

# Critical properties of bipartite permutation graphs

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## Funding information

Engineering and Physical Sciences Research Council

## Abstract

The class of bipartite permutation graphs enjoys many nice and important properties. In particular, this class is critically important in the study of clique- and rank-width of graphs, because it is one of the minimal hereditary classes of graphs of unbounded clique- and rank-width. It also contains a number of important subclasses, which are critical with respect to other parameters, such as graph lettericity or shrub-depth, and with respect to other notions, such as well-quasi-ordering or complexity of algorithmic problems. In the present paper we identify critical subclasses of bipartite permutation graphs of various types.

## KEYWORDS

bipartite permutation graphs, induced subgraph isomorphism, universal graph, well-quasi-ordering

## 1 | INTRODUCTION

A *graph property*, also known as a *class of graphs*, is a set of graphs closed under isomorphism. A property is *hereditary* if it is closed under taking induced subgraphs. The universe of hereditary properties is rich and diverse, and it contains various classes of theoretical or practical importance, such as perfect graphs, interval graphs, permutation graphs, bipartite graphs, planar graphs, threshold graphs, split graphs, graphs of bounded vertex degree, graphs of bounded clique-width and so forth. It also contains all classes closed under taking subgraphs, minors, induced minors, vertex-minors and so forth.

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## 1.1 | The wonderful world of bipartite permutation graphs

The class that we focus on in this paper is that of *bipartite permutation graphs*, that is, graphs that are simultaneously bipartite and permutation graphs. This class was introduced by Spinrad, Brandstädt and Stewart [52] in 1987 and since then it appears frequently in the mathematical and computer science literature. Partly, this is because bipartite permutation graphs have a nice structure allowing solutions to many problems that are notoriously difficult for general graphs. For instance:

- the reconstruction conjecture, which is wide open in general, holds true for bipartite permutation graphs [30];
- many algorithmic problems that are generally NP-hard admit polynomial-time solutions when restricted to bipartite permutation graphs (see, e.g., [35, 42]);
- the MINIMUM COST HOMOMORPHISM problem to a fixed graph  $H$  is an optimisation problem with applications in defence logistics [26]. It is polynomially solvable when  $H$  is a bipartite permutation graph (and in fact, the boundary of solvability is not much further) [25];
- bipartite permutation graphs have bounded contiguity, a complexity measure that is important in biological applications [23];
- bipartite permutation graphs have a universal element of small order: there is a bipartite permutation graph with  $n^2$  vertices (depicted in Figure 3) containing all  $n$ -vertex bipartite permutation graphs as induced subgraphs [41].

On the other hand, in spite of the many attractive properties of bipartite permutation graphs, they represent a complex world. Indeed, in this world some algorithmic problems remain NP-hard, for instance, INDUCED SUBGRAPH ISOMORPHISM, and some parameters that measure the complexity of the graphs take arbitrarily large values, for instance, clique-width. Moreover, the class of bipartite permutation graphs is critical with respect to clique-width in the sense that in every proper hereditary subclass of bipartite permutation graphs, clique-width is bounded by a constant [39]. In other words, the class of bipartite permutation graphs is a minimal hereditary class of graphs of unbounded clique-width. The same is true with respect to the notion of rank-width, because rank-width is bounded in a class of graphs if and only if clique-width is. Moreover, in the terminology of vertex-minors, bipartite permutation graphs constitute the *only* obstacle to bounding rank-width of bipartite graphs, because every bipartite graph of large rank-width contains a large universal bipartite permutation graph as a vertex-minor, see Corollary 3.9 in [51].

This class, however, is not critical with respect to the complexity of the INDUCED SUBGRAPH ISOMORPHISM problem, because the problem remains NP-hard when further restricting to the class of linear forests, a proper subclass of bipartite permutation graphs [15, 22]. One of the results of the present paper is that the class of linear forests is a minimal hereditary class where the problem is NP-hard.

Is it always possible to find minimal “difficult” classes? In the universe of minor-closed classes of graphs the answer to this question is “yes”, because graphs are well-quasi-ordered under the minor relation [50]. In particular, in the family of minor-closed classes of graphs the planar graphs constitute a unique minimal class of unbounded tree-width [49]. However, the induced subgraph relation is not a well-quasi-order, because it contains infinite antichains of graphs. As a result, the universe of hereditary classes contains infinite strictly descending chains of classes. The intersection of all classes in such a chain is called a *limit class* and a

minimal limit class is called a *boundary class*. Unfortunately, limit classes can be found even within bipartite permutation graphs. On the other hand, fortunately, there is only one boundary class in this universe, as we show in the present paper. We also show that this unique boundary class is the only obstacle to finding minimal classes in the universe of bipartite permutation graphs.

## 1.2 | Our results

In this paper, we identify a variety of critical classes with respect to the diverse problems and properties mentioned above. More specifically:

- In Section 3, we study critical classes of bipartite permutation graphs with respect to the notion of well-quasi-ordering by induced subgraphs. In that direction, we show that, for any finite set  $\mathcal{F}$  of graphs, the class of  $\mathcal{F}$ -free bipartite permutation graphs is well-quasi-ordered under induced subgraphs if and only if  $\mathcal{F}$  contains a proper induced subgraph of an  $H$ -graph (see Figure 5).
- In Section 4, we analyse various graph parameters including lettericity and shrub-depth (defined later). We characterise boundedness of those parameters in the universe of bipartite permutation graphs, and prove a conjecture from [53] which, in the language of lettericity, gives a tight upper bound on the lettericity of bipartite permutation graphs.
- In Section 5, we study the INDUCED SUBGRAPH ISOMORPHISM problem for subclasses of bipartite permutation graphs. We identify a minimal “difficult” class for this problem—namely, linear forests—and conjecture that this is the only such class, in the sense that the ISI problem should be easy for bipartite permutation graphs avoiding any fixed path  $P_k$ . We support this conjecture by proving it in the case  $k = 5$  via a reduction to the assignment problem.
- In Section 6, we investigate the minimum number of vertices of  $n$ -universal graphs for subclasses of bipartite permutation graphs. We work towards understanding when a class has an  $n$ -universal bipartite permutation graph with linearly many vertices. We show that such a class must avoid a star forest, since those only have  $n$ -universal constructions on  $\Omega(n \log n)$  vertices, but this is not enough: we also prove that the minimum  $n$ -universal graph for  $3K_{1,6}$ -free bipartite permutation graphs has  $\Omega(n^\alpha)$  vertices for any  $\alpha < 2$ .

## 1.3 | Terminology and notation

All graphs in this paper are *simple*, that is, finite, undirected, without loops and without multiple edges. The vertex set and the edge set of a graph  $G$  are denoted by  $V(G)$  and  $E(G)$ , respectively.

As usual,  $P_n$ ,  $C_n$  and  $K_n$  denote the chordless path, the chordless cycle and the complete graph with  $n$  vertices, respectively. Also,  $K_{n,m}$  is the complete bipartite graphs with parts of size  $n$  and  $m$ , and we write  $S_n$  for the star  $K_{1,n}$ . Finally, we denote the graph represented in Figure 1 by  $S_{i,j,k}$ .

The disjoint union of two graphs  $G_1$  and  $G_2$  will be denoted by  $G_1 + G_2$ , and the disjoint union of  $k$  copies of  $G$  will be denoted by  $kG$ .

An *induced subgraph* of a graph  $G$  is a subgraph obtained from  $G$  by a (possibly empty) sequence of vertex deletions. We say that  $G$  contains a graph  $H$  as an induced subgraph if  $H$  is isomorphic to an induced subgraph of  $G$ . Otherwise, we say that  $G$  is  *$H$ -free* and that  $H$  is a forbidden induced subgraph for  $G$ .

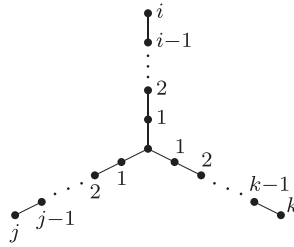


FIGURE 1 The graph  $S_{i,j,k}$ .

It is well known (and not difficult to see) that a class of graphs is hereditary if and only if it can be characterised by means of minimal forbidden induced subgraphs, that is, graphs that do not belong to the class and which are minimal with this property (with respect to the induced subgraph relation). The class of graphs containing no induced subgraphs from a set  $M$  will be denoted  $\text{Free}(M)$ , and given graphs  $G_1, G_2, \dots$ , we will write  $\text{Free}(G_1, G_2, \dots)$  to mean  $\text{Free}(\{G_1, G_2, \dots\})$ .

## 2 | BIPARTITE PERMUTATION GRAPHS AND THEIR SUBCLASSES

Throughout the paper we denote the class of all bipartite permutation graphs by  $\mathcal{BP}$ . This class was introduced in [52] and also appeared in the literature under various other names, such as proper interval bigraphs [28], monotone graphs [18] and Parikh word representable graphs [12]. This class admits various characterisations, many of which can be found in [52]. For the purpose of the present paper, we are interested in the induced subgraph characterisation and the structure of a “typical” graph in this class.

In terms of forbidden induced subgraphs the class of bipartite permutation graphs is precisely the class of

$(S_{2,2,2}, \text{Sun}_3, \Phi, C_3, C_5, C_6, C_7, \dots)$ -free graphs,

where  $\text{Sun}_3$  and  $\Phi$  are the graphs represented in Figure 2. We remark that the first three graphs in the list above are also known as the bipartite claw, bipartite net and bipartite tent, respectively.

A “typical” bipartite permutation graph is represented in Figure 3. We emphasise that this figure contains two representations of the *same* graph. In most of our considerations the square representation is more preferable and we denote a graph of this form with  $n$  columns and  $n$  rows by  $H_{n,n}$ . Explicitly,  $H_{n,n}$  has vertex set  $\{v_{i,j} : 1 \leq i, j \leq n\}$  and edge set  $\{v_{i,j}v_{i+1,t} : t \geq j\}$ . The graph  $H_{n,n}$  is typical in the sense that it contains all  $n$ -vertex bipartite permutation graphs as induced subgraphs, that is, this is an  $n$ -universal bipartite permutation graph, as shown in [41].

Below, we list a number of subclasses of bipartite permutation graphs that play an important role in this paper.

