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Bifurcation of a disappearance of a non-compact heteroclinic curve

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Abstract

In the present paper, we describe a scenario of a disappearance of a non-compact heteroclinic curve for a three-dimensional diffeomorphism. As a consequence, it is established that 3-diffeomorphisms with a unique heteroclinic curve and fixed points of pairwise different Morse indices exist only on the 3-sphere. The described scenario is directly related to the reconnection processes in the solar corona, the mathematical essence of which, from the point of view of the magnetic charging topology, consists of a disappearance or a birth of non-compact heteroclinic curves.

Keywords Saddle-node bifurcation · Heteroclinic curve · Stable arc · Morse-Smale systems · Hyperbolic dynamics

Mathematics Subject Classification 37D15

1 Statement of results

Consider a class G of orientation-preserving Morse–Smale diffeomorphisms f defined on a closed manifold M^3, the non-wandering set of which consists of exactly four

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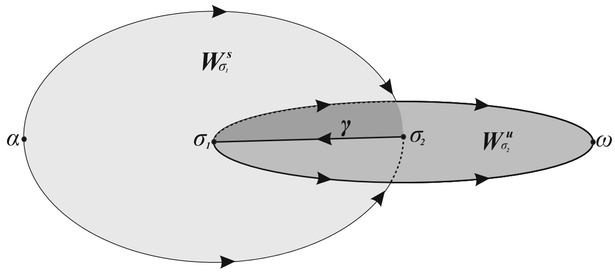


Fig. 1 A diffeomorphism $f : S^3 \to S^3$ from class G

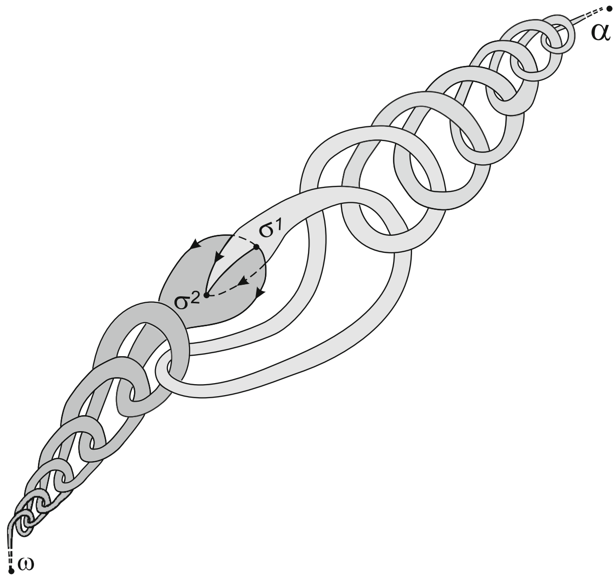


Fig. 2 Phase portrait of a diffeomorphism $f \in G$

points $\omega, \sigma_1, \sigma_2, \alpha$ with positive types of orientation and with Morse indices (dimensions of unstable manifolds) 0, 1, 2, 3, respectively. It was established in [7] that if $f \in G$ is the one-time shift of a gradient flow of a Morse function then the admitting f manifold M^3 is a lens space, moreover, every lens spaces admits a gradient-like flow with exactly four critical points of pairwise different indices. Also, by [2, 10], we know that two-dimensional saddle separatrices of such f always intersect (see Fig. 1), except the case when M^3 is homeomorphic to $S^2 \times S^1$.

Notice, that in general case, Morse–Smale diffeomorphisms are not embeddable even in a topological flow [6]. In particular, despite the simple structure of the non-wandering set of $f \in G$, the class under consideration contains diffeomorphisms with wildly embedded saddle separatrices [3, 14] (see Fig. 2), that is an obstruction to the embedding to a flow. Thus, the question of the complete list of ambient manifolds for diffeomorphisms $f \in G$ is open.

In the present paper the following fact will be established.

Theorem 1 *Let $f \in G$ and the set $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ consists of a unique curve (see Fig. 1, 2). Then M^3 is diffeomorphic to the 3-sphere S^3.*

Proof of the Theorem 1 based on the construction of the following arc of diffeomorphisms.

Theorem 2 *Let $f \in G$ and the set $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ consists of a unique curve. Then f is connected by a smooth arc $\varphi_t : M^3 \to M^3, t \in [0, 1]$ with a "source-sink" diffeomorphism. Moreover, this arc contains a unique bifurcation point which is a saddle-node.*

2 Required definitions and facts

Definition 1 (*Morse–Smale diffeomorphism*) A diffeomorphism $f : M^n \to M^n$, given on a smooth closed connected orientable n-dimensional manifold ($n \geq 1$) M^n is called a Morse–Smale diffeomorphism if

1. its non-wandering set Ω_f consists of a finite number of hyperbolic orbits;
2. manifolds W^s_p, W^u_q intersect transversally for any non-wandering points p, q.

Definition 2 (*Smooth arc*) A smooth arc in the space of diffeomorphisms $Diff(M^n)$ is a family of diffeomorphisms $\varphi_t(x) : M^n \to M^n, t \in [0; 1]$, generated by a smooth map $\Phi : M^n \times [0, 1] \to M^n$ with $\varphi_t(x) = \Phi(x, t)$.

Definition 3 (*Smooth product of arcs*) A smooth arc φ_t is called a *smooth* product of the smooth arcs ϕ_t and ψ_t such that $\phi_1 = \psi_0$, if $\varphi_t = \begin{cases} \phi_{\tau(2t)}, & 0 \leq t \leq \frac{1}{2}, \\ \psi_{\tau(2t-1)}, & \frac{1}{2} \leq t \leq 1, \end{cases}$ where $\tau : [0, 1] \to [0, 1]$ is a smooth monotone map such that $\tau(t) = 0$ for $0 \leq t \leq \frac{1}{3}$ and $\tau(t) = 1$ for $\frac{2}{3} \leq t \leq 1$. We will write $\varphi_t = \phi_t * \psi_t$.

