

Chow dilogarithm and strong Suslin reciprocity law

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ABSTRACT

We prove a conjecture of A. Goncharov concerning strong Suslin reciprocity law. The main idea of the proof is the construction of the norm map on so-called lifted reciprocity maps. This construction is similar to the construction of the norm map on Milnor K -theory. As an application, we express Chow dilogarithm in terms of Bloch-Wigner dilogarithm. Also, we obtain a new reciprocity law for four rational functions on an arbitrary algebraic surface with values in the pre-Bloch group.

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1. Introduction

Everywhere we work over \mathbb{Q} . So any abelian group is supposed to be tensored by \mathbb{Q} . For example, when we write $\Lambda^2 k^\times$ this actually means $(\Lambda^2 k^\times) \otimes \mathbb{Q}$. All exterior powers and tensor products are over \mathbb{Q} .

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Let k be a field and X be a smooth projective curve over k . For a field F denote by $K_n(F)$ the n -th algebraic K -theory of F . For any closed point $z \in X$ one can define the residue map $\widehat{\partial}_z: K_n(k(X)) \rightarrow K_{n-1}(k(z))$ (we use this notation to distinguish this map from the residue map on polylogarithmic complexes which will be defined below), where $k(z)$ is the residue field of the point z (see [Wei13, V.5]). Denote by $tr_{k(z)/k}$ the push-forward map $K_{n-1}(k(z)) \rightarrow K_{n-1}(k)$ associated to the natural projection $\text{Spec}(k(z)) \rightarrow \text{Spec}(k)$. It follows from the basic properties of algebraic K -theory that for any $a \in K_n(k(X))$ and all but finitely many $z \in X$, we have $\widehat{\partial}_z(a) = 0$ and moreover the following sum is equal to zero:

$$\sum_{z \in X^{(1)}} tr_{k(z)/k} \circ \widehat{\partial}_z(a) = 0.$$

In this formula $X^{(1)}$ denotes the set of closed points of the curve X .

On the other hand, for any field F , A. Goncharov [Gon95] defined so-called polylogarithmic complexes $\Gamma(F, n)$, $n \in \mathbb{N}$ and conjectured that these complexes compute the graded pieces of the algebraic K -theory of F . More precisely the cohomology $H^i(\Gamma(F, n) \otimes \mathbb{Q})$ should be isomorphic to $gr_\gamma^n K_{2n-i}(F) \otimes \mathbb{Q}$. Here $gr_\gamma^n K_{2n-i}(F) \otimes \mathbb{Q}$ is the associated graded space with respect to γ -filtration (see for example [Wei13]).

The complex $\Gamma(F, n)$ looks as follows:

$$\Gamma(F, n): \mathcal{B}_n(F) \xrightarrow{\delta_n} \mathcal{B}_{n-1}(F) \otimes F^\times \xrightarrow{\delta_n} \dots \xrightarrow{\delta_n} \mathcal{B}_2(F) \otimes \Lambda^{n-2} F^\times \xrightarrow{\delta_n} \Lambda^n F^\times.$$

This complex is concentrated in degrees $[1, n]$. The group $\mathcal{B}_n(F)$ is the quotient of the free abelian group generated by symbols $\{x\}_n, x \in \mathbb{P}^1(F)$ by some explicitly defined subgroup $\mathcal{R}_n(F)$ (see [Gon94]). In the next section we will present the generators for the group $\mathcal{R}_2(F)$. The differential is defined as follows: $\delta_n(\{x\}_k \otimes y_{k+1} \wedge \dots \wedge y_n) = \{x\}_{k-1} \otimes x \wedge y_{k+1} \wedge \dots \wedge y_n$ for $k > 2$ and $\delta_n(\{x\}_2 \otimes y_3 \wedge \dots \wedge y_n) = x \wedge (1-x) \wedge y_3 \wedge \dots \wedge y_n$.

Let us assume that the field k is algebraically closed. In this case A. Goncharov constructed the morphism of complexes $\partial_z^{(n)}: \Gamma(F, n) \rightarrow \Gamma(k, n-1)[-1]$ which should correspond to the residue map on the algebraic K -theory. So, it is natural to suppose that there is a homotopy between the map $\sum_{z \in X^{(1)}} \partial_z^{(n)}$ and the zero map. In this paper we will deal only with the case $n = 3$. In this case the existence of such a homotopy was proved in [Rud21].

It turns out that this story is connected with so-called Chow dilogarithm defined by A. Goncharov in [Gon05]. For any smooth projective curve X over \mathbb{C} and three non-zero rational functions f_1, f_2, f_3 on X , the Chow dilogarithm $\mathcal{P}_2(X; f_1, f_2, f_3)$ is defined by the formula

$$\mathcal{P}_2(X; f_1, f_2, f_3) = (2\pi i)^{-1} \int_{X(\mathbb{C})} r_2(f_1, f_2, f_3),$$

where

$$r_2(f_1, f_2, f_3) = \frac{1}{6} \sum_{\sigma \in S_3} \text{sgn}(\sigma) \widetilde{r}_2(f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3})$$

$$\widetilde{r}_2(f_1, f_2, f_3) = \log |f_1| d \log |f_2| \wedge d \log |f_3| - 3 \log |f_1| d \arg(f_2) \wedge d \arg(f_3).$$

On the other hand there is the canonical map $\widetilde{\mathcal{L}}_2: \mathcal{B}_2(\mathbb{C}) \rightarrow \mathbb{R}$, given by the Bloch-Wigner dilogarithm (see [Gon95]). A. Goncharov conjectured that for any algebraically closed field k and any smooth projective curve X over k , there should exist the canonical map $\mathcal{H}_{k(X)}: \Lambda^3 k(X) \rightarrow \mathcal{B}_2(k)$ such that for $k = \mathbb{C}$ we have $\mathcal{P}_2(X; f_1, f_2, f_3) = \widetilde{\mathcal{L}}_2(\mathcal{H}_{\mathbb{C}(X)}(f_1 \wedge f_2 \wedge f_3))$. The word

“canonical” means that this map should be functorial under non-constant morphisms of curves. Moreover, motivated by the analytic properties of Chow dilogarithm, he conjectured that the map $\mathcal{H}_{k(X)}$ should additionally satisfy to the following two properties:

- (i) It should vanish on the elements of the form $c \wedge f_2 \wedge f_3, c \in k, f_2, f_3 \in k(X)$,
- (ii) This map should give a homotopy between the map $\sum_{z \in X^{(1)}} \partial_z^{(3)}$ and the zero map:

$$\begin{array}{ccc}
 \mathcal{B}_3(k(X)) & \xrightarrow{\delta_3} & \mathcal{B}_2(k(X)) \otimes k(X)^\times & \xrightarrow{\delta_3} & \Lambda^3 k(X)^\times \\
 & & \downarrow \sum_{z \in X^{(1)}} \partial_z^{(3)} & \swarrow \mathcal{H}_{k(X)} & \downarrow \sum_{z \in X^{(1)}} \partial_z^{(3)} \\
 & & \mathcal{B}_2(k) & \xrightarrow{-\delta_2} & \Lambda^2(k^\times).
 \end{array} \tag{1}$$

In this paper we assume that the field k is an algebraically closed field of characteristic zero. In this case we prove the above conjecture. That is for any smooth projective curve over k , we construct the map $\mathcal{H}_{k(X)}: \Lambda^3 k(X)^\times \rightarrow \mathcal{B}_2(k)$ such that all these maps satisfy the conditions stated above.

- REMARK 1.1. (i) We impose the condition on characteristic of the field k only for simplicity. It seems that the results of this paper can be generalized to the case of arbitrary characteristic. Meanwhile, the condition that k is algebraically closed is essential. If the field k were not algebraically closed, then in the case $k \subsetneq k(z)$ there would be no natural morphism of complexes $\Gamma(F, n) \rightarrow \Gamma(k, n-1)[-1]$. The reason is that while there is the natural map $\Gamma(F, n) \rightarrow \Gamma(k(z), n-1)[-1]$, the push-forward map $\Gamma(k(z), n-1)[-1] \rightarrow \Gamma(k, n-1)[-1]$ cannot be defined on the level of complexes.
- (ii) Let k_0 be some subfield of k . It can be deduced from our main result, that if the curve X together with three functions f_1, f_2, f_3 are defined over k_0 then the element $\mathcal{H}_{k(X)}(f_1 \wedge f_2 \wedge f_3)$ lies in the invariants $\mathcal{B}_2(k)^{Gal(k/k_0)}$. However, it seems that in general the group $\mathcal{B}_2(k)^{Gal(k/k_0)}$ is strictly bigger than $\mathcal{B}_2(k_0)$.
 - (iii) By theorem of A. Suslin (Corollary 5.7 from [Sus91]) when the field k is algebraically closed, the group $\mathcal{B}_2(k)$ is uniquely divisible. So it seems that the restriction that we work only \mathbb{Q} -linearly is not essential.

1.1 Definitions

We recall that everywhere we work \mathbb{Q} -linearly. Let F be an arbitrary field. We repeat the definition of the complex $\Gamma(F, n)$ for convenience.

DEFINITION 1.2. Define the complex $\Gamma(F, n)$ as follows:

$$\Gamma(F, n): \mathcal{B}_n(F) \xrightarrow{\delta_n} \mathcal{B}_{n-1}(F) \otimes F^\times \xrightarrow{\delta_n} \dots \xrightarrow{\delta_n} \mathcal{B}_2(F) \otimes \Lambda^{n-2} F^\times \xrightarrow{\delta_n} \Lambda^n F^\times.$$

This complex is concentrated in degrees $[1, n]$. The group $\mathcal{B}_n(F)$ is the quotient of the free abelian group generated by symbols $\{x\}_n, x \in \mathbb{P}^1(F)$ by some explicitly defined subgroup $\mathcal{R}_n(F)$ (see [Gon94]). The differential is defined as follows: $\delta_n(\{x\}_k \otimes y_{k+1} \wedge \dots \wedge y_n) = \{x\}_{k-1} \otimes x \wedge y_{k+1} \wedge \dots \wedge y_n$ for $k > 2$ and $\delta_n(\{x\}_2 \otimes y_3 \wedge \dots \wedge y_n) = x \wedge (1-x) \wedge y_3 \wedge \dots \wedge y_n$.

- REMARK 1.3. (i) It is not known whether the definitions of the group $\mathcal{R}_n(F)$ from [Gon95] and [Gon94] are equivalent. While it is believed to be the case, this statement relies on

the so-called Suslin rigidity conjecture. In this paper we use the later definition, that is the definition from [Gon94].

- (ii) Everywhere in this paper we can replace the complex $\Gamma(F, n)$ with its canonical truncation $\tau_{\geq n-1}\Gamma(F, n)$. Therefore, only the definition of the group $\mathcal{R}_2(F)$ is relevant for us. As it was noted in Section 4.2 of [Gon94] this group is generated by the following elements:

$$\sum_{i=1}^5 (-1)^i \{c.r.(x_1, \dots, \hat{x}_i, \dots, x_5)\}_2, \{0\}_2, \{1\}_2, \{\infty\}_2.$$

In this formula x_i are five different points on \mathbb{P}^1 and $c.r.(·)$ is the cross ratio.

Let F be an arbitrary field. We recall that the n -th Milnor K -theory $K_n^M(F)$ of the field F is defined as the quotient of the vector space $\Lambda^n F^\times$ by the elements of the form $a_1 \wedge (1 - a_1) \wedge a_3 \wedge \dots \wedge a_n$, where $a_i \in F^\times$ and $a_1 \neq 1$. We have the canonical identification $H^n(\Gamma(F, n)) \cong K_n^M(F)$. If $j: F_1 \hookrightarrow F_2$ is an embedding of fields, denote by $j_*: K_n^M(F_1) \rightarrow K_n^M(F_2)$ the natural map given by the formula $j_*(a_1 \wedge \dots \wedge a_n) = j(a_1) \wedge \dots \wedge j(a_n)$. Bass and Tate [BT73] constructed the norm map $\widehat{N}_{F_2/F_1}: K_n^M(F_2) \rightarrow K_n^M(F_1)$ which a priori depends on the choice of generators of F_2 over F_1 . A. Suslin [Sus79] proved that the norm map \widehat{N}_{F_2/F_1} is independent of the choice of generators and is determined only by the embedding j .

Let (F, ν) be a discrete valuation field. Denote $\mathcal{O}_\nu = \{x \in F | \nu(x) \geq 0\}$, $m_\nu = \{x \in F | \nu(x) > 0\}$ and $\overline{F}_\nu = \mathcal{O}_\nu/m_\nu$. We recall that an element $a \in F^\times$ is called a *uniformiser* if $\nu(a) = 1$ and a *unit* if $\nu(a) = 0$. For $u \in \mathcal{O}_\nu$ denote by \overline{u} its residue class in \overline{F}_ν .

