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Pythagorean fuzzy multi-attribute decision making approach with incomplete weight information

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Abstract

Pythagorean fuzzy (PF) set is widely employed in multi-attribute decision making (MADM) process as it provides excellent convenience in describing the inference information of decision makers. In this paper, we develop a new PF MADM framework with incomplete weight information. Firstly, a linear programming model based on PF decision matrix and PF judgment matrix is introduced to determine the weights of attribute. Secondly, a new additive consistency checking index based on the order σ discriminant PF information is proposed. Then, we present the PF Yager weighted averaging aggregation operator to aggregate the PF decision information based on Yager triangular norms, which has the desirable property of being monotonic with respect to the total order. Finally, a comprehensive MADM framework is constructed by integrating the above elements, and a case of purchasing air conditioner is given to illustrate the applicability of the proposed approach.

Keywords: Pythagorean fuzzy sets, Incomplete weights, Multi-attribute decision making, Additive consistency, Aggregation operator;

1. Introduction

Multi-attribute decision making (MADM) is the process by which the decision maker (DM) evaluates the alternatives and selects the superior alternative from them, with the constraints implicit in the attributes [1]. Since in most cases there is no alternative optimizing all criteria, compromised solution has to be made [2, 3, 4, 5]. In today's complex socio-economic environment, MADM problems have been introduced into the fuzzy and imprecise decision-making environment [6, 7, 8].

The quantitative study of fuzzy environments originated from fuzzy set theory [9], and subsequently various fuzzy sets were derived to express fuzzy information [10, 11]. The concept of Pythagorean fuzzy sets (PFSs) was proposed by Yager [12],which features that the square sum of the membership degree and the non-membership degree is equal to or less than one. Compared to intuitionistic fuzzy sets (IFS) [10], it has greater advantages in membership and non-membership grades space. Pythagorean fuzzy preference relation (PFPR) is the extension of PFS, which inherits the advantages of PFS and allows DMs to express their preferences through a two-by-two comparison between alternatives [13]. This property enables that PFPR has an excellent performance in

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capturing the motivation of DMs [14, 15]. Consistency is a key focus in PFPR studies, as lacking of consistency for PFPRs can lead to unreasonable or incorrect conclusions. Many scholars have worked on developing consistency checking indices and iterative algorithms to ensure that the judgment matrices provided by DMs are not self-contradictory based on entropy [16], distance[17], similarity [18], etc. However, they ignore the possibility that the information gain of an event is neither bounded at both ends nor defined at all points, and that the additive property of independent events does not carry any additional weight when measuring information uncertainty.

The aggregation of Pythagorean fuzzy (PF) information is particularly important [19, 20] for PF MADM. Based on different t-norms (t-NMs) and t-conorms (t-CMs), a sea of aggregation operators has been developed in the existing studies [19, 20, 21, 22]. A problem is that these aggregation operators are monotone with respect to the partial order $\leq_L$ defined as $< u_1, v_1 > \leq_L < u_2, v_2 > \Longleftrightarrow \; u_1, \leq u_2$, and $v_1 \geq v_2$ rather than total order, which means these aggregation operators are not suitable for aggregating general PF information [23]. The monotonicity of these PF aggregation operators maybe is inconsistent with their application in practical MADM problems. Therefore, there is a need to develop some aggregation operators that are monotone with respect to total order.

Since it is difficult to know the weight of each attribute when the DMs do not have particular knowledge of the prioritization of the attributes in practical decision making, the study of techniques for assigning attribute weights when the attribute weights are incomplete is drawing attention [24], such as the Analytic Hierarchy Process (AHP) [25], Delphi [26], the maximizing deviations apporach[27] and the entropy weight method [28], etc. However, all of the above apporaches are only simply consider the decision matrix or judgment matrix as the information base to calculate the weights of the attributes, which fails to take adequately advantage of the initiative of DMs.

From the above analysis, there are still three research gaps in the existing PF MADM research, including 1) lacking of capacity of existing consistency test indices; 2) lacking of aggregation operators that can aggregate general PF information and 3) the initiative and motivation of DMs is underutilized in assigning attribute weights.Therefore, the core of this paper lies in constructing a PF MADM framework with incomplete weight information, which is innovative in that it contains three elements: an iterative algorithm to improve the consistency of PFPRs; a PF aggregation operator with the ability to aggregate general PF decision information and a linear programming model to determine the attribute weights.

The paper is structured as follows. In Section 2, relevant terminology and basic concepts are reviewed. Section 3 presents an intergating MADM framework by incorporating a linear programming model for determining the weights of attribute, an iterative algorithm of improving the consistency for self-contradictory PFPRs and the PF Yager weighted averaging operator on total order. The next section provides an application of the proposed method through a numerical example. In the end, some concluding comments are added.

2. Preliminaries

This section reviews some indispensable concepts and definitions about PFSs and PFPR.