Proposition 2.1 (Thom's isotopy extension theorem, [12], Theorem 5.8) *Let Y be a smooth manifold without boundary, X be a smooth compact submanifold Y and $\{f_t : X \to Y, t \in [0, 1]\}$ be a smooth isotopy such that f_0 is the inclusion map X into Y. Then there is a smooth isotopy $\{g_t \in Diff(Y), t \in [0, 1]\}$ such that $g_0 = id$, $g_t|_X = f_t|_X$ for every $t \in [0, 1]$ and the identity outside some compact subset of Y.*

Proposition 2.2 (Fragmentation lemma, [1]) *Let $U = \{U_i, i = 1, \dots, q\}$ be an open cover of a closed manifold M^n and $\varphi : M^n \to M^n$ be a diffeomorphism smoothly isotopic to the identity. Then there exist diffeomorphisms $\varphi_i : M^n \to M^n, i = 1, ..., q$ smoothly isotopic to the identity and such that:*

i) $supp\{\varphi_{i,t}\} \subset U_i$[1] *for a smooth arc $\varphi_{i,t}$, connecting the identity map and φ_i for every $i \in \{1, \dots, q\}$;*
ii) $\varphi = \varphi_1 \circ ... \circ \varphi_q$.

Definition 4 (*Saddle-node bifurcation*) An arc φ_t, connecting two Morse–Smale diffeomorphisms, *unfolds generically through a saddle-node bifurcation* (see Fig. 3), if

[1] *A support $supp\{f_t\}$ of an isotopy* $\{f_t\}$ is the closure of the set $\{x \in X : f_t(x) \neq f_0(x)$ for some $t \in [0, 1]\}$.

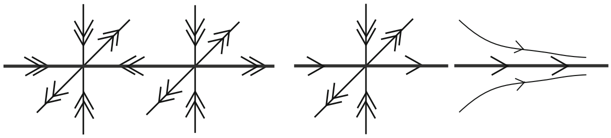


Fig. 3 Saddle-node bifurcation

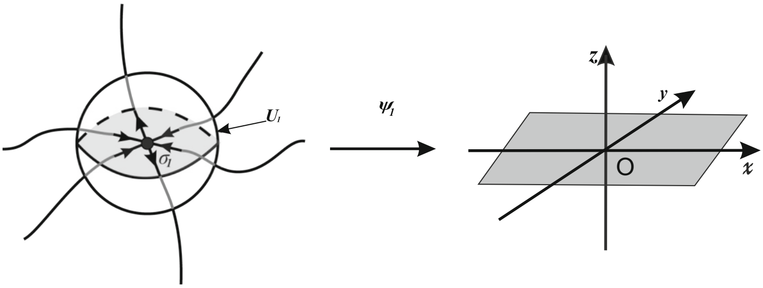


Fig. 4 Local chart in the saddle point σ_1

all elements of the arc are Morse–Smale diffeomorphisms with the exception of a diffeomorphism $\varphi_b,\ b \in (0, 1)$, which has a unique non-hyperbolic fixed point p such that in some neighbourhood of the point (p, b) the arc φ_t is conjugate with

$$\tilde{\varphi}_{\tilde{t}}(x_1, x_2, \dots, x_{1+n_u}, x_{2+n_u}, \dots, x_n) = \left(x_1 + \frac{x_1^2}{2} + \tilde{t}, \pm 2x_2, \dots, \pm 2x_{1+n_u}, \frac{\pm x_{2+n_u}}{2}, \dots, \frac{\pm x_n}{2}\right),$$

where $(x_1, \dots, x_n) \in \mathbb{R}^n,\ |x_i| < 1,\ |\tilde{t}| < 1/10$.

3 Disappearance of the heteroclinic curve

In this section, we outline the proof of the Theorem 2 with references to statements that will be proved in the following sections.

Let $f \in G$, that is the set $H_f = W^s_{\sigma_1} \cap W^u_{\sigma_2}$ consists of a unique curve. Let us prove that the diffeomorphism f is connected by a stable arc $\varphi_t : M^3 \to M^3, t \in [0, 1]$ with a "source-sink" diffeomorphism, moreover φ_t has a unique bifurcation point and it is saddle-node.

Proof Let $f \in G$, then the set H_f consists of one non-compact heteroclinic curve γ_f. Due to Lemma 1, without loss of generality, we can assume that the diffeomorphism f in a neighbourhood of the saddle point σ_1 has a local chart (see Fig. 4) $(U_1, \psi_1),\ \psi_1 : U_1 \to \mathbb{R}^3$ such that $\sigma_1 \in U_1$, $\psi_1(\sigma_1) = O$ and diffeomorphism $\psi_1 f \psi_1^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear diffeomorphism Q, given by the matrix $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

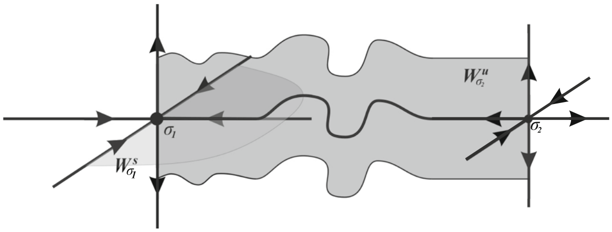


Fig. 5 Laying of the heteroclinic curve on a coordinate axis

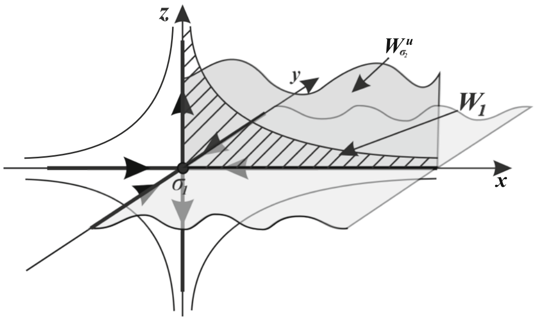


Fig. 6 Laying of the manifold $W^u_{\sigma_2}$ on a coordinate plane

Due to Lemma 2, diffeomorphism f is connected by an arc without bifurcations (it means that all elements of the arc are pairwise topologically conjugate diffeomorphisms) with the diffeomorphism $f_1 : M^3 \to M^3$ with the following properties:

1. diffeomorphism f_1 coincides with the diffeomorphism f outside U_1 and in some neighbourhood $V_1 \subset U_1$ of σ_1;
2. $\psi_1(\gamma_{f_1} \cap V_1) \subset Ox$, where γ_{f_1} is the heteroclinic curve of diffeomorphism f_1.