The proof of the following proposition can be found in [Gon95]:

PROPOSITION 1.4. *Let (F, ν) be a discrete valuation field and $n \geq 3$. There is a unique morphism of complexes $\partial_\nu^{(n)}: \Gamma(F, n) \rightarrow \Gamma(\overline{F}_\nu, n-1)[-1]$ satisfying the following conditions:*

- (i) *For any uniformiser π and units $u_2, \dots, u_n \in F$ we have $\partial_\nu^{(n)}(\pi \wedge u_2 \wedge \dots \wedge u_n) = \overline{u_2} \wedge \dots \wedge \overline{u_n}$.*
- (ii) *For any $a \in F \setminus \{0, 1\}$ with $\nu(a) \neq 0$, an integer k satisfying $2 \leq k \leq n$ and any $b \in \Lambda^{n-k} F^\times$ we have $\partial_\nu^{(n)}(\{a\}_k \otimes b) = 0$.*
- (iii) *For any unit u , an integer k satisfying $2 \leq k \leq n$ and $b \in \Lambda^{n-k} F^\times$ we have $\partial_\nu^{(n)}(\{u\}_k \otimes b) = -\{\overline{u}\}_k \otimes \partial_\nu^{(n-k)}(b)$.*

We will call the map $\partial_\nu^{(n)}$ from the previous proposition *the tame symbol map*. The proof of this proposition can be found in [Gon95, Section 14].

We will need the following lemma which easily follows from the definition of the tame-symbol:

LEMMA 1.5. *Let (F, ν) be a discrete valuation field. Let k, n be two natural numbers satisfying the condition $k < n$. Let $a_1, \dots, a_n \in F^\times$ such that $\nu(a_{k+1}), \dots, \nu(a_n) = 0$. Then the following formula holds:*

$$\partial_\nu^{(n)}(a_1 \wedge \dots \wedge a_n) = \partial_\nu^{(k)}(a_1 \wedge \dots \wedge a_k) \wedge \overline{a_{k+1}} \wedge \dots \wedge \overline{a_n}.$$

When D is an irreducible divisor on a smooth variety X , we denote by ν_D the corresponding discrete valuation of the field $k(X)$. For any field F denote by $\nu_{\infty, F}$ the discrete valuation of $F(t)$ given by the point $\infty \in \mathbb{P}^1(F)$.

We recall that we have fixed some algebraically closed field k of characteristic zero. Denote by \mathbf{Fields}_d the category of finitely generated extensions of k of transcendence degree d . Any

morphism in this category is a finite extension. For $F \in \mathbf{Fields}_d$, denote by $\text{dval}(F)$ the set of discrete valuations given by an irreducible Cartier divisor on some birational model of F . When $F \in \mathbf{Fields}_1$ this set is equal to the set of all discrete valuations that are trivial on k . In this case, we denote this set simply by $\text{val}(F)$. If X is an algebraic variety together with the isomorphism $k(X) \rightarrow F$ we denote by $\text{dval}(F)_X \subset \text{dval}(F)$ the subset of divisorial valuations coming from divisors on X .

Let $j: K \hookrightarrow F$ be an extension from \mathbf{Fields}_d and $\nu \in \text{dval}(K)$. Denote by $\text{ext}(\nu, F)$ the set of extensions of the valuation ν to F . Let $\nu' \in \text{ext}(\nu, F)$. Denote by $j_{\nu'|\nu}$ the natural embedding $\overline{K}_\nu \hookrightarrow \overline{F}_{\nu'}$. The inertia degree $f_{\nu'|\nu}$ is defined as $\deg j_{\nu'|\nu}$. The ramification index $e_{\nu'|\nu}$ is defined by the formula $\pi_K = u\pi_F^{e_{\nu'|\nu}}$, where π_K, π_F are uniformisers of K, F and u is some unit. By [Neu13, Chapter II, §8] the set $\text{ext}(\nu, F)$ is finite and, moreover, the following formula holds:

$$\sum_{\nu' \in \text{ext}(\nu, F)} e_{\nu'|\nu} f_{\nu'|\nu} = [F : K]. \quad (2)$$

By the Theorem of O. Zariski [ZS13, Chapter VI, §14, Theorem 31] a discrete valuation on F is divisorial if and only if the corresponding residue field is finitely generated over k and has transcendence degree 1. This implies that for any $\nu \in \text{dval}(K)$, we have $\text{ext}(\nu, F) \subset \text{dval}(F)$.

For any $n \geq 0$ there is the natural map $j_*: \Lambda^n K^\times \rightarrow \Lambda^n F^\times$ given by the formula $j_*(a) = a$. It is easy to see that for any $\nu' \in \text{ext}(\nu, F)$ the following formula holds:

$$\partial_{\nu'}^{(n)} j_*(a) = e_{\nu'|\nu} \cdot (j_{\nu'|\nu})_*(\partial_\nu^{(n)}(a)). \quad (3)$$

1.2 Lifted reciprocity maps

We recall that we work \mathbb{Q} -linearly.

DEFINITION 1.6. Let $F \in \mathbf{Fields}_1$. A *lifted reciprocity map* on the field F is a \mathbb{Q} -linear map $h: \Lambda^3 F^\times \rightarrow \mathcal{B}_2(k)$ satisfying the following conditions:

(i) The following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{B}_3(F) & \xrightarrow{\delta_3} & \mathcal{B}_2(F) \otimes F^\times & \xrightarrow{\delta_3} & \Lambda^3 F^\times \\ & & \downarrow \sum_{\nu \in \text{val}(F)} \partial_\nu^{(3)} & \swarrow h & \downarrow \sum_{\nu \in \text{val}(F)} \partial_\nu^{(3)} \\ & & \mathcal{B}_2(k) & \xrightarrow{-\delta_2} & \Lambda^2(k^\times). \end{array} \quad (4)$$

(ii) The map h vanishes on the image of the multiplication map $\Lambda^2 F^\times \otimes k^\times \rightarrow \Lambda^3 F^\times$.

REMARK 1.7. The set of all lifted reciprocity maps has a structure of affine space over \mathbb{Q} as any set of homotopies.

Denote by **Set** the category of sets. Define a contravariant functor

$$\text{RecMaps}: \mathbf{Fields}_1 \rightarrow \mathbf{Set}$$

as follows. For any $F \in \mathbf{Fields}_1$ the set $\text{RecMaps}(F)$ is equal to the set of all lifted reciprocity maps on F . If $j: K \hookrightarrow F$ then $\text{RecMaps}(j)(h_F)$ is defined by the formula $h_K(a) := \frac{1}{\deg j} h_F(j_*(a))$. It is not difficult to show that the assignment preserves identities. We will present the detailed proof that RecMaps is indeed a functor in Section 2.1.

1.3 Main results

The following theorem is a solution of Conjecture 6.2 from [Gon05].

THEOREM 1.8. *For any field $F \in \mathbf{Fields}_1$ one can choose a lifted reciprocity map \mathcal{H}_F on the field F such that for any embedding $j: F_1 \rightarrow F_2$ we have $\text{RecMaps}(j)(\mathcal{H}_{F_2}) = \mathcal{H}_{F_1}$. Such a collection of lifted reciprocity maps is unique.*

REMARK 1.9. One of the main results from [Rud21] states that for any field $F \in \mathbf{Fields}_1$ there is a map $\Lambda^3 F^\times \rightarrow \mathcal{B}_2(k)$ satisfying the first condition of Definition 1.6. However, it is not clear why this map can be chosen functorial. The functoriality is our new result.

We remark that even the proof of existence of a homotopy is simpler because it does not rely on complicated lemmas 5.2 - 5.7 from [Rud21].

REMARK 1.10. In [Gon05, Section 6], A. Goncharov proved that for any elliptic curve E over k there is a lifted reciprocity map on the field $k(E)$. From the proof of Theorem 1.8 it is not difficult to show that his map coincides with ours. Therefore, Theorem 1.8 generalizes A. Goncharov's construction to curves of arbitrary genus.

1.3.1 Chow dilogarithm The definition of Chow dilogarithm can be found in Section 6 of [Gon05]. This function associates to any smooth projective curve X over \mathbb{C} and three non-zero rational functions f_1, f_2, f_3 on X the value $\mathcal{P}_2(X; f_1, f_2, f_3) \in \mathbb{R}$. The remark after Conjecture 6.2 in loc. cit. implies that Theorem 1.8 has the following corollary:

COROLLARY 1.11. *For any smooth projective curve X over \mathbb{C} and three non-zero rational functions f_1, f_2, f_3 on X the following formula holds:*

$$\mathcal{P}_2(X; f_1, f_2, f_3) = -\tilde{\mathcal{L}}_2(\mathcal{H}_{\mathbb{C}(X)}(f_1 \wedge f_2 \wedge f_3)).$$

Here $\tilde{\mathcal{L}}_2: \mathcal{B}_2(\mathbb{C}) \rightarrow \mathbb{R}$ is a map given on the generators $\{x\}_2$ by the formula

$$\tilde{\mathcal{L}}_2(\{x\}_2) = \mathcal{L}_2(x),$$

where \mathcal{L}_2 is Bloch-Wigner dilogarithm.

REMARK 1.12. (i) The sign comes from the fact that we use a little bit different definition of the map $\partial_\nu^{(3)}$.

(ii) This statement is similar to Corollary 1.5 from [Rud21]. However the proof from loc. cit. is not correct: it relies on a remark after Conjecture 6.2 from [Gon05], which uses the functorial property. So Corollary 1.11 is new.

1.3.2 Two-dimensional reciprocity law From the proof of Theorem 1.8 we get the following corollary:

COROLLARY 1.13. *Let $L \in \mathbf{Fields}_2$. For any $b \in \Lambda^4 L^\times$ and all but finitely many $\nu \in \text{dval}(L)$ we have $\mathcal{H}_{L_\nu} \partial_\nu^{(4)}(b) = 0$. Moreover, the following sum is equal to zero:*

$$\sum_{\nu \in \text{dval}(L)} \mathcal{H}_{L_\nu} \partial_\nu^{(4)}(b) = 0. \tag{5}$$

Applying $\tilde{\mathcal{L}}_2$ to both sides of (5) and using Corollary 1.11, we recover the functional equation for Chow dilogarithm proved by A. Goncharov in [Gon05, Section 1.4], see also [BGKLL18]. Actually Corollary 1.13 was our motivation behind the construction of the map \mathcal{H} .

1.4 The outline of the paper

In Section 2.1 we will show that there is the unique lifted reciprocity map on the field of rational functions $k(t)$. Denote it by $\mathcal{H}_{k(t)}$. Let $F \in \mathbf{Fields}_1$ be any field. Choose an embedding $j: k(t) \hookrightarrow F$. To define \mathcal{H}_F , we extend a lifted reciprocity map $\mathcal{H}_{k(t)}$ from the field $k(t)$ to the field F . For this we solve the more general problem: for any finite extension $j': F_1 \hookrightarrow F_2$ in \mathbf{Fields}_1 we construct the canonical map $N_{F_2/F_1}: \text{RecMaps}(F_1) \rightarrow \text{RecMaps}(F_2)$. More precisely we will prove the following theorem:

THEOREM 1.14. *For any embedding of fields $j: F_1 \hookrightarrow F_2$ one can define the canonical map $N_{F_2/F_1}: \text{RecMaps}(F_1) \rightarrow \text{RecMaps}(F_2)$ satisfying the following properties:*

- (i) $\text{RecMaps}(j) \circ N_{F_2/F_1} = \text{id}$.
- (ii) If $F_1 \subset F_2 \subset F_3$ is a tower of extension from \mathbf{Fields}_1 then $N_{F_3/F_1} = N_{F_3/F_2} \circ N_{F_2/F_1}$.

Item (i) shows that N_{F_2/F_1} is indeed an extension, while item (ii) shows that this extension is functorial.

Sections 2 and 3 are devoted to the proof of this theorem. The details will be given below. Now the proof of Theorem 1.8 is easy: the existence follows from Theorem 1.14 together with the fact that the element $N_{F/k(t)}(\mathcal{H}_{k(t)})$ does not depend on the embedding $j: k(t) \hookrightarrow F$. The uniqueness follows from standard arguments. We will present this proof in Section 4.1. The proof of Corollary 1.13 will be given in Section 4.2.

Let us outline the proof of Theorem 1.14. The proof of this theorem is in many respect similar to the construction of the norm map on Milnor K -theory. (See [BT73, Sus79, Mil70, Kat80]). That is the reason why we denote it by the letter N . (Note that compared to the norm map in Milnor K -theory, in our case, the norm map is directed in the opposite direction. The reason for this is that while Milnor K -theory gives a covariant functor, the functor RecMaps is contravariant.) In Section 3.1 for any field $F \in \mathbf{Fields}_1$ and any $\nu \in \text{dval}(F(t))$ we will construct the map $\mathcal{N}_\nu: \text{RecMaps}(F) \rightarrow \text{RecMaps}(\overline{F(t)}_\nu)$. Using this map, for any extension $F_1 \hookrightarrow F_2$ with a generator a we will define the norm map $N_{F_2/F_1, a}: \text{RecMaps}(F_1) \rightarrow \text{RecMaps}(F_2)$ (see Definition 3.8). Using ideas from [Sus79] we will show that this map does not depend on a and will have finished the proof of Theorem 1.14. This will be done in Section 3.2 and Section 3.3.

