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Smale Regular and Chaotic A-Homeomorphisms and A-Diffeomorphisms

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Abstract—We introduce Smale A-homeomorphisms that include regular, semichaotic, chaotic, and superchaotic homeomorphisms of a topological n-manifold M^n, $n \geqslant 2$. Smale A-homeomorphisms contain axiom A diffeomorphisms (in short, A-diffeomorphisms) provided that M^n admits a smooth structure. Regular A-homeomorphisms contain all Morse–Smale diffeomorphisms, while semichaotic and chaotic A-homeomorphisms contain A-diffeomorphisms with trivial and nontrivial basic sets. Superchaotic A-homeomorphisms contain A-diffeomorphisms whose basic sets are nontrivial. The reason to consider Smale A-homeomorphisms instead of A-diffeomorphisms may be attributed to the fact that it is a good weakening of nonuniform hyperbolicity and pseudo-hyperbolicity, a subject which has already seen an immense number of applications.

We describe invariant sets that determine completely the dynamics of regular, semichaotic, and chaotic Smale A-homeomorphisms. This allows us to get necessary and sufficient conditions of conjugacy for these Smale A-homeomorphisms (in particular, for all Morse–Smale diffeomorphisms). We apply these necessary and sufficient conditions for structurally stable surface diffeomorphisms with an arbitrary number of expanding attractors. We also use these conditions to obtain a complete classification of Morse–Smale diffeomorphisms on projective-like manifolds.

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Dedicated to the memory of A. M. Stepin

1. INTRODUCTION

Diffeomorphisms satisfying Smale's axiom A (in short, A-diffeomorphisms) were introduced by Smale [46] as a magnificent and natural generalization of structurally stable diffeomorphisms. By definition, a nonwandering set of an A-diffeomorphism has a uniform hyperbolic structure and is the topological closure of periodic orbits. For an A-diffeomorphism, Smale proved that the nonwandering set splits into closed, transitive, and invariant pieces called basic sets. A basic set is *trivial* if it is an isolated periodic orbit. A good example of an A-diffeomorphism with trivial basic sets is a Morse–Smale diffeomorphism [38, 45]. Such diffeomorphisms exhibit regular dynamics. Due to Bowen [9], A-diffeomorphisms with nontrivial basic sets exhibit chaotic dynamics since any such diffeomorphism has a positive entropy. The most familiar nontrivial basic sets are Plykin's

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attractor [39] and codimension-one expanding attractors introduced by Williams [48, 49]. Such basic sets appeared in various applications, see, for example, [15, 26, 47].

Keeping in mind that there are manifolds that do not admit smooth structures [35], we introduce Smale A-homeomorphisms with nonwandering sets having a hyperbolic type (see a precise definition below). Such homeomorphisms naturally appear in topological dynamical systems. For example, in [11], the existence of topological Morse functions with three critical points on topological (including nonsmoothable) closed manifolds was proved. Starting with these examples, one can construct topological (perhaps only topological) Morse – Smale flows and Morse – Smale homeomorphisms with a nonwandering set consisting of three fixed points of hyperbolic type. A profound theory of topological dynamical systems was developed in [2, 3].

Another reason to consider Smale A-homeomorphisms instead of A-diffeomorphisms may be attributed to the fact that it is a good weakening of nonuniform hyperbolicity and pseudo-hyperbolicity, a subject which has already seen an immense number of applications [1, 47].

The challenging problem in the theory of dynamical systems is that of classifying, up to conjugacy, dynamical systems with regular and chaotic dynamics. Recall that homeomorphisms $f_1,\ f_2 : M^n \to M^n$ are called *conjugate* if there is a homeomorphism $h : M^n \to M^n$ such that $h \circ f_1 = f_2 \circ h$. To check whether given f_1 and f_2 are conjugate, one usually constructs an invariant of conjugacy which is a dynamical characteristic that is preserved under a conjugacy homeomorphism. Normally, such an invariant is constructed within the framework of a special class of dynamical systems. The famous invariant is Poincaré's rotation number for the class of transitive circle homeomorphisms [40]. This invariant is effective, i. e., two transitive circle homeomorphisms are conjugate if and only if they have the same Poincaré rotation number (see [37] and [6], Ch. 7, concerning invariants of low dimensional dynamical systems). Anosov [4] and Smale [46] were the first to realize the fundamental role of hyperbolicity for dynamical systems. Numerous topological invariants were constructed for special classes of A-diffeomorphisms including Anosov systems [12, 30, 36], A-flows [34], and Morse – Smale systems, see the books [7, 13] and the surveys [14, 31].

Within the framework of Smale A-homeomorphisms, we introduce regular, semichaotic, chaotic, and superchaotic homeomorphisms. We get necessary and sufficient conditions of conjugacy for regular, semichaotic, and chaotic Smale A-homeomorphisms on a closed topological n-manifold M^n, $n \geqslant 2$. Automatically, this gives necessary and sufficient conditions of conjugacy for Morse – Smale diffeomorphisms and a wide class of A-diffeomorphisms with nontrivial basic sets provided that M^n admits a smooth structure. We apply our conditions for structurally stable surface diffeomorphisms with an arbitrary number of one-dimensional expanding attractors. We classify Morse – Smale diffeomorphisms (up to iterations) with three periodic points on projective-like manifolds (such manifolds were introduced by the authors in [33]).

