let now ψ be a matrix from L such that $\psi^{-1} \in L$. Then the linear operator on $U^* \oplus U$ with matrix $\overline{\psi}$ is the restriction to $U^* \oplus U$ of the operator dual to the operator defined by ψ . Indeed,

$$\begin{aligned} \langle \overline{\psi}(\gamma, y)^T, (\gamma', y')^T \rangle &= \overline{\overline{\psi}(y, \gamma)^T(\gamma', y')^T} = \overline{(\gamma, y)^T \psi(\gamma', y')^T} \\ &= (\gamma, y)^T \psi(\gamma', y')^T = \langle (\gamma, y)^T, \psi(\gamma', y')^T \rangle, \end{aligned} \tag{29}$$

where the three middle terms are products of two or respectively three matrices. Moreover, the invertibility of $\overline{\psi}$ follows from the equality $\overline{\psi^{-1}} = \overline{\psi}^{-1}$ which is a consequence of the relation $\psi_1\psi_2 = \overline{\psi_2}\psi_1$. This shows that ψ determines an operator from G_U . □

Remark A.9 Finally, the reader will notice that if in the definition of J we replace the ordered set $\mathbb{Z} \setminus \{0\} = \mathbb{Z}_{<0} \sqcup \mathbb{Z}_{>0}$ by any ordered set which is the disjoint union $O_1 \sqcup O_2$ of two linearly ordered countable sets with the condition $r < s$ for $r \in O_1, s \in O_2$, and define a group by imposing the conditions (28), we will obtain a group isomorphic to G_U . Of course, in such a setting the matrix of the operator

$\bar{\varphi} = \varphi^*|_{U^* \oplus U}$, for a given $\varphi \in G_U$ with matrix ψ , will not be $\bar{\psi}$ as defined above, and its form will depend on the choice of O_1 and O_2 . Moreover, there will also be analogues of the operator Sh , and we leave it to the reader to define one.

The above remark is used in the proof of Theorem 2.1 presented in Section 6.

Finally, using the matrix form of the group G_U given by Theorem A.8, it is straightforward to check that G_U is nothing but the group of continuous automorphisms of $U^* \oplus U$ as a Tate space, see, for instance, [9]. In other words, G_U is the group known as Japanese $GL(\infty)$, and $G^0(U, U^*)$ is simply the connected component of the identity in $G(U, U^*) = G_U$.

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Declarations

Conflict of Interest The authors declare no competing interests.

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References

1. Akhiezer, D.: Lie Group Actions in Complex Analysis. Asp. of Math. 27 Vieweg + Teubner Verlag (1995)
2. Chow, W.-L.: On the Geometry of Algebraic Homogeneous Spaces. Ann. Math. **50**, 32–67 (1949)
3. Dimitrov, I., Penkov, I.: Ind-varieties of generalized flags as homogeneous spaces for classical ind-groups. Int. Math. Res. Notices **55**, 2935–2953 (2004)
4. Dimitrov, I., Penkov, I., Wolf, J.: A Bott-Borel-Weil theory for direct limits of algebraic groups. Am. J. Math. **124**, 955–998 (2002)
5. Gatto, L., Salehyan, P.: The cohomology of the Grassmannian is a \mathfrak{gl}_n -module. Commun. Algebra **48**, 274–290 (2020)
6. Huckleberry, A.: Hyperbolicity of cycle spaces and automorphism groups of flag domains. American J. Math. **135**(2), 291–310 (2010)
7. Huckleberry, A.: Cycle connectivity and automorphism groups of flag domains. In: Mason, G., Penkov, I., Wolf, J.A. (eds.) Developments and Retrospectives in Lie Theory. Geometric and Analytic Methods, pp. 113–126. Springer (2014)

8. Ignatyev, I., Penkov, I.: Ind-varieties of generalized flags: a survey of results. *J. Math. Sci.* **248**, 255–302 (2020)
9. Kapranov, M.: Semiinfinite symmetric powers. Preprint, arXiv:[math.QA/0107089](https://arxiv.org/abs/math.QA/0107089) (2001)
10. Mackey, G.W.: On infinite-dimensional linear spaces. *Trans. Amer. Math. Soc.* **57**(2), 155–207 (1945)
11. Natarajan, L., Rodríguez-Carrington, E., Wolf, J.A.: The Bott–Borel–Weil Theorem for direct limit Lie groups. *Trans. Amer. Math. Soc.* **353**, 4583–4622 (2001)
12. Onishchik, A.: Transitive compact transformation groups. *Math. Sb. (N.S.)* **60**, 447–485 (1963). English Transl.: *AMS Transl. (2)* **55** (1966), 5–58
13. Penkov, I.: Examples of automorphism groups of ind-varieties of generalized flags. *J. Geom. Symmetry Phys.* **50**, 71–77 (2018)
14. Penkov, I., Tikhomirov, A.S.: Linear ind-Grassmannians. *Pure Appl. Math. Quarterly* **10**, 289–323 (2014)
15. Penkov, I., Tikhomirov, A.S.: An algebraic-geometric construction of ind-varieties of generalized flags. Preprint, arXiv:[math.AG/1910.12005](https://arxiv.org/abs/math.AG/1910.12005) (2019)
16. Sato, M.: The KP hierarchy and infinite-dimensional Grassmann manifolds. Theta functions — Bowdoin 1987, Part 1 (Brunswick, ME, 1987). In: *Proc. Sympos. Pure Math.*, vol. 49, pp. 51–66. Amer. Math. Soc., Providence (1989)
17. Tarigradschi, M.: On the Automorphism Groups of Certain Flag Ind-varieties. Jacobs University Bremen, Bachelor's thesis (2019)

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