The most non-trivial part of this paper is the construction of the map \mathcal{N}_ν . Let us give the outline of this construction. Let $F \in \mathbf{Fields}_1$. It is useful to divide the discrete valuations of the field $F(t)$ into two classes, namely the general valuations and the special ones (see Definition 2.3). For the special valuations the definition of the map \mathcal{N}_ν is straightforward. To reduce the definition of the map \mathcal{N}_ν when ν is general to the previous case, we use the notion of *the lift*. Let $\nu \in \text{dval}(F(t))$ be a general valuation and $n, j \in \mathbb{N}$. A lift of an element $a \in \Gamma(\overline{F(t)}_\nu, n)_j$ is an element $b \in \Gamma(F(t), n+1)_{j+1}$, such that the tame-symbol $\partial_\nu^{(n+1)}(b)$ is equal to a and the tame-symbol of b at any other general valuation vanishes. The set of all lifts of the element a is denoted by $\mathcal{L}(a)$. In Section 2.2 we will show that in the case $n = 3, j \in \{2, 3\}$, for any $a \in \Gamma(\overline{F(t)}_\nu, n)_j$ the set $\mathcal{L}(a)$ is non-empty. Now, when $\nu \in \text{dval}(F(t))$ is a general valuation, $h \in \text{RecMaps}(F)$ and $a \in \Lambda^3 \overline{F(t)}_\nu$, we can choose some lift $b \in \mathcal{L}(a)$ and define the element $\mathcal{N}_\nu(h)(a)$ by the following formula:

$$\mathcal{N}_\nu(h)(a) = - \sum_{\mu \in \text{dval}(F(t))_{sp}} \mathcal{N}_\mu(h)(\partial_\mu^{(4)}(b)).$$

Here $\text{dval}(F(t))_{sp}$ denotes the set of all special valuations. In this formula the lifted reciprocity

maps $\mathcal{N}_\nu(h)$ are already defined because μ are special. It remains to show that this expression does not depend on the choice of b and for fixed h gives a lifted reciprocity map on the field $\overline{F}(t)_\nu$. This can be done using the properties of the lift established in Section 2.2 and some version of the Parshin reciprocity law which will be proved in Section 2.3.

1.5 Conventions

If C is a chain complex denote by C_d the elements lying in degree d . The symbol δ_n means the differential in the polylogarithmic complex $\Gamma(F, n)$. Although it depends on the field F we will omit the corresponding sign from the notation. In the same way, when (F, ν) is a discrete valuation field we denote by $\partial_\nu^{(n)}$ the tame-symbol map $\Gamma(F, n) \rightarrow \Gamma(\overline{F}_\nu, n-1)[-1]$.

2. Preliminary results

2.1 Lifted reciprocity maps

PROPOSITION 2.1. *RecMaps is indeed a functor.*

Proof. If j_1, j_2 are some embeddings from **Fields**₁ then the formula

$$\text{RecMaps}(j_2 \circ j_1) = \text{RecMaps}(j_1) \circ \text{RecMaps}(j_2)$$

follows from the fact that the ramification index is multiplicative. So it is enough to show that for any embedding $j: K \hookrightarrow F$ and $h_F \in \text{RecMaps}(F)$ the map

$$h_K := \text{RecMaps}(j)(h_F) = \frac{1}{\deg j} h_F \circ j_*: \Lambda^3 K^\times \rightarrow \mathcal{B}_2(k)$$

is a lifted reciprocity map on K .

The statement that h_K is zero on the image of the map $\Lambda^2 K^\times \otimes k^\times \rightarrow \Lambda^3 K^\times$ follows from the corresponding statement for h_F . Let us prove that diagram (4) is commutative.

For any $\nu \in \text{val}(K)$ and any $\nu' \in \text{ext}(\nu, F)$ we have $f_{\nu'|\nu} = 1$. Therefore, formula (2) becomes $\sum_{\nu' \in \text{ext}(\nu, F)} e_{\nu'|\nu} = [F : K]$. Since in our case $\overline{K}_\nu \cong \overline{F}_{\nu'} \cong k$, the formula (3) takes the form $e_{\nu'|\nu} \partial_\nu^{(3)}(a) = \partial_{\nu'}^{(3)} j_*(a)$.

For any $a \in \Lambda^3 K^\times$, we have:

$$\begin{aligned} \delta_2(h_K(a)) &= \frac{1}{[F : K]} \delta_2(h_F(j_*(a))) = \frac{1}{[F : K]} \sum_{\nu' \in \text{val}(F)} \partial_{\nu'}^{(3)} j_*(a) = \\ &= \frac{1}{[F : K]} \sum_{\nu \in \text{val}(K)} \sum_{\nu' \in \text{ext}(\nu, F)} \partial_{\nu'}^{(3)} j_*(a) = \\ &= \frac{1}{[F : K]} \sum_{\nu \in \text{val}(K)} \sum_{\nu' \in \text{ext}(\nu, F)} e_{\nu'|\nu} \partial_\nu^{(3)}(a) = \sum_{\nu \in \text{val}(K)} \partial_\nu^{(3)}(a). \end{aligned}$$

Here in the fourth equality we have used the formula $\partial_{\nu'}^{(3)}(j_*(a)) = e_{\nu'|\nu} \partial_\nu^{(3)}(a)$ and in the last formula we have used the formula $\sum_{\nu' \in \text{ext}(\nu, F)} e_{\nu'|\nu} = [F : K]$. So the lower right triangle is commutative. The commutativity of the upper left triangle is similar. \square

PROPOSITION 2.2. *On the field $k(t)$ there is the unique lifted reciprocity map. We will denote it by $\mathcal{H}_{k(t)}$*

Proof. Elementary calculation shows that the group $\Lambda^3 k(t)^\times$ is generated by the image of the multiplication map $k(t)^\times \otimes \Lambda^2 k^\times \rightarrow \Lambda^3 k(t)^\times$ and by the image of δ_3 . Uniqueness follows from this statement. Existence was proved in [Gon95, Theorem 6.5]. Let us give two remarks:

- (i) Because we use another sign convention in the definition of $\partial_\nu^{(3)}$ (see Proposition 1.4) we need to multiply the map h (see Proposition 6.6) from [Gon05] by -1 .
- (ii) Although the proof of Proposition 6.6 from [Gon95] uses rigidity argument, this proposition can be easily deduced from [DP82], where it was proved that $\mathcal{B}_2(k(t))$ is generated by elements of the form $\{at + b\}_2, a, b \in k$.

□

2.2 The construction of the lift

DEFINITION 2.3. Let $F \in \mathbf{Fields}_1$. A valuation $\nu \in \text{dval}(F(t))$ is called *general* if it corresponds to some irreducible polynomial over F . The set of general valuations are in bijection with the set of all closed points on the affine line over F , which we denote by $\mathbb{A}_{F,(0)}^1$. A valuation is called *special* if it is not general. Denote the set of general (resp. special) valuations by $\text{dval}(F(t))_{gen}$ (resp. $\text{dval}(F(t))_{sp}$).

REMARK 2.4. Let us realize F as a field of fractions on some smooth projective curve X over k . Set $S = X \times \mathbb{P}^1$. It can be checked that a valuation $\nu \in \text{dval}(k(S))$ is special in the following two cases:

- (i) There is a birational morphism $p: \tilde{S} \rightarrow S$, and the valuation ν corresponds to some irreducible divisor $D \subset \tilde{S}$ contracted under p .
- (ii) The valuation ν corresponds to some of the divisors $X \times \{\infty\}, \{a\} \times \mathbb{P}^1, a \in X$.

Otherwise, the valuation ν is general. It follows from this description that if ν is a special valuation different from $X \times \{\infty\}$, then the residue field $\overline{F(t)}_\nu$ is isomorphic to $k(t)$.

DEFINITION 2.5. Let $F \in \mathbf{Fields}_1$, $j, n \in \mathbb{N}$, $\nu \in \text{dval}(F(t))_{gen}$ and $a \in \Gamma(\overline{F(t)}_\nu, n)_j$. A *lift* of the element a is an element $b \in \Gamma(F(t), n+1)_{j+1}$ satisfying the following two properties:

- (i) $\partial_\nu^{(n+1)}(b) = a$ and
- (ii) for any general valuation $\nu' \in \text{dval}(F(t))_{gen}$ different from ν , we have $\partial_{\nu'}^{(n+1)}(b) = 0$.

The set of all lifts of the element a is denoted by $\mathcal{L}(a)$.

The main result of this section is the following statement:

THEOREM 2.6. *Let $F \in \mathbf{Fields}_1$ and $j \in \{2, 3\}$. For any $\nu \in \text{dval}(F(t))_{gen}$ and $a \in \Gamma(\overline{F(t)}_\nu, 3)_j$, the following statements hold:*

- (i) *The set $\mathcal{L}(a)$ is non-empty.*
- (ii) *Let us assume that $j = 3$. For any $b_1, b_2 \in \mathcal{L}(a)$, the element $b_1 - b_2$ can be represented in the form $a_1 + \delta_4(a_2)$, where $a_1 \in \Lambda^4 F^\times$ and $a_2 \in \Gamma(F(t), 4)_3$ such that for any $\nu \in \text{dval}(F(t))_{gen}$ the element $\partial_\nu^{(4)}(a_2)$ lies in the image of the map $\delta_3: \Gamma(F(t), 3)_1 \rightarrow \Gamma(F(t), 3)_2$.*

REMARK 2.7. Item (i) shows that any element has some lift, while item (ii) shows that in the case $j = 3$ up to some specific elements the lift is unique.

The proof of this theorem was inspired by exact sequences in [Rud21]. We need two lemmas.

LEMMA 2.8. *Let $m \geq 3$ be an integer. The following map is surjective in degrees $m - 1, m$:*

$$\Gamma(F(t), m) \xrightarrow{(\partial_{\nu_p}^{(m)})} \bigoplus_{p \in \mathbb{A}_{F, (0)}^1} \Gamma(F(p), m - 1)[-1].$$

In this formula ν_p is the valuation corresponding to the point p and $F(p) \cong \overline{F(t)}_{\nu_p}$ is the residue field of this point.

The proof of this lemma is completely similar to the proof of surjectivity in the exact sequence of Bass and Tate from [BT73] describing the Milnor K -theory of rational function field in one variable.

Proof. For a point $p \in \mathbb{A}_{F, (0)}^1$ denote by $f_p \in F[t]$ the corresponding monic irreducible polynomial. Define an increasing filtration \mathcal{F} on the complex $\Gamma(F(t), m)$ as follows: the subspace $\mathcal{F}_d(\Gamma(F(t), m))$ is equal to the set of elements lying in the kernels of all the maps $\partial_{\nu_p}^{(m)}$ with $\deg p > d$. It is enough to prove that for any $d \geq 0$ the following map is surjective:

$$\mathcal{F}_d(\Gamma(F(t), m)) \rightarrow \bigoplus_{\substack{p \in \mathbb{A}_{F, (0)}^1 \\ \deg p \leq d}} \Gamma(F(p), m - 1)[-1].$$

The proof is by induction on d . The case $d = -1$ is trivial. Let us prove the inductive step. It is enough to show that for any $a \in \Gamma(F(p), m - 1)_{j-1}, j \in \{m - 1, m\}$ there is an element $\tilde{a} \in \Gamma(F(t), m)_j$ with the following properties:

- (i) for any $p' \neq p$ with $\deg p' \geq \deg p$ we have

$$\partial_{\nu_{p'}}^{(m)}(\tilde{a}) = 0.$$

- (ii) We have $\partial_{\nu_p}^{(m)}(\tilde{a}) = a$.

For an element $\xi \in F(p)$ there is a unique polynomial $l_p(\xi)$ of degree $< \deg p$ such that the image of $l_p(\xi)$ under the natural projection $F[t] \rightarrow F[t]/f_p \cong F(p)$ is equal to ξ .

The following formulas for \tilde{a} are taken from [Rud21, Section 5.2].