- *Chain graphs* are bipartite graphs for which the vertices in each part are linearly ordered under the inclusion of their neighbourhoods. These are precisely the  $2K_2$ -free bipartite graphs.
- *Graphs of vertex degree at most 1* are graphs in which every connected component is either  $K_2$  or  $K_1$ . Alternatively, they can be described as  $(P_3, K_3)$ -free graphs.



FIGURE 2 Graphs  $Sun_3$  (left) and  $\Phi$  (right).

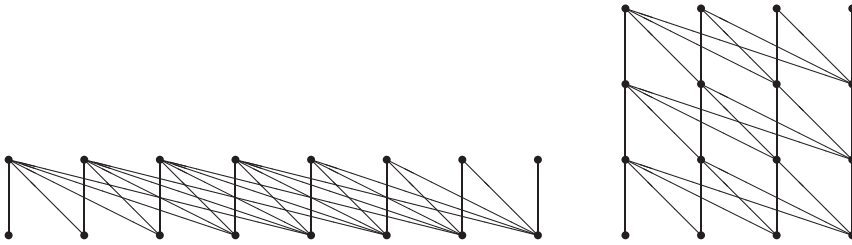


FIGURE 3 Universal bipartite permutation graph  $H_{n,n}$  for  $n = 4$ .

As we shall see in Section 4, both these classes are critical with respect to a parameter known as neighbourhood diversity. Graphs of degree at most 1 admit two important extensions.

- *Linear forests*, also known as *path forests*, are graphs in which every connected component is a path  $P_k$ , for some  $k$ . These are precisely  $(K_{1,3}, C_3, C_4, C_5, \dots)$ -free graphs, or alternatively,  $(K_{1,3}, C_4)$ -free bipartite permutation graphs.
- *Star forests* are graphs in which every connected component is a star  $K_{1,p}$ , for some  $p$ . In the terminology of forbidden induced subgraphs this class can be described as the class of  $(P_4, C_4)$ -free bipartite permutation graphs.

Both linear forests and star forests are special cases of caterpillar forests.

- *Caterpillar forests* are graphs in which every connected component is a caterpillar, that is, a tree containing a dominating path. In terms of forbidden induced subgraphs caterpillar forests can be described as the class of  $C_4$ -free bipartite permutation graphs.

An interesting class between star forests and bipartite permutation graphs is the class of  $P_3$ -free bipartite graphs: these are graphs in which every connected component is a chain graph. The inclusion relationships between the above-listed classes are represented in Figure 4.

### 3 | WELL-QUASI-ORDERING AND BOUNDARY CLASSES

In this section, we look for critical classes of bipartite permutation graphs with respect to the notion of well-quasi-ordering by induced subgraphs. We start with basic definitions.

A binary relation  $\leq$  on a set  $W$  is a *quasi-order* (also known as *preorder*) if it is reflexive and transitive. Two elements  $x, y \in W$  are said to be *comparable* with respect to  $\leq$  if either  $x \leq y$  or

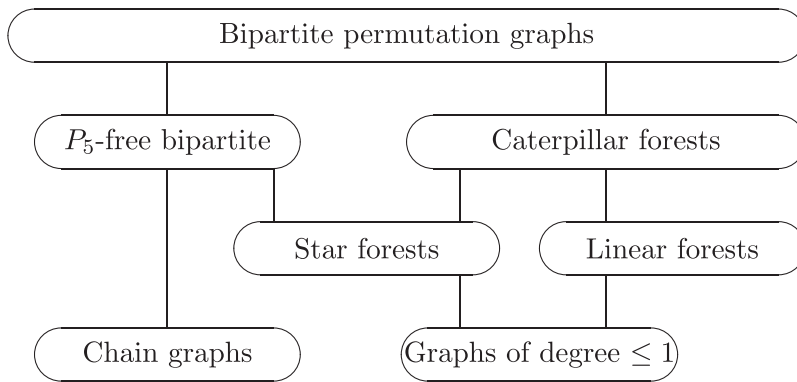


FIGURE 4 Inclusion relationships between subclasses of bipartite permutation graphs.

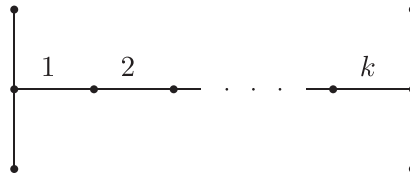


FIGURE 5 The graph  $H_k$ .

$y \leq x$ . Otherwise,  $x$  and  $y$  are *incomparable*. A set of pairwise comparable elements is called a *chain*, and a set of pairwise incomparable elements is an *antichain*. If  $x \leq y$  and  $y \not\leq x$ , we write  $x < y$ . A chain  $\{x_1, x_2, \dots\}$  with  $x_1 > x_2 > \dots$  is called *strictly descending*. A quasi-order  $(W, \leq)$  is a *well-quasi-order* (“wqo”, for short) if it contains neither infinite strictly descending chains nor infinite antichains.

The celebrated result of Robertson and Seymour [50] states that the set of all simple graphs is well-quasi-ordered with respect to the minor relation. However, the induced subgraph relation is not a wqo, as the cycles create an infinite antichain with respect to this relation. This example does not apply to the class of bipartite permutation graphs, since all chordless cycles are forbidden in this class, except for  $C_4$ . Nonetheless, graphs in this class are not well-quasi-ordered under induced subgraphs, because they contain the infinite antichain of *H-graphs*, that is, graphs of the form  $H_k$  represented in Figure 5.

On the other hand, if we restrict ourselves to the class of chain graphs, we find ourselves in the well-quasi-ordered world, because chain graphs have lettericity at most 2, as we shall see in Section 4.2, and graphs of bounded lettericity are known to be well-quasi-ordered by induced subgraphs [48]. Unfortunately, the boundary separating wqo classes from non-wqo ones cannot be described in the terminology of minimal non-wqo classes, because such classes do not exist. Indeed, if  $\mathcal{X}$  is a non-wqo class, then it contains an infinite antichain of graphs. Excluding these graphs one by one, we obtain an infinite strictly descending sequence of subclasses of  $\mathcal{X}$ , none of which is wqo.

To overcome this difficulty, we employ the notion of boundary classes, which can be viewed as a relaxation of the notion of minimal classes. This notion was introduced in [2] to study the maximum independent set problem in hereditary classes. Later, this notion was applied to

some other graph problems of both algorithmic [3, 4, 32, 43–46] and combinatorial [33, 38] nature. In particular, in [33] it was applied to the study of well-quasi-ordered classes and was defined as follows.

To simplify the discussion, we use the term *bad* to refer to classes of graphs that are not well-quasi-ordered under the induced subgraph relation and the term *good* to refer to those classes that are well-quasi-ordered under the induced subgraph relation.

**Definition 1.** We say that  $\mathcal{X}$  is a *limit class* if  $\mathcal{X}$  is the intersection of some sequence  $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \mathcal{X}_3 \supseteq \dots$  of bad classes.

In [33], it was shown that every bad class contains a minimal limit class.

**Definition 2.** A minimal limit class is called a *boundary class*.

The importance of this notion is due to the following theorem, also proved in [33].

**Theorem 1.** *A class of graphs defined by finitely many forbidden induced subgraphs is good if and only if it contains no boundary classes.*

One of the boundary classes identified in [33] is the class of linear forests. It is the limit of the sequence  $\text{Free}(K_{1,3}, C_3, \dots, C_k)$  with  $k$  tending to infinity. Each class of this sequence is bad, since each of them contains infinitely many cycles. However, the class of linear forests is not boundary in the universe of bipartite permutation graphs, since none of the classes  $\text{Free}(K_{1,3}, C_3, \dots, C_k)$  belongs to this universe.

To identify a boundary class  $\mathcal{X}$  in the universe of bipartite permutation graphs, we need to construct a sequence of bad subclasses of bipartite permutation graphs converging to  $\mathcal{X}$  and to show that  $\mathcal{X}$  is a minimal limit class. To this end, we consider the following sequence:

$$BP \cap \text{Free}(C_4, K_{1,4}, S_{1,2,2}, 3K_{1,3}, H_1, \dots, H_k).$$

We denote the limit class of this sequence with  $k$  tending to infinity by  $\mathcal{L}$  and show that  $\mathcal{L}$  is the only boundary class in the universe of bipartite permutation graphs. We start with the following helpful lemma.

**Lemma 1.**  *$\mathcal{L}$  is the class of graphs in which every connected component is a path, except possibly for at most two components of the form  $S_{1,1,k}$  for some  $k$ .*

*Proof.* By forbidding  $C_4$  in the universe of bipartite permutation graphs we restrict ourselves to caterpillar forests. By forbidding  $K_{1,4}$  we further restrict ourselves to caterpillar forests of vertex degree at most 3. If, additionally, we forbid an  $S_{1,2,2}$ , then every connected graph becomes an  $H_k$  or an  $S_{1,1,k}$  or a  $P_k$ . Since all graphs of the form  $H_k$  are forbidden, every connected graph is either an  $S_{1,1,k}$  or a  $P_k$ . Finally, since  $3K_{1,3}$  is forbidden, at most two components have the form  $S_{1,1,k}$ .  $\square$

To show the minimality and uniqueness of the class  $\mathcal{L}$ , we use the following two results proved in [33] and [40], respectively.



**Lemma 2.** *A limit class  $\mathcal{X} = \text{Free}(\mathcal{M})$  is minimal (i.e., boundary) if and only if for every graph  $G \in \mathcal{X}$  there is a finite set  $\mathcal{T} \subseteq \mathcal{M}$  such that  $\text{Free}(\{G\} \cup \mathcal{T})$  is good.*

**Theorem 2.** *The antichain of  $H$ -graphs is canonical in the universe of bipartite permutation graphs, that is, a hereditary subclass of bipartite permutation graphs is well-quasi-ordered under the induced subgraph relation if and only if it contains finitely many  $H$ -graphs.*

**Theorem 3.**  *$\mathcal{L}$  is the only boundary class in the universe of bipartite permutation graphs.*

*Proof.* According to Lemma 1, by forbidding any graph  $G$  from  $\mathcal{L}$  we destroy all  $H$ -graphs, except possibly finitely many of them, and hence, by Theorem 2, we obtain a subclass  $\mathcal{BP} \cap \text{Free}(G)$  of bipartite permutation graphs which is well-quasi-ordered. Now applying Lemma 2 with  $\mathcal{T} = \emptyset$  we conclude that  $\mathcal{L}$  is a minimal limit class, that is, a boundary class.