Definition 1. *[29] Considering U to be a discourse universe, a PFS P in U is a form specified by $P = \{\langle x, \alpha_P(x), \beta_P(x)\rangle : x \in U\}$, where $\alpha_P(x) : U \to [0,1]$, $\beta_P(x) : U \to [0,1]$ incorporating the condition: $0 \leq \alpha_P^2(x) + \beta_P^2(x) \leq 1$ for each x in U.*

For every PFS P in U, the numbers $\alpha_P(x)$ and $\beta_P(x)$ serve as the "membership degree" and the "non-membership degree" of the element x in the set P, respectively. The number $\pi(x)$ is considered as the "indeterminacy degree" of the element x in the set P with the relationship of $\pi(x) = \sqrt[2]{1 - \alpha_P^2(x) - \beta_P^2(x)}$. The pairs $P = \langle \alpha_P, \beta_P \rangle$ is called an Pythagorean fuzzy number (PFN) for convenience.

In order to visually measure the value of PFNs, Zhang et al.[30] defined the score function and the accuracy function of a PFN as $S(P) = \alpha^2 - \beta^2$ and $H(P) = \alpha^2 + \beta^2$, where $S(P) \in [-1, 1]$ and $H(P) \in [0, 1]$.

Dependent upon the score and accuracy functions of PFSs, the ranking of total order is developed.

Definition 2. *[30] Let $P_1 = \langle \alpha_{P_1}, \beta_{P_1} \rangle$ and $P_2 = \langle \alpha_{P_2}, \beta_{P_2} \rangle$ be two any PFNs, then the total order is described as: (1) If $S(P_1) < S(P_2)$, then $P_1 < P_2$; (2) If $S(P_1) > S(P_2)$, then $P_1 > P_2$; (3) If $S(P_1) = S(P_2)$, then ; (a) If $H(P_1) < H(P_2)$, then $P_1 < P_2$; (b) If $H(P_1) > H(P_2)$, then $P_1 > P_2$; (c) If $H(P_1) = H(P_2)$, then $P_1 = P_2$.*

As an extension of PFS, the relevant studies of PFPR are defined below.

Definition 3. *[31] A PFPR Q on a finite set of alternatives $X = \{x_1, x_2, \cdots, x_n\} (n \geq 2)$ is characterized by a membership function $u_q : X \times X \to [0, 1]$ and a non-membership function $v_q : X \times X \to [0, 1]$ such that $0 \leq u_q^2(x_i, x_j) + v_q^2(x_i, x_j) \leq 1$ with $u_q^2(x_i, x_j) = u_{ij}$ interpreted as the certainty degree up to which x_i is preferred to x_j; $v_q^2(x_i, x_j) = v_{ij}$ interpreted as the certainty degree up to which x_i is non-preferred to x_j.*

Definition 4. *[32] Let $A = \left([a_{ij}^-, a_{ij}^+]\right)_{n\times n}$ be a PF judgment matrix, where $a_{ij}^- = u_{ij}, a_{ij}^+ = \sqrt{1 - v_{ij}^2}, i, j \in \{1, 2, \cdots, n\}$, if there exists a normalized priority vector $\omega = (\omega_1, \omega_2, \cdots \omega_n)^T$, such that $u_{ij} \leq \sqrt{2}/2\ (\omega_i - \omega_j + 1) \leq \sqrt{1 - v_{ij}^2}$, where $\omega_i \geq 0,\ i = 1, 2, \cdots, n, \sum_{i=1}^n \omega_i = 1$, then A is called an additive consistent PF judgment matrix.*

Definition 5. *[32] For two PFPRs $Q = (q_{ij})_{n\times n}$ with $q_{ij} = \langle u_{ij}, v_{ij}\rangle$ and $\bar{Q} = (\bar{q}_{ij})_{n\times n}$ with $\bar{q}_{ij} = \langle \bar{u}_{ij}, \bar{v}_{ij}\rangle$, if there exists a normalized PF weight vector $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots \tilde{\omega}_n)^T$ such that*

$$\bar{q}_{ij} = (\bar{u}_{ij}, \bar{v}_{ij}) = \begin{cases} (\sqrt{2}/2, \sqrt{2}/2) & if\ i = j \\ \left(\sqrt{\frac{(\omega_i^u)^2 + (\omega_j^v)^2}{2}}, \sqrt{\frac{(\omega_i^v)^2 + (\omega_j^u)^2}{2}}\right) & if\ i \neq j \end{cases} \tag{1}$$

where $\tilde{\omega}_i = \langle \omega_i^u, \omega_i^v\rangle, \omega_i^u, \omega_i^v \in [0, 1], (\omega_i^u)^2 + (\omega_i^v)^2 \leq 1$, and $\sum_{\substack{j=1\\ j\neq i}}^n \omega_j^u + \sqrt{1 - (\omega_i^v)^2} \leq 1, \sum_{\substack{j=1\\ j\neq i}}^n \sqrt{1 - (\omega_j^v)^2} + \omega_i^u \geq 1$ for $i = 1, 2, \cdots, n$, then $\bar{Q} = (\bar{q}_{ij})_{n\times n}$ is called the additive consistent PFPR of $Q = (q_{ij})_{n\times n}$.