Case $j = m - 1$ Choose a representation $a = \sum_{\alpha} n_{\alpha} \cdot (\{\xi_1^{\alpha}\}_2 \otimes \xi_3^{\alpha} \wedge \cdots \wedge \xi_{m-1}^{\alpha})$. Define the element \tilde{a} by the formula

$$\tilde{a} = - \sum_{\alpha} n_{\alpha} \cdot (\{l_p(\xi_1^{\alpha})\}_2 \otimes f_p \wedge l_p(\xi_3^{\alpha}) \wedge \cdots \wedge l_p(\xi_{m-1}^{\alpha})).$$

Case $j = m$ Choose a representation $a = \sum_{\alpha} n_{\alpha} \cdot (\xi_1^{\alpha} \wedge \cdots \wedge \xi_{m-1}^{\alpha})$. The element \tilde{a} is defined by the formula

$$\tilde{a} = \sum_{\alpha} n_{\alpha} \cdot (f_p \wedge l_p(\xi_1^{\alpha}) \wedge \cdots \wedge l_p(\xi_{m-1}^{\alpha})).$$

It is easy to see that these elements satisfy the conditions stated above. □

PROPOSITION 2.9. *The following sequence is exact for $j = m$ and exact in the third term for $j = m - 1$:*

$$0 \rightarrow H^j(\Gamma(F, m)) \rightarrow H^j(\Gamma(F(t), m)) \xrightarrow{(\partial_{\nu_p}^{(m)})} \bigoplus_{p \in \mathbb{A}_{F, (0)}^1} H^{j-1}(\Gamma(F(p), m-1)) \rightarrow 0. \quad (6)$$

Proof. We recall that for any field K we have the canonical identification $H^m(\Gamma(K, m)) \cong K_m^M(K)$. So in the case $j = m$ the statement of the proposition is equivalent to the following exact sequence:

$$0 \rightarrow K_m^M(F) \rightarrow K_m^M(F(t)) \xrightarrow{(\partial_{\nu_p}^{(m)})} \bigoplus_{p \in \mathbb{A}_{F, (0)}^1} K_{m-1}^M(F(p)) \rightarrow 0.$$

This is the exact sequence of Bass and Tate from [BT73] describing the Milnor K -theory of rational function field in one variable. The exactness in the last term for $j = m - 1$ is a particular case of the main result from [Rud21]. \square

The proof of Theorem 2.6. (i) Follows from Lemma 2.8 for $m = 4$.

(ii) Denote $b = b_1 - b_2$. The element b satisfies $\partial_{\nu}^{(4)}(b) = 0$ for any $\nu \in \text{dval}(F(t))_{gen}$. Denote by \bar{b} the corresponding element in $H^4(\Gamma(F(t), 4)) = K_4^M(F(t))$. Proposition 2.9 for $j = m = 4$ implies that \bar{b} lies in the image of the map $H^4(\Gamma(F, 4)) \rightarrow H^4(\Gamma(F(t), 4))$. This implies that b can be represented in the form $a_1 + \delta_4(\tilde{a}_2)$, where $a_1 \in \Lambda^4 F^\times$ and $\tilde{a}_2 \in \Gamma(F(t), 4)_3$. It remains to show that there is $a_2 \in \Gamma(F(t), 4)_3$ satisfying the following two conditions:

- (a) $\delta_4(a_2) = \delta_4(\tilde{a}_2)$ and
- (b) $\partial_{\nu}^{(4)}(a_2)$ lies in the image of δ_3 .

For any $\nu \in \text{dval}(F(t))_{gen}$ we set $b_{\nu} = \partial_{\nu}^{(4)}(\tilde{a}_2)$. We have:

$$\delta_3(b_{\nu}) = \delta_3(\partial_{\nu}^{(4)}(\tilde{a}_2)) = -\partial_{\nu}^{(4)}(\delta_4(\tilde{a}_2)) = -\partial_{\nu}^{(4)}(b - a_1) = -\partial_{\nu}^{(4)}(b) + \partial_{\nu}^{(4)}(a_1) = 0.$$

So b_{ν} lies in the kernel of δ_3 and gives the element $\overline{b_{\nu}} \in H^2(\Gamma(\overline{F(t)}_{\nu}, 3))$. Consider the element

$$(\overline{b_{\nu}})_{\nu \in \text{dval}(F(t))_{gen}} \in \bigoplus_{\nu \in \text{dval}(F(t))_{gen}} H^2(\Gamma(\overline{F(t)}_{\nu}, 3)).$$

Proposition 2.9 for $j = 3, m = 4$ shows that there is an element $a_3 \in H^3(\Gamma(F(t), 4))$ such that for any $\nu \in \text{dval}(F(t))_{gen}$ we have $\partial_{\nu}^{(4)}(a_3) = \overline{b_{\nu}}$. Let \tilde{a}_3 be arbitrary lift of a_3 to $\Gamma(F(t), 4)_3$ and $a_2 = \tilde{a}_2 - \tilde{a}_3$. By construction, for any $\nu \in \text{dval}(F(t))_{gen}$, the element $\partial_{\nu}^{(4)}(a_2)$ is zero in $H^2(\Gamma(\overline{F(t)}_{\nu}, 3))$ and hence lies in the image of δ_3 . Since \tilde{a}_3 lies in the kernel of δ_4 , we have $\delta_4(a_2) = \delta_4(\tilde{a}_2)$. So the two conditions above hold. \square

2.3 Parshin reciprocity law

The goal of this section is to prove the following theorem:

THEOREM 2.10. *Let $L \in \mathbf{Fields}_2$ and $j \in \{3, 4\}$. For any $b \in \Gamma(L, 4)_j$ and all but finitely many $\mu \in \text{dval}(L)$ the following sum is zero:*

$$\sum_{\mu' \in \text{val}(\overline{L}_{\mu})} \partial_{\mu'}^{(3)} \partial_{\mu}^{(4)}(b) = 0.$$

Moreover the following sum is zero:

$$\sum_{\mu \in \text{dval}(L)} \sum_{\mu' \in \text{val}(\bar{L}_\mu)} \partial_{\mu'}^{(3)} \partial_{\mu}^{(4)}(b) = 0.$$

Let X be a smooth algebraic variety. The definition of a simple normal crossing divisor can be found in [Kol09]. Denote by $X^{(1)}$ the set of all closed irreducible subsets of X of codimension 1. For a divisor $D = \sum_{D \in X^{(1)}} n_D [D]$ on X , denote by $|D|$ its support defined by the formula $\sum_{\substack{D \in X^{(1)} \\ n_D \neq 0}} [D]$.

A divisor D is called *supported on a simple normal crossing divisor* if $|D|$ is a simple normal crossing divisor. Let $x \in X$. A divisor D is called *supported on a simple normal crossing divisor locally at x* , if the restriction of this divisor to some open affine neighborhood of the point x is supported on a simple normal crossing divisor.

We have the following statement:

THEOREM 2.11. *Let X be a variety over an algebraically closed field of characteristic zero and D an effective Weil divisor on X . There is a birational morphism $f: \tilde{X} \rightarrow X$ such that \tilde{X} is smooth and $f^*(D)$ is supported on a simple normal crossing divisor.*

For a rational function f on X , denote by (f) its divisor. For the definition of the complexes $\Gamma(F, n)$, see Definition 1.2.

DEFINITION 2.12. Let X be a smooth algebraic variety and $x \in X$. An element of the vector space $\Gamma(k(X), 4)_3$ (resp. $\Gamma(k(X), 4)_4$) is called *strictly regular at $x \in X$* if it can be represented as a linear combination of elements of the form $\{\xi_1\}_2 \otimes \xi_3 \wedge \xi_4$ (resp. $\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4$) such that all the divisors $|(\xi_1)| + |(\xi_3)| + |(\xi_4)|$ (resp. $|(\xi_1)| + |(\xi_2)| + |(\xi_3)| + |(\xi_4)|$) are supported on a simple normal crossing divisor locally at x .

Theorem 2.11 has the following corollary:

COROLLARY 2.13. *Let S be a smooth surface and $j \in \{3, 4\}$. For any element $a \in \Gamma(k(S), 4)_j$ there is a birational morphism $p: \tilde{S} \rightarrow S$ such that the element $p^*(a)$ is strictly regular at all points.*

The following lemma characterizes strictly regular elements:

LEMMA 2.14. *Let S be a smooth algebraic surface and $x \in S$.*

(i) *The subgroup of strictly regular elements of $\Gamma(k(S), 4)_3$ is generated by elements of the following form:*

- (a) $\{\pi_1^n \pi_2^m \xi_1\} \otimes \pi_1 \wedge \pi_2$.
- (b) $\{\pi_1^n \pi_2^m \xi_1\} \otimes \pi_1 \wedge \xi_4$.
- (c) $\{\pi_1^n \pi_2^m \xi_1\} \otimes \xi_3 \wedge \xi_4$.

Here all the functions ξ_i take non-zero values at x and π_i is a regular system of parameters.

(ii) *The subgroup of strictly regular elements of $\Gamma(k(S), 4)_4$ is generated by elements of the following form:*

- (a) $\pi_1 \wedge \pi_2 \wedge \xi_3 \wedge \xi_4$.
- (b) $\pi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4$.
- (c) $\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4$.

The functions ξ_i, π_i satisfy the same conditions as in item (i).

Proof. Follows from the fact that if π_1, π_2 is a regular system of parameters at x then any function $f \in k(S)$ can be written in the form $\pi_1^{n_1} \pi_2^{n_2} \xi$, where $n_i \in \mathbb{Z}$ and ξ is a regular function at x such that $\xi(x) \neq 0$. \square

The following result is a version of the classical Parshin reciprocity law for strictly regular elements (see [Par75, HK14, OZ11]).

THEOREM 2.15. *Let S be a surface smooth at some point $x \in S$ and $j \in \{3, 4\}$. For any strictly regular element b of the group $\Gamma(k(S), 4)_j$ at x the following sum is equal to zero:*

$$\sum_{\substack{C \subset S \\ C \ni x}} \partial_{\nu_{x,C}}^{(3)} \partial_{\nu_C}^{(4)}(b) = 0.$$

Here the sum is taken over all irreducible curves $C \subset S$ containing the point x and smooth at it, ν_C is the valuation corresponding to C and $\nu_{x,C}$ is a valuation of the residue field $\overline{k(S)}_{\nu_C} \cong k(C)$ corresponding to $x \in C$.

Proof. It is enough to prove this theorem for any of the generators from Lemma 2.14. We will only consider the most interesting case (i), (a). We can assume that S is a smooth surface, $x \in S$ and π_1, π_2 is a system of regular parameters at x . Passing to some open affine neighborhood of the point x , we can assume that the following conditions hold:

- (i) the function ξ_1 is invertible and regular,
- (ii) the functions π_i are regular,
- (iii) for any $i \in \{1, 2\}$ the divisor of the function π_i is equal to some irreducible curve C_i passing through x .

In general, if X is a subvariety of an algebraic variety Y and f is a regular function on Y we denote its restriction to X by $f|_X$.

Let $C \subset S$ be a curve and f_1, f_2, f_3 regular functions on S . Assume that $\nu_C(f_3) = 0$. It follows from Proposition 1.4 that the tame-symbol $\partial_C^{(4)}(\{f_1\}_2 \otimes f_2 \wedge f_3)$ is equal to zero if $\nu_C(f_1) \neq 0$ and is equal to $-\nu_C(f_2) \cdot (\{f_1|_C\}_2 \otimes (f_3|_C))$ if $\nu_C(f_1) = 0$.

Let b be $\{\pi_1^n \pi_2^m \xi_1\}_2 \otimes \pi_1 \wedge \pi_2$ and C be an irreducible curve on X . It follows from the last paragraph that the only curves on S satisfying $\partial_{\nu_C}^{(4)}(b) \neq 0$ are C_1 and C_2 . Consider the following three cases:

Case $n, m \neq 0$ In this case the integers $\nu_{C_i}(\pi_1^n \pi_2^m \xi_1), i = 1, 2$ are non-zero and so both of tame-symbols $\partial_{\nu_{C_1}}^{(4)}(b)$ and $\partial_{\nu_{C_2}}^{(4)}(b)$ vanish. The statement follows.

Case $n \neq 0, m = 0$ or $m \neq 0, n = 0$ Consider, say, the first case. As in the previous item,

$$\partial_{\nu_{C_1}}^{(4)}(b) = 0.$$

So it is enough to prove that $\partial_{\nu_{x,C_2}}^{(3)} \partial_{\nu_{C_2}}^{(4)}(b) = 0$. We have:

$$\partial_{\nu_{C_2}}^{(4)}(b) = \{(\pi_1^n \xi_1)|_{C_2}\}_2 \otimes \pi_1|_{C_2}.$$

Now the statement follows from the following formula: $\text{ord}_x((\pi_1^n \xi_1)|_{C_2}) = n \neq 0$.

Case $n = m = 0$ We have: $\partial_{\nu_{C_1}}^{(4)}(b) = -\{\xi_1|_{C_1}\}_2 \otimes \pi_2|_{C_1}$. So $\partial_{\nu_{x,C_1}}^{(3)} \partial_{\nu_{C_1}}^{(4)}(b) = \{\xi_1(x)\}_2$. Similarly, $\partial_{\nu_{x,C_2}}^{(3)} \partial_{\nu_{C_2}}^{(4)}(b) = -\{\xi_1(x)\}_2$. The statement follows.