Let us give the main definitions and formulate the main results. In [33], the authors introduced the notion of equivalent embedding as follows. Let $M_1^k,\ M_2^k \subset M^n$ be topologically embedded k-manifolds, $1 \leqslant k \leqslant n-1$. We say they *have the equivalent embedding* if there are neighborhoods $U(\operatorname{clos} M_1^k)$, $U(\operatorname{clos} M_2^k)$ of $\operatorname{clos} M_1^k$, $\operatorname{clos} M_2^k$, respectively, and a homeomorphism $h : U(\operatorname{clos} M_1^k) \to U(\operatorname{clos} M_2^k)$ such that $h(M_1^k) = M_2^k$. Here, $\operatorname{clos} N$ means the topological closure of N. This notion allows one to classify Morse – Smale topological flows with nonwandering sets consisting of three equilibria [33]. To be precise, it was proved that two such flows f_1^t, f_2^t are topologically equivalent if and only if the stable (or unstable) separatrices of saddles of f_1^t and f_2^t, respectively, have the equivalent embedding. Remark that the notion of equivalent embedding goes back to a scheme introduced by Leontovich and Maier [27, 28] to attack the classification problem for flows on a 2-sphere. Solving the conjugacy problem for homeomorphisms, we have to add conjugacy relations to the equivalent embedding. The modification of (global) conjugacy is a local conjugacy when the conjugacy holds in some neighborhoods of compact invariant sets. We introduce the intermediate notion, the so-called locally equivalent dynamical embedding (in short, dynamical embedding), as follows.

Let $f_i : M_i^n \to M_i^n$ be a homeomorphism of a closed topological n-manifold M_i^n, $n \geqslant 2$, $i = 1, 2$, and $N_1 \subset M_1^n$, $N_2 \subset M_2^n$ invariant sets of f_1 and f_2, respectively, i.e., $f_i(N_i) = N_i$, $i = 1, 2$. We say that the sets N_1 and N_2 *have the same dynamical embedding* if there are neighborhoods δ_1 and δ_2 of $\operatorname{clos} N_1$ and $\operatorname{clos} N_2$, respectively, and a homeomorphism $h_0 : \delta_1 \cup f_1(\delta_1) \to M_2^n$ on its image such that

$$h_0(\delta_1) = \delta_2, \qquad h_0(\operatorname{clos} N_1) = \operatorname{clos} N_2, \qquad h_0 \circ f_1|_{\delta_1} = f_2 \circ h_0|_{\delta_1}. \tag{1.1}$$

Recall that $F : L^n \to L^n$ is an A-diffeomorphism of a smooth manifold L^n provided that the nonwandering set $NW(F)$ is hyperbolic, and the periodic orbits of F are dense in $NW(F)$ [46]. The hyperbolicity implies that every point $z_0 \in NW(F)$ has the stable $W^s(z_0)$ and unstable $W^u(z_0)$ manifolds formed by points $y \in L^n$ such that $\varrho_L(F^{pk} z_0, F^{pk} y) \to 0$ as $k \to +\infty$ and $k \to -\infty$, respectively, where ϱ_L is a metric on L^n [18, 19, 21, 24, 41, 46]. Moreover, $W^s(z_0)$ and $W^u(z_0)$ are homeomorphic (in the interior topology) to Euclidean spaces $\mathbb{R}^{\dim W^s(z_0)}$ and $\mathbb{R}^{\dim W^u(z_0)}$, respectively. Note that $\dim W^s(z_0) + \dim W^u(z_0) = n$. The nonwandering set $NW(F)$ is a finite union of pairwise disjoint F-invariant closed sets $\Omega_1, \ldots, \Omega_k$ such that every restriction $F|_{\Omega_i}$ is topologically transitive. These Ω_i are called *basic sets* of F. A basic set is *nontrivial* if it is not a periodic isolated orbit. Set $W^{s(u)}(\Omega_i) = \cup_{x \in \Omega_i} W^{s(u)}(x)$. One says that Ω_i is a *sink* (*source*) *basic set* provided that $W^u(\Omega_i) = \Omega_i$ ($W^s(\Omega_i) = \Omega_i$). A basic set Ω_i is a *saddle basic set* if it is neither a sink nor a source basic set.

A homeomorphism $f : M^n \to M^n$ is called a *Smale A-homeomorphism* if there is an A-diffeomorphism $F : L^n \to L^n$ such that the nonwandering sets $NW(f)$, $NW(F)$ have the same dynamical embedding. As a consequence, $NW(f)$ is a finite union of pairwise disjoint f-invariant closed sets $\Lambda_1, \ldots, \Lambda_k$ called *basic sets* of f such that every restriction $f|_{\Lambda_i}$ is topologically transitive (see Proposition 3). Each basic set Λ has the stable manifold $W^s(\Lambda)$ and the unstable manifold $W^u(\Lambda)$. Similarly, one introduces the families of sink basic sets $\omega(f)$, source basic sets $\alpha(f)$, and saddle basic sets $\sigma(f)$.

A Smale A-homeomorphism f is called *regular* if all basic sets $\omega(f)$, $\sigma(f)$, $\alpha(f)$ are trivial.

A Smale A-homeomorphism f is called *semichaotic* if exactly one family from the families $\omega(f)$, $\sigma(f)$, $\alpha(f)$ consists of nontrivial basic sets.

A Smale A-homeomorphism f is called *chaotic* if exactly two families from the families $\omega(f)$, $\sigma(f)$, $\alpha(f)$ consists of nontrivial basic sets.

A Smale A-homeomorphism f is called *superchaotic* if all basic sets $\omega(f)$, $\sigma(f)$, $\alpha(f)$ are nontrivial.

Denote by $SsH(M^n)$ the set of either regular, or semichaotic, or chaotic Smale A-homeomorphisms $f : M^n \to M^n$ of a closed topological n-manifold M^n, $n \geqslant 2$. If f is chaotic, we'll assume that $\omega(f)$ or $\alpha(f)$ consists of trivial basic sets.

In Section 2, we give examples of all types above of Smale A-homeomorphisms. Actually, all examples are A-diffeomorphisms.

Now let us introduce invariant sets that determine the dynamics of Smale homeomorphisms. Given any Smale A-homeomorphism $f : M^n \to M^n$, denote by $A(f)$ (resp., $R(f)$) the union of $\omega(f)$ (resp., $\alpha(f)$) and unstable (resp., stable) manifolds of saddle basic sets $\sigma(f)$:

$$A(f) = \omega(f) \bigcup_{\nu \in \sigma(f)} W^u(\nu), \quad R(f) = \alpha(f) \bigcup_{\nu \in \sigma(f)} W^s(\nu).$$

The following statement gives the necessary and sufficient conditions of conjugacy for three types of the Smale A-homeomorphisms. This statement is a generalization of the main result in [51].