To prove the uniqueness of  $\mathcal{L}$ , assume there is another boundary class  $\mathcal{X}$  different from  $\mathcal{L}$ . Then for any graph  $G \in \mathcal{L} - \mathcal{X}$  (which must exist due to the minimality of  $\mathcal{X}$ ),  $\mathcal{BP} \cap \text{Free}(G)$  contains  $\mathcal{X}$  and hence is not wqo by Theorem 1 (applied to the universe of bipartite permutation graphs). This contradicts the first paragraph of the proof and shows that  $\mathcal{L}$  is the only boundary class in the universe of bipartite permutation graphs.  $\square$

**Theorem 4.** *Let  $\text{Free}(\mathcal{L})$  denote the family of hereditary subclasses of bipartite permutation graphs, none of which contains  $\mathcal{L}$  as a subclass. Then  $\text{Free}(\mathcal{L})$  is well-founded with respect to inclusion, that is, it contains no strictly descending infinite chains of classes.*

*Proof.* Note that any class  $\mathcal{X} \in \text{Free}(\mathcal{L})$  is wqo by Theorem 2, since it contains only finitely many  $H$ -graphs. It is folklore (and not difficult to see) that a class of graphs is wqo if and only if the set of its subclasses is well-founded under inclusion. This immediately implies that  $\text{Free}(\mathcal{L})$  itself is well-founded under inclusion.  $\square$

Theorem 4 shows that in the universe of bipartite permutation graphs the unique boundary class  $\mathcal{L}$  is the only obstacle to finding minimal “difficult” classes. Note, however, that the number of minimal classes may in principle be infinite: an example of an infinite antichain of classes with respect to inclusion in  $\text{Free}(\mathcal{L})$  is given by the sequence  $(\mathcal{X}_i)_{i \geq 1}$ , with  $\mathcal{X}_i := \mathcal{BP} \cap \text{Free}(P_{4+i}, H_1, \dots, H_{i-1})$  (in particular, we have  $\mathcal{X}_1 = \mathcal{BP} \cap \text{Free}(P_5)$ ). We remark that the above antichain is constructed in such a way that  $H_i \in \mathcal{X}_i \setminus \bigcup_{j \neq i} \mathcal{X}_j$ . The dependence of the construction on the canonical antichain of  $H$ -graphs raises the following question.

**Open problem 1.** Is the antichain  $(\mathcal{X}_i)_{i \geq 1}$  canonical in  $\text{Free}(\mathcal{L})$ ?

One more open problem related to the notion of well-quasi-ordering comes from the fact that the set of  $H$ -graphs, which forms a (canonical) antichain with respect to induced subgraphs, is a chain with respect to induced minors, where an induced minor of a graph  $G$  is any graph obtained from  $G$  by a (possibly empty) sequence of vertex deletions and edge contractions.

Well-quasi-orderability under induced minors has been studied before—see, for instance, [17], where it was shown that interval graphs are not wqo under the relation, but chordal graphs of bounded clique number are. We note that the class of bipartite permutation graphs is



not closed under taking induced minors, but this should not stop us from investigating its orderability with respect to the induced minor relation, especially given its similarity with the class of unit interval graphs (see, e.g., [40]). Indeed, the latter is closed under the relation, and its well-quasi-orderability is an open problem which follows naturally from the results in [17]. In particular, we ask the following question.

**Open problem 2.** Is the class of bipartite permutation graphs well-quasi-ordered under the induced minor relation?

## 4 | GRAPH PARAMETERS AND MINIMAL CLASSES

Many important parameters are bounded in the class of bipartite permutation graphs, which is the case for the size of a maximum clique, chromatic number, contiguity and so forth. Many other parameters can be arbitrarily large in this class. In the present section, we characterise several such parameters in terms of minimal subclasses of bipartite permutation graphs where these parameters are unbounded. We start by reporting in Section 4.1 some known results, which will be helpful in the subsequent sections.

### 4.1 | Neighbourhood diversity, distinguishing number and uniformicity

The notion of *neighbourhood diversity* was introduced in [34] and can be defined as follows.

**Definition 3.** Two vertices  $x$  and  $y$  are *weak twins* if  $N(x) = N(y)$ , and they are *strong twins* if  $N[x] = N[y]$  (where  $N[x]$  denotes the closed neighbourhood of  $x$ ). We say that  $x$  and  $y$  are *similar* if they are weak twins or strong twins. Equivalently,  $x$  and  $y$  are similar if there is no vertex  $z \neq x, y$  distinguishing them, that is, if there is no vertex  $z \neq x, y$  adjacent to exactly one of  $x$  and  $y$ . Vertex similarity is an equivalence relation. We denote by  $nd(G)$  the number of similarity classes in  $G$  and call it the *neighbourhood diversity* of  $G$ .

Neighbourhood diversity was characterised in [37] by means of nine minimal hereditary classes of graphs where this parameter is unbounded. Three of these classes are subclasses of bipartite graphs:

- the class  $\mathcal{X}_1$  of chain graphs,
- the class  $\mathcal{X}_2$  of graphs of vertex degree at most 1,
- the class  $\mathcal{X}_3$  of bipartite complements of graphs of vertex degree at most 1, that is, bipartite graphs in which every vertex has at most one nonneighbour in the opposite part.

Three other classes are complements of graphs in  $\mathcal{X}_1$ ,  $\mathcal{X}_2$  and  $\mathcal{X}_3$ , and the remaining three classes are subclasses of split graphs obtained by creating a clique in one of the parts in a bipartition of graphs in  $\mathcal{X}_1$ ,  $\mathcal{X}_2$  and  $\mathcal{X}_3$ . The subclass of split graphs obtained in this way from graphs in  $\mathcal{X}_1$  is known as the class of *threshold graphs*. Only two of the nine listed classes are subclasses of bipartite permutation graphs, which allows us to make the following conclusion.

**Theorem 5.** *The classes of chain graphs and graphs of vertex degree at most 1 are the only two minimal hereditary subclasses of bipartite permutation graphs of unbounded neighbourhood diversity.*

The notion of distinguishing number appeared implicitly in [10] and was given its name in [7]. To define this notion, consider a graph  $G$ , a subset  $U \subseteq V(G)$ , a collection of pairwise disjoint subsets  $U_1, \dots, U_m$  of  $V(G)$ , also disjoint from  $U$ . We will say that  $U$  *distinguishes* the sets  $U_1, U_2, \dots, U_m$  if for each  $i$ , all vertices of  $U_i$  have the same neighbourhood in  $U$ , and for each  $i \neq j$ , vertices  $x \in U_i$  and  $y \in U_j$  have different neighbourhoods in  $U$ .

**Definition 4.** The *distinguishing number* of  $G$  is the maximum  $k$  such that  $G$  contains a subset  $U \subseteq V(G)$  that distinguishes a collection of at least  $k$  subsets of  $V(G)$ , each of size at least  $k$ .

The paper [10] provides a complete description of minimal classes of unbounded distinguishing number, of which there are precisely 13:

- the class of graphs every connected component of which is a complete graph,
- the class of chain graphs,
- the class of threshold graphs,
- the class of star forests,
- the class of graphs obtained from star forests by creating a clique on the leaves of the stars,
- the class of graphs obtained from star forests by creating a clique on the centres of the stars,
- the class of graphs obtained from star forests by creating a clique on the leaves of the stars and a clique on the centres of the stars,
- the classes of complements of graphs in the above-listed classes (note that the complements of threshold graphs are threshold graphs).

The global structure of graphs of bounded distinguishing number can informally be described as follows: the vertex set of every graph admits a partition into finitely many subsets such that each subset induces a graph of bounded degree or codegree (i.e., degree in the complement) and the edges between any two subsets form a bipartite graph of bounded degree or codegree. Note that bounded neighbourhood diversity is the special case where we require the degree or codegree to equal 0, so that bounded neighbourhood diversity implies bounded distinguishing number (the same conclusion can be obtained by comparing the respective lists of minimal classes).

There is a parameter between neighbourhood diversity and distinguishing number, known as *uniformity*, and introduced in [31]. One can define it as follows. Let  $k$  be a natural number,  $F$  a simple graph on the vertex set  $\{1, 2, \dots, k\}$ , and  $K$  a graph on  $\{1, 2, \dots, k\}$  with loops allowed. Let  $U(F)$  be the disjoint union of infinitely many copies of  $F$ , and for  $i = 1, \dots, k$ , let  $V_i$  be the subset of vertices of  $U(F)$  containing vertex  $i$  from each copy of  $F$ . Now we construct from  $U(F)$  an infinite graph  $U(F, K)$  on the same vertex set by connecting two vertices  $u \in V_i$  and  $v \in V_j$  if and only if

- either  $uv \in E(U(F))$  and  $ij \notin E(K)$ ,
- or  $uv \notin E(U(F))$  and  $ij \in E(K)$ .

Intuitively, we start with independent sets  $V_i$  corresponding to the vertices of  $F$ , and for  $i \neq j$ , the sets  $V_i$  and  $V_j$  have an induced matching between them wherever  $F$  has an edge. We then apply complementations according to the edges of  $K$ : if  $ij \in E(K)$ , we complement the edges between  $V_i$  and  $V_j$  (replacing  $V_i$  with a clique if  $i = j$ ).

Finally, we let  $\mathcal{P}(F, K)$  be the hereditary class consisting of all the finite induced subgraphs of  $U(F, K)$ . As an example, if  $k = 2$ ,  $F = K_2$  and  $E(K) = \emptyset$ , then  $\mathcal{P}(F, K)$  is the class of graphs of vertex degree at most 1, and if  $E(K) = \{12\}$  instead, then  $\mathcal{P}(F, K)$  is the class of bipartite complements of graphs of vertex degree at most 1.

**Definition 5.** A graph  $G$  is called  $k$ -uniform if there is a number  $k$  such that  $G \in \mathcal{P}(F, K)$  for some  $F$  and  $K$  with  $V(F) = V(K) = \{1, \dots, k\}$ . The minimum  $k$  such that  $G$  is  $k$ -uniform is the *uniformicity* of  $G$ .