Given that, Zhang, Li and Zhou et al [32] proposed a linear model to solve for PF weight vector $\tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots \tilde{\omega}_n)^T$.

3. Pythagorean fuzzy MADM approach

For a MADM problem under PF environment, let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of alternatives to be selected, $C = \{C_1, C_2, \cdots, C_m\}$ be a set of attribute to be evaluated, and Λ be the set of possible weights of the attributes determined by the known partial weight information. The performance of alternative $x_i (i = 1, 2, \cdots, n)$ under the criterion $C_j (j = 1, 2, \cdots, m)$ is expressed as a PFN $d_{ij} = (\alpha_{ij}, \beta_{ij})$. When all the performances of the alternatives are provided, the PF decision matrix are obtained as $D = (d_{ij})_{n\times m} = \left((\alpha_{ij}, \beta_{ij})\right)_{n\times m}$. In addition, in order to take advantage of the initiative and motivation of the DM, the DM provides two-by-two comparison information for alternatives $x_i (i = 1, 2, \cdots, n)$ and constructs the PF judgment matrix $Q = (q_{ij})_{n\times n} = \left((u_{ij}, v_{ij})\right)_{n\times n}$.

3.1. Linear programming model for determining the weights of attributes

According to the score function, the score matrix of the Pythagorean fuzzy decision matrix D can be calculated as $S = \left(s(d_{ij})\right)_{n\times m}$, where $s(d_{ij}) \in [-1, 1] (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m)$. Normalizing the score matrix $S = \left(s(d_{ij})\right)_{n\times m}$ to $\bar{S} = \left(\bar{s}(d_{ij})\right)_{n\times m}$ using the following equation

$$\bar{s}(d_{ij}) = \frac{s(d_{ij}) - \min_i \{s(d_{ij})\}}{\max_i \{s(d_{ij})\} - \min_i \{s(d_{ij})\}}, \quad (i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m), \tag{2}$$

we can further get the combined weighted score value of each alternative as $\bar{s}(d_i) = \sum_{j=1}^m w_j \bar{s}(d_{ij}), (i = 1, 2, \cdots, n)$, where $w = (w_1, w_2, \cdots w_m)^T$ is the weight vector of the attribute, satisfying $w_j > 0, j = 1, 2, \cdots, m, \sum_{j=1}^m w_j = 1$. Then, the additive consistency complementary judgment matrix $\bar{Q} = (\bar{q}_{ij})_{n\times n}$ can be constructed as $\bar{q}_{ij} = \sqrt{2}/2\ (\bar{s}(d_i) - \bar{s}(d_j) + 1), (i, j = 1, 2, \cdots, n)$.

Considering the additive consistency complementary judgement matrix $\bar{Q}$ has some deviation from the PF judgement matrix Q of the DM, the inequality $u_{ij} - \varepsilon_{ij}^- \leq \sqrt{2}/2\ \left(\sum_{k=1}^m w_k \left(\bar{s}(d_{ik}) - \bar{s}(d_{jk})\right) + 1\right) \leq \sqrt{1 - v_{ij}^2} + \varepsilon_{ij}^+$ holds with the relaxation variables ε_{ij}^- and $\varepsilon_{ij}^+ (i=1, 2, \cdots, n-1\ j = i+1, \cdots, n)$, where ε_{ij}^- and ε_{ij}^+ are both non-negative real numbers.

Clearly, the smaller the deviation variables ε_{ij}^{-} and ε_{ij}^{+}, the closer the additive consistency complementary judgement matrix $\bar{Q}$ is to the Pythagorean judgement matrix Q of the DM. Therefore, the following optimization model is developed.

(M-1)

$$\varphi = \min \sum\nolimits_{i=1}^{n-1} \sum\nolimits_{j=i+1}^{n} \left(\varepsilon_{ij}^{-} + \varepsilon_{ij}^{+}\right)$$
$$s.t. \begin{cases} \sqrt{2}/2 \, \left(\sum\nolimits_{k=1}^{m} w_k \left(\bar{s}(d_{ik}) - \bar{s}(d_{jk})\right) + 1\right) + \varepsilon_{ij}^{-} \geq u_{ij} \\ \sqrt{2}/2 \, \left(\sum\nolimits_{k=1}^{m} w_k \left(\bar{s}(d_{ik}) - \bar{s}(d_{jk})\right) + 1\right) - \varepsilon_{ij}^{+} \leq \sqrt{1 - v_{ij}^2} \\ w = (w_1, w_2, \cdots, w_m)^T \in \Lambda, \;\; w_k \geq 0, \;\; (k = 1, 2, \cdots, m), \quad \sum\nolimits_{k=1}^{m} w_i = 1 \\ \varepsilon_{ij}^{-}, \varepsilon_{ij}^{+} \geq 0, \quad (i{=}1, 2, \cdots, n-1 \, j = i+1, \cdots, n) \end{cases}$$

Solving the model yields the optimal deviation quantities $\dot{\varepsilon}_{ij}^{-}$ and $\dot{\varepsilon}_{ij}^{+}(i{=}1, 2, \cdots, n-1;\, j = i+1, \cdots, n)$.

