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MATHEMATICAL PROBLEMS OF NONLINEARITY

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Topology of Ambient 3-Manifolds of Non-Singular Flows with Twisted Saddle Orbit

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In the present paper, nonsingular Morse – Smale flows on closed orientable 3-manifolds are considered under the assumption that among the periodic orbits of the flow there is only one saddle and that it is twisted. An exhaustive description of the topology of such manifolds is obtained. Namely, it is established that any manifold admitting such flows is either a lens space or a connected sum of a lens space with a projective space, or Seifert manifolds with a base homeomorphic to a sphere and three singular fibers. Since the latter are prime manifolds, the result obtained refutes the claim that, among prime manifolds, the flows considered admit only lens spaces.

Keywords: nonsingular flows, Morse – Smale flows, Seifert fiber space

1. Introduction and formulation of results

In the present paper, we consider *NMS-flows* f^t, that is, *nonsingular* (without fixed points) Morse – Smale flows defined on closed orientable 3-manifolds M^3. The nonwandering set of such flows consists of a finite number of periodic hyperbolic orbits. It is known from Asimov's work [1] that the ambient manifold in this case has a round handle decomposition. However, in the case of a small number of periodic orbits, the topology of the manifold can be significantly refined. For example, only lens spaces are ambient for NMS-flows with exactly two periodic orbits. Moreover, in [2] it is proved that for every lens space there are exactly two equivalence classes of such flows, except for the 3-sphere $\mathbb{S}^3$ and the projective space $\mathbb{R}P^3$, on which there is one equivalence class.

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In [3], it is stated that the lens space is also the only *prime* (homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ or *irreducible* — any cylindrically embedded 2-sphere bounds the 3-ball) 3-manifold which is ambient for NMS-flows with a unique saddle periodic orbit. However, this is incorrect. In the previous work of one of the authors [4], NMS-flows with exactly three periodic orbits (attractive, repelling and saddle) are constructed on a countable set of pairwise nonhomeomorphic mapping tori that are not lens spaces. Moreover, in [5] necessary and sufficient conditions for the topological equivalence of such flows are obtained.

In this paper, we recognize the topology of all orientable 3-manifolds that admit NMS-flows with exactly one saddle periodic orbit, assuming that it is *twisted* (its invariant manifolds are nonorientable).

Let us proceed to the formulation of the results.

Let M^3 be a connected closed orientable 3-manifold, $f^t\colon M^3 \to M^3$ an NMS-flow and $\mathcal{O}$ — its periodic orbit. In the neighborhood of the hyperbolic periodic orbit $\mathcal{O}$, the flow can be simply described (up to topological equivalence). Namely, there exist a linear diffeomorphism of the plane, given by the matrix with positive determinant and real eigenvalues with absolute value different from one, and a tubular neighborhood $V_{\mathcal{O}}$ homeomorphic to the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$, in which the flow is topologically equivalent to the suspension over this diffeomorphism (see, for example, [6]). If both eigenvalues are greater (less) than one in absolute value, then the corresponding periodic orbit is called *repelling* (*attractive*) and *saddle* otherwise. In this case, a saddle orbit is called *twisted* if both eigenvalues are negative and *untwisted* otherwise.

Let $T_{\mathcal{O}} = \partial V_{\mathcal{O}}$. Let us choose *meridian* $M_{\mathcal{O}} \subset T_{\mathcal{O}}$ (a null-homotopic curve on $V_{\mathcal{O}}$ and essential on $T_{\mathcal{O}}$) and *longitude* $L_{\mathcal{O}} \subset T_{\mathcal{O}}$ (the curve homologous in the $V_{\mathcal{O}}$ to the orbit $\mathcal{O}$). We assume that the meridian $M_{\mathcal{O}}$ is oriented so that the pair of oriented curves $M_{\mathcal{O}}$, $L_{\mathcal{O}}$ determines the outer side of the solid torus boundary. Thus, the homotopy types $\langle L_{\mathcal{O}}\rangle = \langle 1,\, 0\rangle$, $\langle M_{\mathcal{O}}\rangle = \langle 0,\, 1\rangle$ of knots $L_{\mathcal{O}}$, $M_{\mathcal{O}}$ are generators of the homotopy types $\langle K\rangle$ of oriented knots K on torus $T_{\mathcal{O}}$, that is,

$$\langle K\rangle = \langle l_{\mathcal{O}},\, m_{\mathcal{O}}\rangle = l_{\mathcal{O}}\langle L_{\mathcal{O}}\rangle + m_{\mathcal{O}}\langle M_{\mathcal{O}}\rangle, \tag{1.1}$$

where $l_{\mathcal{O}}, m_{\mathcal{O}} \in \mathbb{Z}$ are numbers of twists of the oriented knot K around the parallel and the meridian, respectively. Note that the choice of the longitude does not influence the following reasoning, since only the remainder of the division of $l_{\mathcal{O}}$ by $m_{\mathcal{O}}$ matters in the subsequent discussion.

Consider the class $G_3^-\left(M^3\right)$ of NMS-flows $f^t\colon M^3 \to M^3$ with a unique saddle orbit, assuming that it is twisted. Since the ambient manifold M^3 is the union of the stable (unstable) manifolds of all its periodic orbits, the flow $f^t \in G_3^-\left(M^3\right)$ must have at least one attracting and at least one repelling orbit. In Section 3, we will prove the following fact.