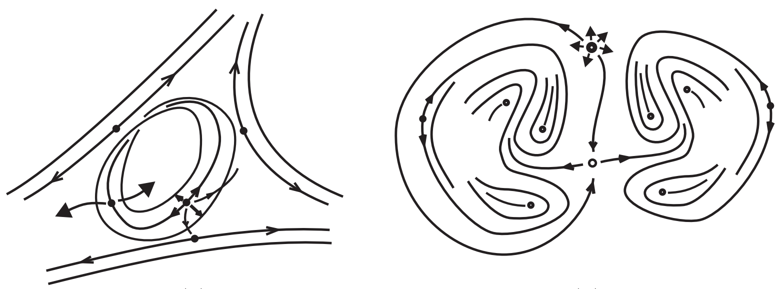
Theorem 1. *Let M^n be a closed topological n-manifold M^n, $n \geqslant 2$ and $f_i : M^n \to M^n$ is either a regular, or semichaotic, or chaotic Smale A-homeomorphism such that $\omega(f_i)$ or $\alpha(f_i)$ consists of trivial basic sets ($i = 1, 2$, respectively). Two homeomorphisms f_1 and f_2 are conjugate if and only if one of the following conditions holds:*

- *the basic sets $\alpha(f_1)$ and $\alpha(f_2)$ are trivial, while the sets $A(f_1)$ and $A(f_2)$ have the same dynamical embedding;*

- *the basic sets $\omega(f_1)$ and $\omega(f_2)$ are trivial, while the sets $R(f_1)$ and $R(f_2)$ have the same dynamical embedding.*

Note that the Smale A-homeomorphisms in Theorem 1 could be A-diffeomorphisms provided that M^n is a smooth manifold.

In Section 5, we apply Theorem 1 to consider the conjugacy for structurally stable surface diffeomorphisms $M^2 \to M^2$ with one-dimensional (orientable and nonorientable) attractors $\Lambda_1, \ldots, \Lambda_k$, $k \geqslant 2$, and to classify Morse–Smale diffeomorphisms with three periodic points on projective-like manifolds. Note that the one-dimensional attractors $\Lambda_1, \ldots, \Lambda_k$ are expanding (recall that an attractor Λ is expanding if its topological dimension equals $\dim W(x)$ for any point $x \in \Lambda$ [49]).



(a) (b)

Fig. 1. (a) One isolated saddle and one expanding attractor on a nonoriented surface; (b) one isolated saddle and two Plykin attractors.

First, we prove the following statement interesting in itself (remark that a structurally stable diffeomorphism is an A-diffeomorphism [29]).

Proposition 1. *Let $f : M^2 \to M^2$ be an A-diffeomorphism with the nonwandering set $NW(f)$ consisting of one-dimensional expanding attractors $\Lambda_1, \ldots, \Lambda_k$, and $s_0 \geqslant 0$ isolated saddle periodic points, and an arbitrary number of isolated nodal periodic orbits. Then $k \leqslant s_0 + 1$.*

The case $k = s_0 = 1$ is given in Fig. 1a, while the case $k = 2$ and $s_0 = 1$ is represented in Fig. 1b with two Plykin attractors.

The following statement shows that the dynamical embedding of unstable manifolds of isolated saddles (trivial basic sets) determines completely global dynamics of structurally stable surface diffeomorphisms with one-dimensional expanding attractors (nontrivial basic sets).

Theorem 2. *Let $f_i : M^2 \to M^2$, $i = 1, 2$, be a structurally stable diffeomorphism of a closed 2-manifold M^2 such that the spectral decomposition of f_i consists of $k \geqslant 2$ one-dimensional expanding attractors $\Lambda_1^{(i)}, \ldots, \Lambda_k^{(i)}$, and isolated source periodic orbits, and $k-1$ isolated saddle periodic points denoted by $\sigma_1^{(i)}, \ldots, \sigma_{k-1}^{(i)}$. Then f_1 and f_2 are conjugate if and only if the sets $\cup_{j=1}^{j=k-1} W^u(\sigma_j^{(1)})$, $\cup_{j=1}^{j=k-1} W^u(\sigma_j^{(2)})$ have the same dynamical embedding.*

Denote by $SRH(M^n)$ the class of Smale regular homeomorphisms $M^n \to M^n$. Note that it is possible that $f \in SRH(M^n)$ has the empty set $\sigma(f)$ of saddle periodic points. In this case the set $\alpha(f)$ consists of a unique source and the set $\omega(f)$ consists of a unique sink, and $M^n = S^n$ is the n-sphere. Later on, we'll assume that $f \in SRH(M^n)$ has a nonempty set $\sigma(f)$ of saddle periodic points.

Clearly, $SRH(M^n)$ contains all Morse–Smale diffeomorphisms provided that M^n admits a smooth structure. Note that the class $SRH(M^n)$ is an essential extension of the class of Morse–Smale diffeomorphisms because the diffeomorphisms from $SRH(M^n)$ can contain nonhyperbolic periodic points, tangencies, and separatrix connections, Fig. 2.

As a consequence of Theorem 1, one gets the following statement (in particular, one gets the necessary and sufficient conditions of conjugacy for any Morse–Smale diffeomorphisms on smooth closed manifolds).

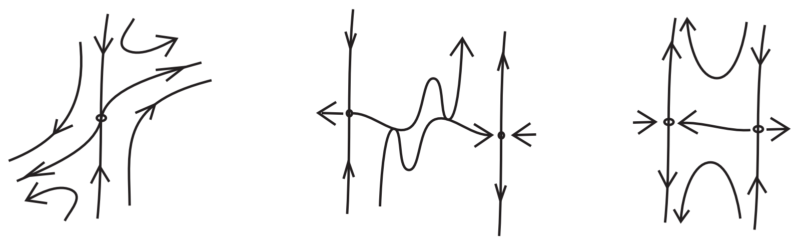


Fig. 2. Examples of regular Smale diffeomorphisms.