□

The proof of Theorem 2.10. Let S be an algebraic surface with $k(S) \cong L$. The definition of the set $\text{dval}(L)_S$ was given in the Section 1.1. We recall that this is the set of all discrete valuations coming from divisors on S . Choose S in such a way that b would be strictly regular at all points of S . This is possible by Corollary 2.13. Theorem 2.15 implies the following formula:

$$\sum_{\mu \in \text{dval}(L)_S} \sum_{\mu' \in \text{val}(\bar{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b) = 0.$$

It remains to prove that for any $\mu \in \text{dval}(L) \setminus \text{dval}(L)_S$ the following sum vanishes:

$$\sum_{\mu' \in \text{val}(\bar{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b) = 0.$$

There is a birational morphism $p: \tilde{S} \rightarrow S$ such that μ is given by a divisor on \tilde{S} contracted under p . The morphism p is a sequence of blow-ups $p_m \circ \dots \circ p_1, p_i: S_i \rightarrow S_{i-1}, S_m = \tilde{S}, S_0 = S$. Let $D_i \subset S_i$ be the corresponding exceptional curve. Denote by μ_i the corresponding valuation. It is enough to show that for any i the following formula holds:

$$\sum_{\mu' \in \text{val}(\bar{L}_{\mu_i})} \partial_{\mu'}^{(3)} \partial_{\mu_i}^{(4)}(b) = 0.$$

This formula follows from Theorem 2.15 for the element b and the surfaces S_i and S_{i-1} . □

2.4 Lemma about finiteness

The goal of this section is to prove the following lemma which we will need later:

LEMMA 2.16. *Let $L \in \mathbf{Fields}_2$. For any $b \in \Lambda^4 L^\times$ and all but a finite number of $\nu \in \text{dval}(L)$ the element $\partial_\nu^{(4)}(b)$ belongs to the image of the multiplication map $\bar{L}_\nu^\times \otimes \Lambda^2 k^\times \rightarrow \Lambda^3 \bar{L}_\nu^\times$.*

Proof. By Corollary 2.13, there is a smooth proper algebraic surface S such that $L = k(S)$ and b is strictly regular at all points of S . We recall that the set $\text{dval}(L)_S$ was defined in the Section 1.1.

We can assume that the element b has the form $f_1 \wedge f_2 \wedge f_3 \wedge f_4$. Let $W = \sum_{j=1}^4 |(f_j)|$. According to Proposition 1.4, the tame-symbol $\partial_{\nu_C}^{(4)}(b)$ vanishes if the curve C does not belong to divisor W . So for all but finitely many $\nu \in \text{dval}(L)_S$ the tame-symbol $\partial_\nu^{(4)}(b)$ vanishes. It remains to show that for any $\nu \in \text{dval}(L) \setminus \text{dval}(L)_S$, the element $\partial_\nu^{(4)}(b)$ lies in the image of the map $\bar{L}_\nu^\times \otimes \Lambda^2 k^\times \rightarrow \Lambda^3 \bar{L}_\nu^\times$. (See also the proof of Theorem 2.10).

Let $p: \tilde{S} \rightarrow S$ be a birational morphism, such that ν correspond to some irreducible divisor D on \tilde{S} contracted under p . Denote by $x \in S$ the image of D under p . Lemma 2.14 for the surface S , the point x and the element b implies that b can be represented in the form $\sum_i m_i \cdot (f_1^{(i)} \wedge f_2^{(i)} \wedge \xi_3^{(i)} \wedge \xi_4^{(i)})$, such that the divisors of the functions $\xi_j^{(i)}$ do not contain the point x . This implies that $\nu_D(\xi_j^{(i)}) = 0$ and moreover the restrictions of the functions $\xi_j^{(i)}$, considered as functions on \tilde{S} , to D lie in k . Now the statement follows from Lemma 1.5 for $k = 2, n = 4$. □

3. The norm map

3.1 The definition of \mathcal{N}_ν

Let $F \in \mathbf{Fields}_1$, $\nu \in \text{dval}(F(t))$. Denote the field $F(t)$ by L . The goal of this section is to construct the map $\mathcal{N}_\nu: \text{RecMaps}(F) \rightarrow \text{RecMaps}(\overline{L}_\nu)$. We will do this in the following three steps:

- (i) We will define this map when ν is a special valuation (see Definition 2.3).
- (ii) Using the construction of the lift from Section 2.2, for any general valuation $\nu \in \text{dval}(L)$, we will define the map $\mathcal{N}_\nu: \text{RecMaps}(F) \rightarrow \text{Hom}(\Lambda^3 \overline{L}_\nu^\times, \mathcal{B}_2(k))$. (Proposition 3.2).
- (iii) Using Theorem 2.10 from Section 2.3, we will show that for any $h \in \text{RecMaps}(F)$, the map $\mathcal{N}_\nu(h): \Lambda^3 \overline{L}_\nu^\times \rightarrow \mathcal{B}_2(k)$ is a lifted reciprocity map on the field \overline{L}_ν . (Proposition 3.4). So, \mathcal{N}_ν gives a map $\text{RecMaps}(F) \rightarrow \text{RecMaps}(\overline{L}_\nu)$.

We recall that the discrete valuation $\nu_{\infty, F} \in \text{dval}(F(t))$ was defined in Section 1.1. Let ν be special. If $\nu = \nu_{\infty, F}$ then define $\mathcal{N}_\nu(h) = h$ (here we have used the identification of $\overline{L}_{\nu_{\infty, F}}$ with F). In the other case we have $\overline{L}_\nu \simeq k(t)$ (see Remark 2.4). In this case define $\mathcal{N}_\nu(h)$ to be the unique lifted reciprocity map from Proposition 2.2. We have defined \mathcal{N}_ν for any $\nu \in \text{dval}(L)_{sp}$.

Let $h \in \text{RecMaps}(F)$. Define the map $H_h: \Lambda^4 L^\times \rightarrow \mathcal{B}_2(k)$ by the following formula:

$$H_h(b) = - \sum_{\mu \in \text{dval}(L)_{sp}} \mathcal{N}_\mu(h)(\partial_\mu^{(4)}(b)).$$

This sum is well-defined by Lemma 2.16. Recall that we defined the notion of the lift in Section 2.2.

DEFINITION 3.1. Let $\nu \in \text{dval}(L)_{gen}$. Define the map $\mathcal{N}_\nu: \text{RecMaps}(F) \rightarrow \text{Hom}(\Lambda^3 \overline{L}_\nu^\times, \mathcal{B}_2(k))$ as follows. Let $h \in \text{RecMaps}(F)$ and $a \in \Lambda^3 \overline{L}_\nu^\times$. Choose some lift $b \in \mathcal{L}(a)$ and define the element $\mathcal{N}_\nu(h)(a)$ by the formula $H_h(b)$.

PROPOSITION 3.2. *The previous definition is well-defined i.e. for any $b_1, b_2 \in \mathcal{L}(a)$ we have $H_h(b_1) = H_h(b_2)$.*

Proof. We need to show that the element $H_h(b_1) - H_h(b_2) = H_h(b_1 - b_2)$ is equal to zero. By item (ii) of Theorem 2.6, it is enough to show that the map H_h vanishes on the elements of the form $a_1, \delta_4(a_2)$, where $a_1 \in \Lambda^4 F^\times$ and $a_2 \in \Gamma(L, 4)_3$ such that for any $\nu \in \text{dval}(L)_{gen}$ the element $\partial_\nu^{(4)}(a_2) \in \Gamma(\overline{L}_\nu, 3)_2$ lies in the image of the map $\delta_3: \Gamma(\overline{L}_\nu, 3)_1 \rightarrow \Gamma(\overline{L}_\nu, 3)_2$.

- (i) Direct computation shows that for any $a_1 \in \Lambda^4 F^\times$ and any $\mu \in \text{dval}(L)$ the element $\partial_\mu^{(4)}(a_1)$ lies in the subgroup $\Lambda^3 k^\times \subset \Lambda^3 \overline{L}_\mu^\times$. It follows that $H_h(a_1) = 0$.
- (ii) We need to show that $H_h(\delta_4(a_2)) = 0$. Let μ be a special valuation. By Proposition 1.4 and the fact that the map $\mathcal{N}_\mu(h)$ is a lifted reciprocity map (see Definition 1.6) we have $\mathcal{N}_\mu(h) \partial_\mu^{(4)} \delta_4(a_2) = -\mathcal{N}_\mu(h) \delta_3 \partial_\mu^{(4)}(a_2) = - \sum_{\mu' \in \text{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(a_2)$. So by the definition of the map H_h we get:

$$\begin{aligned} H_h(\delta_4(a_2)) &= - \sum_{\mu \in \text{dval}(L)_{sp}} \mathcal{N}_\mu(h) \partial_\mu^{(4)} \delta_4(a_2) = \sum_{\mu \in \text{dval}(L)_{sp}} \sum_{\mu' \in \text{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(a_2) = \\ &= \sum_{\mu \in \text{dval}(L)} \sum_{\mu' \in \text{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(a_2) = 0. \end{aligned} \tag{7}$$

Here the third equality holds because for any general valuation μ the element $\partial_\mu^{(4)}(a_2) \in \Gamma(\overline{L}_\mu, 3)_2$ lies in the image of the map $\delta_3: \Gamma(\overline{L}_\mu, 3)_1 \rightarrow \Gamma(\overline{L}_\mu, 3)_2$ and so it lies in the kernel of all the maps $\partial_{\mu'}^{(3)}, \mu' \in \text{val}(\overline{L}_\mu)$. This follows from the fact that $\partial_{\mu'}^{(3)}$ is a morphism of complexes. The fourth equality follows from Theorem 2.10. \square

It remains to prove that for any $h \in \text{RecMaps}(F)$, the map $\mathcal{N}_\nu(h)$ is a lifted reciprocity map on the field \overline{L}_ν . For this we need the following lemma:

LEMMA 3.3. *Let $j \in \{2, 3\}$, $\nu \in \text{dval}(L)_{\text{gen}}$, $a \in \Gamma(\overline{L}_\nu, 3)_j$ and $b \in \mathcal{L}(a)$. The following formula holds:*

$$\sum_{\nu' \in \text{val}(\overline{L}_\nu)} \partial_{\nu'}^{(3)}(a) = - \sum_{\mu \in \text{dval}(L)_{\text{sp}}} \sum_{\mu' \in \text{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b).$$

This lemma allows us to reduce some statement about the field \overline{L}_ν for $\nu \in \text{dval}(L)_{\text{gen}}$ to the corresponding statements for the fields \overline{L}_μ for special μ .

Proof. Theorem 2.10 implies the following formula:

$$\sum_{\mu \in \text{dval}(L)} \sum_{\mu' \in \text{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b) = 0.$$

On the other hand:

$$\begin{aligned} & \sum_{\mu \in \text{dval}(L)} \sum_{\mu' \in \text{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b) = \\ &= \sum_{\nu' \in \text{val}(\overline{L}_\nu)} \partial_{\nu'}^{(3)}(a) + \sum_{\mu \in \text{dval}(L)_{\text{sp}}} \sum_{\mu' \in \text{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b). \end{aligned}$$

The statement of the lemma follows. \square

PROPOSITION 3.4. *Let $h \in \text{RecMaps}(F)$. The map $\mathcal{N}_\nu(h)$ is a lifted reciprocity map.*

Proof. Let us show that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{B}_2(\overline{L}_\nu) \otimes \overline{L}_\nu^\times & \xrightarrow{\delta_3} & \Lambda^3 \overline{L}_\nu^\times \\ \downarrow \sum_{\nu' \in \text{val}(\overline{L}_\nu)} \partial_{\nu'}^{(3)} & \swarrow \mathcal{N}_\nu(h) & \downarrow \sum_{\nu' \in \text{val}(\overline{L}_\nu)} \partial_{\nu'}^{(3)} \\ \mathcal{B}_2(k) & \xrightarrow{-\delta_2} & \Lambda^2(k^\times). \end{array}$$

The lower right triangle Let $a \in \Lambda^3 \overline{L}_\nu^\times$. Choose some $b \in \mathcal{L}(a)$. We have:

$$\begin{aligned} & \sum_{\nu' \in \text{dval}(\overline{L}_\nu)} \partial_{\nu'}^{(3)}(a) = - \sum_{\mu \in \text{dval}(L)_{\text{sp}}} \sum_{\mu' \in \text{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b) \quad (\text{by Lemma 3.3}) \\ &= \sum_{\mu \in \text{dval}(L)_{\text{sp}}} \delta_2 \mathcal{N}_\mu(h) \partial_\mu^{(4)}(b) \quad (\text{because } \mathcal{N}_\mu(h) \text{ is a lifted reciprocity map}) \\ &= -\delta_2 H_h(b) \quad (\text{follows from the definition of the map } H_h) \\ &= -\delta_2 \mathcal{N}_\nu(h)(a). \end{aligned}$$