From the results in [9] and [10] it follows that classes of bounded uniformicity are precisely the classes whose speed (i.e., the number of  $n$ -vertex labelled graphs) is below the Bell number. The set of minimal hereditary classes with speed at least the Bell number, and hence of unbounded uniformicity, consists of the 13 minimal classes of unbounded distinguishing number together with an infinite collection of classes, which can be described as follows.

Let  $A$  be a finite alphabet,  $w = w_1w_2\dots$  an infinite word over  $A$ , and  $K$  an undirected graph with loops allowed and with vertex set  $V(K) = A$ . We define the graph  $U(w, K)$  as follows: the vertex set of  $U(w, K)$  is the set of natural numbers and the edge set consists of pairs of distinct numbers  $i, j$  such that

- either  $|i - j| = 1$  and  $w_iw_j \notin E(K)$ ,
- or  $|i - j| > 1$  and  $w_iw_j \in E(K)$ .

To illustrate this notion, consider the case of  $A = \{a\}$  and  $E(K) = \emptyset$ . Then the infinite word  $w = aaa\dots$  defines the infinite path  $U(w, K)$ . In fact, if  $E(K) = \emptyset$ , then  $U(w, K)$  is an infinite path for any  $w$ .

Define  $\mathcal{P}(w, K)$  to be the hereditary class consisting of all the finite induced subgraphs of  $U(w, K)$ . In particular, if  $E(K) = \emptyset$ , then  $\mathcal{P}(w, K)$  is the class of linear forests.

A *factor* in a word  $w$  is a contiguous subword, that is, a subword whose letters appear consecutively in  $w$ . A word  $w$  is called *almost periodic* if for any factor  $f$  of  $w$  there is a constant  $k_f$  such that any factor of  $w$  of size at least  $k_f$  contains  $f$  as a factor.

**Theorem 6** (Atminas et al. [7]). *Let  $X$  be a class of graphs with finite distinguishing number. Then  $X$  is a minimal hereditary class with speed at least the Bell number if and only if there exists a finite graph  $K$  with loops allowed and an infinite almost periodic word  $w$  over  $V(K)$  such that  $X = \mathcal{P}(w, K)$ .*

**Definition 6.** Any class of the form  $\mathcal{P}(w, K)$ , where  $K$  is a finite graph with loops allowed and  $w$  is an infinite almost periodic word over  $V(K)$ , will be called a *class of folded linear forests*.

We observe that each class of folded linear forests is a boundary class for well-quasi-orderability by induced subgraphs, and in the world of classes with finite distinguishing number there are no other boundary classes; both of these statements were shown in [6].

Summarising the discussion, we make the following conclusion.

**Theorem 7.** *The list of minimal hereditary classes of unbounded uniformicity consists of the 13 classes of unbounded distinguishing number and all the classes of folded linear forests.*

With the restriction to bipartite permutation graphs, distinguishing number and uniformicity can be characterised in terms minimal classes as follows.

**Theorem 8.** *In the universe of bipartite permutation graphs, the list of minimal hereditary subclasses of unbounded distinguishing number consists of star forests and chain graphs, and the list of minimal hereditary subclasses of unbounded uniformicity consists of star forests, chain graphs and linear forests.*

## 4.2 | Lettericity and Parikh word representability

Parikh word representable graphs were introduced in [12] as follows. Let  $A = \{a_1 < a_2 < \dots < a_k\}$  be an ordered alphabet, and let  $w = w_1 w_2 \dots w_n$  be a word over  $A$ . The *Parikh graph* of  $w$  has  $\{1, 2, \dots, n\}$  as its vertex set and two vertices  $i < j$  are adjacent if and only if there is  $p \in \{1, 2, \dots, k-1\}$ , such that  $w_i = a_p$  and  $w_j = a_{p+1}$ .

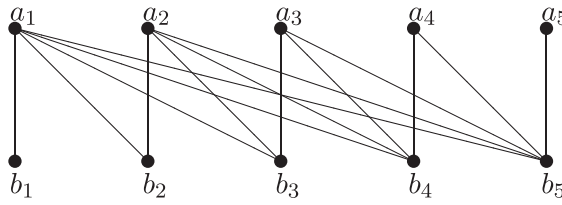
In [53], it was shown that the class of Parikh word representable graphs coincides with the class of bipartite permutation graphs and was conjectured that every bipartite permutation graph with  $n$  vertices admits a Parikh word representation over an alphabet of  $\lfloor \frac{n}{2} \rfloor + 1$  letters. In this section, we prove the conjecture. To this end, we use the related notion of letter graphs, which was introduced in [48] and can be defined as follows.

As before,  $A$  is a finite alphabet, but this time we do not assume any order on the letters of  $A$ . Let  $D$  be a subset of  $A^2$  and  $w = w_1 w_2 \dots w_n$  a word over  $A$ . The *letter graph*  $G(D, w)$  associated with  $w$  has  $\{1, 2, \dots, n\}$  as its vertex set, and two vertices  $i < j$  are adjacent if and only if the ordered pair  $(w_i, w_j)$  belongs to  $D$ . A graph  $G$  is said to be a *k-letter graph* if there exist an alphabet  $A$  with  $|A| = k$ , a subset  $D \subseteq A^2$  and a word  $w = w_1 w_2 \dots w_n$  over  $A$  such that  $G$  is isomorphic to  $G(D, w)$ .

The role of  $D$  is to decode (transform) a word into a graph and therefore we refer to  $D$  as a *decoder*. Every graph  $G$  is trivially a  $|V(G)|$ -letter graph over the alphabet  $A = V(G)$  with the decoder  $D = \{(v, w), (w, v) : \{v, w\} \in E(G)\}$ . The *lettericity* of  $G$ , denoted by  $\ell(G)$ , is the minimum  $k$  such that  $G$  is a  $k$ -letter graph.

To give a less trivial example, consider the alphabet  $A = \{a, b\}$  and the decoder  $D = \{(a, b)\}$ . Then, the word  $abababab$  describes the graph represented in Figure 6. Clearly, this is a chain graph. Moreover, a chain graph of this form with  $n$  vertices in each part, which we denote by  $Z_n$ , is  $n$ -universal, that is, it contains every  $n$ -vertex chain graph as an induced subgraph, as was shown in [41]. Therefore a graph is a chain graph if and only if it is a letter graph over the alphabet  $A = \{a, b\}$  with the decoder  $D = \{(a, b)\}$ .

The  $n$ -universal bipartite permutation graph  $H_{n,n}$  (Figure 3) can be viewed as a sequence of  $n-1$  copies of the  $n$ -universal chain graphs and hence it can be expressed as a letter graph on letters  $a_1, \dots, a_n$  with decoder  $\{(a_i, a_{i+1}) : 1 \leq i \leq n-1\}$ . The word representing  $H_{n,n}$  consists of the concatenation of  $n$  copies of  $a_1 a_2 \dots a_n$ . The same word represents  $H_{n,n}$  as a Parikh graph over



**FIGURE 6** The letter graph of the word *abababab*. We use indices to indicate in which order the *a*-letters and the *b*-letters appear in the word.

the ordered alphabet  $A = \{a_1 < a_2 < \dots < a_n\}$ . This discussion provides an alternative proof of the fact that the class of Parikh word representable graphs is precisely the class of bipartite permutation graphs. Indeed, every bipartite permutation graph is induced in some  $H_{n,n}$  and thus Parikh word representable by the above, while any Parikh word can be embedded in an appropriately long concatenation of  $a_1 a_2 \dots a_n$ , showing that the corresponding graph is induced in some  $H_{n,n}$ .

**Theorem 9.** *Let  $G$  be a bipartite permutation graph with  $n$  vertices. Then  $G$  admits a letter graph representation with  $k \leq \frac{n}{2} + 1$  letters  $a_1, \dots, a_k$  and decoder  $\{(a_i, a_{i+1}) : 1 \leq i \leq k - 1\}$ . Equivalently,  $G$  admits a Parikh word representation with  $k \leq \frac{n}{2} + 1$  letters.*

*Proof.* We first deal with the case when  $G$  is connected, and assume  $n \geq 2$ . We know that  $G$  can be expressed as a letter graph on letters  $a_1, \dots, a_n$  with decoder  $\{(a_i, a_{i+1}) : 1 \leq i \leq n - 1\}$ .

Among all expressions  $w_1 w_2 \dots w_n$  for  $G$  with that decoder, writing  $l(j)$  for the index of the letter in position  $j$  of the word, pick one that minimises  $\sum_{j=1}^n l(j)$  (i.e., an expression that minimises the sum of the indices of the letters in  $w$ ). Let us state some properties of this expression:

- $a_1$  appears at least once in  $w$ . If not, we can shift all indices down by 1.
- The last letter is  $a_2$ . Indeed, the last letter cannot be  $a_1$ , since that would mean  $G$  has an isolated vertex. If the last letter is  $a_t$ , for some  $t \geq 3$ , we can remove it, and add an  $a_{t-2}$  at the beginning of  $w$ : this new expression still represents  $G$ , but the sum of indices is smaller.
- Denote by  $r$  the largest index occurring in  $w$ . Let  $2 < t \leq r$ , and let  $w_j = a_t$  be the rightmost appearance of  $a_t$  in  $w$ . Then there is an  $a_{t-1}$  to the right of  $w_j$  in  $w$ . Note that this holds for  $t = r$ —otherwise, as before, we may replace this rightmost  $a_r$  with an  $a_{r-2}$  in the beginning of the word. Recursively, for  $r > t > 2$ , the rightmost  $a_t$  is thus to the right of the rightmost  $a_{t+1}$ , and must have an  $a_{t-1}$  to its right by the same argument as above. It follows that the rightmost occurrence of  $a_t$  has to its right at least one  $a_i$  for each  $2 \leq i < t$ , and no  $a_i$  for  $i > t$ .
- Let  $2 \leq t \leq r$ , and let  $w_j = a_t$  be the rightmost occurrence of  $a_t$ . Then there is an  $a_{t-1}$  to the left of  $w_j$  in  $w$ . Indeed, since  $G$  is connected, the vertex  $j$  has a neighbour. But as shown above, there are no  $a_{t+1}$ s to the right of  $w_j$ , hence that neighbour must be an  $a_{t-1}$  to its left.

The above discussion implies that  $w$  uses letters  $a_1, a_r$  at least once, and letters  $a_2, \dots, a_{r-1}$  at least twice (since for  $2 \leq t < r$ , letter  $a_t$  appears after the rightmost  $a_{t+1}$  by the third bullet point, and before the rightmost  $a_{t+1}$  by the fourth). Since  $G$  has  $n$  vertices, this implies  $r \leq \frac{n}{2} + 1$ .