Making a similar construction in a local chart (U_2, ψ_2), $\psi_2 : U_2 \to \mathbb{R}^3$ such that $\sigma_2 \in U_2$, $\psi_2(\sigma_2) = O$ and $\psi_2 f_1 \psi_2^{-1} = Q^{-1}$, we get a diffeomorphism $f_2 : M^3 \to M^3$, which is connected by an arc without bifurcations with the diffeomorphism $f_1 : M^3 \to M^3$ and has the following properties:

1. diffeomorphism f_2 coincides with the diffeomorphism f_1 outside U_2 and in some neighbourhood $V_2 \subset U_2$ of σ_2;
2. $\psi_2(\gamma_{f_2} \cap V_2) \subset Ox$, where γ_{f_2} is the heteroclinic curve of diffeomorphism f_2 and $V_2 \subset U_2$ neighbourhood of point σ_2 (see Fig. 5).

Due to Lemma 3, diffeomorphism f_2 is connected by an arc without bifurcations with the diffeomorphism $f_3 : M^3 \to M^3$ with the following properties:

1. diffeomorphism f_3 coincides with the diffeomorphism f_2 outside some neighbourhood U_A of $A = cl\, W^u_{\sigma_1}$ and in some neighbourhood $W_1 \subset V_1$ of σ_1;
2. $\psi_1(W^u_{\sigma_2} \cap W_1) \subset Oxz$ (see Fig. 6).

Making a similar construction in a neighbourhood of σ_2, we get an arc without bifurcations connecting the diffeomorphism f_3 with diffeomorphism $f_4 : M^3 \to M^3$ with the following properties:

1. diffeomorphism f_4 coincides with the diffeomorphism f_3 outside some neighbourhood U_R of $R = cl\, W^s_{\sigma_2}$ and in some neighbourhood $W_2 \subset V_2$ of σ_2;
2. $\psi_2(W^s_{\sigma_1} \cap W_2) \subset Oxz$.

Due to Lemma 4, diffeomorphism f_4 is connected by an arc with one saddle-node bifurcation with a source-sink diffeomorphism on the manifold M^3. Which implies that M^3 is three-dimensional sphere. □

4 Reduction of a structurally stable diffeomorphism to a linear diffeomorphism in neighborhoods of hyperbolic fixed points

Let p be a hyperbolic fixed point of a diffeomorphism $f : M^n \to M^n$. *A type of point p* is the set of parameters (q_p, ν_p, μ_p), where $q_p = \dim\ W^u_p$, $\nu_p = +1\ (-1)$, if $f|_{W^u_p}$ preserves (reverses) orientation and $\mu_p = +1\ (-1)$, if $f|_{W^s_p}$ preserves (reverses) orientation. According to [13, Theorem 5.5], the diffeomorphism f in some neighborhood of a point p of type (q_p, ν_p, μ_p) is topologically conjugate to a linear diffeomorphism of the space $\mathbb{R}^n$, defined by the matrix

$$A_p = \begin{pmatrix} \nu_p \cdot 2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 & 0 & 0 & \dots & 0 \\ & & \ddots & & & & & \\ 0 & 0 & \dots & 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \mu_p \cdot 1/2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1/2 & \dots & 0 \\ & & & & & & \ddots & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1/2 \end{pmatrix},$$

the number of the rows of A_p, containing 2 (including $\nu_p \cdot 2$), equals q_p. Denote by $\bar{A}_p : \mathbb{R}^n \to \mathbb{R}^n$ a linear diffeomorphism defined by A_p. Let $\mathbb{R}^u = Ox_1 \dots x_{q_p}$, $\mathbb{R}^s = Ox_{q_p+1} \dots x_n$, $\bar{A}^u_p = \bar{A}_p|_{\mathbb{R}^u}$ and $\bar{A}^s_p = \bar{A}_p|_{\mathbb{R}^s}$. Then in local coordinates $x^u = (x_1, \dots, x_{q_p}) \in \mathbb{R}^u$, $x^s = (x_{q_p+1}, \dots, x_n) \in \mathbb{R}^s$ diffeomorphism $\bar{A}_p$ has a form

$$\bar{A}_p(x^u, x^s) = (\bar{A}^u_p(x^u), \bar{A}^s_p(x^s)).$$

Lemma 1 *Let a structurally stable diffeomorphism $\varphi_0 : M^n \to M^n$ has an isolated hyperbolic fixed point p, let (U_0, ψ_0) be a local chart of M^n such that $p \in U_0$, $\psi_0(p) = O$ and U_0 does not contain non-wandering points of the diffeomorphism φ_0 other than p. Then there exists neighborhoods U_1, U_2 of the point p, $U_2 \subset U_1 \subset U_0$ and the arc $\varphi_t : M^n \to M^n$, $t \in [0, 1]$ without bifurcations such that:*

1) *the diffeomorphism* φ_t, $t \in [0, 1]$ *coincides with the diffeomorphism* φ_0 *outside the set* U_1*;*
2) p *is an isolated hyperbolic point for each* φ_t*;*
3) $W^s_p(\varphi_t) = W^s_p(\varphi_0)$ *and* $W^u_p(\varphi_t) = W^u_p(\varphi_0)$ *outside the set* U_1*;*
4) *the diffeomorphism* $\psi_0\varphi_1\psi_0^{-1}$ *coincides with the diffeomorphism* $\bar{A}_p$ *on the set* $\psi_0(U_2)$.