A sufficient condition for the complementary judgment matrix $\bar{Q}$ of additive consistency to be identical to the Pythagorean judgment matrix Q is $\varphi = 0$. If $\varphi \neq 0$, then the following linear programming models are further developed based on the optimal deviation quantities $\dot{\varepsilon}_{ij}^{-}$ and $\dot{\varepsilon}_{ij}^{+}(i{=}1, 2, \cdots, n-1\, j = i+1, \cdots, n)$.

(M-2)

$$w_k^{-} = \min w_k \quad (w_k^{+} = \max w_k)$$
$$s.t. \begin{cases} \sqrt{2}/2 \, \left(\sum\nolimits_{k=1}^{m} w_k \left(\bar{s}(d_{ik}) - \bar{s}(d_{jk})\right) + 1\right) + \dot{\varepsilon}_{ij}^{-} \geq u_{ij} \\ \sqrt{2}/2 \, \left(\sum\nolimits_{k=1}^{m} w_k \left(\bar{s}(d_{ik}) - \bar{s}(d_{jk})\right) + 1\right) - \dot{\varepsilon}_{ij}^{+} \leq \sqrt{1 - v_{ij}^2} \\ w = (w_1, w_2, \cdots, w_m)^T \in \Lambda, \quad w_k \geq 0, \quad (k = 1, 2, \cdots, n), \quad \sum\nolimits_{k=1}^{m} w_k = 1 \\ i{=}1, 2, \cdots, n-1 \, j = i+1, \cdots, n \end{cases}$$

Solving model (M-2) yields the set of attribute weight vectors.

$$\Upsilon_1 = \left\{ w = (w_1, w_2, \cdots, w_m)^T \mid w_k \in [w_k^{-}, w_k^{+}], w_k \geq 0 (k = 1, 2, \cdots, m), \sum\nolimits_{k=1}^{m} w_k = 1 \right\} \tag{3}$$

At this point, the weighting interval for the attributes are arrived. In order to acquire the optimal weight vector of attribute from the weight interval, we construct the following linear programming model based on the Pythagorean fuzzy decision matrix $D = (d_{ij})_{n\times m} = \left(\langle \alpha_{ij}, \beta_{ij} \rangle\right)_{n\times m}$.

(M-3)

$$\varphi^* = \max \sum\nolimits_{i=1}^{n} \sum\nolimits_{j=1}^{m} (\sqrt{1 - \beta_{ij}^2} - \alpha_{ij}) w_j$$
$$s.t. \quad w = (w_1, w_2, \cdots, w_m)^T \in \bar{\Upsilon}$$

The optimal attribute weight vector $w^* = (w_1^*, w_2^*, \cdots, w_m^*)^T$ is determined by solving the model.

3.2. The additive consistency checking index

In solving the minimum deviation model for the weight interval of the attributes, we ignore the acceptable range of minimum deviations. If the additive consistency complementary judgement matrix constructed using the linear transformation function is too far away from the Pythagorean judgement matrix of the DM, this will allow the DM's decision information to be weakened in the decision making process. Obviously, such a decision process is not reasonable, which will lead to inaccurate decision results.

Accordingly, in this subsection, we present a consistency checking index based on the order σ discriminant information of PFSs for judgment of whether the DM's PFPR satisfies acceptable consistency.

Let $Q = (q_{ij})_{n\times n}$ be a PFPR with $q_{ij} = \langle u_{ij}, v_{ij}\rangle$, and its additive consistent PFPR $\bar{Q} = (\bar{q}_{ij})_{n\times n}$ with $\bar{q}_{ij} = \langle \bar{u}_{ij}, \bar{v}_{ij}\rangle$. To make Q approximate $\bar{Q}$ as much as possible, we define $CI(Q)$ as a consistency index (CI) as