If  $G$  is disconnected, writing  $G_i, 1 \leq i \leq s$  for its connected components, we can produce words  $w(G_i)$  as above for each  $G_i$ , where  $G_1$  uses letters  $a_1$  to  $a_{r_1}$ ,  $G_2$  uses letters  $a_{r_1}$  to  $a_{r_2}$ , and so on. We then obtain a word representing  $G$  by concatenating  $w(G_s), w(G_{s-1}), \dots, w(G_1)$  in that order. The resulting word uses once more each letter twice, except for possibly the first and the last one.  $\square$

The upper bound in Theorem 9 is tight and attained on graphs of vertex degree at most 1 (see [53] for arguments given in the terminology of Parikh word representability or [1] for arguments given in the terminology of lettericity). In the rest of this section we show that, within the universe of bipartite permutation graphs, the class of graphs of vertex degree at most 1 is the only obstacle for bounded lettericity, that is, it is the unique minimal subclass of bipartite permutation graphs of unbounded lettericity.

**Theorem 10.** *For each  $p$ , the lettericity of  $pK_2$ -free bipartite permutation graphs is at most  $3p^2 - 5p + 3$ .*

*Proof.* Let  $G$  be a  $pK_2$ -free bipartite permutation graph. Then  $G$  has at most  $p - 1$  nontrivial connected components, that is, components of size at least 2. Each component can be embedded, as an induced subgraph, into the universal graph  $H_{n,n}$  with at most  $3p - 2$  consecutive rows, since any connected induced subgraph of the universal graph occupying at least  $3p - 1$  consecutive rows contains an induced  $P_{3p-1}$  and, hence, an induced  $pK_2$ . Therefore, any component of  $G$  requires at most  $3p - 2$  letters to represent it. Altogether, we need at most  $(p - 1)(3p - 2) + 1 = 3p^2 - 5p + 3$  letters to represent  $G$ .  $\square$

### 4.3 | Shrub-depth and related parameters

In this section, we analyse various width and depth parameters and characterise their boundedness in the class of bipartite permutation graphs by exhibiting, within this universe, the minimal hereditary classes where these parameters are unbounded.

We start by repeating that the class of all bipartite permutation graphs is a minimal hereditary class of unbounded clique-width. *Tree-width* is a restriction of clique-width in the sense that bounded tree-width implies bounded clique-width, but not necessarily vice versa.

**Proposition 1.** *The class of complete bipartite graphs is the only minimal hereditary subclass of bipartite permutation graphs of unbounded tree-width.*

*Proof.* It is well known that tree-width can be arbitrarily large for complete bipartite graphs. On the other hand, it was shown in [13] that for graphs that do not contain  $K_{n,n}$  as a subgraph, tree-width is bounded by a function of  $n$  and its clique-width. Therefore, for every subclass of bipartite permutation graphs excluding at least one complete

bipartite graph tree-width is bounded, since clique-width is bounded (as the class of bipartite permutation graphs is minimal of unbounded clique-width [39]).  $\square$

*Tree-depth* is a restriction of tree-width in the sense that bounded tree-depth implies bounded tree-width, but not necessarily vice versa. Let us define the *path number* of  $G$  to be the length of a longest (not necessarily induced) path in  $G$ . In the terminology of minimal classes, the two parameters are equivalent, as shown in the following proposition.

**Proposition 2.** *The classes of complete graphs, complete bipartite graphs and linear forests are the only three minimal hereditary classes of graphs of unbounded tree-depth and path number. In the universe of bipartite permutation graphs, complete bipartite graphs and linear forests are the only two minimal hereditary classes of graphs of unbounded tree-depth and path number.*

*Proof.* It is known [47] that tree-depth is unbounded in a class  $\mathcal{X}$  if and only if graphs in  $\mathcal{X}$  contain arbitrarily long paths as subgraphs. Also, it was shown in [8] that for every  $t, p, s$ , there exists a  $z = z(t, p, s)$ , such that every graph with a (not necessarily induced) path of length at least  $z$  contains either an induced path of length  $t$  or an induced complete bipartite graph with colour classes of size  $p$  or a clique of size  $s$ . The proposition then follows.  $\square$

*Shrub-depth* is an extension of tree-depth for dense graphs. A drawback of the original definition of this parameter (which can be found in [21]) is that it applies to classes of graphs and not to individual graphs. To overcome this difficulty, the authors of [21] propose a related depth-like parameter called *SC-depth*, which is asymptotically equivalent to shrub-depth. Denoting by  $SC(n)$  the set of graphs of SC-depth  $n$ , we define this set recursively as follows:

- $SC(0) = \{K_1\}$ ,
- if  $G_1, \dots, G_p \in SC(n)$  and  $G$  is the disjoint union of  $G_1, \dots, G_p$ , then for any subset  $U \subseteq V(G)$ , the graph obtained from  $G$  by complementing all edges with both endpoints in  $U$  is in  $SC(n + 1)$ .

*Rank-depth* is a related parameter, which is equivalent to shrub-depth in the following sense.

**Theorem 11** (DeVos et al. [16]). *A class of graphs has bounded rank-depth if and only if it has bounded shrub-depth.*

Our next result characterises shrub-depth and rank-depth in the terminology of minimal classes within the universe of bipartite permutation graphs.

**Theorem 12.** *The classes of chain graphs and linear forests are the only two minimal hereditary subclasses of bipartite permutation graphs of unbounded shrub-depth and rank-depth.*

*Proof.* It was shown in [16] that paths, and therefore linear forests, have unbounded rank-depth.

To show that the parameters are unbounded in the class of chain graphs, we use the notions of local complementations and local equivalence. A *local complementation* is the operation of complementing the subgraph induced by the neighbourhood of a vertex, and



two graphs  $G$  and  $H$  are said to be *locally equivalent* if  $G$  can be obtained from  $H$  by a sequence of local complementations (and vice versa). It was also pointed out in [16] that, as a consequence of a result from [51], rank-depth is invariant under local complementations, hence it suffices to show that there are arbitrarily long paths that are locally equivalent to chain graphs.

There is one specific sequence of local complementations known as a pivot. Applied to an edge  $uv$  of a graph, the pivot consists of complementing the neighbourhoods of  $u$ ,  $v$  and then  $u$  again. Its overall effect is to complement the edges between  $N(u) \setminus \{v\}$  and  $N(v) \setminus \{u\}$ ; if the starting graph is bipartite, it remains so after the pivot.

We observe that the universal chain graph  $Z_n$  (Figure 6) can be pivoted to a path  $P_{2n}$  and vice versa. To transform  $Z_n$  into  $P_{2n}$ , one can apply pivoting on the edges  $a_2b_2, a_3b_3, \dots, a_{n-1}b_{n-1}$ . Therefore, rank- and shrub-depth are unbounded in the class of chain graphs as claimed.

It remains to show that by excluding a path  $P_k$  and a chain graph  $Z_t$  within the universe of bipartite permutation graphs, we obtain a class  $\mathcal{X}$  of bounded rank- and shrub-depth. Immediately from the definitions, we note that bounded neighbourhood diversity implies bounded SC-depth (and hence bounded shrub-depth), and that it is enough to prove that SC-depth (and hence shrub-depth) is bounded for connected graphs in  $\mathcal{X}$ .

Let  $G$  be a connected graph in  $\mathcal{X}$ . Since  $P_k$  is forbidden,  $G$  has diameter less than  $k$ . In particular, it can be embedded into the universal construction  $H_{n,n}$  using at most  $k - 1$  consecutive rows (since the number of rows used bounds the diameter from below). Every two consecutive rows induce a chain graph, and since  $Z_t$  is forbidden, this chain graph has neighbourhood diversity bounded by a constant, by Theorem 5. It follows that the neighbourhood diversity of  $G$  is bounded, as required.  $\square$

Outside of the universe of bipartite permutation graphs there exist other minimal classes of unbounded shrub-depth and rank-depth related to linear forests and chain graphs, such as classes of folded linear forests (defined in Section 4.1), the class of complements of chain graphs and the class of threshold graphs. Theorem 12 suggests the following conjecture.

**Conjecture 1.** *Shrub-depth and rank-depth are unbounded in a hereditary class  $\mathcal{X}$  if and only if  $\mathcal{X}$  contains a minimal hereditary class of unbounded shrub-depth and rank-depth. The set of minimal classes is infinite and consists of all the classes of folded linear forests, as well as the classes of chain graphs, complements of chain graphs and threshold graphs.*

## 5 | ALGORITHMIC PROBLEMS AND MINIMAL CLASSES

The simple structure of bipartite permutation graphs allows efficient solutions for many algorithmic problems that are generally NP-hard. However, some NP-hard problems remain intractable in this class, which is the case, for instance, for PAIR-COMPLETE COLOURING (also known as ACHROMATIC NUMBER) [5], HARMONIOUS COLOURING [5] and INDUCED SUBGRAPH ISOMORPHISM [27]. In this section, we focus on the latter of these problems and identify a minimal hereditary subclass of bipartite permutation graphs where the problem is NP-hard.

The INDUCED SUBGRAPH ISOMORPHISM (ISI, for short) problem can be stated as follows: given two graphs  $H$  and  $G$ , decide whether  $H$  is isomorphic to an induced subgraph of  $G$  or not. A

hereditary class  $\mathcal{X}$  will be called *ISI-easy* if, for every instance  $(G, H)$  with  $G \in \mathcal{X}$ , the problem can be solved in time polynomial in  $|V(G)| + |V(H)|$ . If, however, solving the problem in  $\mathcal{X}$  is NP-hard, then accordingly,  $\mathcal{X}$  will be called *ISI-hard*. An inclusion-wise minimal *ISI-hard* class will be called *minimal ISI-hard*.

The INDUCED SUBGRAPH ISOMORPHISM is known to be NP-complete in the class of linear forests, which is a proper subclass of bipartite permutation graphs [15, 22]. In this section, we show that the class of linear forests is a minimal *ISI-hard* hereditary class. To do so, we express the ISI problem in proper subclasses of linear forests as an integer linear programming problem.