Proof For $r > 0$ let $B_r = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 \leq r^2\}$, $B^u_r = \{(x_1, \dots, x_{q_p}) \in \mathbb{R}^u : \sum_{i=1}^{q_p} x_i^2 \leq r^2\}$ and $B^s_r = \{(x_{q_p+1}, \dots, x_n) \in \mathbb{R}^s : \sum_{i=q_p+1}^{n} x_i^2 \leq r^2\}$.

By virtue of the structural stability of the diffeomorphism φ_0 any diffeomorphism sufficiently close to φ_0 in the C^1-topology can be joined with φ_0 by an arc without bifurcations. According to Franks' lemma [4][2] we can assume that the diffeomorphism $\bar{\varphi}_0 = \psi_0\varphi_0\psi_0^{-1}$ in some ball $B_{r_0} \subset \psi_0(U_0)$ coincides with the linear diffeomorphism $\bar{\Phi}_p : \mathbb{R}^n \to \mathbb{R}^n$ given by a matrix Φ_p all of whose eigenvalues are pairwise different. Then the diffeomorphism $\bar{\Phi}_p$ is smoothly conjugate to the linear diffeomorphism $\bar{Q}_p$ given by the normal Jordan form Q_p of the matrix Φ_p (see, for example, [5, Chapter 3]). That is, there exists an orientation-preserving diffeomorphism $\xi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\bar{Q}_p = \xi\bar{\Phi}_p\xi^{-1}$. According to [12, section 6, Lemma 2], ξ is isotopic to the identity, then there is an isotopy ξ_t from $\xi_0 = id$ to $\xi_1 = \xi$. According to Proposition 2.1, there is an isotopy $\Xi_t : \mathbb{R}^2 \to \mathbb{R}^2$ between the identity map $\Xi_0 = id$ and the diffeomorphism Ξ_t coincides with ξ_t on B_{r_2} and is the identity map outside B_{r_1} for some $r_2 < r_1 < r_0$.

Thus, the arc $\bar{\eta}_t = \Xi_t\bar{\Phi}_p\Xi_t^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ connects the diffeomorphism $\bar{\eta}_0 = \bar{\Phi}_p$ with a diffeomorphism $\bar{\eta}_1$, coinciding with $\bar{Q}_p$ on B_{r_2} and with $\bar{\Phi}_p$ outside B_{r_1}. Additionally, $\bar{\eta}_t$ is an arc without bifurcations O, an isolated hyperbolic point for each $\bar{\eta}_t$ and $W^s_O(\bar{\eta}_t) = W^s_O(\bar{\Phi}_p)$, $W^u_O(\bar{\eta}_t) = W^u_O(\bar{\Phi}_p)$ outside the set B_{r_1}.

If $Q_p = A_p$, then the lemma is proved. Otherwise, due to the fact that the eigenvalues of the matrix Q_p are pairwise different, it has a quasi-diagonal form with blocks consisting of either eigenvalues or of matrices of the form $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, where $0 < \alpha^2 + \beta^2 < 1$ or $\alpha^2 + \beta^2 > 1$. Then the diffeomorphism $\bar{Q}_p$ has the form

$$\bar{Q}_p(x^u, x^s) = (\bar{Q}^u_p(x^u), \bar{Q}^s_p(x^s)),$$

where $(\bar{Q}^u_p)^{-1}(B^u_r) \subset int\, B^u_r$ for every disk B^u_r and $\bar{Q}^s_p(B^s_r) \subset int\, B^s_r$ for every disk B^s_r. Choose $r_3 < r_2$ in such a way that $B^u_{r_3} \times (\bar{Q}^s_p)^{-1}(B^s_{r_3}) \subset int\, B_{r_2}$. Choose $r^u_4, r^s_4 \in (r_3/2, r_3)$ in such a way that $(\bar{Q}^u_p)^{-1}(B^u_{r_3}) \subset int\, B^u_{r^u_4}$ and $\bar{Q}^s_p(B^s_{r_3}) \subset int\, B_{r^s_4}$.

[2] In Franks' lemma, in a neighborhood U_p of the fixed point p of the diffeomorphism $f : M^n \to M^n$ we consider the local chart (U_p, ψ_p) where $\psi_p^{-1} = exp : T_x M^n \to U_p$ – exponential map. Then in these local coordinates the diffeomorphism f has the form $\hat{f} = exp^{-1} \circ f \circ exp : \mathbb{R}^n \to \mathbb{R}^n$. The Franks lemma is that in any neighborhood of a diffeomorphism f there exists a diffeomorphism g having a fixed point p and a linear local representation $\hat{g} = exp^{-1} \circ g \circ exp$ if it is close enough to Df_p. Thus, in any neighborhood of the diffeomorphism f there exists a diffeomorphism g, having a fixed point p and a linear local representation given by a matrix all of whose eigenvalues are pairwise different.

In the proof of the proposition 5.4 of monography [13] arcs $\bar{\tau}^u_t : \mathbb{R}^u \to \mathbb{R}^u$, $\bar{\tau}^s_t : \mathbb{R}^s \to \mathbb{R}^s$ are constructed composing by linear hyperbolic contractions such that

- $(\bar{\tau}^u_t)^{-1}(B^u_r) \subset int\, B^u_r$ for any disk B^u_r and $\bar{\tau}^s_t(B^s_r) \subset int\, B^s_r$ for any disk B^s_r;
- $\bar{\tau}^u_0 = \bar{Q}^u_p$, $\bar{\tau}^u_1 = \bar{A}^u_p$ and $\bar{\tau}^s_0 = \bar{Q}^s_p$, $\bar{\tau}^s_1 = \bar{A}^s_p$.

