Available online at www.sciencedirect.com

ScienceDirect

Procedia Computer Science 221 (2023) 434–441

Procedia Computer Science

www.elsevier.com/locate/procedia

Information Technology and Quantitative Management (ITQM 2023)

Cops and Robbers: A Game of Pursuit on Graphs

Andrea Olsen[a]

[a] California Institute of Technology, Adresse: 1200 E California Blvd, Pasadena, CA 91125, USA

Abstract

Cops and Robbers is an exciting game that can be mathematically modelled and investigated with the use of graph theory, counting principles and probability. A discrete version of this game is investigated in the present study. A Cop and a Robber are two opponents placed upon the battlefield. The goal of the Cop is to capture the Robber by landing on the same vertex as him. The goal of the Robber is to avoid the Cop. Simple graph examples are used to showcase the principles of this discrete game. The notion of pitfalls for the Robber is introduced, and their role in determining whether a graph is a Cop-win or Robber-win is explained. Then, more difficult graph examples are presented together with variations in the number of Cops on the battlefield and the number of ways the Cops and the Robbers can travel upon a graph are calculated. Finally, the probability of a Cop and Robber meeting on a specific vertex is found for a three-dimensional graph.

Peer-review under responsibility of the scientific committee of the Tenth International Conference on Information Technology and Quantitative Management

Keywords: Graph, Pitfall, Robber-win, Cop-win

1. Introduction

There is a given finite and connected graph $G = (V, E)$; where G is the graph, V are vertices and E are edges. There is 1 Cop and 1 Robber. At the beginning of the game, the Cop always starts first by choosing his initial vertex and the Robber chooses a starting vertex after the Cop. The Cop and Robber start moving one by one on the graph along the edges. In 1 move they can only move along 1 edge; alternatively, they can choose to skip their move.

The Cop's goal is to catch the Robber, which means landing on the same vertex as him. The Robber tries to avoid the Cop by always staying on a **non-adjacent** vertex, which is a vertex that does not share an edge with the Cop's vertex. A graph is called Cop-win if despite any behaviour of the Robber, the Cop will still be able to catch him on that graph. Otherwise, the graph is called Robber-win.

2. Simple graph examples

A graph is a set of **vertices** (V) and **edges** (E), such that $G = (V, E)$. The **degree** of a vertex is the number of edges attached to that vertex, for example, there could be multiple roads leading from the same supermarket. Now let us consider some simple graph examples.

*Corresponding author. Tel.: +447983839333
E-mail address: viandove@gmail.com.

Peer-review under responsibility of the scientific committee of the Tenth International Conference on Information Technology and Quantitative Management
10.1016/j.procs.2023.07.058

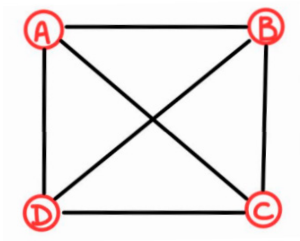

Fig. 1.

Figure 1 is a complete graph with 4 vertices and 6 edges. If the Cop starts on C, he will be able to move to any of the remaining 3 vertices in his next move. This means that no matter where the Robber places himself at the start, the Cop will catch him in 1 move. This graph is Cop-win, because if 1 Cop and 1 Robber play on it, the Cop will always triumph.

Consider Figure 2, a cyclic graph. This is a graph where a player's path starts and ends in the same vertex if he does not go on the same vertex more than once:

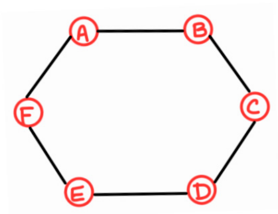

Fig. 2.

Assume the Cop starts the game on vertex C and that the Robber starts on vertex F. The Cop is the first to make a move, for example, clockwise. The Robber will respond by moving anticlockwise. Since both the Cop and Robber can only move one vertex at a time and they move alternately, the Cop will continue following the Robber in a circle and will never catch him. Consequently, this graph is Robber-win, since the Robber will always win here, as long as he moves away from the Cop or skips his turn if the Cop skips his turn.

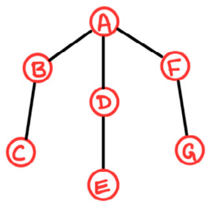

Fig. 3.

Tree graphs, such as Figure 3, are always Cop-win, because if the Cop starts the game at the 'root' of the tree (the point from which all the branches start) — in this case A — he can trap the Robber on any one of the branches and follow him on it until he catches the Robber. If the Cop starts on one of the branches, he should move to the root of the tree and carry out the same strategy.