Lemma 1. *The nonwandering set of any flow $f^t \in G_3^-\left(M^3\right)$ consists of exactly three periodic orbits S, A, R, saddle, attracting and repelling, respectively.*

Since the flow f^t in the neighborhood of a periodic orbit is equivalent to a suspension over the linear diffeomorphism, the stable and unstable manifolds of these orbits have the following topology:

- $W^u_S \cong W^s_S \cong \mathbb{R} \mathbin{\widetilde{\times}} \mathbb{S}^1$ (open Moebius strip);
- $W^s_A \cong W^u_R \cong \mathbb{R}^2 \times \mathbb{S}^1$;
- $W^u_A \cong W^s_R \cong \mathbb{S}^1$.

This fact and Lemma 1 immediately imply the following proposition (for more details, see [5]).

Proposition 1. *The ambient manifold M^3 of any flow $f^t \in G_3^-\left(M^3\right)$ is represented as the union of three solid tori*:

$$M^3 = \mathcal{V}_A \cup V_S \cup \mathcal{V}_R$$

with disjoint interiors being tubular neighborhoods of A, S, R orbits, respectively, with the following properties:

- *$T_S = \partial V_S$ is the union of tubular neighborhoods T_S^u, T_S^s of knots $K_S^u = W_S^u \cap T_S$, $K_S^s = W_S^s \cap T_S$, respectively, such that $T_S^u \cap T_S^s = \partial T_S^u \cap \partial T_S^s$*;
- *the torus $\mathcal{T}_A = \partial \mathcal{V}_A$ is the union of the annulus T_S^u and a compact surface $\mathcal{T}$ (an annulus or disjoint union of a handle with a disk) with disjoint interiors, and the knot K_S^u has homotopy type*
$$\langle K_S^u \rangle = \langle l_A, m_A \rangle$$
with generators L_A, M_A, respectively;
- *the torus $\mathcal{T}_R = \partial \mathcal{V}_R$ is the union of the annulus T_S^s and the surface $\mathcal{T}$ with disjoint interiors, and the knot K_S^s has homotopy type*
$$\langle K_S^s \rangle = \langle l_R, m_R \rangle$$
with generators L_R, M_R, respectively.

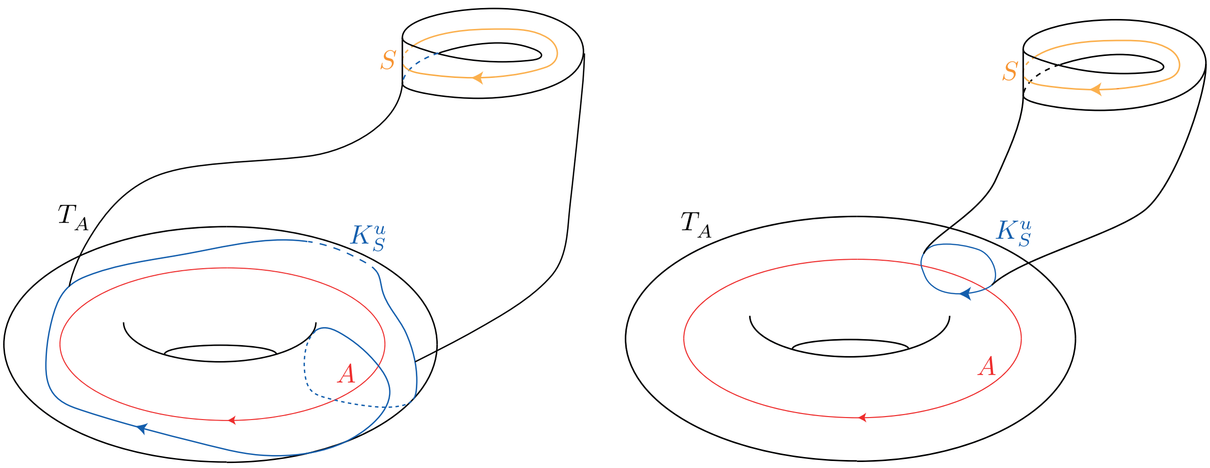

Fig. 1. Knot K_S^u

Thus, both knots $K_S^u \subset \mathcal{T}_A$, $K_S^s \subset \mathcal{T}_R$ are either inessential or essential (see Fig. 1). For every flow $f^t \in G_3^-\left(M^3\right)$ we determine a quadruple of integers

$$C_{f^t} = (l_1, m_1, l_2, m_2)$$

as follows:

- if the knots K^u_S, K^s_S are essential on tori $\mathcal{T}_A$, $\mathcal{T}_R$, then
$$C_{f^t} = (l_R,\, m_R,\, l_A,\, m_A);$$
- if the knots K^u_S, K^s_S are inessential on tori $\mathcal{T}_A$, $\mathcal{T}_R$, then
$$C_{f^t} = (0,\, 2,\, l_2,\, m_2),$$
where $\langle l_2,\, m_2\rangle$ is the homotopy type of the knot on torus $\mathcal{T}_R$ which is the meridian on torus $\mathcal{T}_A$.