$$\begin{aligned} CI(Q) = &\frac{1}{2^{\sigma}n - 2n}\left\{\exp\left\{(\sigma-1)\sum_{i=1}^{n}\left[u_{ij}^2 \ln\frac{2u_{ij}^2}{u_{ij}^2+\bar{u}_{ij}^2} + v_{ij}^2 \ln\frac{2v_{ij}^2}{v_{ij}^2+\bar{v}_{ij}^2} + \pi_{ij}^2 \ln\frac{2\pi_{ij}^2}{\pi_{ij}^2+\bar{\pi}_{ij}^2}\right]\right\}\right. \\ &\left. + \exp\left\{(\sigma-1)\sum_{i=1}^{n}\left[\bar{u}_{ij}^2 \ln\frac{2\bar{u}_{ij}^2}{u_{ij}^2+\bar{u}_{ij}^2} + \bar{v}_{ij}^2 \ln\frac{2\bar{v}_{ij}^2}{v_{ij}^2+\bar{v}_{ij}^2} + \bar{\pi}_{ij}^2 \ln\frac{2\bar{\pi}_{ij}^2}{\pi_{ij}^2+\bar{\pi}_{ij}^2}\right]\right\} - 2\right\} \end{aligned} \tag{4}$$

Trivially, the smaller the $CI(Q)$, more consistent the PFPR Q. Especially, $CI(Q) = 0$ if and only if Q is an additive consistent PFPR. In most cases, the construction of an additive-consistent PFPR is impractical as DMs are susceptible to many factors in the decision making process. Accordingly, an acceptable additive consistent PFPR will be further developed to allow a certain degree of deviation.

Let $Q = (q_{ij})_{n\times n}$ be a PFPR, and there is a threshold value $\overline{CI}$, if the additive consistency index satisfies $CI(Q) \le \overline{CI}$, then we call a PFPR Q with acceptably additive consistency. The value of $\overline{CI}$ can be determined according to the preferences of the DM or the actual situation of the problem. For the PFPR that do not reach an acceptable consistency, the following algorithm can be applied to adjust or repair the inconsistent PFPR $Q^{(t)} = (q_{ij}^{(t)})_{n\times n}$ until it has acceptable additive consistency.

Algorithm 1 Consistency checking and improving process

1: **Input:** The original PFPR $Q = (q_{ij})_{n\times n} = \langle u_{ij}, v_{ij}\rangle_{n\times n}$, the parameter $\sigma \in (0,1)$ that is the trade-off parameter between the inconsistent preference relation and the corresponding consistent preference relation, the maximum number of iterations t^*, and the threshold value $\overline{CI} \in (0,1]$.
2: **Output:** The adjusted PFPR $\tilde{Q} = (\tilde{q}_{ij})_{n\times n} = \langle \tilde{u}_{ij}, \tilde{v}_{ij}\rangle_{n\times n}$, and the consistency index $CI(\tilde{Q})$.
3: **step1:** Let $Q^{(0)} = \langle u_{ij}^{(0)}, v_{ij}^{(0)}\rangle_{n\times n} = Q = \langle u_{ij}, v_{ij}\rangle_{n\times n}$, $t = 0$. Construct the additive consistent PFPR $\bar{Q}^{(0)} = \langle \bar{u}_{ij}^{(0)}, \bar{v}_{ij}^{(0)}\rangle_{n\times n}$, where $\bar{Q}^{(0)} = \bar{Q}^{(i)}, i = 1, 2, \cdots, t+1$.
4: **step2:** Compute the consistency index $CI(Q^{(t)})$.
5: **step3:** If $CI(Q^{(t)}) \le \overline{CI}ort \ge t^*$, then go to the Step5; otherwise, go to the Step4.
6: **step4:** Let $Q^{(t+1)} = (q_{ij}^{(t+1)})_{n\times n} = \langle u_{ij}^{(t+1)}, v_{ij}^{(t+1)}\rangle_{n\times n}$, where $u_{ij}^{(t+1)} = \sqrt{(1-\sigma)(u_{ij}^{(t)})^2 + \sigma(\bar{u}_{ij}^{(t)})^2}$, $v_{ij}^{(t+1)} = \sqrt{(1-\sigma)(v_{ij}^{(t)})^2 + \sigma(\bar{v}_{ij}^{(t)})^2}$, $\pi_{ij}^{(t+1)} = \sqrt{(1-\sigma)(\pi_{ij}^{(t)})^2 + \sigma(\bar{\pi}_{ij}^{(t)})^2}$. Set $t = t+1$ and go to Step2.
7: **step5:** Let $\tilde{Q} = Q^{(t)}$. Output the modified PFPR $\tilde{Q}$ and its consistency index $CI(\tilde{Q})$.
8: **step6:** End

3.3. The aggregation operator based on Yager triangular norms

To aggregate the weights of attributes and decision information, we will propose an aggregation operator based on Yager t-NMs and t-CMs [8] , in this subsection, which have desirable monotonic properties in total ordering.

For any two PFNs $P_1 = \langle u_{P_1}, v_{P_1}\rangle$ and $P_2 = \langle u_{P_2}, v_{P_2}\rangle$, the generalized intersection $P_1 \oplus P_2$ based on Yager t-NMs T_p^Y and its dual t-CMs S_p^Y can be written as $P_1 \oplus P_2 = \langle T_p^Y(u_{P_1}, u_{P_2}), S_p^Y(v_{P_1}, v_{P_2})\rangle$, where the family T_p^Y of Yager t-NMs and the family S_p^Y of Yager t-CMs can be find in reference [8] Given that, the PF Yager weighted averaging (PFYWA) operator can be defined.