3. Pitfalls

How could we determine whether an arbitrarily large graph is Cop-win or Robber-win? To answer this, we must understand what has to happen for the Cop to catch the Robber. At the end of the game, the Cop must move along an edge to land on the same vertex as the Robber for the Robber to be caught. The Robber must not have had a way out on his last move, assuming he tries to avoid capture. This means that any vertex that the Robber could have gone to using an edge must have been connected to the Cop's vertex. Therefore, on his last move, the Robber stands on a pitfall point. A pitfall is a vertex (P) whose neighbouring vertices and that vertex itself are all connected to one other vertex (K) by edges. K is therefore called a dominating vertex.

To understand this better, imagine that the Robber is on point P on Figure 4. It is sufficient for the Cop to occupy either E or D to block off all exits off for the Robber, which makes E and D dominating vertices to P.

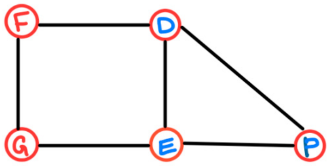

Fig. 4.

Based on this information, a Robber must never land on P. Therefore, he must never use any of the edges extending from P. Consequently, if we remove all edges connected to P on the graph, the situation will not change for the Robber. He will never end up in P because he does not use the edges connected to this vertex in the first place. This simple example shows realise that if there is a pitfall, the edges connecting to it can be removed from the graph. If the pitfall vertex remains isolated, it can clearly also be removed, as the Robber will have no means of getting there. This means that the removal or addition of a pitfall point does not change the outcome of the graph. Let us prove this now through mathematical induction:

Proposition 1: Let graph $H(V_H, E_H)$ be a subgraph of $G(V_G, E_G)$. H is formed by removing a certain number of pitfalls from G. If G is Robber-win, then H is also a Robber-win. If G is Cop-win, H is also Cop-win.

Basis: Let us take a simple graph: $H(5, 5)$, from Figure 5.

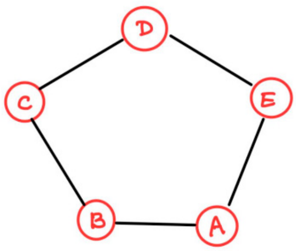

Fig. 5.

Let's add a pitfall point and several edges, to make the graph into $G(6, 7)$, from Figure 6:

Since F is a pitfall point, the Robber will not go on it, because this would lead to his capture. Evidently, the addition of a pitfall point to H to make it into G does not change the outcome of the game (Robber-win), since the Robber will stay on the original H anyway.

Assume that for all Robber-win H graphs, the addition of n pitfalls does not change the outcome of the game (i.e. whether it is Cop-win or Robber-win). Investigate the addition of $n + 1$ pitfalls to graph H.

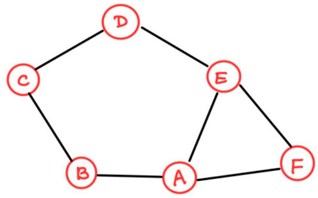

Fig. 6.

Let's iteratively add pitfalls to the initial graph H to make graph G, by creating a chain of intermediate graphs H_i, where i corresponds to the number of iterations made.

Lemma 1: The outcome of the game on graph H_i is the same as the outcome of the game on graph H_{i+1}. In other words: $w(H - i) = w(H_{i+1})$, where w is the function showing the outcome of the game.

Proof 1: Let K be the dominating vertex to a pitfall. K is found on both H_i and H_{i+1}.

If H_i was Cop-win, the Cop can extend his winning strategy of driving the Robber into pitfalls from H_i to all of H_{i+1} by pretending the Robber is on K whenever the Robber enters the pitfall and moves accordingly to his strategy in H_{i+1}. If H_i was Robber-win, the Robber can extend his winning strategy at Hi to all of H_{i+1} by simply avoiding the new pitfalls. According to Lemma 1, we can iteratively add pitfalls to the graph H_0, where every H_{i+1}. graph has an additional pitfall to the previous graph. $w(H_0) = w(H_1) = w(H_2) = w(H_3) = ... = w(H_n)$, where $n \leq \infty$

4. Cop-number

What happens when we increase the number of Cops on a graph? Let us investigate the Cop-number, which is the minimum number of Cops required to catch a Robber on a Robber-win graph. The Cop-number of a Cop-win graph is always 1. However, the Cop-number of Robber-win graphs can vary. For example, in a simple cyclic graph like Figure 7.