Note that the class $G_3^-\left(M^3\right)$ is not empty, because by [5] every quadruple $C = (l_1,\, m_1,\, l_2,\, m_2)$ with $\gcd(l_i,\, m_i) = 1$, $i = 1,\, 2$ and quadruple $C = (0,\, 2,\, l_2,\, m_2)$ with $\gcd(l_2,\, m_2) = 1$ are realizable by a flow $f^t \in G_3\left(M^3\right)$ such that $C = C_{f^t}$.

The main result of the paper is the following theorem (all the necessary information about the objects mentioned below is given in Section 2).

Theorem 1. *Ambient manifolds of the flows in $G_3^-\left(M^3\right)$ are lens spaces $L_{p,q}$, connected sums of the form $L_{p,q}\#\mathbb{R}P^3$ and Seifert manifolds of the form $M\left(\mathbb{S}^2,\, (2,\, 1),\, (\alpha_1,\, \beta_1),\, (\alpha_2,\, \beta_2)\right)$. Namely, let the flow $f^t \in G_3^-\left(M^3\right)$ correspond to the collection $C_{f^t} = (l_1,\, m_1,\, l_2,\, m_2)$. Then*

1) *if $l_1 = 0$ and $l_2 \neq 0$, then M^3 is homeomorphic to the manifold $L_{l_2,m_2}\#\mathbb{R}\mathrm{P}^3$;*
2) *if $l_1 \neq 0$ and $l_2 = 0$, then M^3 is homeomorphic to the manifold $L_{l_1,m_1}\#\mathbb{R}\mathrm{P}^3$;*
3) *if $l_1 = 0$ and $l_2 = 0$, then M^3 is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1\#\mathbb{R}\mathrm{P}^3$;*
4) *if $|l_1| = 1$ and $|l_2| > 1$, then M^3 is homeomorphic to the lens space $L_{p,q}$, where $p = 2\beta_2 - l_2 b$, $q = \frac{l_2(b+1)}{2} - \beta_2$, $\beta_2 m_2 \equiv 1 \pmod{l_2}$, $b \equiv 1 \pmod 2$;*
5) *if $|l_2| = 1$ and $|l_1| > 1$, then M^3 is homeomorphic to the lens space $L_{p,q}$, where $p = 2\beta_1 - l_1 b$, $q = \frac{l_1(b+1)}{2} - \beta_1$, $\beta_1 m_1 \equiv 1 \pmod{l_1}$, $b \equiv 1 \pmod 2$;*
6) *if $|l_1 l_2| = 1$, then M^3 is homeomorphic to the lens space $L_{b,2}$, $b \equiv 1 \pmod 2$;*
7) *if $|l_1| > 1$ and $|l_2| > 1$, then M^3 is homeomorphic to the prime Seifert manifold $M\left(\mathbb{S}^2,\, (2,\, 1),\, (l_1,\, \beta_1),\, (l_2,\, \beta_2)\right)$, $\beta_i m_i \equiv 1 \pmod{l_i}$, $i = 1,\, 2$ and is not homeomorphic to any lens space.*

2. Necessary information on the topology of 3-manifolds

2.1. Lens spaces

Everywhere below we assume that generators of homotopy types of knots on boundary $\partial\mathbb{V}$ of the standard solid torus $\mathbb{V} = \mathbb{D}^2 \times \mathbb{S}^1$ are meridian $\mathbb{M} = \left(\partial\mathbb{D}^2\right) \times \{y\}$, $y \in \mathbb{S}^1$ with homotopy type $\langle 0,\, 1\rangle$ and parallel $\mathbb{L} = \{x\} \times \mathbb{S}^1$, $x \in \partial\mathbb{D}^2$ with homotopy type $\langle 1,\, 0\rangle$.

Lens space is a three-dimensional manifold $L_{p,q} = V_1 \underset{j}{\cup} V_2$, which is the result of gluing together two copies of the solid torus $V_1 = \mathbb{V}$, $V_2 = \mathbb{V}$ by some homeomorphism $j\colon \partial V_1 \to \partial V_2$ such that $j_*(\langle 0,\, 1\rangle) = \langle p,\, q\rangle$.

Proposition 2 ([7]). *Two lens spaces $L_{p,q}$, $L_{p',q'}$ are homeomorphic (up to preserving the numbering of copies) if and only if $p = \pm p'$, $q \equiv \pm q' \pmod{|p|}$.*

2.2. Dehn surgery along knots and links

Suppose the following data are given:

1) a closed 3-manifold M;

2) a knot $\gamma \subset M$;

3) a tubular neighborhood U_γ of γ with standard generators on ∂U_γ: meridian M_γ and longitude L_γ;

4) a homeomorphism $h\colon \partial\mathbb{V} \to \partial U_\gamma$ inducing an isomorphism such that $h_*(\langle 0,\, 1\rangle) = \langle \beta,\, \alpha\rangle$.

A manifold

$$M_{\gamma,h} = (M \setminus \operatorname{int} U_\gamma) \underset{h}{\cup} \mathbb{V}$$

is called the *manifold obtained from M by Dehn surgery along the knot γ*.