Definition 6. *Let $P_\phi = \langle u_{P_\phi}, v_{P_\phi}\rangle(\phi = 1, 2, \cdots, \rho)$ be an arrangement of PFNs and $\theta = (\theta_1, \theta_2, \cdots, \theta_\rho)^T$ be the weight vector of $P_\phi(\phi = 1, 2, \cdots, \rho)$, with $\theta_\phi > 0$ and $\sum_{\phi=1}^{\rho}\theta_\phi = 1$. we can obtain the PFYWA aggregation operator as*

$$PFYWA_\theta(P_1, P_2, \cdots, P_\rho) = \overset{\rho}{\underset{\phi=1}{\oplus}} \theta_\phi P_\phi = \left\langle \left(\sum_{\phi=1}^{\rho}\theta_\phi u_{P_\phi}^p\right)^{1/p}, 1 - \left(\sum_{\phi=1}^{\rho}\theta_\phi (1-v_{P_\phi})^p\right)^{1/p}\right\rangle \tag{5}$$

where $p \in (0, \infty)$. In particular, if $p = 1$, then the PFYWA aggregation operator is reduced to the following:

$$PFYWA_\theta(P_1, P_2, \cdots, P_\rho) = \overset{\rho}{\underset{\phi=1}{\oplus}} \theta_\phi P_\phi = \left\langle \sum_{\phi=1}^{\rho}\theta_\phi u_{P_\phi}, \sum_{\phi=1}^{\rho}\theta_\phi v_{P_\phi}\right\rangle$$

3.4. The intergated Pythagorean fuzzy MADM framework

Step 1: Calculate the score matrix $S = \left(s(d_{ij})\right)_{n\times m}$ of decision matrix by the score function, and normalize it to $\bar{S} = \left(\bar{s}(d_{ij})\right)_{n\times m}$ using Eq.(2).

Step 2: Construct the additive consistency complementary judgment matrix $\bar{Q} = (\bar{q}_{ij})_{n\times n}$ and test the consistency of PF judgment matrix $Q = (q_{ij})_{n\times n} = \left(\langle u_{ij}, v_{ij}\rangle\right)_{n\times n}$ according to the proposed additive consistency checking index. For the unacceptable PF judgment matrix, it should be adjusted to an acceptable consistent PFPR $\tilde{Q} = (\tilde{q}_{ij})_{n\times n} = \left(\langle \tilde{u}_{ij}, \tilde{v}_{ij}\rangle\right)_{n\times n}$ by applying the proposed iterative algorithm.

Step 3: Calculate the optimal weight vector $w^* = (w_1^*, w_2^*, \cdots, w_m^*)^T$ of attributes by solving models (M-3).

Step 4: Aggregate PF decision matrix by PFYWA operator and produce a combined aggregated value X_i for each alternative x_i.

Step 5: Calculate and rank the scores $S_i (i = 1, 2, \cdots, n)$ of each alternative by means of the score function.

4. An illustrative numerical example

This section presents a numerical example to implement the proposed approach. In this paper, since $\sqrt{2}/2$ is irrational number, we make $\sqrt{2}/2 \approx 0.7071$.

A family desires to purchase an air conditioner and there are now four brands $X_i (i = 1, 2, 3, 4)$ to consider. There are six main evaluation indicators: Security (C_1); Cooling capacity and heat production (C_2); Structural properties (C_3); Reliability (C_4); Economical (C_5); Aesthetics (C_6). The weighting information for each indicator is known to be $\Lambda = \{w_1 \leq 0.3, 0.2 \leq w_3 \leq 0.5, w_2 \leq 0.2, w_3 - w_2 \geq w_5 - w_4, 0.1 \leq w_5 \leq 0.4, w_4 \leq w_1, w_4 \leq 0.1, w_6 \geq 0.2\}$. Using statistical methods, the degrees of membership α_{ij} and non-membership $\beta_{ij} (i = 1, 2, 3, 4; j = 1, 2, \cdots, 6)$ of alternative X_i to criterion C_j were obtained and noted as the Pythagorean fuzzy number $d_{ij} = (\alpha_{ij}, \beta_{ij})$. This is shown in Table 1.