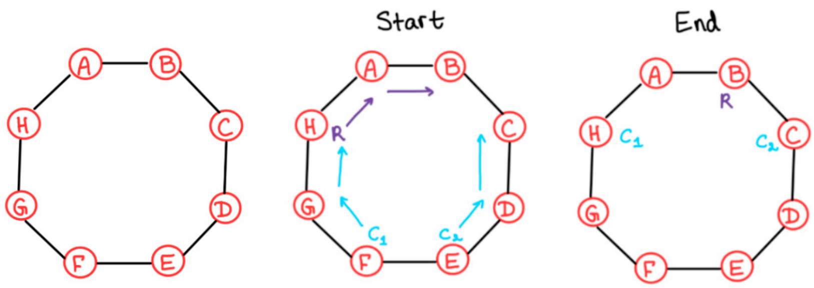

Fig. 7.

2 Cops are enough to catch the Robber, as one follows from behind and the other moves at him from the other side. For example, let Cop 1 start on F and move clockwise, Cop 2 start on E and move anticlockwise and the Robber start on H and move clockwise. Assume that the Cops move simultaneously. In just three moves the Robber will be caught by Cop 2 on B.

It is important to note that only cycles with four or more vertices are Robber-win, because an isolated cycle from three vertices will always be a Cop-win, as the Cop can catch the Robber no matter where the Robber moves, making every vertex in a triangular cycle a pitfall.

Let's investigate the Cop-number on a more complicated graph, such as the planar representation of the dodecahedron:

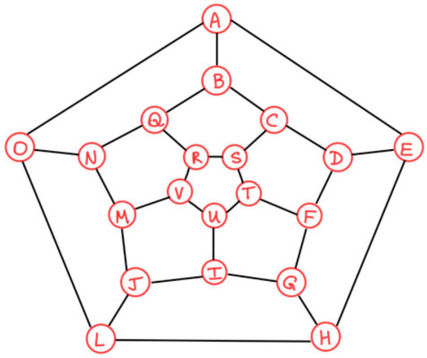

Fig. 8.

Several different theorems have been proposed for finding the Cop-number. Meyniel's conjecture, on the one hand, proposes that the Cop-number for any large connected graph is $n^{\frac{1}{2}}$, when n tends towards infinity and where n is the number of vertices in the graph. In this case, the dodecahedron should have a Cop-number of $20^{\frac{1}{2}} = 4.47 \sim 5$. Fromme and Aigner (1984)[1], on the other hand, state a lower upper bound for the Cop-number, however, only for a specific subtype of graphs. They say that for any planar graph the Cop-number is ≤ 3. Let us test which theorem holds true for the dodecahedron, which is a planar graph.

Clearly, 1 Cop is not enough to catch a Robber, since there are no pitfalls. What about a Cop-number of 2? There are a total of C_{20}^{2} ways for the 2 Cops to choose starting positions, as there are 20 vertices and 2 cops.

$$C_{20}^{2} = \frac{20!}{2!(20-2)!} = 190$$

We could also look at the different types of starting positions. For example, how many ways are there for both Cops to start on the outer pentagon ($AEHLO$)?

$$C_{5}^{2} = \frac{5!}{2!(5-2)!} = 190$$

How many ways are there for one Cop to start on the middle contour ($BCDFGIJMNQ$) and for the other to be on the inner pentagon ($RSTUV$)?

$$C_{10}^{1} \times C_{5}^{1} = \frac{10!}{1!(10-1)!} \times \frac{5!}{1!(5-1)!} = 50$$

If we could distinguish between the two Cops, for instance if they were wearing different uniforms, we would instead have to use permutations when counting the number of possible starting positions for the Cops:

$$P_{20}^{2} = \frac{20!}{(20-2)!} = 380$$

How many ways are there for the Robber to start on the dodecahedron after both differently clothed Cops have selected their starting vertices?

$$P_{20}^{2} \times C_{18}^{1} = \frac{20!}{(20-2)!} \times \frac{18!}{1!(18-1)!} = 6840$$

The starting positions can be investigated further, however, it is important to understand that if we can show at least one example of 2 Cops losing on the planar dodecahedron no matter their starting points, it is enough to prove that the graph remains a Robber-win as the Rovver always selects the most optimal strategy. We will now investigate two random starting points for the Cops.