Corollary 1. *Let M^n be a closed topological n-manifold M^n, $n \geqslant 2$. Homeomorphisms f_1, $f_2 \in SRH(M^n)$ are conjugate if and only if one of the following conditions holds:*

- *the sets $A(f_1)$ and $A(f_2)$ have the same dynamical locally equivalent embedding;*
- *the sets $R(f_1)$ and $R(f_2)$ have the same dynamical locally equivalent embedding.*

Denote by $MS(M^n; a, b, c)$ the set of Morse – Smale diffeomorphisms $f : M^n \to M^n$ whose nonwandering set consists of a sinks, b sources, and c saddles. In [32], the authors proved that for $MS(M^n; 1, 1, 1)$ the only values of n possible are $n \in \{2, 4, 8, 16\}$. Moreover, the supporting manifolds for $MS(M^n; 1, 1, 1)$ are projective-like provided that $n \in \{2, 8, 16\}$ [32, 33].

First, to illustrate the applicability of Corollary 1, we consider a very simple class $MS(M^2; 1, 1, 1)$. In this case, the supporting manifold M^2 is the projective plane $M^2 = \mathbb{P}^2$ [32]. Below, we define a type for a unique saddle of $f \in MS(\mathbb{P}^2, 1, 1, 1)$. Using Corollary 1 we'll show how to get the following complete classification of Morse – Smale diffeomorphisms $MS(\mathbb{P}^2, 1, 1, 1)$.

Proposition 2. *Two diffeomorphisms f_1, $f_2 \in MS(\mathbb{P}^2, 1, 1, 1)$ are conjugate if and only if the types of their saddles coincide. There are four types T_i, $i = 1, 2, 3, 4$ of a saddle. Given any type T_i, $i = 1, 2, 3, 4$, there is a diffeomorphism $f \in MS(\mathbb{P}^2, 1, 1, 1)$ with a saddle $\sigma(f)$ of type T_i.*

Thus, up to conjugacy, there are four classes of Morse – Smale diffeomorphisms $MS(\mathbb{P}^2, 1, 1, 1)$.

The most essential application is a complete classification of Morse – Smale diffeomorphisms $MS(M^8; 1, 1, 1)$ and $MS(M^{16}; 1, 1, 1)$. The supporting $2k$-manifolds for diffeomorphisms from the set $MS^{2k}(1, 1, 1)$ will be denoted by $M^{2k}(1, 1, 1)$.

Remark that the manifolds $M^{2k}(1, 1, 1)$, $k = 1, 2, 4, 8$, are unique ones which admit Morse – Smale diffeomorphisms with the nonwandering set consisting of three fixed points [32]. Moreover, every set $M^{2k}(1, 1, 1)$, $k = 1, 2, 4, 8$, contains a smooth manifold supporting a Morse – Smale diffeomorphism from the set $MS^{2k}(1, 1, 1)$ [11, 14].

Recall that S^k is a k-sphere. Below, α_f, σ_f, and ω_f mean the source, the saddle, and the sink of $f \in MS^{2k}(1, 1, 1)$, respectively.

An embedding $\varphi : S^k \to M^{2k}(1, 1, 1)$ is called *basic* if

- $\varphi(S^k)$ is a locally flat k-sphere;
- $M^{2k}(1, 1, 1) \setminus \varphi(S^k)$ is an open $2k$-ball, $M^{2k}(1, 1, 1) = B^{2k} \sqcup \varphi(S^k)$.

It was proved in [32] that every supporting manifold $M^{2k}(1, 1, 1)$, $k = 4, 8$, admits a basic embedding.

Theorem 3. *Let $f : M^{2k}(1,1,1) \to M^{2k}(1,1,1)$ be a diffeomorphism from the set $MS^{2k}(1,1,1)$, $k = 4, 8$. Then the following claims hold:*

1) *for any $f \in MS^{2k}(1,1,1)$, there are basic embeddings*

$$\varphi_u(f) : S^k \to M^{2k}(1,1,1), \qquad \varphi_s(f) : S^k \to M^{2k}(1,1,1)$$

such that $\varphi_u(f)(S^k) = W^u_{\sigma_f} \cup \{\omega_f\}$ and $\varphi_s(f)(S^k) = W^s_{\sigma_f} \cup \{\alpha_f\}$;

2) *given any basic embedding $\varphi : S^k \to M^{2k}(1,1,1)$, there is $f \in MS^{2k}(1,1,1)$ such that one of the following equalities holds:*

$$\varphi(S^k) = W^u_{\sigma_f} \cup \{\omega_f\} \quad \text{or} \quad \varphi(S^k) = W^s_{\sigma_f} \cup \{\alpha_f\};$$

3) *two Morse – Smale diffeomorphisms $f_1, f_2 \in MS^{2k}(1,1,1)$ are conjugate if and only if one of the following conditions holds:*

- *the basic embeddings $\varphi_u(f_1)(S^k) = W^u_{\sigma_{f_1}} \cup \{\omega_{f_1}\}$, $\varphi_u(f_2)(S^k) = W^u_{\sigma_{f_2}} \cup \{\omega_{f_2}\}$ have the same dynamical embedding;*
- *the basic embeddings $\varphi_s(f_1)(S^k) = W^s_{\sigma_{f_1}} \cup \{\alpha_{f_1}\}$, $\varphi_s(f_2)(S^k) = W^s_{\sigma_{f_2}} \cup \{\alpha_{f_2}\}$ have the same dynamical embedding.*

Thus, every $f \in MS^{2k}(1,1,1)$ corresponds to the basic embedding $\varphi(f) : S^k \to M^{2k}(1,1,1)$. Given any basic embedding φ, there is $f \in MS^{2k}(1,1,1)$ such that $\varphi(f) = \varphi$. Finally, a dynamical embedding of basic embedding defines completely a conjugacy class in $MS^{2k}(1,1,1)$. We see that the set of basic embedding (up to isotopy) forms the admissible set of conjugacy invariants for the Morse – Smale diffeomorphisms $MS^{2k}(1,1,1)$, $k = 4, 8$. As to the class $MS^4(1,1,1)$, the existence of a realizable and effective conjugacy invariant is still an open problem.