Table 1: Pythagorean fuzzy decision matrix D

	C_1	C_2	C_3	C_4	C_5	C_6
X_1	$\langle 0.3, 0.5\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.6, 0.4\rangle$	$\langle 0.8, 0.2\rangle$	$\langle 0.4, 0.5\rangle$	$\langle 0.5, 0.3\rangle$
X_2	$\langle 0.7, 0.3\rangle$	$\langle 0.5, 0.3\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.7, 0.1\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.4, 0.1\rangle$
X_3	$\langle 0.4, 0.3\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.6, 0.3\rangle$	$\langle 0.4, 0.3\rangle$	$\langle 0.3, 0.2\rangle$
X_4	$\langle 0.6, 0.2\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.3, 0.2\rangle$	$\langle 0.5, 0.4\rangle$	$\langle 0.7, 0.3\rangle$

The assessor provides two-by-two comparison information for four brands of air conditioners and constructs the following Pythagorean judgment matrix Q.

$$Q = \begin{pmatrix} \langle 0.7071, 0.7071\rangle & \langle 0.3500, 0.6500\rangle & \langle 0.4000, 0.7000\rangle & \langle 0.8000, 0.3000\rangle \\ \langle 0.6500, 0.3500\rangle & \langle 0.7071, 0.7071\rangle & \langle 0.7000, 0.3000\rangle & \langle 0.5000, 0.6000\rangle \\ \langle 0.7000, 0.4000\rangle & \langle 0.3000, 0.7000\rangle & \langle 0.7071, 0.7071\rangle & \langle 0.9000, 0.2000\rangle \\ \langle 0.3000, 0.8000\rangle & \langle 0.6000, 0.5000\rangle & \langle 0.2000, 0.9000\rangle & \langle 0.7071, 0.7071\rangle \end{pmatrix}$$

Step 1: Calculate the normalized score matrix and accordingly bulid the additive consistency PFPR $\bar{Q}$ as

$$\bar{Q} = \begin{pmatrix} \langle 0.7071, 0.7071\rangle & \langle 0.3500, 0.6500\rangle & \langle 0.4000, 0.6381\rangle & \langle 0.7071, 0.6381\rangle \\ \langle 0.6500, 0.3500\rangle & \langle 0.7071, 0.7071\rangle & \langle 0.4188, 0.3500\rangle & \langle 0.7179, 0.3500\rangle \\ \langle 0.6381, 0.4000\rangle & \langle 0.3500, 0.4188\rangle & \langle 0.7071, 0.7071\rangle & \langle 0.7071, 0.4000\rangle \\ \langle 0.6381, 0.7071\rangle & \langle 0.3500, 0.4188\rangle & \langle 0.4000, 0.7071\rangle & \langle 0.7071, 0.7071\rangle \end{pmatrix}.$$

Set $\sigma = 0.5, \overline{CI} = 0.1$ and the maximum number of iterations $t^* = 10$. The consistency index $CI(Q)$ for PFPR Q is calculated via Eq. (4) as $CI(Q) = 0.07330748 \;\; 0.1 = \overline{\mathrm{CI}}$. It proves that PFPR Q satisfies the acceptable consistent.

Step 3: Based on Model (M-1), the following linear goal program is established $Min\ \varphi$, as shown below.

$$\varphi = \min(\varepsilon_{12}^{-} + \varepsilon_{12}^{+} + \varepsilon_{13}^{-} + \varepsilon_{13}^{+} + \varepsilon_{14}^{-} + \varepsilon_{14}^{+} + \varepsilon_{23}^{-} + \varepsilon_{23}^{+} + \varepsilon_{24}^{-} + \varepsilon_{24}^{+} + \varepsilon_{34}^{-} + \varepsilon_{34}^{+})$$