The upper left triangle Let $a \in \Gamma(\overline{L}_\nu, 3)_2$. Choose $b \in \mathcal{L}(a)$. We have:

$$\begin{aligned}
 \sum_{\nu' \in \text{val}(\overline{L}_\nu)} \partial_{\nu'}^{(3)}(a) &= - \sum_{\mu \in \text{dval}(L)_{sp}} \sum_{\mu' \in \text{val}(\overline{L}_\mu)} \partial_{\mu'}^{(3)} \partial_\mu^{(4)}(b) \quad (\text{by Lemma 3.3}) \\
 &= - \sum_{\mu \in \text{dval}(L)_{sp}} \mathcal{N}_\mu(h) \delta_3 \partial_\mu^{(4)}(b) \quad (\text{because } \mathcal{N}_\mu(h) \text{ is a lifted reciprocity map}) \\
 &= \sum_{\mu \in \text{dval}(L)_{sp}} \mathcal{N}_\mu(h) \partial_\mu^{(4)} \delta_4(b) \quad (\text{because } \partial_\mu^{(4)} \text{ is a morphism of complexes}) \\
 &= -H_h(\delta_4(b)) \quad (\text{follows from the definition of the map } H_h) \\
 &= \mathcal{N}_\nu(h) \delta_3(a) \quad (\text{because } -\delta_4(b) \in \mathcal{L}(\delta_3(a))).
 \end{aligned}$$

To prove that $\mathcal{N}_\nu(h)$ is a lifted reciprocity map it remains to show that it vanishes on elements of the form $a \wedge c$, $a \in \Lambda^2 \overline{L}_\nu^\times$, $c \in k^\times$. Let $b \in \mathcal{L}(a)$. Then $b \wedge c \in \mathcal{L}(a \wedge c)$ and we have:

$$\begin{aligned}
 \mathcal{N}_\nu(h)(a \wedge c) &= H_h(b \wedge c) \quad (\text{because } b \wedge c \in \mathcal{L}(a \wedge c)) \\
 &= - \sum_{\mu \in \text{dval}(L)_{sp}} \mathcal{N}_\mu(h) \partial_\mu^{(4)}(b \wedge c) \quad (\text{by the definition of } H_h) \\
 &= - \sum_{\mu \in \text{dval}(L)_{sp}} \mathcal{N}_\mu(h) (\partial_\mu^{(3)}(b) \wedge c) \quad (\text{by the property of } \partial_\mu^{(4)}) \\
 &= 0 \quad (\text{because } \mathcal{N}_\mu(h) \text{ is a lifted reciprocity map}).
 \end{aligned}$$

So we have proved that $\mathcal{N}_\nu(h)$ is a lifted reciprocity map. \square

3.2 Property of \mathcal{N}_ν under extensions of scalars

The goal of this section is to prove the following statement:

PROPOSITION 3.5. *Let $j_0: F \hookrightarrow K$ be an embedding from **Fields**₁ and $\nu \in \text{dval}(F(t))$. Denote by n the degree $[K : F]$ and by $j: F(t) \hookrightarrow K(t)$ the unique extension of j_0 satisfying $j(t) = t$. For any $h \in \text{RecMaps}(K)$, the following formula holds:*

$$(\mathcal{N}_\nu \circ \text{RecMaps}(j_0))(h) = \frac{1}{n} \sum_{\nu' \in \text{ext}(\nu, K(t))} e_{\nu'|\nu} f_{\nu'|\nu}(\text{RecMaps}(j_{\nu'|\nu}) \circ \mathcal{N}_{\nu'})(h). \quad (8)$$

This proposition is similar to Lemma 1.9 from [Sus79]. (See also exercise III.7.7 from [Wei13]).

DEFINITION 3.6. Denote the right hand-side of formula (8) by $\tilde{\mathcal{N}}_\nu(h)$.

Set $h_0 = \text{RecMaps}(j_0)(h)$. We need to show that $\mathcal{N}_\nu(h_0) = \tilde{\mathcal{N}}_\nu(h)$. To do this, we need the following lemma:

LEMMA 3.7. *The following statements hold:*

- (i) *For any $\nu \in \text{dval}(F(t))$ the map $\tilde{\mathcal{N}}_\nu(h)$ is a lifted reciprocity map on the field $\overline{F(t)}_\nu$.*
- (ii) *Let $\tilde{h} \in \text{RecMaps}(F)$. For any $b \in \Lambda^4 F(t)^\times$ and all but finitely many $\nu \in \text{dval}(F(t))$ we have $\mathcal{N}_\nu(\tilde{h})(\partial_\nu^{(4)}(b)) = 0$ and moreover*

$$\sum_{\nu \in \text{dval}(F(t))} \mathcal{N}_\nu(\tilde{h})(\partial_\nu^{(4)}(b)) = 0.$$

(iii) For any $b \in \Lambda^4 F(t)^\times$ and all but finitely many $\nu \in \text{dval}(F(t))$ we have $\tilde{\mathcal{N}}_\nu(h)(\partial_\nu^{(4)}(b)) = 0$ and moreover

$$\sum_{\nu \in \text{dval}(F(t))} \tilde{\mathcal{N}}_\nu(h)(\partial_\nu^{(4)}(b)) = 0.$$

The deduction of Proposition 3.5 from Lemma 3.7. We will prove the statement in two steps: first, we will check it when ν is special, then we reduce the case when ν is general to the previous case using Lemma 3.7.

- (i) Let $\nu = \nu_{\infty, F}$. We recall that this is the valuation associated to the point $\infty \in \mathbb{P}_F^1$. In this case it is easy to see that the set $\text{ext}(\nu, K(t))$ consists of only 1 element, namely $\nu' = \nu_{\infty, K}$. We have $f_{\nu'|\nu} = n$, $e_{\nu'|\nu} = 1$. Let us identify $\overline{F(t)}_{\nu_{\infty, F}}$ with F and $\overline{K(t)}_{\nu_{\infty, K}}$ with K . Then the map $j_{\nu'|\nu}$ is identified with j_0 . Now, Definition 3.6 gives: $\tilde{\mathcal{N}}_\nu(h) = \text{RecMaps}(j_{\nu'|\nu})(\mathcal{N}_{\nu'}(h)) = \text{RecMaps}(j_0)(h) = \mathcal{N}_\nu(\text{RecMaps}(j_0)(h)) = \mathcal{N}_\nu(h_0)$. Here we have used the definition of \mathcal{N}_ν and $\mathcal{N}_{\nu'}$ on special valuations (see the previous section).
- (ii) Let ν be a special valuation different from $\nu_{\infty, F}$. By the item (i) of the previous Lemma $\tilde{\mathcal{N}}_\nu(h)$ is a lifted reciprocity map on the field $\overline{F(t)}_\nu$. Since $\overline{F(t)}_\nu \cong k(t)$, by Proposition 2.2 any two lifted reciprocity maps on the field $\overline{F(t)}_\nu$ are equal. So $\mathcal{N}_\nu(h_0) = \tilde{\mathcal{N}}_\nu(h)$.
- (iii) Let ν be a general valuation. We need to show that for any $a \in \Lambda^3 \overline{F(t)}_\nu^\times$, the following formula holds: $\mathcal{N}_\nu(h_0)(a) = \tilde{\mathcal{N}}_\nu(h)(a)$. Choose some $b \in \mathcal{L}(a)$. We have:

$$\begin{aligned} \mathcal{N}_\nu(h_0)(a) &= \\ &= \sum_{\nu \in \text{dval}(F(t)_{\text{gen}})} \mathcal{N}_\nu(h_0)(\partial_\nu^{(4)}(b)) \quad (\text{Because } b \in \mathcal{L}(a)) \\ &= - \sum_{\nu \in \text{dval}(F(t)_{\text{sp}})} \mathcal{N}_\nu(h_0)(\partial_\nu^{(4)}(b)) \quad (\text{By item (ii) of Lemma 3.7}) \\ &= - \sum_{\nu \in \text{dval}(F(t)_{\text{sp}})} \tilde{\mathcal{N}}_\nu(h)(\partial_\nu^{(4)}(b)) \quad (\text{By item (i) and (ii) of this proof}) \\ &= \sum_{\nu \in \text{dval}(F(t)_{\text{gen}})} \tilde{\mathcal{N}}_\nu(h)(\partial_\nu^{(4)}(b)) \quad (\text{By item (iii) of Lemma 3.7}) \\ &= \tilde{\mathcal{N}}_\nu(h)(a) \quad (\text{Because } b \in \mathcal{L}(a)). \end{aligned}$$

So $\mathcal{N}_\nu(h_0) = \tilde{\mathcal{N}}_\nu(h)$. □

The proof of Lemma 3.7. (i) The set of all lifted reciprocity maps on some field has a structure of an affine set over \mathbb{Q} , see Remark 1.7. This means that if X is a finite set and $h_\mu, \mu \in X$ are some lifted reciprocity maps on the field $\overline{F(t)}_\nu$, then for any $\alpha_\mu \in \mathbb{Q}$ satisfying $\sum_{\mu \in X} \alpha_\mu = 1$ the map defined by the formula $\sum_{\mu \in X} \alpha_\mu h_\mu$ is a lifted reciprocity map on the field $\overline{F(t)}_\nu$. Applying this statement to $X = \text{ext}(\nu, K(t))$, $\alpha_\mu = \frac{e_{\mu|\nu} f_{\mu|\nu}}{[K:F]}$ and $h_\mu = \text{RecMaps}(j_{\mu|\nu})(\mathcal{N}_\mu(h))$, we get the statement of the lemma. (The formula $\sum_{\mu \in X} \alpha_\mu = 1$ follows from formula (2)).

- (ii) The first statement follows from Lemma 2.16. Let us prove the second statement. It follows from Theorem 2.6 that the sets $\mathcal{L}(a), a \in \Lambda^3 \overline{L}_\mu^\times, \mu \in \text{dval}(F(t))$ generate $\Lambda^4 F(t)^\times$ as a vector space. So we can assume that $b \in \mathcal{L}(a)$ for some $a \in \Lambda^3 \overline{L}_\mu^\times, \mu \in \text{dval}(F(t))$. We need to show the following equality:

$$\sum_{\nu \in \text{dval}(F(t))_{\text{gen}}} \mathcal{N}_\nu(\tilde{h})(\partial_\nu^{(4)}(b)) = - \sum_{\nu \in \text{dval}(F(t))_{\text{sp}}} \mathcal{N}_\nu(\tilde{h})(\partial_\nu^{(4)}(b)).$$

By the definition of the set $\mathcal{L}(a)$ (see Definition 2.5) the left hand-side is equal to $\mathcal{N}_\mu(\tilde{h})(a)$. On the other hand the right hand side is equal to $H_{\tilde{h}}(b)$ which is exactly the definition of $\mathcal{N}_\mu(\tilde{h})(a)$ (see Definition 3.1).

- (iii) The first statement of the lemma follows from Proposition 2.16 and item (i) of this lemma. (We recall that any lifted reciprocity map on the field $\overline{F(t)}_\nu$ is zero on the image of the multiplication map $\overline{F(t)}_\nu^\times \otimes \Lambda^2 k^\times \rightarrow \Lambda^3 \overline{F(t)}_\nu^\times$).

To prove the second statement, let us rewrite the element $\tilde{\mathcal{N}}_\nu(h)(\partial_\nu^{(4)}(b))$ as follows:

$$\begin{aligned} & \tilde{\mathcal{N}}_\nu(h)(\partial_\nu^{(4)}(b)) \\ &= \frac{1}{n} \sum_{\nu' \in \text{ext}(\nu, K(t))} e_{\nu'|\nu} f_{\nu'|\nu} \text{RecMaps}(j_{\nu'|\nu})(\mathcal{N}_{\nu'}(h)(\partial_{\nu'}^{(4)}(b))) \quad (\text{By the definition of } \tilde{\mathcal{N}}_\nu(h)) \\ &= \frac{1}{n} \sum_{\nu' \in \text{ext}(\nu, K(t))} e_{\nu'|\nu} \mathcal{N}_{\nu'}(h)((j_{\nu'|\nu})_*(\partial_{\nu'}^{(4)}(b))) \quad (\text{By the definition of } \text{RecMaps}(j_{\nu'|\nu})) \\ &= \frac{1}{n} \sum_{\nu' \in \text{ext}(\nu, K(t))} \mathcal{N}_{\nu'}(h)(\partial_{\nu'}^{(4)}(j_*(b))) \quad (\text{By formula (3)}). \end{aligned}$$

So we have:

$$\begin{aligned} \sum_{\nu \in \text{dval}(F(t))} \tilde{\mathcal{N}}_\nu(h)(\partial_\nu^{(4)}(b)) &= \sum_{\nu \in \text{dval}(F(t))} \frac{1}{n} \sum_{\nu' \in \text{ext}(\nu, K(t))} \mathcal{N}_{\nu'}(h)(\partial_{\nu'}^{(4)}(j_*(b))) \\ &= \frac{1}{n} \sum_{\nu' \in \text{dval}(K(t))} \mathcal{N}_{\nu'}(h)(\partial_{\nu'}^{(4)}(j_*(b))). \end{aligned}$$

The last expression is zero by item (ii) of this lemma applied to the field K , the lifted reciprocity map $h \in \text{RecMaps}(K)$ and the element $j_*(b) \in \Lambda^4 K(t)$. □

3.3 The proof of Theorem 1.14

In this section we will use results of Section 3.1 to construct the norm map on lifted reciprocity maps. We follow ideas from [Sus79, §1] (see also [Mil70, BT73, Kat80]).