The structure of the paper is as follows. In Section 3, we give some preliminary results. In Section 4, we prove Theorem 1. In Section 5, we prove Proposition 1, Theorems 2, 3, and Proposition 2.

2. EXAMPLES OF A-DIFFEOMORPHISMS

1) *Regular A-diffeomorphisms*. An obvious example of a regular A-diffeomorphism is a Morse – Smale diffeomorphism. Note that there are regular A-diffeomorphisms that do not belong to the set of Morse – Smale diffeomorphisms. For example, they can belong to the boundary of the set of Morse – Smale diffeomorphisms in the space of diffeomorphisms. There are regular A-diffeomorphisms which cannot be approximated by Morse – Smale diffeomorphisms [42].

2) *Semichaotic A-diffeomorphisms*. A good example of a semichaotic diffeomorphism is the so-called DA-diffeomorphism obtained from Anosov automorphism after Smale surgery [46], see Fig. 3.

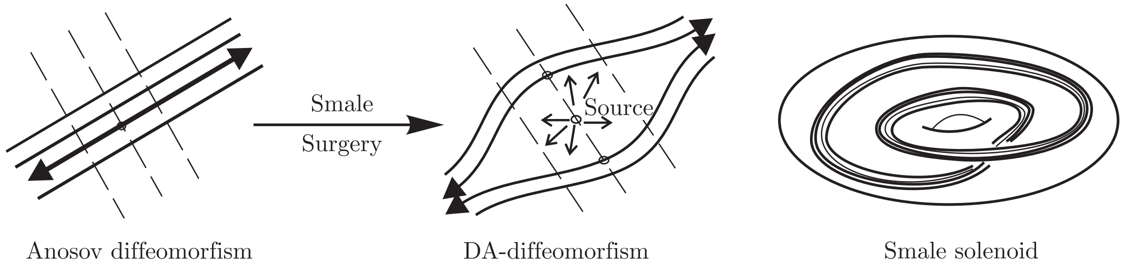


Fig. 3. Examples of semichaotic Smale diffeomorphisms.

A classical DA-diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ contains a nontrivial attractor $\omega(f)$, a trivial repeller $\alpha(f)$, and an empty set $\sigma(f)$. A generalized DA-diffeomorphism contains a nonempty set $\sigma(f)$ [17].

Taking f^{-1}, one gets other examples. One more example of the semichaotic A-diffeomorphism is a classical Smale horseshoe $g_s : \mathbb{S}^2 \to \mathbb{S}^2$. It is well known that there is g_s with trivial attractor $\omega(g_s)$ and repeller $\alpha(g_s)$, and nontrivial $\sigma(g_s)$.

Starting with DA-diffeomorphisms, Williams [48] constructed an open domain $\mathcal{N} \subset \mathrm{Diff}^1(\mathbb{T}^2)$ consisting of structurally unstable diffeomorphisms. It is easy to see that $\mathcal{N}$ contains semichaotic A-diffeomorphisms.

One more example of the semichaotic A-diffeomorphism is shown in Fig. 4 with a DA-attractor and Plykin attractor on a torus.

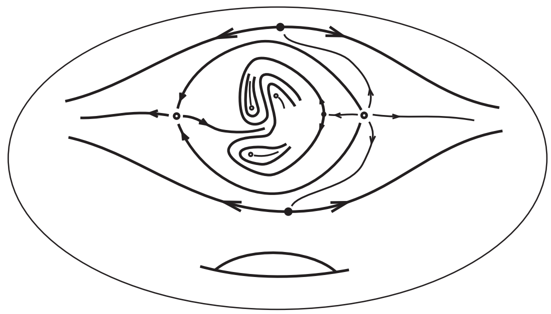


Fig. 4. One isolated saddle and two expanding attractors on a torus.

3) *Chaotic A-diffeomorphisms*. Take the classical DA-diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ with the nonwandering set consisting of a source α and one-dimensional expanding attractor Λ_a. The diffeomorphism f^{-1} defined on a copy $\mathbb{T}^2$ has the nonwandering set consisting of a sink ω and one-dimensional contracting repeller Λ_r. Let us delete a small neighborhood U_a (resp., U_s) of α (resp., ω) homeomorphic to a disk. Take an orientation reversing diffeomorphism $h : \partial U_a \to \partial U_r$. Then the surface $M^2 = \big(\mathbb{T}^2 \setminus U_a\big) \cup_h \big(\mathbb{T}^2 \setminus U_r\big)$ is a pretzel (closed orientable surface of genus 2). Following [44], one can construct an A-diffeomorphism $g : M^2 \to M^2$ with the nonwandering set consisting of $\Lambda_a \cup \Lambda_r$ such that $g|_{\Lambda_a} = f$ and $g|_{\Lambda_r} = f^{-1}$. Thus, $\alpha(g) = \Lambda_r$ and $\omega(g) = \Lambda_a$. Clearly, g is a chaotic A-diffeomorphism. Due to [44], there is a construction such that g has a closed simple curve consisting of the tangencies of the invariant stable manifolds of Λ_a and the invariant unstable manifolds of Λ_r.

One gets another example starting with a Smale solenoid [46], see Fig. 3. This mapping can be extended to an Ω-stable diffeomorphism $f_s : M^3 \to M^3$ with a one-dimensional expanding attractor, say Ω_1, and one-dimensional contracting repeller, say Ω_2, where M^3 is a 3-sphere or lens space [8, 25]. This chaotic diffeomorphism is similar to the Robinson-Williams diffeomorphism g considered above. There is a bifurcation of Ω_1 into a zero-dimensional saddle type basic set and isolated attracting periodic orbits [50]. As a result, one gets a chaotic Smale diffeomorphism $f_0 : M^3 \to M^3$ with trivial basic sets $\omega(f_0)$, and the nontrivial source basic set $\alpha(f_0) = \Omega_2$, and the nontrivial zero-dimensional saddle basic set $\sigma(f_0)$. Taking f^{-1}, one gets other examples.