Consider isotopies $\bar{\lambda}^u_t = \bar{Q}^u_p(\bar{\tau}^u_t)^{-1}$, $\bar{\lambda}^s_t = (\bar{Q}^s_p)^{-1}\bar{\tau}^s_t$, which joints the identity maps $\bar{\lambda}^u_0 = \bar{\lambda}^s_0 = id$ with diffeomorphism $\bar{\lambda}^u_1 = \bar{Q}^u_p(\bar{A}^u_p)^{-1}$, $\bar{\lambda}^s_1 = (\bar{Q}^s_p)^{-1}\bar{A}^s_p$, respectively. By construction, $\bar{\lambda}^u_t(B^u_{r_3}) \subset \bar{Q}^u_p(B_{r^u_4})$ and $\bar{\lambda}^s_t(B^s_{r_3}) \subset (\bar{Q}^s_p)^{-1}(B_{r^s_4})$ for each $t \in [0, 1]$. Thus, by virtue of Proposition 2.1, there exist isotopies $\bar{\Lambda}^u_t : \mathbb{R}^u \to \mathbb{R}^u$, $\bar{\Lambda}^s_t : \mathbb{R}^s \to \mathbb{R}^s$ starting from the identity map $\bar{\Lambda}^u_0 = \bar{\Lambda}^s_0 = id$, coinciding with $\bar{\lambda}^u_t$, $\bar{\lambda}^s_t$ on $B^u_{r_3}$, $B^s_{r_3}$ and are exactly identity outside $\bar{Q}^u_p(B_{r^u_4})$, $(\bar{Q}^s_p)^{-1}(B_{r^s_4})$, respectively. Let

$$\bar{\Lambda}_t(x^u, x^s) = ((\bar{\Lambda}^u_t)^{-1}\bar{Q}^u_p(x^u), \bar{Q}^s_p\bar{\Lambda}^s_t(x^s)).$$

Let us denote an arc coinciding with $\bar{\eta}_1$ outside B_{r_2} and with $\bar{\Lambda}_t$ on B_{r_2} by $\bar{\zeta}_t : \mathbb{R}^n \to \mathbb{R}^n$. Choose $r_5 < r_4$, such that $B_{r_5} \subset \bar{Q}^u_p(B^u_{r_3}) \times B^s_{r_3}$. Let $\bar{U}_2 = B_{r_5}$, $\bar{U}_1 = B_{r_1}$ and

$$\bar{\varphi}_t = \bar{\eta}_t * \bar{\zeta}_t.$$

Then $\bar{\varphi}_t$ is an arc without bifurcations, which coincide with $\bar{\varphi}_0$ outside $\bar{U}_1$, O is an isolated hyperbolic point for each $\bar{\varphi}_t$, $W^s_O(\bar{\varphi}_t) = W^s_O(\bar{\varphi}_0)$, $W^u_O(\bar{\varphi}_t) = W^u_O(\bar{\varphi}_0)$ outside the set $\bar{U}_1$ and the diffeomorphism $\bar{\varphi}_1$ coinsedes with the diffeomorphism $\bar{A}_p$ on $\bar{U}_2$. So, the arc $\varphi_t : M^n \to M^n$ which coincides with $\psi_0^{-1}\bar{\varphi}_1\psi_0$ on U_0 and with φ_0 outside U_0 satisfies all the conditions of the lemma in $U_1 = \psi_0^{-1}(\bar{U}_1)$ and $U_2 = \psi_0^{-1}(\bar{U}_2)$. □

5 Straightening of the heteroclinic curve

In this section, the diffeomorphism φ_0 belongs to the set G and in the saddle point σ_1 has a local chart (U_1, ψ_1), $\psi_1 : U_1 \to \mathbb{R}^3$ such that $\sigma_1 \in U_1$, $\psi_1(\sigma_1) = O$ and diffeomorphism $\psi_1\varphi_0\psi_1^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear diffeomorphism Q given by the matrix $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$. In the next lemma, we move the heteroclinic curve γ_{φ_0} into a canonical position required for further bifurcation.

Lemma 2 *There is a neighborhood $V_1 \subset U_1$ of σ_1 and an arc $\varphi_t : M^3 \to M^3, t \in [0, 1]$ without bifurcations with the following properties:*

1) diffeomorphism φ_t, $t \in [0, 1]$ coincides with the diffeomorphism φ_0 outside U_1 and in the neighbourhood V_1;

2) $\psi_1(\gamma_{\varphi_1} \cap V_1) \subset Ox$, where γ_{φ_1} is the heteroclinic curve of the diffeomorphism φ_1.

Proof The construction of the required arc will be done in local coordinates in $\mathbb{R}^3$ in two steps, first on the plane Oxy, then we will continue it along the axis Oz.

Step 1. Construction on the plane Oxy. Let $g = Q|_{Oxy}$. Then $g(x, y) = (x/2, y/2)$. Let $\bar{\gamma} = \psi_1(\gamma_{\varphi_0} \cap U_1)$. For $r > 0$ we state $B_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\}$. Let $K_0 = B_2 \setminus int\, B_1$. Let us denote by E_g the set of contractions (diffeomorphisms topologically conjugate to the diffeomorphism g) $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, coinciding with g out of B_1 and in some neighbourhood B_{r_ϕ} of the initial. For $\phi \in E_g$ let

$$\gamma_\phi = \bigcup_{k \in \mathbb{Z}} \phi^k(\bar{\gamma} \cap K_0).$$

By the construction ϕ-invariant curve γ_ϕ coinciding with g-invariant curve $\bar{\gamma}$ outside B_1. We construct an arc of contractions $\phi_t : \mathbb{R}^2 \to \mathbb{R}^2$, $t \in [0, 1]$ such that

(1) diffeomorphism ϕ_t, $t \in [0, 1]$ coincides with the diffeomorphism g outside B_1;

(2) $(\gamma_{\phi_1} \cap B_{r_{\phi_1}}) \subset Ox$.

To construct the arc ϕ_t we introduce the following notation for any diffeomorphism $\phi \in E_g$.