We represent a  $P_{n+1}$ -free linear forest as  $\alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_{n-1} P_{n-1} + \alpha_n P_n$ , where  $\alpha_i \in \mathbb{N}_{\geq 0}$ , and the forest contains exactly  $\alpha_i$  connected components isomorphic to  $P_i$ .

If  $G$  is a  $P_{n+1}$ -free linear forest, then all its induced subgraphs are also  $P_{n+1}$ -free linear forests, and the recognition problem for the class of  $P_{n+1}$ -free linear forests can be solved in polynomial time. So, we may assume that

$$H = \alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_{n-1} P_{n-1} + \alpha_n P_n \quad \text{and}$$

$$G = \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_{n-1} P_{n-1} + \beta_n P_n$$

for some coefficients  $\alpha_1, \dots, \alpha_n$  and  $\beta_1, \dots, \beta_n$ .

Now, for each  $1 \leq i \leq n$  we construct an integer matrix  $A_i$  whose columns correspond to induced subgraphs of the path  $P_i$ . Specifically,  $A_i$  has a column  $(\gamma_1, \gamma_2, \dots, \gamma_n)$  for each linear forest  $\gamma_1 P_1 + \gamma_2 P_2 + \dots + \gamma_n P_n$  that is an induced subgraph of the path  $P_i$ . Write  $m_i$  for the number of columns of  $A_i$ , and let  $A = (a_{ij}) := (A_1 | A_2 | \dots | A_n)$  be the horizontal concatenation of the  $A_i$ , with  $m := m_1 + \dots + m_n$  columns.

Finally, consider the following system of linear constraints:

$$\begin{cases} \sum_{j=1}^m a_{ij} x_j \geq \alpha_i, & \forall i \in \{1, 2, \dots, n\}, \\ \sum_{j=m_1+\dots+m_{i-1}+1}^{m_1+\dots+m_i} x_j \leq \beta_i, & \forall i \in \{1, 2, \dots, n\}. \end{cases} \quad (1)$$

**Lemma 3.** *The graph  $H$  is an induced subgraph of  $G$  if and only if the system (1) has a solution in  $\mathbb{N}^m$ .*

*Proof.* We interpret each column vector  $(x_1, \dots, x_m)^T \in \mathbb{N}^m$  as an attempt to embed, for each  $j$ ,  $x_j$  copies of the forest  $a_{1,j} P_1 + \dots + a_{n,j} P_n$  into different components of  $G$ . A sufficient condition for a simultaneous embedding to exist where each copy (across all  $j$ ) uses a different component of  $G$  is that, for each  $i \in \{1, \dots, n\}$ ,

$$\sum_{j=m_1+\dots+m_{i-1}+1}^{m_1+\dots+m_i} x_j \leq \beta_i.$$

Note that, if such an embedding exists,  $G$  must contain the forest

$$\sum_{j=1}^m a_{1j} x_j P_1 + \sum_{j=1}^m a_{2j} x_j P_2 + \dots + \sum_{j=1}^m a_{nj} x_j P_n.$$

If, in addition, we have for each  $i \in \{1, \dots, n\}$ ,

$$\sum_{j=1}^m a_{ij}x_j \geq \alpha_i,$$

then the above forest, and hence  $G$ , must contain  $H$  as an induced subgraph. This shows that a feasible solution to the system implies the existence of an embedding of  $H$  into  $G$ .

Conversely, assume that  $H$  has an induced subgraph embedding  $\iota : H \hookrightarrow G$ . To produce a feasible solution to the system, write  $G_1, \dots, G_t$  for the connected components of  $G$ . For  $1 \leq s \leq t$ , let  $v_s$  be the standard basis (column) vector in  $\mathbb{N}^m$  described as follows: if  $G_s$  is isomorphic to  $P_i$ , then  $v_s$  has a 1 in the row between  $m_1 + \dots + m_{i-1} + 1$  and  $m_1 + \dots + m_i$  corresponding to the forest induced by  $\iota(H) \cap V(G_s)$ .

Now let  $x := \sum_{s=1}^t v_s$ . It can be checked that, by construction,  $x$  is a feasible solution (with equality for the constraints involving the  $\alpha_i$ ).  $\square$

**Lemma 4.** *For any fixed  $n$ , the ISI problem can be solved in polynomial time for  $P_{n+1}$ -free linear forests.*

*Proof.* Clearly,  $m_i \leq 2^i$ , for any  $i$ . Therefore,  $m \leq n2^n$ . The integer linear programming problem is known to be polynomial for a fixed number of variables [14, 24, 29, 36, 54]. More precisely, its best-known complexity bound for  $k$  variables is  $O(k^k)$ , multiplied by a polynomial in the length of the input data [14, 24, 54]. By these facts and Lemma 3, the result holds.  $\square$

The following statement is the main result of this section.

**Theorem 13.** *Unless  $P = NP$ , the class of linear forests is a minimal ISI-hard class.*

*Proof.* The class of all linear forests is known to be hard for the ISI problem [15, 22]. For any proper hereditary subclass  $\mathcal{X}$  of this class, a linear forest  $F$  is forbidden. If  $F$  has  $n$  vertices, then  $F$  is an induced subgraph of  $P_{2n}$  and, hence, all graphs in  $\mathcal{X}$  are  $P_{2n}$ -free. The theorem then follows from Lemma 4.  $\square$

If there exist ISI-hard subclasses of bipartite permutation graphs not containing the class of linear forests, then, according to Theorem 4, each such subclass contains a minimal ISI-hard class. In other words, in the universe of bipartite permutation graphs the ISI problem can be characterised by a set of minimal ISI-hard classes. We believe that this set consists of a single class.

**Conjecture 2.** *Unless  $P = NP$ , the INDUCED SUBGRAPH ISOMORPHISM problem is NP-hard in a hereditary subclass  $\mathcal{X}$  of bipartite permutation graphs if and only if  $\mathcal{X}$  contains all linear forests. Equivalently, for any fixed  $k$ , the problem can be solved in polynomial time for  $P_k$ -free bipartite permutation graphs.*

To support the conjecture, we solve the problem for  $P_5$ -free bipartite graphs.

**Proposition 3.** *If both  $G$  and  $H$  are  $P_5$ -free bipartite graphs, then the INDUCED SUBGRAPH ISOMORPHISM problem can be solved for the pair  $(G, H)$  in polynomial time.*

*Proof.* We reduce the problem to finding a maximum weight one-sided-perfect matching in an auxiliary edge-weighted complete bipartite graph  $B$  with order linear in  $|V(G)|$ . This is equivalent to the assignment problem, which is well known to be solvable in polynomial time. We first describe the graph  $B$ , then show how it can be constructed in polynomial time.

Note that every connected component  $G'$  of  $G$  can accommodate at most one nontrivial component  $H'$  of  $H$  as an induced subgraph, and potentially some isolated vertices of  $H$ . The graph  $B = (V_G \cup V_H, E = V_G \times V_H, \omega : E \rightarrow \mathbb{Z} \cup \{-\infty\})$  is defined as follows:  $V_G$  represents nontrivial connected components of  $G$ , while  $V_H$  represents nontrivial connected components of  $H$ . The weight  $\omega((G', H'))$  of the edge between components  $G'$  and  $H'$  indicates the maximum number of isolated vertices that can be accommodated by  $G'$  in addition to  $H'$ , with  $-\infty$  if  $H'$  cannot be embedded into  $G'$  (as is usual in these situations,  $-\infty$  is to be replaced, in practice, with a large enough negative number, say  $-(|V(G)|(|V_G||V_H| + 1))$ ), that guarantees the weight of an optimal matching immediately tells us if  $H$  does not embed into  $G$ . For notational reasons, we will keep it as  $-\infty$ ). With this setup, one easily checks that  $H$  has an induced embedding in  $G$  if and only if  $B$  has a matching of size  $|V_H|$  whose weight is at least the number of isolated vertices of  $H$  minus the number of isolated vertices of  $G$ .

It remains to show that  $B$  can be constructed in polynomial time. The only nontrivial part is determining the edge weights. In other words, for each of the  $O(|V(G)||V(H)|)$  pairs  $(G', H')$  of connected components, we must determine in polynomial time whether  $H'$  can be embedded into  $G'$ , and if yes, we must find the maximum number of isolated vertices that can be embedded into  $G'$  in addition to  $H'$ .

Given an embedding  $\iota : H' \hookrightarrow G'$ , write  $\mu(\iota)$  for the size of a maximum independent set in  $G' \setminus N[\iota(H')]$ , where  $N[A]$  denotes the closed neighbourhood of a set  $A$  (i.e.,  $A$  together with all vertices having at least one neighbour in  $A$ ). Our edge weight  $\omega((G', H'))$  is thus  $\max_{\iota : H' \hookrightarrow G'}(\mu(\iota))$  (with  $-\infty$  for a maximum over an empty set).

To compute the edge weights, we note that any connected  $P_5$ -free bipartite graph is  $2K_2$ -free (i.e., a chain graph), so we may encode it as a letter graph on the alphabet  $\{a, b\}$  with decoder  $\{(a, b)\}$  (see Section 4.2 for details). Any connected chain graph admits at most two such representations (depending on which letter represents which side). Given components  $G'$  and  $H'$ , write  $w_{H'}^1$  and  $w_{H'}^2$  for the two words representing  $H'$ , and write  $w_{G'}$  for one of the two words representing  $G'$ . This gives a surjective mapping from the set of induced subgraph embeddings  $\iota : H' \hookrightarrow G'$  onto the union of the sets of subword embeddings  $\iota_w : w_{H'}^i \hookrightarrow w_{G'}$  for  $i = 1, 2$ . In particular,  $H'$  has an induced subgraph embedding  $\iota$  into  $G'$  if and only if one of  $w_{H'}^1$  and  $w_{H'}^2$  has a corresponding subword embedding  $\iota_w$  into  $w_{G'}$ .

Determining whether such an embedding  $\iota_w$  exists can be done greedily in linear time, but that only tells us whether the corresponding edge in  $B$  has weight  $-\infty$  or not. We claim that, for any embedding  $\iota$ ,  $\mu(\iota)$  can be easily computed from the corresponding subword embedding  $\iota_w$ . To see this, note that, since  $H'$  is nontrivial and connected, the first letter of  $w_{H'}^i$  is an  $a$ , while the last letter is a  $b$ . Then  $\mu(\iota)$  is the number of  $b$ 's in  $w_{G'}$  before the first  $a$  in  $\iota_w(w_{H'}^i)$  plus the number of  $a$ 's after the last  $b$ . Indeed, those  $b$ 's in the

prefix  $as$  in the suffix correspond to an independent set, and every other vertex is adjacent to at least one of the initial  $a$  and final  $b$  of  $t_w(w_{H'}^i)$ .