4) *Superchaotic A-diffeomorphisms*. Let $g_s : \mathbb{S}^2 \to \mathbb{S}^2$ be the classical Smale horseshoe and $f : \mathbb{T}^2 \to \mathbb{T}^2$ the classical DA-diffeomorphism considered above. Delete small neighborhoods U_1, U_2 of the sink $\omega(g_s)$ and the source $\alpha(g_s)$, respectively, each homeomorphic to a disk. There are reversing orientation diffeomorphisms $h_1 : \partial U_1 \to \partial U_a$ and $h_2 : \partial U_2 \to \partial U_r$. Then the surface $M^2 = \big(\mathbb{T}^2 \setminus U_a\big) \bigcup_{h_1} \big(S^2 \setminus U_1 \cup U_2\big) \bigcup_{h_2} \big(\mathbb{T}^2 \setminus U_r\big)$ is a pretzel. Similarly to Robinson–Williams's method developed in [44], one can construct a diffeomorphism $g_0 : M^2 \to M^2$ with $\alpha(g_0) = \Lambda_r$, $\omega(g_0) = \Lambda_a$, and $\sigma(g_0)$ homeomorphic to the Smale horseshoe $\sigma(g_s)$. Thus, g_0 is a superchaotic A-diffeomorphism. In a similar way, one can get other examples starting with semichaotic A-diffeomorphisms.

Let us clarify the structure of the nonwandering set for a Smale A-homeomorphism.

Proposition 3. *Let $f : M^n \to M^n$ be a Smale homeomorphism, and $F : L^n \to L^n$ an A-diffeomorphism such that the nonwandering sets $NW(f)$, $NW(F)$ have the same dynamical embedding under a homeomorphism $h : NW(f) \to NW(F)$. Let Ω_1, …, Ω_k be the basic sets of F. Then $NW(f) = \Lambda_1 \cup \cdots \cup \Lambda_k$ where $\Lambda_i = h^{-1}(\Omega_i$, $i = 1, \ldots, k$, are pairwise disjoint closed f-invariant and transitive sets. Moreover,*

$$M^n = \bigcup_{i=1}^{k} W^s(\Lambda_i) = \bigcup_{i=1}^{k} W^u(\Lambda_i).$$

Proof. By definition, $h\bigl(NW(f)\bigr) = NW(F)$. Since h is a homeomorphism in some neighborhood of $NW(f)$, the sets $\Lambda_i = h^{-1}(\Omega_i$, $i = 1, \ldots, k$, are pairwise disjoint and closed. Due to the conjugacy relation

$$F \circ h|_{NW(f)} = h \circ f|_{NW(f)}$$

every Λ_i is f-invariant and transitive because every Ω_i is F-invariant and transitive.

Take a point $x \in M^n$. Since the limit set of the positive semi-orbit $\cup_{i\geqslant 0} f^i(x)$ belongs to $NW(f)$, $f^i(x) \to NW(f)$ as $i \to \infty$. Let δ be a neighborhood of $NW(f)$ with hyperbolic structure. To be precise, the relation $F \circ h|_\delta = h \circ f|_\delta$ holds in the neighborhood δ. According to [24] (see also [23]), $h(f^i(x))$ belongs to $W^s(y)$ for some $y \in \Omega_j$. Since h is a homeomorphism, $f^i(x) \in W^s(h^{-1}(y))$ for $h^{-1}(y) \in \Lambda_j$. Hence, $x \in \bigcup_{i=1}^k W^s(\Lambda_i)$ and $M^n = \bigcup_{i=1}^k W^s(\Lambda_i)$. Similarly, one can prove $M^n = \bigcup_{i=1}^k W^u(\Lambda_i)$. □

3. PROPERTIES OF SMALE HOMEOMORPHISMS

We begin by recalling several definitions. Further details may be found in [6, 7, 46]. Denote by $\mathrm{Orb}(x)$ the orbit of point $x \in M^n$ under a homeomorphism $f : M^n \to M^n$. The ω-limit set $\omega(x)$ of the point x consists of the points $y \in M^n$ such that $f^{k_i}(x) \to y$ for some sequence $k_i \to \infty$. Clearly, any points of $\mathrm{Orb}(x)$ have the same ω-limit. Replacing f with f^{-1}, one gets an α-limit set. Obviously, $\omega(x) \cup \alpha(x) \subset NW(f)$ for every $x \in M^n$.

Recall that we denote by $SsH(M^n)$ the set of either regular, or semichaotic, or chaotic Smale A-homeomorphisms $f : M^n \to M^n$ of the closed topological n-manifold M^n, $n \geqslant 2$. If f is chaotic, we'll assume that $\omega(f)$ or $\alpha(f)$ consists of trivial basic sets. Thus, $f \in SsH(M^n)$ satisfies the condition of Theorem 1.