Naturally, the manifold M is restored from $M_{\gamma,h}$ by inverse surgery. Namely, we denote by $p_{\gamma,h}\colon (M \setminus \operatorname{int} U_\gamma) \sqcup \mathbb{V} \to M_{\gamma,h}$ the natural projection. Let $\widetilde{\gamma} = p_{\gamma,h}\left(\{0\} \times \mathbb{S}^1\right)$, $U_{\widetilde{\gamma}} = p_{\gamma,h}(\mathbb{V})$, $\widetilde{h} = p_{\gamma,h} h^{-1}\colon \partial U_\gamma \to \partial U_{\widetilde{\gamma}}$. Then

$$M \cong (M_{\gamma,h})_{\widetilde{\gamma},\widetilde{h}}. \tag{2.1}$$

The following assertions follow directly from the relation (2.1).

Proposition 3. *Let $\gamma \subset M$ be equipped with β, α. Then*

$$M \cong (M_\gamma)_{\widetilde{\gamma}},$$

where $\widetilde{\gamma}$ is equipped with $-\beta$, ξ satisfying $\xi\alpha + \nu\beta = 1$.

Proposition 4. *Let $L_{p,q} = V_1 \underset{j}{\cup} V_2$, where $j_*(\langle 0,\, 1\rangle) = \langle p,\, q\rangle$ and $\mathbb{S}^3 = V_1 \underset{j_0}{\cup} V_2$, where $j_{0*}(\langle 0,\, 1\rangle) = \langle 1,\, 0\rangle$. Then*

$$L_{p,q} \cong \mathbb{S}^3_{M_1},$$

where M_1 is the meridian of the torus V_1, equipped with q, p.

Dehn surgery is naturally generalized to the case where $\gamma = \gamma_1 \sqcup \ldots \sqcup \gamma_r \subset M$ is a disjoint union (link) of equipped knots. The resulting manifold M_γ in this case is called the *manifold obtained from the manifold M^3 by Dehn surgery along the equipped link γ*. A link $\gamma = \gamma_1 \sqcup \ldots \sqcup \gamma_r \subset M$ is called *trivial* if knots $\gamma_1, \ldots, \gamma_r$ bound pairwise disjoint 2-discs $d_1, \ldots, d_r \subset M$.

Proposition 5 ([7]). *Let $\gamma = \gamma_1 \sqcup \ldots \sqcup \gamma_r \subset M$ be a trivial link equipped with q_1, $p_1, \ldots, q_r$, p_r. Then*

$$M_\gamma \cong M \# L_{p_1,q_1} \# \ldots L_{p_r,q_r}.$$

2.3. Seifert fiber space

A solid torus $\mathbb{V}$ split into fibers of the form $\{x\} \times \mathbb{S}^1$ is called a *trivially foliated solid torus*. Consider the solid torus $\mathbb{V} = \mathbb{D}^2 \times \mathbb{S}^1$ as the cylinder $\mathbb{D}^2 \times [0, 1]$ with the bases glued due to the $\frac{2\pi\nu}{\alpha}$ angle rotation for coprime integers α, ν, $\alpha > 1$. The partition of the cylinder into segments of the form $\{x\} \times [0, 1]$ determines the partition of this solid torus into circles called *fibers*. The segment $\{0\} \times [0, 1]$ generates a fiber which we call *exceptional*, all other (*ordinary*) fibers of the solid torus wrap α times around the exceptional fiber and ν times around the solid torus meridian. The number α is called the *multiplicity* of the exceptional fiber. A solid torus with such a partition into fibers is called a *nontrivially fibered solid torus* with *orbital invariants* (α, ν).

A *Seifert manifold* is a compact, orientable 3-manifold M decomposed into disjoint simple closed curves (fibers) in such a way that every fiber has a neighborhood consisting of fibers, fiberwise homeomorphic to a foliated solid torus. Such a partition is called *Seifert fibration*. The fibers which correspond to the exceptional fiber under such homeomorphism of a nontrivially foliated solid torus are called *exceptional*.

Two Seifert fibrations M, M' are called *isomorphic* if there exists a homeomorphism $h\colon M \to M'$ such that the image of each fiber of one bundle is a fiber of the second bundle. It is easy to show (see, for example, [8, Proposition 10.1]) that two bundles of a solid torus with orbital invariants (α_1, ν_1); (α_2, ν_2) are isomorphic (preserving the orientation of fibers) if and only if $\alpha_1 = \alpha_2$ $(= \alpha)$; $\nu_1 \equiv \nu_2 \pmod{\alpha}$.

The *base* of a Seifert manifold M is a compact surface $\Sigma = M/{\sim}$, where $\sim$ is an equivalence relation such that $x \sim y$ if and only if x and y belong to the same fiber. It is easy to show (see, for example, [8, Proposition 10.2]) that the base of any solid torus bundle is a disc. The base of any Seifert manifold is a compact surface and Seifert bundles with nonhomeomorphic bases are not isomorphic (see, for example, [8]).