DEFINITION 3.8. Let $j: F \hookrightarrow K$ be an extension of some fields from \mathbf{Fields}_1 . Let a be some generator of K over F . Denote by $p_a \in F[t]$ the minimal polynomial of a over F . Denote by ν_a the corresponding valuation. The residue field $\overline{F(t)}_{\nu_a}$ is canonically isomorphic to K . So we get the map $\mathcal{N}_{\nu_a}: \text{RecMaps}(F) \rightarrow \text{RecMaps}(K)$, which we denote by $N_{K/F,a}$. This map is called *the norm map*.

The proof of Theorem 1.14. The proof of Theorem 1.14 goes as follows. First of all we will show that the map $N_{K/F,a}$ is well behaved with respect to extension of scalars (Lemma 3.9). This will

follow directly from Proposition 3.5. Then we will prove Lemma 3.10 stating that $N_{K/F,a}$ is a right inverse for $\text{RecMaps}(j)$. This will show that $\text{RecMaps}(j)$ is surjective and that in the case $K = F$ the map $N_{K/F,a}$ does not depend on a . Then we will prove Proposition 3.11 stating that the map $N_{K/F,a}$ does not depend on a . By that moment, for any field extension $F \hookrightarrow K$ we will have constructed the canonical norm map $N_{K/F}$ and will have proved item (i) of Theorem 1.14. The item (ii) will follow from Proposition 3.12. \square

LEMMA 3.9. *Let $j: F_1 \hookrightarrow K, F_1 \hookrightarrow F_2$, be extensions and $F_2 \otimes_{F_1} K = \bigoplus_{i=1}^m F_{2,i}$. Denote by j_i the natural embedding $F_2 \hookrightarrow F_{2,i}$. Let $n = [K : F_1]$ and $n_i = [F_{2,i} : F_2]$. Let a be a generator of F_2 over F_1 . Denote by a_i the corresponding generators of $F_{2,i}$ over K . The following diagram is commutative:*

$$\begin{array}{ccc}
 \text{RecMaps}(F_1) & \xrightarrow{N_{F_2/F_1,a}} & \text{RecMaps}(F_2) \\
 \uparrow \text{RecMaps}(j) & & \uparrow \sum_{i=1}^m \frac{n_i}{n} \text{RecMaps}(j_i) \\
 \text{RecMaps}(K) & \xrightarrow{(N_{F_{2,i}/K,a_i})} & \bigoplus_{i=1}^m \text{RecMaps}(F_{2,i}).
 \end{array} \tag{9}$$

Proof. It follows from Proposition 3.5 that for any $\nu \in \text{dval}(F_1(t))$ the following diagram is commutative:

$$\begin{array}{ccc}
 \text{RecMaps}(F_1) & \xrightarrow{\mathcal{N}_\nu} & \text{RecMaps}(\overline{F_1(t)}_\nu) \\
 \uparrow \text{RecMaps}(j) & & \uparrow \sum_{\nu' \in \text{ext}(\nu, K(t))} \frac{e_{\nu'|\nu} f_{\nu'|\nu}}{n} \text{RecMaps}(j_{\nu'|\nu}) \\
 \text{RecMaps}(K) & \xrightarrow{(\mathcal{N}_{\nu'})} & \bigoplus_{\nu' \in \text{ext}(\nu, K(t))} \text{RecMaps}(\overline{K(t)}_{\nu'}).
 \end{array} \tag{10}$$

Let us apply this statement in the case when ν is equal to ν_a . Let $p_a = \prod_i^m p_{a,i}$ be the decomposition of p_a in the field $K(t)$. The set $\text{ext}(\nu, K(t))$ is in bijection with the irreducible factors of p_a in $K(t)$. Denote by $\nu_i \in \text{ext}(\nu, K(t))$ the valuation corresponding to $p_{a,i}$. We have $F_{2,i} \cong \overline{K(t)}_{\nu_i}$. The embeddings j_i correspond to the embeddings $j_{\nu_i|\nu}$. Since the polynomial p_a is separable, we have $f_{\nu'|\nu} = 1$, and so $e_{\nu'|\nu} f_{\nu'|\nu} = [F_{2,i} : F_2]$. So the diagram (10) can be identified with (9). \square

LEMMA 3.10. *For any embedding $j: F_1 \hookrightarrow F_2$, we have $\text{RecMaps}(j) \circ N_{F_2/F_1,a} = \text{id}$. In particular, the map $\text{RecMaps}(j)$ is surjective and in the case $F_1 = F_2$, the map $N_{F_2/F_1,a}$ is the identity map.*

Proof. Let $n = [F_2 : F_1]$. We need to show that for any $h \in \text{RecMaps}(F_1)$ and $x \in \Lambda^3 F_1^\times$ the following formula holds: $N_{F_2/F_1,a}(h)(j_*(x)) = n \cdot h(x)$. Consider the element $b = p_a \wedge x \in \Lambda^4 F_1(t)^\times$, where p_a is the minimal polynomial of a over F_1 . By item (ii) of Lemma 3.7 we have:

$$\sum_{\nu \in \text{dval}(F_1(t))_{\text{gen}}} \mathcal{N}_\nu(h)(\partial_\nu^{(4)}(b)) + \sum_{\nu \in \text{dval}(F_1(t))_{\text{sp}}} \mathcal{N}_\nu(h)(\partial_\nu^{(4)}(b)) = 0.$$

We have $\partial_{\nu_{p_a}}^{(4)}(b) = x$ and this is the only general valuation satisfying $\partial_\nu^{(4)}(b) \neq 0$. So the first

term is equal to $\mathcal{N}_{\nu_{pa}}(h)(x)$ which is equal to $N_{F_2/F_{1,a}}(h)(x)$. So we have:

$$N_{F_2/F_{1,a}}(h)(x) = - \sum_{\nu \in \text{dval}(F_1(t))_{sp}} \mathcal{N}_{\nu}(h) \partial_{\nu}^{(4)}(b).$$

It is easy to see that there is only one special valuation ν such that $\mathcal{N}_{\nu}(h)(\partial_{\nu}^{(4)}(b)) \neq 0$, namely $\nu_{F_1,\infty}$. We have $\partial_{\nu_{F_1,\infty}}^{(4)}(b) = -nx$. Since $\mathcal{N}_{\nu_{F_1,\infty}}(h)$ can be identified with h , we have $N_{F_2/F_{1,a}}(h)(x) = -\mathcal{N}_{\nu_{F_1,\infty}}(h)(\partial_{\nu_{F_1,\infty}}^{(4)}(b)) = n \cdot h(x)$. \square

PROPOSITION 3.11. *The map $N_{F_2/F_{1,a}}$ does not depend on a .*

We denote the map $N_{F_2/F_{1,a}}$ simply by N_{F_2/F_1} .

Proof. Let $j: F_1 \hookrightarrow K$ be a field extension of F_1 satisfying $F_2 \otimes_{F_1} K \cong K^{\oplus [F_2:F_1]}$. We apply Lemma 3.9. By definition of K for any n we have $F_{2,i} \cong K$. By Lemma 3.10 the maps $N_{F_{2,i}/K, a_i}$ are the identity maps. We conclude that in the diagram from Lemma 3.9 all the maps except maybe $N_{F_2/F_{1,a}}$ do not depend on a . So the map $N_{F_2/F_{1,a}}$ does not depend on a on the image of $\text{RecMaps}(j)$. By the previous lemma this image coincides with $\text{RecMaps}(F_1)$. \square

PROPOSITION 3.12. *If $F_1 \subset F_2 \subset F_3$ is a tower of extensions from **Fields**₁ then $N_{F_3/F_1} = N_{F_3/F_2} \circ N_{F_2/F_1}$.*

Proof. Let $j: F_1 \hookrightarrow K$ be a field extension. Denote $F_2 \otimes_{F_1} K \cong \bigoplus_{i=1}^{n_2} F_{2,i}$ and $F_3 \otimes_{F_2} F_{2,i} \cong \bigoplus_{s=1}^{n_{3,i}} F_{3,i,s}$. By associativity of tensor product we have $F_3 \otimes_{F_1} K \cong \bigoplus_{i,s} F_{3,i,s}$. Denote by $j_{i,s}$ the natural embeddings $F_3 \hookrightarrow F_{3,i,s}$. Let $n_{i,s} = [F_{3,i,s} : F_3]$. Let n be the degree of F_3 over F_1 . Repeated application of Lemma 3.9 together with Proposition 3.11 shows that the following diagram is commutative:

$$\begin{array}{ccc} \text{RecMaps}(F_1) & \xrightarrow{N_{F_3/F_2} \circ N_{F_2/F_1}} & \text{RecMaps}(F_3) \\ \text{RecMaps}(j) \uparrow & & \uparrow \sum_{i,s} \frac{n_{i,s}}{n} \text{RecMaps}(j_{i,s}) \\ \text{RecMaps}(K) & \xrightarrow{(N_{F_{3,i,s}/F_{2,i}} \circ N_{F_{2,i}/K})} & \bigoplus_{i,s} \text{RecMaps}(F_{3,i,s}). \end{array}$$

Choose K such that $F_3 \otimes_{F_1} K \cong K^{\oplus [F_3:F_1]}$. It follows that $F_{3,i,s} \cong F_{2,i} \cong K$. So the bottom maps in the above diagram are the identity maps. Let us compare this diagram with diagram (9) for $F_2 = F_3$. We see that the left, right and bottom maps are the same. Since $\text{RecMaps}(j)$ is surjective, the statement of the proposition follows. \square

4. The proofs of the main results

4.1 The proof of Theorem 1.8

Let F_0 be the field $k(x)$ and $L = F_0(y)$. For any $\nu \in \text{dval}(L)$ denote by $\sigma_{\nu} \in \text{RecMaps}(\overline{L}_{\nu})$ the lifted reciprocity map given by the formula $\mathcal{N}_{\nu}(\mathcal{H}_{k(x)})$. ($\mathcal{H}_{k(x)}$ is a lifted reciprocity map from Proposition 2.2). Denote by λ the involution of L fixing k and interchanging x and y . The map λ induces the natural map $\text{dval}(L) \rightarrow \text{dval}(L)$ given by the formula $\lambda(\nu)(f) = \nu(\lambda(f))$. Denote by $\overline{\lambda}_{\nu}$ the natural map $\overline{\lambda}_{\nu}: \overline{L}_{\nu} \rightarrow \overline{L}_{\lambda(\nu)}$.

LEMMA 4.1. For any $\nu \in \text{dval}(L)$ we have $\sigma_\nu = \text{RecMaps}(\overline{\lambda}_\nu)(\sigma_{\lambda(\nu)})$.

Proof. For any $\nu \in \text{dval}(L)$ denote $\tilde{\sigma}_\nu = \text{RecMaps}(\overline{\lambda}_\nu)(\sigma_{\lambda(\nu)})$. We need to show that $\sigma_\nu = \tilde{\sigma}_\nu$.

- (i) Let us assume that ν is special. In this case $\overline{L}_\nu \cong k(t)$ and the statement follows from Proposition 2.2.
- (ii) Let ν be general. It is easy to see that $\tilde{\sigma}$ satisfies item (ii) of Lemma 3.7. Namely, for any $b \in \Lambda^4 L^\times$ and for all but finitely many $\nu \in \text{dval}(L)$ we have $\tilde{\sigma}_\nu(\partial_\nu^{(4)}(b)) = 0$ and moreover the following formula holds:

$$\sum_{\nu \in \text{dval}(L)} \tilde{\sigma}_\nu(\partial_\nu^{(4)}(b)) = 0.$$

Now, the case of a general valuation can be reduced to the case of a special one in the same way as it was done in the proof of Proposition 3.5.

□

PROPOSITION 4.2. For any $a, b \in F \setminus k$ we have

$$N_{F/k(a)}(\mathcal{H}_{k(a)}) = N_{F/k(b)}(\mathcal{H}_{k(b)}).$$

In particular, the element $\mathcal{H}_F := N_{F/k(a)}(\mathcal{H}_{k(a)})$ does not depend on the choice of $a \in F \setminus k$.

Proof. We first prove the statement in the case when a and b generate F over k .