	Cop 1	Cop 2	Robber
Starting point	N	T	J
1st move	M	U	L
2nd move	J	I	H
3rd move	L	G	E

Once the Robber has gone to E, he has a large enough distance between himself and the two Cops, and will not be caught, because he will be able to keep moving out of their reach on the outer pentagon ($AEHLO$). In the case that both Robbers try to circle him like on Figure 7, he will simply move inwards to the central pentagon ($RSTUV$). This example alone proves that the dodecahedron is a Robber-win for 1 and 2 cops in the game, because the Robber will always select the most optimal strategy, which is a set of rules, that he will follow to avoid the Cops. Now let us explore a Cop-number of 3 for the full dodecahedron. First of all, the 3 Cops, assuming they are identical, have $C^3_{20} = 1140$ ways to select their starting positions on the graph.

When Cop 1 stands on B (guarding vertices A, C, Q), Cop 2 stands on J (guarding vertices L, M, I) and Cop 3 stands on T (guarding vertices S, F, U) the Cops have a very strong position. Future explorations of this game may wish to further investigate whether this is the most optimal initiation point for the Cops. The Robber will be placed on a random vertex that is not being guarded by any of the Cops (otherwise the game would be over in 1 move).

	Cop 1	Cop 2	Cop 3	Robber
Starting point	B	J	T	N
1st move	pass	pass	U	O
2nd move	pass	pass	V	pass
3rd move	A	pass	M	caught

Cops 1 and 2 do not move from their positions on their first move, as they are guarding the path through vertices A and L. Cop 3 moves towards the Robber, trying to push him onto the path between the two other Cops. The Robber attempts to escape through the outer pentagon ($AEHLO$), but Cop 1 and 2 prevent him from moving further along it. Meanwhile, Cop 3 keeps moving at the Robber. If the Robber skips his move, Cop 1 or 2 will walk upon either A or L respectively, forcing the Robber to move back to N. By that time, Cop 3 is already at M and will catch the Robber within the next move.

This example demonstrates that 3 Cops can catch a Robber on the dodecahedron, so this graph's Cop-number is 3. This result shows that Fromme and Aigner's theorem is a better Cop-number approximation than Meyniel's conjecture for this graph.

5. Random graph

New variations of "Cops and Robbers" can be created for further mathematical exploration. We will take a cube (*Figure 9a*), and look at it in planar view (*Figure 9b*). We will transform it into a random graph by assigning each edge a probability. Let's consider a game of blindfolded Cops and Robbers. Both the Cop and Robber are blindfolded, so they choose which edge to move onto with an equal probability (the probability of them using an edge is normalised to 1 from every vertex for simplicity). What is the probability of them selecting such a path that they meet at one vertex?

We will add the condition that the vertices in the paths that the Cop and Robber select must not be repeated, so that both players only use unique vertices on their way. This condition is added to remove the possibility of going back and forth between several vertices endlessly, which would have made the number of possible paths between two points on the graph infinite. **This condition also means that if the Cop or Robber lands on a vertex (X) that is adjacent to a vertex he has already been on (Y), the probability of going on that old vertex Y is 0, and so the probability is evenly split among going to the remaining adjacent vertices from X.** This is important to consider, because this is the only way for the probability of leaving the vertex (X) to be equal to the probability of entering that vertex (X).

We will also add the condition that the players cannot skip their moves. Therefore, the number of steps that each player takes to get to the same point will be the same. If both players meet at the end of the particular path investigated, the game ends. If they have not met by then, the game is considered to start anew, but now from their new positions, which are the final vertices from the previously investigated path.

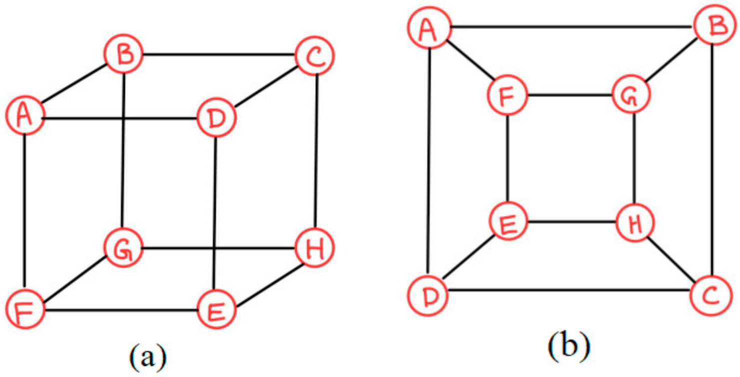

Fig. 9.