Thus, any Seifert fibering M with a given base Σ and orbital invariants $(\alpha_1, \nu_1), \ldots, (\alpha_r, \nu_r)$, $r \in \mathbb{N}$ is obtained from the manifold $\Sigma \times \mathbb{S}^1$ by Dehn surgery along the link $\gamma = \bigsqcup\limits_{i=1}^{r} \gamma_i$, where $\gamma_i = \{s_i\} \times \mathbb{S}^1$, $s_i \in \Sigma$ is a knot with equipment β_i, α_i, $\nu_i\beta_i \equiv 1 \pmod{\alpha_i}$. Therefore, the conventional notation for such a Seifert fibration is

$$M(\Sigma, (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)).$$

Proposition 6 ([8, 9]). *Seifert fibrations* $M(\Sigma, (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$ *and* $M'\left(\Sigma', (\alpha'_1, \beta'_1), \ldots, (\alpha'_{r'}, \beta'_{r'})\right)$ *are isomorphic if and only if there exists* $\delta = \pm 1$ *such that*:

- Σ *is homeomorphic to* Σ';
- $r = r'$; $\alpha_i = \alpha'_i$; $\beta_i \equiv \delta\beta'_i \pmod{\alpha_i}$ *for* $i \in \{1, \ldots, r\}$;
- *if the surface* Σ *is closed, then* $\sum\limits_{i=1}^{r} \frac{\beta_i}{\alpha_i} = \delta \sum\limits_{i=1}^{r} \frac{\beta'_i}{\alpha'_i}$.

Proposition 7 ([9, Proposition 1.12]). *All closed orientable Seifert manifolds are prime except* $M\left(\mathbb{S}^2, (2, 1), (2, 1), (2, 1), (2, 1)\right) \cong \mathbb{R}P^3 \# \mathbb{R}P^3$.

Proposition 8 ([10]). *A* 3*-manifold admits a Seifert fibration with sphere base and at most two singular fibers if and only if it is homeomorphic to a lens space, so that*

- *the only manifold which admits fibering without singular fibers is* $\mathbb{S}^2 \times \mathbb{S}^1$;
- $M\left(\mathbb{S}^2, (\alpha, \beta)\right) \cong L_{\beta,\alpha}$;
- $M\left(\mathbb{S}^2, (\alpha_1, \beta_1), (\alpha_2, \beta_2)\right) \cong L_{p,q}$, *where* $p = \beta_1\alpha_2 - \alpha_1\beta_2$, $q = \beta_1\nu_2 - \alpha_1\xi_2$ *and* $\nu_2\beta_2 - \alpha_2\xi_2 = 1$.

It follows from the above statement, in particular, that any lens space admits more than one Seifert fibrations. However, as the result below shows, any such fibration with base sphere cannot have more than two exceptional fibers.

Proposition 9 ([8]). *Any lens space does not admit a Seifert fibration with base homeomorphic to sphere and more than two exceptional fibers.*

3. Dynamics of the flows $G_3^-(M^3)$

This section is devoted to the proof of Lemma 1: the nonwandering set of any flow $f^t \in G_3^-\left(M^3\right)$ consists of exactly three periodic orbits S, A, R, saddle, attracting and repelling, respectively.

Proof. The basis of the proof is the following representation of the ambient manifold M^3 of the NMS flow f^t with the set of periodic orbits Per_{f^t} (see, for example, [11])

$$M^3 = \bigcup \mathcal{O} \in Per_{f^t} W^u_{\mathcal{O}} = \bigcup_{\mathcal{O}\in Per_{f^t}} W^s_{\mathcal{O}}, \tag{3.1}$$

as well as the asymptotic behavior of invariant manifolds

$$cl\,(W^u_{\mathcal{O}}) \setminus W^u_{\mathcal{O}} = \bigcup_{\widetilde{\mathcal{O}}\in Per_{f^t}\,:\, W^u_{\mathcal{O}}\cap W^s_{\widetilde{\mathcal{O}}}\neq\varnothing} W^u_{\widetilde{\mathcal{O}}},$$
$$cl\,(W^s_{\mathcal{O}}) \setminus W^s_{\mathcal{O}} = \bigcup_{\widetilde{\mathcal{O}}\in Per_{f^t}\,:\, W^s_{\mathcal{O}}\cap W^u_{\widetilde{\mathcal{O}}}\neq\varnothing} W^s_{\widetilde{\mathcal{O}}}.$$

In particular, it follows from the above relations that any NMS flow has at least one attracting orbit and at least one repelling one. Moreover, if an NMS flow has a saddle periodic orbit, then the basin of any attracting orbit has a nonempty intersection with an unstable manifold of at least one saddle orbit (see [12, Proposition 2.3]) and a similar situation with the basin of a repelling orbit.