Next, $f \in SsH(M^n)$. Given a family $C = \{c_1, \ldots, c_l\}$ of sets $c_i \subset M^n$, denote by $\widetilde{C}$ the union $c_1 \cup \ldots \cup c_l$. It follows immediately from the definitions that

$$NW(f) = \widetilde{\alpha(f)} \cup \widetilde{\omega(f)} \cup \widetilde{\sigma(f)}, \quad f \in SsH(M^n). \tag{3.1}$$

Lemma 1. *Let $f \in SsH(M^n)$ and $x \in M^n$. Then*

1) *if $\omega(x) \subset \widetilde{\sigma(f)}$, then $x \in W^s(\sigma_*)$ for some saddle basic set $\sigma_* \in \sigma(f)$.*

2) *if $\alpha(x) \subset \widetilde{\sigma(f)}$, then $x \in W^u(\sigma_*)$ for some saddle basic set $\sigma_* \in \sigma(f)$.*

Proof. Suppose that $\omega(x) \subset \widetilde{\sigma(f)}$. Since $\widetilde{\alpha(f)}$ and $\widetilde{\omega(f)}$ are invariant sets, $x \notin \widetilde{\alpha(f)} \cup \widetilde{\omega(f)}$. Therefore, there exist a neighborhood $U(\alpha)$ of $\alpha(f)$ and a neighborhood $U(\omega)$ of $\omega(f)$ such that the positive semi-orbit $\mathrm{Orb}^+(x)$ belongs to the compact set $N = M^n \setminus (U(\omega) \cup U(\alpha))$. Let $V(\sigma_1)$, …, $V(\sigma_m)$ be pairwise disjoint neighborhoods of saddle basic sets σ_1, … and σ_m, respectively, such that $\cup_{i=1}^m V(\sigma_i) \subset N$. Since every $V(\sigma_i)$ does not intersect $\cup_{j\neq i} V(\sigma_j)$ and all saddle basic sets are invariant, one can take the neighborhoods $V(\sigma_1)$, …, $V(\sigma_m)$ so small that every $f(V(\sigma_i))$ does not intersect $\cup_{j\neq i} V(\sigma_j)$. Suppose the contrary, i. e., there is no a unique saddle basic set $\sigma_* \in \sigma(f)$ with $x \in W^s(\sigma_*)$. Thus, there are at least two different saddle basic sets σ_1, σ_2 such that $x \in W^s(\sigma_1)$ and $x \in W^s(\sigma_2)$. Hence, $\omega(x)$ have to intersect σ_1, σ_2. It follows that the compact set $N_0 = N \setminus (\cup_{i=1}^m V(\sigma_i))$ contains infinitely many points of the semi-orbit $\mathrm{Orb}^+(x)$. This implies $\omega(x) \cap N_0 \neq \emptyset$ that contradicts (3.1). The second assertion is proved similarly. □

A set U is a *trapping region* for f if $f(\operatorname{clos} U) \subset \operatorname{int} U$. A set A is an *attracting set* for f if there exists a trapping set U such that

$$A = \bigcap_{k \geqslant 0} f^k(U).$$

A set A^* is a *repelling set* for f if there exists a trapping region U for f such that

$$A^* = \bigcap_{k \leqslant 0} f^k(M^n \setminus U).$$

In other words, A^* is an attracting set for f^{-1} with the trapping region $M^n \setminus U$ for f^{-1}. When we wish to emphasize the dependence of an attracting set A or a repelling set A^* on the trapping region U from which it arises, we denote it by A_U or A^*_U, respectively.

Let A be an attracting set for f. The *basin* $B(A)$ of A is the union of all open trapping regions U for f such that $A_U = A$. One can similarly define the notion of basin for a repelling set.

Let N be an attracting or repelling set and $B(N)$ the basin of N. A closed set $G(N) \subset B(N) \setminus N$ is called a *generating set* for the domain $B(N) \setminus N$ if

$$B(N) \setminus N = \cup_{k \in \mathbb{Z}} f^k\big(G(N)\big).$$

Moreover,

1) every orbit from $B(N) \setminus N$ intersects $G(N)$; 2) if an orbit from $B(N) \setminus N$ intersects the interior of $G(N)$, then this orbit intersects $G(N)$ at a unique point; 3) if an orbit from $B(N) \setminus N$ intersects the boundary of $G(N)$, then the intersection of this orbit with $G(N)$ consists of two points; 4) the boundary of $G(N)$ is the union of finitely many compact codimension-one topological submanifolds.

Lemma 2. *Let $f \in SsH(M^n)$.*

1) *Suppose that all basic sets $\alpha(f)$ are trivial. Then $\widetilde{\alpha(f)}$ is a repelling set, while $A(f)$ is an attracting set with*

$$B\left(\widetilde{\alpha(f)}\right) \setminus \widetilde{\alpha(f)} = B\big(A(f)\big) \setminus A(f).$$

Moreover,

- *there is a trapping region $T(\alpha)$ for f^{-1} of the set $\widetilde{\alpha(f)}$ consisting of pairwise disjoint open n-balls $b_1, \ldots, b_r$ such that each b_i contains a unique periodic point from $\alpha(f)$;*

- *the regions $B\big(\widetilde{\alpha(f)}\big) \setminus \widetilde{\alpha(f)}$, $B\big(A(f)\big) \setminus A(f)$ have the same generating set $G(\alpha)$ consisting of pairwise disjoint closed n-annuli $a_1, \ldots, a_r$ such that $a_i = \operatorname{clos} f^{p_i}(b_i) \setminus b_i$ where $p_i \in \mathbb{N}$ is a minimal period of a periodic point belonging to b_i, $i = 1, \ldots, r$:*

$$G(\alpha) = \cup_{i=1}^r a_i = \cup_{i=1}^r \big(\operatorname{clos} f^{p_i}(b_i) \setminus b_i\big);$$

- $B(A(f)) \setminus A(f) = \cup_{k \in \mathbb{Z}} f^k\big(G(\alpha)\big)$.

2) *Suppose that all basic sets $\omega(f)$ are trivial. Then $\widetilde{\omega(f)}$ is an attracting set, while $R(f)$ is a repelling set with*

$$B\big(\widetilde{\omega(f)}\big) \setminus \widetilde{\omega(f)} = B\big(R(f)\big) \setminus R(f).$$

Moreover,

- *there is a trapping region $T(\omega)$ for f of the set $\widetilde{\omega(f)}$ consisting of pairwise disjoint open n-balls $b_1, \ldots, b_l$ such that each b_i contains a unique periodic point from $\omega(f)$;*

- *the regions* $B\big(\widetilde{\omega(f)}\big) \setminus \widetilde{\omega(f)}$, $B\big(R(f)\big) \setminus R(f)$ *have the same generating set* $G(\omega)$ *consisting of pairwise disjoint closed* n*-annuli* $a_1, \ldots, a_l$ *such that* $a_i = b_i \setminus \mathrm{int}\, f^{p_i}(b_i)$ *where* $p_i \in \mathbb{N}$ *is a minimal period of a periodic point belonging to* b_i, $i = 1, \ldots, l$ *:*

$$G(\omega) = \cup_{i=1}^{r} a_i = \cup_{i=1}^{r} \big(b_i \setminus \mathrm{int}\, f^{p_i}(b_i)\big);$$

- $B\big(R(f)\big) \setminus R(f) = \cup_{k\in\mathbb{Z}} f^k\big(G(\omega)\big)$.