We represent the two-dimensional torus $\mathbb{T}^2$ as the orbit space of the diffeomorphism g action on $\mathbb{R}^2 \setminus O$ and denote by $p : \mathbb{R}^2 \setminus O \to \mathbb{T}^2$ the natural projection. We fix on the torus $\mathbb{T}^2$ generators $\hat{a} = p(Ox_+)$ and $\hat{b} = p(\mathbb{S}^1)$. Let $K_\phi = B_{r_\phi} \setminus B_{r_\phi/2}$ and $\hat{\gamma}_\phi = p(\gamma_\phi \cap K_\phi)$. Then the curve $\hat{\gamma}_\phi$ is a knot on the torus $\mathbb{T}^2$ with the homotopy type $\langle 1, -n_\phi \rangle$, $n_\phi \in \mathbb{Z}$ in the basis $\hat{a}, \hat{b}$ (see, for example, [8]).

The arc ϕ_t will be a smooth product of arcs η_t and ζ_t, where

I) the arc η_t, $t \in [0, 1]$ consists of contractions coinciding with the diffeomorphism g outside B_1 and joints the diffeomorphism $\eta_0 = g$ with some diffeomorphism $\eta_1 \in E_g$ such that the knot $\hat{\gamma}_{\eta_1}$ has the gomotopy type $\langle 1, 0 \rangle$ in the basis $\hat{a}, \hat{b}$;

II) the arc $\zeta_t \in E_g$, $t \in [0, 1]$ joints the diffeomorphism $\zeta_0 = \eta_1$ with a diffeomorphism ζ_1 such that $\hat{\gamma}_{\zeta_1} = \hat{a}$.

I) If $n_\phi = 0$, then we set $\eta_t = g$ for all $t \in [0, 1]$. Otherwise, we define a diffeomorphism θ_t, $t \in [0, 1] : \mathbb{R}^2 \to \mathbb{R}^2$ in the polar coordinates (ρ, ψ) so that $\theta_t(O) = O$ and $\theta_t(\rho e^{i\psi}) = \rho e^{i(\psi + \psi_t(\rho))}$, where $\psi_t(\rho)$ is a smooth monotonically decreasing function equal to $2n_\phi \pi t$ for $\rho \leq \frac{1}{2}$ and equal to 0 for $\rho \geq 1$.

Then $\eta_t = \theta_t g : \mathbb{R}^2 \to \mathbb{R}^2$ is the required arc.

II) By construction, the diffeomorphism η_1 belongs to E_g and the knot $\hat{\gamma}_{\eta_1}$ has the homotopy type $\langle 1, 0 \rangle$ in the basis $\hat{a}, \hat{b}$. According to [15], there exists a diffeomorphism smoothly isotopic to the identity $\hat{h} : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\hat{h}(\hat{\gamma}_{\eta_1}) = \hat{a}$. For $0 < r < 1/2$ let $K_r = B_r \setminus B_{r/2}$. Let's choose an open cover $D = \{D_1, \ldots, D_q\}$ of the torus $\mathbb{T}^2$ such that a connected component $\bar{D}_i$ of the set $p^{-1}(D_i)$ is a subset of K_{r_i} for some $r_i < \frac{r_{i-1}}{2}$ and $r_1 \leq r_0/2$. According to Proposition 2.2 there exist smoothly isotopic to the identity diffeomorphisms $\hat{w}_1, \ldots, \hat{w}_q : \mathbb{T}^2 \to \mathbb{T}^2$ with the following properties:

i) for every $i \in \{1, \ldots, q\}$ there is exists a smooth isotopy $\{\hat{w}_{i,t}\}$, identical outside D_i and connecting the identity map with $\hat{w}_i$;

ii) $\hat{h} = \hat{w}_1 \ldots \hat{w}_q$.

Let $w_{i,t} : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism that coincides with $(p|_{K_{r_i}})^{-1} \hat{w}_{i,t}\, p$ on K_{r_i} and coincides with the identity map outside K_{r_i}. Then the required arc is determined by the formula

$$\zeta_t = w_{1,t} \dots w_{q,t}\eta_1.$$

Step 2. Extension of ϕ_t to $\mathbb{R}^3$.

I) Let's extend the arc $\eta_t : \mathbb{R}^2 \to \mathbb{R}^2$ to the arc $\bar{\eta}_t : \mathbb{R}^3 \to \mathbb{R}^3$ in the following way. We set $C = B_1 \times [-1, 1]$. We define an isotopy $\bar{\theta}_t : \mathbb{R}^3 \to \mathbb{R}^3$ by the formula

$$\bar{\theta}_t(x, y, z) = \begin{cases} (\theta_{t(1-z^2)}(x, y), z), \ (x, y, z) \in C, \\ (x, y, z), \ (x, y, z) \in \mathbb{R}^3 \backslash C. \end{cases}$$

Let $\bar{\eta}_t = \bar{\theta}_t Q : \mathbb{R}^3 \to \mathbb{R}^3$.

II) Let's extend the arc $\zeta_t : \mathbb{R}^2 \to \mathbb{R}^2$ to the arc $\bar{\zeta}_t : \mathbb{R}^3 \to \mathbb{R}^3$ by the following way.

We set $C_i = D_i \times [-1, 1]$. We define an isotopy $\bar{w}_{i,t} : \mathbb{R}^3 \to \mathbb{R}^3$ by the formula

$$\bar{w}_{i,t}(x, y, z) = \begin{cases} (w_{i,t(1-z^2)}(x, y), z), \ (x, y, z) \in C_i \\ (x, y, z), \ (x, y, z) \in \mathbb{R}^3 \backslash C_i. \end{cases}$$

Let $\bar{\zeta}_t = \bar{w}_{1,t} \dots \bar{w}_{q,t}\bar{\eta}_1$.

Then the required arc $\bar{\varphi}_t : \mathbb{R}^3 \to \mathbb{R}^3$ will be a smooth product of the arcs $\bar{\eta}_t$ and $\bar{\zeta}_t$. □

6 Straightening of two-dimensional saddle manifolds

In this section, a diffeomorphism φ_0 belongs to the set G and in the neighbourhood of the saddle point σ_1 has a local chart (V_1, ψ_1), $\psi_1 : V_1 \to \mathbb{R}^3$ such that $\sigma_1 \in V_1$, $\psi_1(\sigma_1) = O$, diffeomorphism $\psi_1\varphi_0\psi_1^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear diffeomorphism Q and $\psi_1(\gamma_{\varphi_0} \cap V_1) \subset Ox$ for heteroclinic curve γ_{φ_0} of the diffeomorphism φ_0. In the next lemma we move $W^u_{\sigma_2}$ into a canonical position required for further bifurcation.