Now let $f^t \in G_3^-\left(M^3\right)$ and S be its only saddle orbit. It follows from the relation (3.1) that $W^u_S \setminus S$ intersects only basins of attracting orbits. Since the set $W^u_S \setminus S$ is connected and the basins of attracting orbits are open, W^u_S intersects exactly one such basin. Denote by A the corresponding attracting orbit. Since there is only one saddle orbit, there is only one attracting orbit. Similar reasoning for W^s_S leads to the existence of a unique repelling orbit R. $\square$

4. Topology of ambient manifolds of flows of class $G_3^-(M^3)$

In this section, we prove Theorem 1: flows of class $G_3^-\left(M^3\right)$ admit all lens spaces $L_{p,q}$, all connected sums of the form $L_{p,q}\#\mathbb{R}P^3$ and all Seifert manifolds of the form $M\left(\mathbb{S}^2, (2,1), (\alpha_1, \beta_1), (\alpha_2, \beta_2)\right)$. Namely, let the flow $f^t \in G_3^-\left(M^3\right)$ have the invariant $C_{f^t} = (l_1, m_1, l_2, m_2)$. Then

1) if $l_1 = 0$ and $l_2 \neq 0$, then M^3 is homeomorphic to the manifold $L_{l_2,m_2}\#\mathbb{R}\mathrm{P}^3$;

2) if $l_1 \neq 0$ and $l_2 = 0$, then M^3 is homeomorphic to the manifold $L_{l_1,m_1}\#\mathbb{R}\mathrm{P}^3$;

3) if $l_1 = 0$ and $l_2 = 0$, then M^3 is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1\#\mathbb{R}\mathrm{P}^3$;

4) if $|l_1| = 1$ and $|l_2| > 1$, then M^3 is homeomorphic to the lens space $L_{p,q}$, where $p = 2\beta_2 - l_2 b$, $q = \frac{l_2(b+1)}{2} - \beta_2$, $\beta_2 m_2 \equiv 1 \pmod{l_2}$, $b \equiv 2 \pmod 2$;

5) if $|l_2| = 1$ and $|l_1| > 1$, then M^3 is homeomorphic to the lens space $L_{p,q}$, where $p = 2\beta_1 - l_1 b$, $q = \frac{l_1(b+1)}{2} - \beta_1$, $\beta_1 m_1 \equiv 1 \pmod{l_1}$, $b \equiv 2 \pmod 2$;

6) if $|l_1 l_2| = 1$, then M^3 is homeomorphic to the lens space $L_{b,2}$, $b \equiv 1 \pmod 2$;

7) if $|l_1| > 1$ and $|l_2| > 1$ then M^3 is homeomorphic to the prime Seifert manifold $M\left(\mathbb{S}^2, (2,1), (l_1, \beta_1), (l_2, \beta_2)\right)$, $\beta_i m_i \equiv 1 \pmod{l_i}$, $i = 1, 2$ and is not homeomorphic to any lens space.

Proof. The idea of the proof is to recognize that the sphere $\mathbb{S}^3$ is obtained by Dehn surgery along a link consisting of a saddle orbit S of the flow f^t and a knot γ from the ambient manifold M^3. Then, due to the relation (2.1), we have $M^3 \cong \mathbb{S}^3_{\widetilde{S}\sqcup\widetilde{\gamma}}$, which allows us to describe the topology of the manifold M^3 using the set $C_{f^t} = (l_1, m_1, l_2, m_2)$. Let us break down the discussion into steps.

1. *Dehn surgery along a saddle orbit S.* Let us show that the following relation is true for a saddle orbit S:

$$M^3_S \cong L_{r,s}.$$

Let us put

$$\mathbb{V}_+ = \{(d_1, d_2, s) \in \mathbb{V} \mid d_1 \geqslant 0\}, \quad \mathbb{T}_+ = \{(d_1, d_2, s) \in \partial\mathbb{V} \mid d_1 \geqslant 0\},$$
$$\mathbb{V}_- = \{(d_1, d_2, s) \in \mathbb{V} \mid d_1 \leqslant 0\}, \quad \mathbb{T}_- = \{(d_1, d_2, s) \in \partial\mathbb{V} \mid d_1 \leqslant 0\}.$$

Let $h\colon \partial\mathbb{V} \to \partial V_S$ be a homeomorphism such that

$$h(\mathbb{T}_+) = T^u_S, \quad h(\mathbb{T}_-) = T^s_S.$$

Then $h_*(\langle 1, 0\rangle) = \langle 2, 1\rangle$, which implies

$$h_* = \begin{pmatrix} 2 & 1 \\ b & c \end{pmatrix}, \quad b, c \in \mathbb{Z}.$$

Consider the Dehn surgery M^3_S on M^3 along the knot S with a neighborhood V_S and equipment b, c. Let $v_S\colon \left(M^3 \setminus \operatorname{int} V_S\right) \sqcup \mathbb{V} \to M^3_S$ be the natural projection. For simplicity, we keep the notation of all objects on $v_S\left(M^3 \setminus \operatorname{int} V_S\right)$ the same as it was on $M^3 \setminus \operatorname{int} V_S$ and set $\widetilde{S} = v_S\left(\{0\} \times \mathbb{S}^1\right)$, $V_{\widetilde{S}} = v_S(\mathbb{V})$. Then $M^3_{\widetilde{S}}$ is the union of two solid tori $\widetilde{\mathcal{V}}_A = \mathcal{V}_A \cup v_S(\mathbb{V}_+)$ and $\widetilde{\mathcal{V}}_R = \mathcal{V}_R \cup v_S(\mathbb{V}_-)$ such that $\widetilde{\mathcal{V}}_A \cap \widetilde{\mathcal{V}}_R = \partial\widetilde{\mathcal{V}}_A \cap \partial\widetilde{\mathcal{V}}_R$ and hence $M^3_{\widetilde{S}} \cong L_{r,s}$ for some coprime integers a, b.