Thus  $\mu(t)$  only depends on the positions of the first and last letters of the embedding  $t_w$ . This gives us the following way of determining  $\omega((G', H'))$  in polynomial time:

1. Set  $\nu = -\infty$ .
2. Check (in linear time) if any of the two  $w_{H'}^i$  embeds as a subword in  $w_{G'}$ . If not, return  $\nu$ .
3. For each pair of letters  $l_j, l_k$  in  $w_{G'} = l_1 l_2 \dots l_t$  with  $j < k$ ,  $l_j = a$  and  $l_k = b$ :
  - (a) Write  $w_{G'}$  as a concatenation  $w_1 w_2 w_3$ , where  $w_2$  is the substring that starts with  $l_j$  and ends with  $l_k$ .
  - (b) Determine (in linear time) whether any of the two  $w_{H'}^i$  embeds in  $w_2$ . If not, continue to the next pair. If yes, let  $\nu'$  be the number of  $bs$  in  $w_1$  plus the number of  $as$  in  $w_3$ , and set  $\nu := \max(\nu, \nu')$ .
4. Return  $\nu$ .

It is routine to check that the above procedure terminates in polynomial time, and that the value returned is  $\max_{t: H' \hookrightarrow G'}(\mu(t))$  as required.  $\square$

Let us repeat that the class of linear forests is one of the infinitely many classes  $\mathcal{P}(w, K)$  of folded linear forests, defined in Section 4.1. We conjecture that all classes of this form are minimal hard for the INDUCED SUBGRAPH ISOMORPHISM problem.

**Conjecture 3.** *Each class of folded linear forests is a minimal hard class for the INDUCED SUBGRAPH ISOMORPHISM problem.*

## 6 | UNIVERSAL GRAPHS AND MINIMAL CLASSES

As we have seen earlier, the class of bipartite permutation graphs contains a universal element of quadratic order, that is, a graph with  $n^2$  vertices that contains all  $n$ -vertex bipartite permutation graphs as induced subgraphs. On the other hand, for the class of chain graphs, we have an  $n$ -universal graph on  $2n$  vertices, that is, a universal graph of linear order. This raises many questions regarding the growth rates of order-optimal universal graphs for subclasses of bipartite permutation graphs. One of the most immediate questions is identifying a boundary separating classes with a universal graph of linear order from classes where the smallest universal graph is super-linear. In this section, we show that the class of star forests is a minimal hereditary class with a super-linear universal graph.

Before we present the result for star forests, let us observe that in general not every hereditary class  $\mathcal{X}$  contains a universal graph, and even if it does, an optimal universal construction for  $\mathcal{X}$  does not necessarily belong to  $\mathcal{X}$ . To circumvent these difficulties (and to ensure downwards closure of the set of classes with, say, linear universal graphs), we will only consider universal constructions consisting of bipartite permutation graphs. In other words, whenever we look for universal constructions for some class  $\mathcal{X} \subseteq \mathcal{BP}$ , we allow *any* bipartite permutation graphs, and *only* bipartite permutation graphs, to appear in our universal constructions.

In the following lemma we first describe a star forest of order  $O(n \cdot \log(n))$  containing all  $n$ -vertex star forests as induced subgraphs, and then we show that this construction is

(asymptotically) order-optimal in the universe of all bipartite permutation graphs. To simplify notation, throughout this section we denote the star  $K_{1,n}$  by  $S_n$ .

**Lemma 5.** *The minimum number of vertices in a bipartite permutation graph containing all  $n$ -vertex star forests is  $\Theta(n \cdot \log(n))$ .*

*Proof.* Let  $F^*$  be the star forest  $S_{\lfloor \frac{n}{4} \rfloor} + S_{\lfloor \frac{n}{2} \rfloor} + \dots + S_{\lfloor \frac{n}{n-1} \rfloor} + S_{\lfloor \frac{n}{n} \rfloor}$ . It is a bipartite permutation graph, and it has  $n$  connected components and  $\sum_{i=1}^n (\lfloor \frac{n}{i} \rfloor + 1)$  vertices. As  $\lfloor x \rfloor < x$ , for any  $x$ ,  $F^*$  has  $O\left(\sum_{i=1}^n \frac{n}{i}\right)$  vertices. Recall that the  $n$ -th harmonic number  $\sum_{i=1}^n \frac{1}{i}$  is equal to  $\ln(n) + \gamma + \epsilon_n$ , where  $\gamma = 0.577\dots$  is the Euler–Mascheroni constant and  $\epsilon_n$  tends to 0 with  $n$  tending to infinity. Therefore,  $F^*$  has  $O(n \cdot \log(n))$  vertices.

Let us show that  $F^*$  is a universal graph for  $n$ -vertex star forests. Indeed, let  $F = S_{a_1} + S_{a_2} + \dots + S_{a_p}$  be an  $n$ -vertex star forest, where  $a_1 \geq a_2 \geq \dots \geq a_p$ . Clearly,  $i \cdot a_i \leq a_1 + \dots + a_i < n$ , for any  $1 \leq i \leq p \leq n$ . Hence,  $a_i < \frac{n}{i}$  and  $a_i \leq \lfloor \frac{n}{i} \rfloor$ , as  $a_i$  is an integer. Therefore, for any  $i$ ,  $S_{a_i}$  is an induced subgraph of  $S_{\lfloor \frac{n}{i} \rfloor}$ . Thus,  $F$  is an induced subgraph of  $F^*$ .

To prove the lower bound, let  $H$  be a bipartite permutation graph containing all  $n$ -vertex star forests. It can be embedded, as an induced subgraph, into  $H_{n',n'}$  for some  $n'$  (see Figure 3). Now let  $n_1, n_2, \dots$  be a list in nonincreasing order of the numbers of vertices of  $H$  embedded in each row of  $H_{n',n'}$  (so that  $n_1$  is the number of vertices of  $H$  in a row of  $H_{n',n}$  with the most vertices of  $H$ , and so on). We show that  $n_i \geq \frac{1}{2} \left( \lfloor \frac{n}{10i} \rfloor - 1 \right)$  for any  $1 \leq i \leq \lfloor \frac{n}{20} \rfloor$ , implying that the graph  $H$  has  $\Omega(n \cdot \log(n))$  vertices.

Let  $t \in \{10i : i \in \mathbb{N}\} \cap \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ . By  $F_t$ , we denote the star forest with  $t$  connected components, each isomorphic to  $S_{\lfloor \frac{n}{t} \rfloor - 1}$ . For any  $t$ , the graph  $F_t$  is an induced subgraph of  $H$  and hence  $F_t$  must embed into  $H_{n',n'}$ . Since any two consecutive rows of  $H_{n',n'}$  induce a chain graph, that is, a  $2K_2$ -free graph, any row of  $H_{n',n'}$  contains the centres of at most two stars of  $F_t$ . Each star intersects at most 3 consecutive rows of  $H_{n',n}$ , hence a star can intersect the same row as at most 9 other stars. It is therefore possible to find  $\frac{t}{10}$  stars  $S_{\lfloor \frac{n}{t} \rfloor - 1}$  in  $F_t$  such that no two of them intersect the same row of  $H_{n',n'}$ , and thus there are at least  $\frac{t}{10}$  pairwise distinct rows in  $H_{n',n'}$ , each of which contains at least  $\frac{1}{2} \left( \lfloor \frac{n}{t} \rfloor - 1 \right)$  vertices of  $H$ . It follows that  $n_{t/10} \geq \frac{1}{2} \left( \lfloor \frac{n}{t} \rfloor - 1 \right)$  or, changing indices, that  $n_i \geq \frac{1}{2} \left( \lfloor \frac{n}{10i} \rfloor - 1 \right)$  for any  $1 \leq i \leq \lfloor \frac{n}{20} \rfloor$  as required.  $\square$

**Theorem 14.** *The class of star forests is a minimal hereditary class that does not admit a universal bipartite permutation graph of linear order.*

*Proof.* Let  $\mathcal{X}$  be any proper hereditary subclass of the class  $\mathcal{SF}$  of star forests. Then,  $\mathcal{X} \subseteq \mathcal{SF} \cap \text{Free}(kS_k)$  for some  $k$ . Therefore, every graph in  $\mathcal{X}$  consists of at most  $k - 1$  stars with at least  $k$  leaves and arbitrarily many stars with at most  $k - 1$  leaves. But then  $(k - 1)S_n + nS_{k-1}$  is an  $n$ -universal bipartite permutation graph for  $\mathcal{X}$  of linear order.  $\square$

The class of star forests is not the only obstruction to admitting a universal graph of linear order. To see this, we show that the class of  $3S_6$ -free bipartite permutation graphs requires a super-linear universal graph.

**Lemma 6.** *Suppose  $H$  is a bipartite permutation graph containing all  $n$ -vertex  $3S_6$ -free bipartite permutation graphs as induced subgraphs. Then  $|V(H)| = \Omega(n^{3/2})$ .*

*Proof.* To prove the statement, we will show there must be  $\Omega(n^3)$  pairs of vertices in  $H$ , from which we immediately get

$$|V(H)|^2 \geq \binom{|V(H)|}{2}$$

(which is in  $\Omega(n^3)$ ), and so  $|V(H)| = \Omega(n^{3/2})$ .

We know  $H$  can be embedded as an induced subgraph into a universal graph  $H_{n',n'}$  for some  $n'$ . The main idea is to construct a structure that is “rigid”, in the sense that we can guarantee that the distance (within the structure) between certain vertices is not much greater than the distance in  $H_{n',n'}$  between the embeddings of those vertices. To this end, we use a result due to Ferguson [20], that states the lettericity of the path  $P_s$ , for  $s \geq 3$ , is precisely  $\lfloor \frac{s+4}{3} \rfloor$  (the bounds from [48] are sufficient for the proof, but make it a bit messier). In our language, it follows that any embedding of a chordless path  $P_s$  into  $H_{n',n'}$  uses at least  $\lfloor \frac{s+4}{3} \rfloor$  layers (rows).