$$s.t.\begin{cases}
\sqrt{2}/2\ (\bar{s}(d_1) - \bar{s}(d_2) + 1) + \varepsilon_{12}^{-} \geq 0.35,\ \sqrt{2}/2\ (\bar{s}(d_1) - \bar{s}(d_3) + 1) + \varepsilon_{13}^{-} \geq 0.4,\\
\sqrt{2}/2\ (\bar{s}(d_1) - \bar{s}(d_4) + 1) + \varepsilon_{14}^{-} \geq 0.8,\ \sqrt{2}/2\ (\bar{s}(d_2) - \bar{s}(d_3) + 1) + \varepsilon_{23}^{-} \geq 0.7,\\
\sqrt{2}/2\ (\bar{s}(d_2) - \bar{s}(d_4) + 1) + \varepsilon_{24}^{-} \geq 0.5,\ \sqrt{2}/2\ (\bar{s}(d_3) - \bar{s}(d_4) + 1) + \varepsilon_{34}^{-} \geq 0.9,\\
\sqrt{2}/2\ (\bar{s}(d_1) - \bar{s}(d_2) + 1) - \varepsilon_{12}^{+} \leq 0.7599,\ \sqrt{2}/2\ (\bar{s}(d_1) - \bar{s}(d_3) + 1) - \varepsilon_{13}^{+} \leq 0.7141,\\
\sqrt{2}/2\ (\bar{s}(d_1) - \bar{s}(d_4) + 1) - \varepsilon_{14}^{+} \leq 0.9539,\ \sqrt{2}/2\ (\bar{s}(d_2) - \bar{s}(d_3) + 1) - \varepsilon_{23}^{+} \leq 0.9539,\\
\sqrt{2}/2\ (\bar{s}(d_2) - \bar{s}(d_4) + 1) - \varepsilon_{24}^{+} \leq 0.8,\ \sqrt{2}/2\ (\bar{s}(d_3) - \bar{s}(d_4) + 1) - \varepsilon_{34}^{+} \leq 0.9798,\\
\bar{s}(d_1)=0.5w_2+0.3056w_3+w_4+0.3143w_6\\
\bar{s}(d_2)=w_1+0.1944w_2+w_3+0.7818w_4+w_5+0.2857w_6\\
\bar{s}(d_3)=0.4107w_1+w_2+0.4w_4+0.8889w_5\\
\bar{s}(d_4)=0.8571w_1+w_3+w_5+w_6\\
\varepsilon_{12}^{-} \geq 0, \varepsilon_{13}^{-} \geq 0, \varepsilon_{14}^{-} \geq 0, \varepsilon_{23}^{-} \geq 0, \varepsilon_{24}^{-} \geq 0, \varepsilon_{34}^{-} \geq 0,\\
\varepsilon_{12}^{+} \geq 0, \varepsilon_{13}^{+} \geq 0, \varepsilon_{14}^{+} \geq 0, \varepsilon_{23}^{+} \geq 0, \varepsilon_{24}^{+} \geq 0, \varepsilon_{34}^{+} \geq 0,\\
w_1+w_2+w_3+w_4+w_5+w_6=1\\
w_1=0.3, w_2=0.2, w_3=0.2, w_3=0.5, w_3\text{-}w_2=w_5\text{-}w_4, w_4=w_1, w_4=0.1,\\
w_5=0.1, w_5=0.4, w_6=0.2, w_1=0, w_2=0, w_3=0, w_4=0, w_5=0, w_6=0,
\end{cases}$$

Solving this model by *LINGO*, it follows that the optimal objective value $\varphi = 0.7155872$, and the optimal deviation variable: $\dot{\varepsilon}_{12}^{-} = \dot{\varepsilon}_{12}^{+} = 0, \dot{\varepsilon}_{13}^{-} = \dot{\varepsilon}_{13}^{+} = 0, \dot{\varepsilon}_{14}^{-} = 0.3386, \dot{\varepsilon}_{14}^{+} = 0, \dot{\varepsilon}_{23}^{-} = \dot{\varepsilon}_{23}^{+} = 0, \dot{\varepsilon}_{24}^{-} = \dot{\varepsilon}_{24}^{+} = 0, \dot{\varepsilon}_{34}^{-} = 0.3770, \dot{\varepsilon}_{34}^{+} = 0$. The weight intervals for the attributes are $w_1 \in [0.1999994, 0.2]$, $w_2 \in [0.2000000, 0.2000000]$, $w_3 \in [0.2000000, 0.2000003]$, $w_4 \in [0.0999999, 0.1]$, $w_5 \in [0.1000000, 0.1000003]$, $w_6 \in [0.2000000, 0.2000001]$, and the optimal attribute weight vector is $w^* = (0.2, 0.2, 0.2, 0.1, 0.1, 0.2)^T$

Step 4: Aggregate the evaluation of each alternative by the PFYWA operator as $X_1 = (0.54037, 0.383406), X_2 = (0.593296, 0.250998), X_3 = (0.50000, 0.289828), X_4 = (0.593296, 0.293258)$.

Step 5: Based on score function, the combined score of each alternative is $S_1 = 0.145, S_2 = 0.289, S_3 = 0.166, S_4 = 0.266$. Since $S_2 > S_4 > S_3 > S_1$, the four brands of air conditioners are ranked as $X_2 > X_4 > X_3 > X_1$.

To better present the effectiveness of the proposed method, we compare it to the method proposed by Khan et al.[33] based on grey relational analysis. The final results of Khan's method are $X_2 > X_3 > X_1 > X_4$. The main reason is that the ranking of alternativesby using the method proposed in Khan et al.[33] is based on gey relational coefficient and relative relational degree, while in our approach we use the aggregation operator for the ranking of alternatives. Moreover, in the proposed approach we utilized the PF decision matrix and judgment matrix to find the unknown attribute weight, which take advantage of the initiative and motivation of DMs.

5. Conclusions

In this paper, we focused on the PF MADM problem with incomplete weight information and proposed a PF MADM framwork. There are three keys to this process. First, a linear programming model is developed for determining the incomplete weights of attribute. Second, an iterative algorithm of improving the consistency for self-contradictory PFPRs is constructed based on the expoential consistency index. Third, we provide the PF Yager weighted averaging operator to aggregate the general PF information. In the end, the validation of the proposed approach is proved by a numercial example and comparsion with the method proposed by Khan et al.[33].

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