2. *Reverse Dehn surgery on lens $L_{r,s}$ along the knot $\widetilde{S}$.* Let $\widetilde{\mathcal{T}}_A = \partial\widetilde{\mathcal{V}}_A$, $\widetilde{\mathcal{T}}_R = \partial\widetilde{\mathcal{V}}_R$ and $L_{r,s} = \widetilde{\mathcal{V}}_A \cup \widetilde{\mathcal{V}}_R$. From Proposition 3 we find that $M^3 = (L_{r,s})_{\widetilde{S}}$, where $\widetilde{S}$ is a knot with equipment $-b$, 2. For knots $\delta \subset \widetilde{\mathcal{T}}_A$ $(= \widetilde{\mathcal{T}}_R)$ denote by $\langle\delta\rangle_A$, $\langle\delta\rangle_R$ the homotopy types of the knot δ on tori $\widetilde{\mathcal{T}}_A$, $\widetilde{\mathcal{T}}_R$, respectively. Then for cases 1)–3) from the definition of C_{ft} we have the following relations.

1. If $l_1 = 0$ and $l_2 \neq 0$, then either $\langle\widetilde{S}\rangle_A = \langle 0,\, 0\rangle$ or $\langle\widetilde{S}\rangle_A = \langle 0,\, 1\rangle$. In the first case $\langle\widetilde{S}\rangle_R = \langle 0,\, 0\rangle$ and $\langle M_A\rangle_R = \langle l_2,\, m_2\rangle$, which means $r = l_2$, $s = m_2$. Then by Proposition 4, $\mathbb{S}^3_{M_A} = L_{l_2,m_2}$, where M_A is the meridian of the torus $\widetilde{\mathcal{T}}_A$ equipped with m_2, l_2. Thus, $M^3 \cong \mathbb{S}^3_{\widetilde{S}\sqcup M_A}$. Since the knots $\widetilde{S} \sqcup M_A$ form a trivial link on sphere $\mathbb{S}^3$ (M_A can be chosen not to intersect $\widetilde{S}$), by virtue of Propositions 5,
$$M^3 = \mathbb{S}^3_{\widetilde{S}\sqcup M_A} \cong L_{l_2,m_2} \# L_{2,1} = L_{l_2,m_2} \# \mathbb{R}\mathrm{P}^3.$$
Similarly, if $\langle\widetilde{S}\rangle_A = \langle 0,\, 1\rangle$, then $\langle M_A\rangle_R = \langle\widetilde{S}\rangle_R = \langle l_2,\, m_2\rangle$ and so $r = l_2$, $s = m_2$. Since M_A can also be chosen to be disjoint from $\widetilde{S}$, it follows that
$$M^3 \cong L_{l_2,m_2} \# \mathbb{R}\mathrm{P}^3.$$
2. If $l_1 \neq 0$ and $l_2 = 0$, then $\langle\widetilde{S}\rangle_R = \langle 0,\, 1\rangle$. Then $\langle M_R\rangle_A = \langle\widetilde{S}\rangle_A = \langle l_1,\, m_1\rangle$ and, hence, $r = l_1$, $s = m_1$, whence, from arguments similar to the above, we obtain
$$M^3 \cong L_{l_1,m_1} \# \mathbb{R}\mathrm{P}^3.$$
3. If $l_1 = l_2 = 0$, then $\langle\widetilde{S}\rangle_R = \langle 0,\, 1\rangle$. Then $\langle M_R\rangle_A = \langle\widetilde{S}\rangle_A = \langle 0,\, 1\rangle$ and, hence, $r = 0$, $s = 1$, whence it follows that
$$M^3 \cong \mathbb{R}\mathrm{P}^3 \# L_{0,1} = \mathbb{R}\mathrm{P}^3 \# \mathbb{S}^2 \times \mathbb{S}^1.$$

3. *Seifert fibration on manifold M^3.* To prove the remaining points, we note that in the case when $l_1 l_2 \neq 0$, the manifold $M^3 = \mathcal{V}_A \cup V_S \cup \mathcal{V}_R$ has a Seifert fibration. Indeed, in this case, the fibration V_S of the solid torus with exceptional fiber S and orbital invariants (2, 1) contains the knots K^u_S and K^s_S as fibers. This fibration extends to a solid torus $\mathcal{V}_A$ and $\mathcal{V}_R$ fibration with fibers A and R (which may or may not be exceptional), respectively, and with orbital invariants $(l_1,\, m_1)$ and $(l_2,\, m_2)$. In this way,
$$M^3 \cong M(\Sigma,\, (2,\, b),\, (l_1,\, \beta_1),\, (l_2,\, \beta_2)), \quad \beta_i m_i \equiv 1 \pmod{l_i}, \quad b \equiv 1 \pmod 2.$$
Let us show that the base Σ of such a bundle is a 2-sphere.