The Cop and Robber can place themselves in only 2 unique ways. Firstly, at the ends of a diagonal of a face of the cube, for example: A and E, A and G, F and H, H and D, etc... There are 12 such variants, because a cube has 6 faces and we consider unique diagonals on every face. Secondly, at the ends of an axial diagonal (a diagonal that passes through the centre of the polyhedron), for example: B and E, G and D, A and H, F and C. There are 4 such variants, because there are 4 axial diagonals.

Let the Cop and Robber start at points B and H respectively. What is the probability that they both meet at a single vertex? Since both players are blindfolded and randomly choose which vertex to go on, it doesn't matter whether we investigate B or H, as the graph is symmetrical, and we can relate any probability results from one letter to the other. So let us investigate the different ways a player can get from B to any other vertex.

The following tables show that if both players' starting points are known (these can be any vertices, because our graph is symmetrical), we can determine the probability of them meeting. Furthermore, from the data displayed, we can determine all the possible path lengths from any one vertex to another (see Table 1).

	A	B	C	D	E	F	G	H
A		1, 3, 5	2, 4	1, 3, 5	2, 4	1, 3, 5	2, 4	3
B	1, 3, 5		1, 3, 5	2, 4	3	2, 4	1, 3, 5	2, 4
C	2, 4	1, 3, 5		1, 3, 5	2, 4	3	2, 4	1, 3, 5
D	1, 3, 5	2, 4	1, 3, 5		1, 3, 5	2, 4	3	2, 4
E	2, 4	3	2, 4	1, 3, 5		1, 3, 5	2, 4	1, 3, 5
F	1, 3, 5	2, 4	3	2, 4	1, 3, 5		1, 3, 5	2, 4
G	2, 4	1, 3, 5	2, 4	3	2, 4	1, 3, 5		1, 3, 5
H	3	2, 4	1, 3, 5	2, 4	1, 3, 5	2, 4	1, 3, 5	

Table 1. Path lengths from any vertex to any other vertex on a cube graph

For example, let player 1 (Cop) start on A and move to F. From Table 1 we deduce that his possible path lengths are of 1, 3 and 5 moves. Let us say player 2 (Robber) starts from E and moves to F. His lengths are also of 1, 3 and 5. The probability of player 1 getting from A to F and player 2 getting from E to F in 1 move is $\frac{1}{3}$, in 3 moves is $\frac{1}{3}$ and in 5 moves is also $\frac{1}{3}$, as shown in Table 1 for vertex B, but now used for vertices A and E. Therefore, the probability that they both meet there in 1 move is: $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ The probability holds true for them meeting at F in 3 and 5 moves.

What if player 2 goes from D to F? Now his path lengths can only by 2 and 4, according to Table 1. Since the players' path lengths are different, the probability of them meeting at point F is 0.

What if player 2 goes from B to F? Now his only possible path length is 3, according to Table 1. So the probability that the two players meet at F only depends on the 3-move path for both of them. For player 1 the probability of making 3 moves from A to F is $\frac{1}{3}$, and for player 2 the probability of going from B to F is 1. So, the total probability of them meeting at F is $\frac{1}{3} \times 1 = \frac{1}{3}$.

In this way, Table 1 is a handy tool for finding the probabilities of the players meeting at any point of the graph just by knowing their starting positions.

In conclusion, this investigation of "Cops and Robbers" can be continued further by adding new challenges to the game and looking at how they affect its outcome. For example, one could introduce more Robbers and see whether that changes the Cops' strategy on the dodecahedron. One could explore non-planar graphs for their Cop number and evaluate other upper bounds proposed for the Cop-number, such as that of Bollobas et al. (2008)[2] and Petrov and Abramovksaia (2012)[3]. One could also alter the probabilities of edges on the cube and other graphs, and calculate the new probability of the Cop and Robber meeting. For instance, one could look at random allocation of probabilities, or instead find a concrete mathematical formula for each edge, like Slettnes et al. (2017)[4] did.

References

[1] M. Aigner, M. Fromme, A game of cops and robbers, Discrete Applied Mathematics 8 (1) (1984) 1–12. doi:https://doi.org/10.1016/0166-218X(84)90073-8.
URL https://www.sciencedirect.com/science/article/pii/0166218X84900738

[2] B. Bollobas, G. Kun, I. Leader, Cops and robbers in random graphs, arXiv preprint arXiv:0805.2709.

[3] T. V. Abramovskaya, N. N. Petrov, The theory of guaranteed search on graphs, Differential equations and control processes (2) (2012) 9–65.

[4] E. Slettnes, C. Quines, S.-F. Tsai, J. Geneson, Variations of the cop and robber game on graphs.