Let us realize the field F as field of fractions on some smooth projective curve X . The elements $a, b \in F$ induce the maps $\varphi_a, \varphi_b: X \rightarrow \mathbb{P}^1$. Define the maps $\psi_{a,b}, \psi_{b,a}: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by the formulas $\psi_{a,b} = (\varphi_a, \varphi_b), \psi_{b,a} = (\varphi_b, \varphi_a)$. Let $S = \mathbb{P}^1 \times \mathbb{P}^1$. We can identify $k(S)$ with $L = k(x)(y)$. Denote by $\mu_{a,b}$ (resp. $\mu_{b,a}$) the valuation of L corresponding to the image of $\psi_{a,b}$ (resp. $\psi_{b,a}$). We have $\lambda(\mu_{a,b}) = \mu_{b,a}$. Denote by $\theta_{a,b}$ (resp. $\theta_{b,a}$) the canonical isomorphism $F \rightarrow \overline{L}_{\nu_{a,b}}$ (resp. $F \rightarrow \overline{L}_{\nu_{b,a}}$). We recall that the definition of σ_ν was given in the beginning of this section. By the definition of the norm map, we have:

$$N_{F/k(a)}(\mathcal{H}_{k(a)}) = \text{RecMaps}(\theta_{a,b})(\sigma_{\nu_{a,b}}), N_{F/k(b)}(\mathcal{H}_{k(b)}) = \text{RecMaps}(\theta_{b,a})(\sigma_{\nu_{b,a}}).$$

So we need to show that $\text{RecMaps}(\theta_{a,b})(\sigma_{\nu_{a,b}}) = \text{RecMaps}(\theta_{b,a})(\sigma_{\nu_{b,a}})$. Denote the map $\overline{\lambda}_{\mu_{a,b}}: \overline{L}_{\mu_{a,b}} \rightarrow \overline{L}_{\mu_{b,a}}$ simply by $\overline{\lambda}$. We have $\overline{\lambda} \circ \theta_{a,b} = \theta_{b,a}$. We get:

$$\begin{aligned} \text{RecMaps}(\theta_{b,a})(\sigma_{\mu_{b,a}}) &= \text{RecMaps}(\overline{\lambda} \circ \theta_{a,b})(\sigma_{\mu_{b,a}}) = (\text{RecMaps}(\theta_{a,b}) \circ \text{RecMaps}(\overline{\lambda}))(\sigma_{\mu_{b,a}}) = \\ &= \text{RecMaps}(\theta_{a,b})(\text{RecMaps}(\overline{\lambda})(\sigma_{\mu_{b,a}})) = \text{RecMaps}(\theta_{a,b})(\sigma_{\mu_{a,b}}). \end{aligned}$$

In the last formula we have used the previous lemma.

So we have proved that for any $a, b \in F \setminus k$ generating F over k we have

$$N_{F/k(a)}(\mathcal{H}_{k(a)}) = N_{F/k(b)}(\mathcal{H}_{k(b)}).$$

Now the first statement of the proposition follows from the following fact: for any $a, b \in F \setminus k$, there is $c \in F \setminus k$ such that the pairs $(a, c), (b, c)$ generate F over k .

The second statement is a reformulation of the first statement.

□

We recall that we work \mathbb{Q} -linearly. The following lemma is well-known:

LEMMA 4.3. Let $j: F_1 \hookrightarrow F_2$ be a Galois extension with the Galois group equal to G . The natural map $j_*: K_n^M(F_1) \rightarrow K_n^M(F_2)$ induces an isomorphism $K_n^M(F_1) \xrightarrow{\sim} K_n^M(F_2)^G$

Proof. There is the norm map $\widehat{N}_{F_2/F_1}: K_n^M(F_2) \rightarrow K_n^M(F_1)$, satisfying the following properties (see [Wei13]):

- (i) The composition $\widehat{N}_{F_2/F_1} \circ j_*: K_n^M(F_1) \rightarrow K_n^M(F_1)$ is the multiplication by the integer $[F_2 : F_1]$.
- (ii) The composition $j_* \circ \widehat{N}_{F_2/F_1}$ is equal to $\sum_{g \in G} g_*$. These statements imply that the kernel and the cokernel of the natural map $K_n^M(F_1) \rightarrow K_n^M(F_2)^G$ is annihilated by the multiplication on $[F_2 : F_1]$. Therefore the map $K_n^M(F_1) \rightarrow K_n^M(F_2)^G$ is a rational isomorphism. \square

The proof of Theorem 1.8. **Existence** Let $F \in \mathbf{Fields}_1$. Choose some embedding

$$j: k(t) \hookrightarrow F.$$

Define the element \mathcal{H}_F by the formula $\mathcal{H}_F := N_{F/k(t)}(\mathcal{H}_{k(t)})$. By Proposition 4.2 this element does not depend on j . We need to show that if $j': F_1 \rightarrow F_2$ is an embedding, then

$$\text{RecMaps}(j')(\mathcal{H}_{F_2}) = \mathcal{H}_{F_1}.$$

This follows from the properties of the norm map (see Theorem 1.14):

$$\begin{aligned} \text{RecMaps}(j')(\mathcal{H}_{F_2}) &= \text{RecMaps}(j')N_{F_2/k(t)}(\mathcal{H}_{k(t)}) = \text{RecMaps}(j')N_{F_2/F_1}N_{F_1/k(t)}(\mathcal{H}_{k(t)}) = \\ &= (\text{RecMaps}(j') \circ N_{F_2/F_1})(N_{F_1/k(t)}\mathcal{H}_{k(t)}) = \mathcal{H}_{F_1}. \end{aligned}$$

Uniqueness Let $\mathcal{H}_F, \mathcal{H}'_F, F \in \mathbf{Fields}_1$ be two families of lifted reciprocity maps such that for any $j: F_1 \hookrightarrow F_2$ we have $\text{RecMaps}(j)(\mathcal{H}_{F_2}) = \mathcal{H}_{F_1}$ and $\text{RecMaps}(j)(\mathcal{H}'_{F_2}) = \mathcal{H}'_{F_1}$. We need to show that $\mathcal{H}_F = \mathcal{H}'_F$ for any $F \in \mathbf{Fields}_1$. By Proposition 2.2, this is true when $F = k(t)$. Let F be any field. There is a field $F' \in \mathbf{Fields}_1$ together with two embeddings $j_1: F \hookrightarrow F', k(t) \hookrightarrow F'$ such that $F'/k(t)$ is Galois. Since $\mathcal{H}_F = \text{RecMaps}(j_1)(\mathcal{H}_{F'})$ and $\mathcal{H}'_F = \text{RecMaps}(j_1)(\mathcal{H}'_{F'})$, it is enough to prove the statement for F' . Denote by G the Galois group of F' over $k(t)$. Since $\mathcal{H}_{F'}$ and $\mathcal{H}'_{F'}$ are invariant under the group G , it is enough to prove that they are equal on the subgroup $(\Lambda^3 F'^{\times})^G$. By Lemma 4.3, we have $(K_3^M(F'))^G = K_3^M(k(t))$. It follows that $(\Lambda^3 F'^{\times})^G$ is generated by the image of δ_3 and by the elements coming from $k(t)$. On the image of δ_3 the maps $\mathcal{H}_{F'}$ and $\mathcal{H}'_{F'}$ coincide because they are lifted reciprocity maps. On the elements coming from $k(t)$ they coincide because $\mathcal{H}_{k(t)} = \mathcal{H}'_{k(t)}$. \square

4.2 The proof of Corollary 1.13

Let $L \in \mathbf{Fields}_2$. Define the map $H_L: \Lambda^4 L^\times \rightarrow \mathcal{B}_2(k)$ by the formula:

$$H_L(b) = \sum_{\nu \in \text{dval}(L)} \mathcal{H}_{L_\nu}(\partial_\nu^{(4)}(b)).$$

This formula is well-defined by Lemma 2.16. The following lemma is a consequence of Theorem 1.8:

LEMMA 4.4. *If $j: L \hookrightarrow M$ is an extension of some fields from \mathbf{Fields}_2 then for any $b \in \Lambda^4 L^\times$ we have $H_L(b) = \frac{1}{[M : L]} H_M(j_*(b))$.*

Proof. Let $\nu \in \text{dval}(L)$. It is enough to show the following formula:

$$\mathcal{H}_{\overline{L}_\nu} \partial_\nu^{(4)}(b) = \frac{1}{[M : L]} \sum_{\nu' \in \text{ext}(\nu, M)} \mathcal{H}_{\overline{M}_{\nu'}} \partial_{\nu'}^{(4)}(j_*(b)).$$

By formula (3), we have:

$$\partial_{\nu'}^{(4)}(j_*(b)) = e_{\nu'|\nu} \cdot j_{\nu'|\nu}(\partial_\nu^{(4)}(b)).$$

If $s: F_1 \hookrightarrow F_2$ is an arbitrary extension from \mathbf{Fields}_1 then for any $x \in \Lambda^3 F_1^\times$, we have:

$$\mathcal{H}_{F_2}(s_*(x)) = [F_2 : F_1] \cdot \mathcal{H}_{F_1}(x).$$

Applying this in the case $s = j_{\nu'|\nu}: \overline{L}_\nu \hookrightarrow \overline{M}_{\nu'}$, we get:

$$\mathcal{H}_{\overline{M}_{\nu'}}((j_{\nu'|\nu})_*(\partial_\nu^{(4)}(b))) = f_{\nu'|\nu} \mathcal{H}_{\overline{L}_\nu}(\partial_\nu^{(4)}(b)).$$

So we get:

$$\mathcal{H}_{\overline{M}_{\nu'}}(\partial_{\nu'}^{(4)}(j_*(b))) = e_{\nu'|\nu} f_{\nu'|\nu} \cdot \mathcal{H}_{\overline{L}_\nu}(\partial_\nu^{(4)}(b)).$$

Using this formula, we obtain:

$$\begin{aligned} & \frac{1}{[M : L]} \sum_{\nu' \in \text{ext}(\nu, M)} \mathcal{H}_{\overline{M}_{\nu'}}(\partial_{\nu'}^{(4)} j_*(b)) = \\ & = \frac{1}{[M : L]} \sum_{\nu' \in \text{ext}(\nu, M)} e_{\nu'|\nu} f_{\nu'|\nu} \mathcal{H}_{\overline{L}_\nu}(\partial_\nu^{(4)}(b)) = \\ & = \mathcal{H}_{\overline{L}_\nu}(\partial_\nu^{(4)}(b)) \frac{1}{[M : L]} \sum_{\nu' \in \text{ext}(\nu, M)} e_{\nu'|\nu} f_{\nu'|\nu} = H_{\overline{L}_\nu}(\partial_\nu^{(4)}(b)). \end{aligned}$$

The last equality follows from the formula $\sum_{\nu' \in \text{ext}(\nu, M)} e_{\nu'|\nu} f_{\nu'|\nu} = [M : L]$. \square

Proof of Corollary 1.13. We need to show that $H_L = 0$ for any $L \in \mathbf{Fields}_2$. Let us prove that it is true when $L = k(x)(y)$.

Let us represent the field L as $F_0(y)$, where $F_0 = k(x)$. For any $\nu \in \text{dval}(F_0(y))$ define $\sigma_\nu = \mathcal{N}_\nu(\mathcal{H}_{k(x)}) \in \text{RecMaps}(\overline{L}_\nu)$. Let us prove that $\sigma_\nu = \mathcal{H}_{\overline{L}_\nu}$. When ν is special it is true by Proposition 2.2. When ν is general it follows from the definition of $\mathcal{H}_{\overline{L}_\nu}$. Now the formula

$$\sum_{\nu \in \text{dval}(L)} \mathcal{H}_{\overline{L}_\nu}(\partial_\nu^{(4)}(b))$$

follows from item (ii) of Lemma 3.7.

Let us prove the statement for an arbitrary L . There is a field L' together with two finite extensions

$$j: k(x)(y) \hookrightarrow L', j': L \hookrightarrow L'$$

such that j is a Galois extension. Lemma 4.4 shows that it is enough to prove the statement for L' . Denote the Galois group of j by G . Since $H_{L'}$ is invariant under G , it is enough to prove that $H_{L'}$ is zero on the subgroup $(\Lambda^4 L'^\times)^G$. By Lemma 4.3, we have $(K_4^M(L'))^G = K_4^M(k(x)(y))$ and so the group $(\Lambda^4 L'^\times)^G$ is generated by the image of δ_4 and by the elements coming from $k(x)(y)$. Vanishing of $H_{L'}$ on the elements coming from $k(x)(y)$ follows from Lemma 4.4 together with the formula $H_{k(x)(y)} = 0$. Let us prove that $H_{L'}$ is zero on the image of the map $\delta_4: \Gamma(L', 4)_3 \rightarrow$

$\Gamma(L', 4)_4$. For any $b \in \Gamma(L', 4)_3$ we have

$$\begin{aligned} H_{L'}(\delta_4(b)) &= \sum_{\nu \in \text{dval}(L')} \mathcal{H}_{\overline{L}'_\nu} \partial_\nu^{(4)} \delta_4(b) = - \sum_{\nu \in \text{dval}(L')} \mathcal{H}_{\overline{L}'_\nu} \delta_3 \partial_\nu^{(4)}(b) = \\ &= - \sum_{\nu \in \text{dval}(L')} \sum_{\nu' \in \text{val}(\overline{L}'_\nu)} \partial_{\nu'}^{(3)} \partial_\nu^{(4)}(b) = 0. \end{aligned}$$

Here the second equality is true because $\partial_\nu^{(4)}$ is morphism of complexes, the third equality is true because $\mathcal{H}_{\overline{L}'_\nu}$ is a lifted reciprocity map and the fourth equality follows from Theorem 2.10. \square

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