Since edges appear only between successive layers, the set of layers used by an embedding of the path is an interval. Moreover, the ends of a chordless path do not appear more than one layer away from the extremal layers in the interval, otherwise a  $2K_2$  forms between two of the layers. This implies that the distance between the ends of a chordless path  $P_s$  must be at least  $\lfloor \frac{s+4}{3} \rfloor - 3 = \lfloor \frac{s-5}{3} \rfloor$ .

For each  $t$ , we construct in two steps a graph  $Q_t$  as depicted in Figure 7.

It is easy to see that  $Q_t$  is a bipartite permutation graph, since we can embed the rows in the figure into successive layers of the universal graph. Moreover, writing  $d_G$  for the distance in a graph  $G$ , we have (using the triangle inequality and the above discussion, and assuming for now that  $Q_t$  embeds into  $H$ ),

$$d_H(x, y) \geq d_H(p, q) - 2 \geq t - 2$$

and

$$d_H(x, y) \leq d_{Q_t}(x, y) = t + 2.$$

We now construct bipartite permutation graphs  $R_{n,t}$  from  $Q_t$  by replacing  $x$  and  $y$  with independent sets  $X, Y$  of twins of size  $\lfloor \frac{n-4t-8}{2} \rfloor$  each, with the same adjacencies as  $x$  and  $y$ , respectively (we only construct those  $R_{n,t}$  for which the above quantity is positive). We note that  $|V(R_{n,t})| \leq n$  by construction, and  $R_{n,t}$  is easily seen to be  $3S_6$ -free, so that  $R_{n,t}$  is an induced subgraph of  $H$ . In addition, like with the original  $x$  and  $y$ , each new pair  $x \in X$  and  $y \in Y$  has  $|d_H(x, y) - t| \leq 2$ .



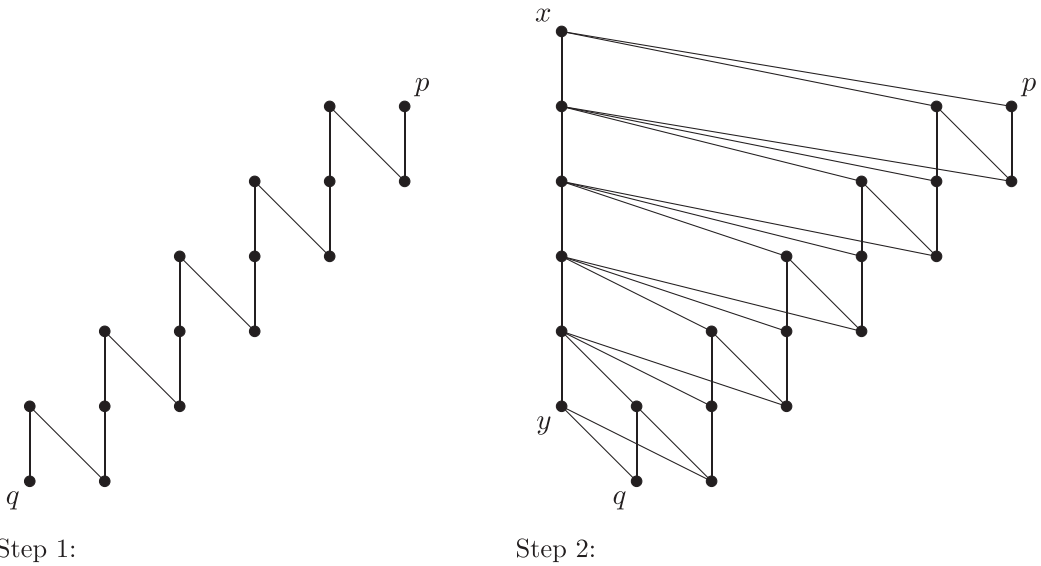


FIGURE 7 The graph  $Q_t$  for  $t = 3$ . (Step 1) Start with a chordless path on  $3t + 7$  vertices. (Step 2) Add a chordless path on  $t + 3$  vertices, connecting it to the previous path as above.

For  $3 \leq t \leq \lfloor \frac{n}{6} \rfloor - 2$ , we have  $|X| = |Y| \geq \lfloor \frac{n}{6} \rfloor$ . In particular, each choice of  $t \in I := \{3, 4, \dots, \lfloor \frac{n}{6} \rfloor - 2\} \cap \{3 + 5i : i \in \mathbb{N}\}$  witnesses the existence in  $H$  of  $|X||Y| \geq \lfloor \frac{n}{6} \rfloor^2$  pairs of vertices, and since the pairs' distance ranges for different  $t \in I$  do not overlap, the sets of pairs are disjoint. Hence  $H$  must contain in total at least

$$|I| \left\lfloor \frac{n}{6} \right\rfloor^2 \geq \frac{1}{5} \left( \left\lfloor \frac{n}{6} \right\rfloor - 5 \right) \left\lfloor \frac{n}{6} \right\rfloor^2 = \Omega(n^3)$$

pairs, as claimed. □

Lemma 6 shows indeed that there are other obstructions to a linear universal graph, but it is not yet clear what those obstructions are. For instance, the existence of a linear universal graph is a nontrivial question even for  $S_t$ -free graphs, that is, bipartite permutation graphs of maximum degree at most  $t - 1$ . In fact, it is not clear whether every class with super-linear universal graphs contains a minimal such class, since the boundary class  $\mathcal{L}$  described in Section 3 has linear universal graphs. We leave the continuation of this study as an open problem.

**Open problem 3.** Characterise the family of hereditary subclasses of bipartite permutation graphs that admit a universal bipartite permutation graph of linear order.

We conclude the paper with one more related open problem. Theorem 9 shows that the graph  $H_{n,n}$  is not an optimal universal construction for the class of bipartite permutation graphs, because all graphs in this class can be embedded into  $H_{n/2+1,n}$  as induced subgraphs. However, this construction is still quadratic. On the other hand, the following result provides an almost quadratic lower bound on the size of a universal graph.

**Theorem 15.** *Suppose  $H$  is a bipartite permutation graph that contains all  $n$ -vertex bipartite permutation graphs as induced subgraphs. Then  $|V(H)| = \Omega(n^\alpha)$  for any  $\alpha < 2$ .*

*Proof.* We show  $|V(H)| = \Omega(n^{2a-1/a})$  for each  $a \in \mathbb{N}$ . This is a generalisation of Lemma 6, which deals with the case  $a = 2$ .

The proof of Lemma 6 generalises as follows. For  $a \in \mathbb{N}$ , we get  $|V(H)| = \Omega(n^{2a-1/a})$  by counting  $a$ -sets of vertices. To do this, we associate to each  $a$ -set the  $\binom{a}{2}$ -multiset consisting of distances between pairs of its vertices; we will refer to this  $\binom{a}{2}$ -multiset as the “distance multiset (in  $H$ )” of the original  $a$ -tuple. To determine that two  $a$ -sets are distinct, it is enough to show they have distinct distance multisets.

We generalise the construction of the graphs  $R_{n,t}$  to graphs  $R_{n,T}$ , where  $T$  is a set of  $a - 1$  natural numbers, each at least 3. To construct  $R_{n,T}$ , we start with  $Q_{\max(T)}$ , but instead of inflating just the endpoints of the second path, we inflate the first vertex, then the  $j + 3$ rd, for each  $j \in T$ . By putting an appropriate upper bound (linear in  $n$ ) on the size of elements in  $T$ , say  $\lambda n$ , we can arrange that each inflated set  $X_j$  has size linear in  $n$ , while ensuring  $|V(R_{n,T})| \leq n$ .

The set  $T$  can be viewed as a condition on the distance multiset in  $R_{n,T}$  of certain  $a$ -sets: an  $a$ -set consisting of one vertex from each inflated set  $X_j$  has  $\{t + 2 : t \in T\}$  as a subset of its distance multiset in  $R_{n,T}$ . The distance multiset in  $H$  might differ from the one in  $R_{n,T}$ , but like before, the rigidity of the structure ensures the two are within a small tolerance of each other. Therefore, as long as we are careful in choosing what sets  $T$  we consider, we can ensure that different choices of  $T$  will give rise to different distance multiset subsets in  $H$ . This is achieved by choosing, like before,  $T \subseteq \{3, \dots, \lambda n\} \cap \{3 + 5i : i \in \mathbb{N}\}$ .

One last hurdle is the following: to decide that two  $a$ -sets of vertices are distinct, we actually need to compare them via their whole distance multisets, not just via the  $a - 1$ -subsets coming from the choice of  $T$ . We notice, however, that the same distance multiset can account (conservatively) for at most  $\binom{a}{a-1}$  different choices of  $T$ .

Altogether, each choice of  $T \subseteq \{3, \dots, \lambda n\} \cap \{3 + 5i : i \in \mathbb{N}\}$  witnesses the existence of  $\Omega(n^a)$   $a$ -sets of vertices in  $H$ , and each  $a$ -set is repeated overall at most a constant number of times. Since there are  $\Omega(n^{a-1})$  choices for  $T$ , this shows  $|V(H)|^a \geq \binom{|V(H)|}{a} = \Omega(n^{2a-1})$ , from which  $|V(H)| = \Omega(n^{2a-1/a})$  as required.  $\square$

We conjecture that the optimal universal graph is, in fact, quadratic.

**Conjecture 4.** *The minimum number of vertices in a bipartite permutation graph containing all  $n$  vertex bipartite permutation graphs is  $\Omega(n^2)$ .*

Establishing the optimal constant would then be a problem analogous to the study of superpatterns from the world of permutations (see, e.g., [11, 19]; we remark that the study of superpatterns is usually done in the universe of *all* permutations, while in this paper we restrict ourselves to *bipartite* permutation graphs—this is the reason behind the apparent discrepancy between the upper bound from [11] and our lower bound from Theorem 15).

## ACKNOWLEDGEMENTS

Bogdan Alecu's work was supported by the Engineering and Physical Sciences Research Council through a Doctoral Training Partnership. The work of Dmitriy S. Malyshev was conducted within the framework of the Basic Research Program at the National Research University Higher School of Economics (HSE).

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**How to cite this article:** B. Alecu, V. Lozin, and D. Malyshev, *Critical properties of bipartite permutation graphs*, *J. Graph Theory.* (2023), 1–27.

<https://doi.org/10.1002/jgt.23011>