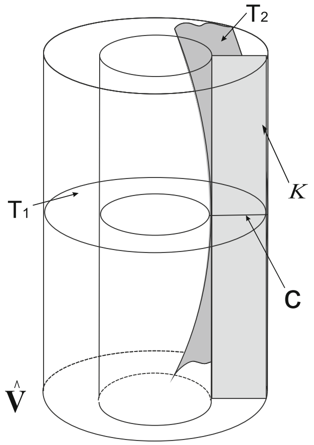
Lemma 3 *There is exist a neighborhood $W_1 \subset V_1$ of σ_1, a neighbourhood U_A of $A = cl\, W^u_{\sigma_1}$ and an arc $\varphi_t : M^3 \to M^3, t \in [0, 1]$ without bifurcations with the following properties:*

1) every diffeomorphism φ_t, $t \in [0, 1]$ coincides with φ_0 outside U_A and in W_1;

2) $\psi_1(W^u_{\sigma_2} \cap W_1) \subset Oxz$ for the diffeomorphism φ_1.

Proof By the condition $A = W^u_{\sigma_1} \cup \omega$. Let $R = W^s_{\sigma_2} \cup \alpha$, $V = M^3 \backslash (A \cup R)$ and $\hat{V} = V/\varphi_0$. Let us denote by $p : V \to \hat{V}$ the natural projection. According to [9], the set A is an attractor and has a closed compact neighbourhood U_A such that $f(U_A) \subset int\, U_A$ and $\bigcap\limits_{n \in \mathbb{N}} f^n(U_A) = A$. The set $B = U_A \backslash int\, \varphi_0(U_A)$ is a fundamental domain of φ_0 restricted to V. The set R is a repeller, the space V consists of the wandering points and the diffeomorphism φ_0 acts on V freely and discontinuously so

Fig. 7 The space orbit $\hat{V}$



that the projection p is a cover and the orbit space $\hat{V}$ is a smooth closed manifold (see, for example, [16]).

Notice, that $(W^u_{\sigma_2} \setminus \sigma_2) \subset V$ and the diffeomorphism $\varphi_0|_{W^u_{\sigma_2}}$ is topologically conjugated to a linear extension. Then the set $T_2 = p(W^u_{\sigma_2})$ is a smoothly embedded in $\hat{V}$ two-dimensional torus. By an analogy, the set $T_1 = p(W^s_{\sigma_1})$ is a two-dimensional torus smoothly embedded in $\hat{V}$. Intersection $W^u_{\sigma_2} \cap W^s_{\sigma_1}$ consists of a unique non-compact heteroclinic curve γ_{φ_0}, where the diffeomorphism φ_0 is topologically conjugate to a shift. Therefore, the intersection $T_1 \cap T_2$ consists of a unique closed curve $c = p(\gamma_{\varphi_0})$.

Let $\mathcal{K} = \{(x, y, z) : |xz| \leq 1, x > 0, y = 0\}$ and $K = p(\psi_1^{-1}(\mathcal{K}))$. By the construction K is a two-dimensional annulus containing the curve c in its interior. Choose a tubular neighbourhood $U_c \subset \hat{V}$ of the curve c such that the sets $K_2 = U_c \cap T_2,\ K_1 = U_c \cap T_1,\ K_0 = U_c \cap K$ are two-dimensional annulus (see Fig. 7).

By the construction the annulus K_2 and K_0 intersect K_1 transversally along the curve c. Then there exists a diffeomorphism $\psi : \hat{V} \to \hat{V}$ isotopic to the identity such that $\psi(T_1) = T_1$ and $\psi(K_2) = K_0$.

Choose an open cover $\{Q_1, \dots, Q_k\}$ of the space $\hat{V}$ such that a connected component $\bar{Q}_i$ of the set $p^{-1}(Q_i)$ is a subset of a fundamental domain $B_i = U^i_A \setminus int\, \varphi_0(U^i_A)$ of the diffeomorphism φ_0 restricted to V, wherein $U^{i+1}_A \subset int\, \varphi_0(U^i_A)$ for $i = 0, \dots, k$, $U^0_A = U_A$. According to Proposition 2.2, there exist smoothly isotopic to the identity diffeomorphisms $\hat{w}_1, \dots, \hat{w}_k : \hat{V} \to \hat{V}$ with the following properties:

i) for each $i \in \{1, \dots, k\}$ there exist a smooth isotopy $\{\hat{w}_{i,t}\}$, identical outside B_i and joining the identity map with $\hat{w}_i$;

ii) $\psi = \hat{w}_1 \dots \hat{w}_k$.

Let $w_{i,t} : V \to V$ be a diffeomorphism, which is the same as $(p|_{\bar{Q}_i})^{-1} \hat{w}_{i,t} p$ on $\bar{Q}_i$ and coincides with the identity map outside $\bar{Q}_i$. Then the arc φ_t is defined by the formula $\varphi_t = w_{1,t} \dots w_{k,t} \varphi_0$. □