Let $\sim$ be an equivalence relation whose equivalence classes are the fibers of this fibration. Figure 2 shows the meridian disks D_A, D_S, D_R of the tori $\mathcal{V}_A$, V_S, $\mathcal{V}_R$, respectively, the segments containing equivalent points are shown in the same color. Gluing the equivalent points in the disks D_A, D_S, D_R, respectively, we obtain the disks $\widehat{D}_A = \mathcal{V}_A/{\sim}$, $\widehat{D}_S = V_S$, $\widehat{D}_R = \mathcal{V}_R$, in which

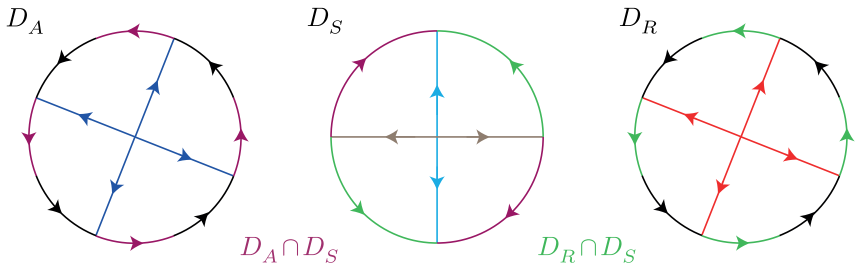

Fig. 2. Disks D_A, D_S, D_R

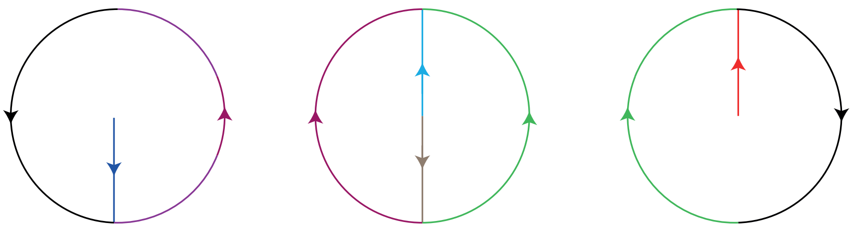

Fig. 3. Disks $\widehat{D}_A$, $\widehat{D}_S$, $\widehat{D}_R$

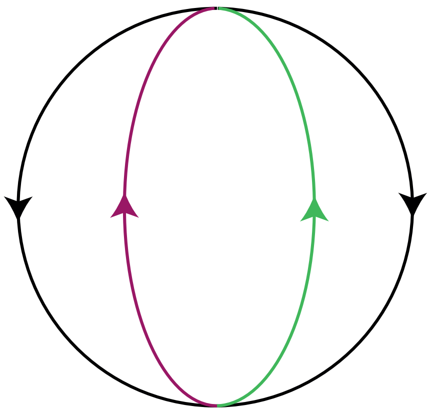

Fig. 4. $\Sigma \cong \mathbb{S}^2$

each fiber, except for the boundary fibers, is represented by one point and each boundary fiber is represented by two points on different disks (see Fig. 3). By gluing the equivalent points in the disks $\widehat{D}_A$, $\widehat{D}_S$, $\widehat{D}_R$ we obtain the sphere $\mathbb{S}^2$ (see Fig. 4), which is the base of the fibration given on M^3. So,

$$M^3 \cong M\left(\mathbb{S}^2, (2, b), (l_1, \beta_1), (l_2, \beta_2)\right), \quad \beta_i m_i \equiv 1 \pmod{l_i}. \tag{4.1}$$

1. If $|l_1| = 1$, $|l_2| > 1$, then the fiber A is ordinary and, by Proposition 8,
$$M^3 \cong M\left(\mathbb{S}^2, (2, b), (l_2, \beta_2)\right) \cong L_{p,q},$$
where $p = 2\beta_2 - l_2 b$, $q = \frac{l_2(b+1)}{2} - \beta_2$.
2. If $|l_1| > 1$, $|l_2| = 1$, then the fiber R is ordinary and, according to Proposition 8,
$$M^3 \cong L_{p,q},$$
where $p = 2\beta_1 - l_1 b$, $q = \frac{l_1(b+1)}{2} - \beta_1$.

3. If $|l_1 l_2| = 1$, then both fibers A, R are ordinary and, hence, by Proposition 8 and Proposition 2,
$$M^3 \cong M\left(\mathbb{S}^2,\, (2,\, b)\right) \cong L_{b,2}.$$

4. If $|l_1| > 1$, $|l_2| > 1$, then M^3 is a Seifert manifold with three exceptional fibers
$$M^3 \cong M\left(\mathbb{S}^2,\, (2,\, b),\, (l_1,\, \beta_1),\, (l_2,\, \beta_2)\right) \cong M\left(\mathbb{S}^2,\, (2,\, 1),\, (l_1,\, \beta_1'),\, (l_2,\, \beta_2')\right),$$
$$\beta_i' m_i \equiv 1 \pmod{l_i}, \quad \frac{\beta_1}{l_1} + \frac{\beta_2}{l_2} + \frac{b}{2} = \frac{\beta_1'}{l_1} + \frac{\beta_2'}{l_2} + \frac{1}{2}.$$

By Proposition 7, M^3 is prime and, by Proposition 9, it is not homeomorphic to a lens space. □

Conflict of interest

The authors declare that they have no conflict of interest.

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