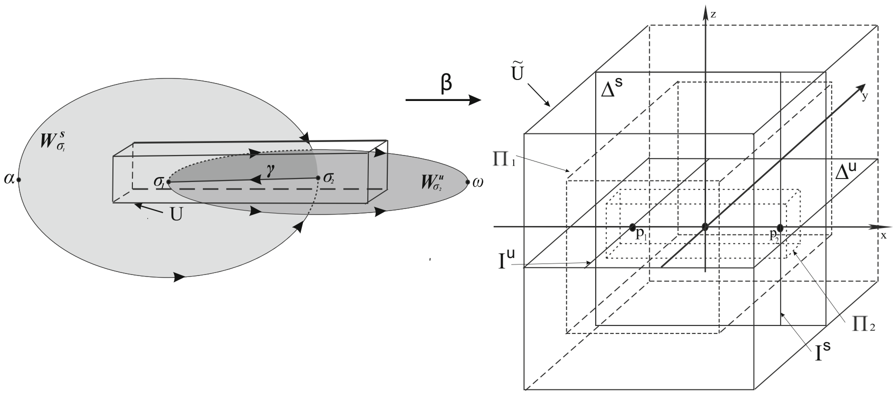


Fig. 8 Confluence of saddle points

7 Confluence of saddle points

In this section a diffeomorphism φ_0 belongs to the set G and has the following properties: in σ_1 there exist a local chart (W_1, ψ_1), $\psi_1 : W_1 \to \mathbb{R}^3$ such that $\sigma_1 \in W_1$, $\psi_1(\sigma_1) = O$, diffeomorphism $\psi_1\varphi_0\psi_1^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ is well-defined and equal to the linear map Q on a neighborhood of the origin, moreover $\psi_1(W^u_{\sigma_2} \cap W_1) \subset Oxz$. Similarly there is a local chart (W_2, ψ_2), $\psi_2 : W_2 \to \mathbb{R}^3$ such that $\sigma_2 \in W_2$, $\psi_2(\sigma_2) = O$, diffeomorphism $\psi_2\varphi_0\psi_2^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ is well-defined and equal to the linear map Q^{-1} on a neighborhood of the origin, moreover $\psi_2(W^s_{\sigma_1} \cap W_2) \subset Oxz$.

In the next lemma we confluence saddle points σ_1, σ_2 along the heteroclinic curve γ_{φ_0}.

Lemma 4 *There exists a neighborhood $U \subset M^3\setminus(\alpha \cup \omega)$ of heteroclinic curve γ_{φ_0} and arc $\varphi_t : M^3 \to M^3, t \in [0, 1]$ with a saddle-node bifurcation such that the diffeomorphism φ_t, $t \in [0, 1]$ coincides with the diffeomorphism φ_0 out of the set U and diffeomorphism φ_1 is a source-sink diffeomorphism.*

Proof Let $\tilde{U} = \{(x, y, z) \in \mathbb{R}^3 : |x| < 1, |y| < 1, |z| < 1\}$. We define an embedding $\tilde{\varphi} : \tilde{U} \to \mathbb{R}^3$ by the formula

$$\tilde{\varphi}(x, y, z) = \left(x + \frac{x^2}{2} - \frac{1}{10}, \frac{y}{2}, 2z\right).$$

By the construction, $\tilde{\varphi}$ has saddle points $P_1(-x_0, 0, 0)$ and $P_2(x_0, 0, 0)$, $x_0 \in (0, 1/2)$. Let $\Delta^s = W^s_{P_1} \cap \tilde{U}$, $\Delta^u = W^u_{P_2} \cap \tilde{U}$, $I^u = W^u_{P_1} \cap \tilde{U}$, $I^s = W^s_{P_2} \cap \tilde{U}$ (see Fig. 8). The properties of φ_0 allow to find a neighborhood U of the heteroclinic curve and an embedding $\beta : U \to \tilde{U}$ inducing a diffeomorphism

$$\tilde{\varphi}_0 = \beta\varphi_0\beta^{-1}|_{\tilde{U}}$$

for which the points P_1, P_2 are hyperbolic fixed saddle points with the local invariant manifolds belonging to Δ^s, I^u; Δ^u, I^s. Let us choose sets $\Pi_1 \subset \beta(U)$ of the form

$\Pi_1 = \{(x, y, z) \in \mathbb{R}^3 : |x| \leq a, |y| \leq b, |z| \leq b\}$, $a \in (x_0, \frac{1}{2})$, $b \in (0, \frac{1}{2})$ and such that $\tilde{\varphi}(\Pi_1) \subset (\tilde{\varphi}_0(\beta(U)) \cap \beta(U))$.

Let us define an embedding $\xi_0 : \Pi_1 \to \beta(U)$ by the formula $\xi_0 = \tilde{\varphi}_0^{-1}\tilde{\varphi}|_{\Pi_1}$. Then there is a family of embeddings $\xi_t : \Pi_1 \to \beta(U)$ such that $\xi_1 = id$, $\xi_t(\Delta^s) \subset \Delta^s$, $\xi_t(I^u) \subset I^u$, $\xi_t(\Delta^u) \subset \Delta^u$, $\xi_t(I^s) \subset I^s$ (see, for example, [11]). The the family of the embeddings $\zeta_t = \tilde{\varphi}_0\xi_{1-t} : \Pi_1 \to \beta(U)$ joints $\tilde{\varphi}_0$ with $\tilde{\varphi}$. Define a family of embeddings $\eta_t : \Pi_1 \to \beta(U)$ by the formula

$$\eta_t(x, y, z) = \left(x + \frac{x^2}{2} + \frac{1}{10}(2t - 1), \frac{y}{2}, 2z\right).$$

Then the isotopy η_t connects $\eta_0 = \tilde{\varphi}$ with an embedding η_1 whose non-wandering set is empty. Let $\theta_t = \zeta_t * \eta_t$. Then the family $\tilde{\psi}_t = \tilde{\varphi}_0^{-1}\theta_t : \Pi_1 \to \beta(U)$ connects the identity map with the embedding $\tilde{\varphi}_0^{-1}\eta_1$. Due to Proposition 2.1, there exist an isotopy $\tilde{\Psi}_t : \beta(U) \to \beta(U)$, coinciding with $\tilde{\psi}_t$ on Π_1, identical on $\partial\beta(U)$ and such that $\tilde{\Psi}_0 = id$. Let $\Psi_t : M^3 \to M^3$ be an isotopy which coincide with $\beta^{-1}\tilde{\Psi}_t\beta$ on U and coincides with the identical map outside U. Then $\varphi_t = \varphi_0\Psi_t$ is the requirement isotopy. □

Data availability Data sharing is not applicable to this article as no new data were created or analysed in this study.

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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