Proof. It is enough to prove the first statement only. Since all basic sets $\alpha(f)$ are trivial and consist of locally hyperbolic source periodic points, there is a trapping region $T(\alpha)$ for f^{-1} of the set $\widetilde{\alpha(f)}$ consisting of pairwise disjoint open n-balls $b_1, \ldots, b_r$ such that each b_i contains a unique periodic point q_i from $\alpha(f)$ [38, 45]. Thus,

$$T(\alpha) = \cup_{i=1}^{r} b_i, \quad \cap_{k\leqslant 0} f^{kp_i}(b_i) = q_i, \quad i = 1, \ldots, r.$$

As a consequence, there is the generating set $G(\alpha) = \cup_{i=1}^{r} \left(\mathrm{clos}\, f^{p_i}(b_i) \setminus b_i\right)$ consisting of pairwise disjoint closed n-annuli $a_i = \mathrm{clos}\, f^{p_i}(b_i) \setminus b_i$, $i = 1, \ldots, r$.

Since the balls $b_1, \ldots, b_r$ are pairwise disjoint and $\mathrm{clos}\, b_i \subset f^{p_i}(b_i)$, the balls $f^{p_1}(b_1), \ldots, f^{p_r}(b_r)$ are pairwise disjoint also. For simplicity of exposition, we'll assume that $\alpha(f)$ consists of fixed points (otherwise, $\alpha(f)$ is divided into periodic orbits each considered like a point). Therefore,

$$f\left(M^n \setminus \cup_{i=1}^{r} b_i\right) = M^n \setminus \cup_{i=1}^{r} f(b_i) \subset M^n \setminus \cup_{i=1}^{r} \mathrm{clos}\, b_i \subset \mathrm{int}\, \left(M^n \setminus \cup_{i=1}^{r} b_i\right).$$

Hence, $M^n \setminus \cup_{i=1}^{r} b_i$ is a trapping region for f. Clearly, $A(f) \subset M^n \setminus \cup_{i=1}^{r} b_i$.

Take a point $x \in M^n \setminus \cup_{i=1}^{r} b_i$. Obviously, $\omega(x) \notin \widetilde{\alpha(f)}$. It follows from (3.1) that $\omega(x) \in \widetilde{\omega(f)} \cup \widetilde{\sigma(f)}$. By Lemma 1, $\omega(x) \in A(f)$. Therefore, $A(f)$ is an attracting set with the trapping region $M^n \setminus \cup_{i=1}^{r} b_i$ for f :

$$A(f) = A_{M^n \setminus \cup_{i=1}^{r} b_i}.$$

Moreover,

$$M^n = \widetilde{\alpha(f)} \cup B\big(A(f)\big)$$

because $\cap_{k\leqslant 0} f^k(b_i) = q_i$, $i = 1, \ldots, r$.

Let us prove the quality $B\left(\widetilde{\alpha(f)}\right) \setminus \widetilde{\alpha(f)} = B\big(A(f)\big) \setminus A(f)$. Take $x \in B\left(\widetilde{\alpha(f)}\right) \setminus \widetilde{\alpha(f)}$. Since $x \notin \widetilde{\alpha(f)}$ and $M^n = \widetilde{\alpha(f)} \cup B\big(A(f)\big)$, $x \in B\big(A(f)\big)$. Since $x \in B\left(\widetilde{\alpha(f)}\right)$, $\alpha(x) \subset \alpha(f)$. Hence, $x \notin A(f)$ and $x \in B\big(A(f)\big) \setminus A(f)$. Now, set $x \in B\big(A(f)\big) \setminus A(f)$. Then $x \notin \alpha(f)$. Since $x \notin A(f)$, $\alpha(x) \subset \widetilde{\sigma(f)} \cup \widetilde{\alpha(f)}$. If one assumes that $\alpha(x) \subset \widetilde{\sigma(f)}$, then according to Lemma 1, $x \in W^u(\nu)$ for some saddle basic set ν. Thus, $x \in A(f)$ which contradicts $x \notin A(f)$. Therefore, $\alpha(x) \subset \widetilde{\alpha(f)}$. Hence, $x \in B\left(\widetilde{\alpha(f)}\right)$. As a consequence, $x \in B\left(\widetilde{\alpha(f)}\right) \setminus \widetilde{\alpha(f)}$.

The last assertion of the first statement follows from the previous ones. This completes the proof. $\square$

In the next statement, we keep the notation of Lemma 2.

Lemma 3. *Let* $f \in SsH(M^n)$.

1) *Suppose that all basic sets* $\alpha(f)$ *are trivial. Then, given any neighborhood* $V_0(A)$ *of* $A(f)$*, there is* $n_0 \in \mathbb{N}$ *such that*

$$\cup_{k\geqslant n_0} f^k\left(G(\alpha)\right) \subset V_0(A),$$

where $G(\alpha)$ *is the generating set of the region* $B\big(\widetilde{\alpha(f)}\big) \setminus \widetilde{\alpha(f)}$.

2) *A similar statement holds when all basic sets* $\omega(f)$ *are trivial.*

Proof. It is enough to prove the first statement only. Take a closed trapping neighborhood U of $A(f)$ for f. Since $\cap_{k\in\mathbb{N}} f^k(U) = A(f) \subset V_0(A)$, there is $k_0 \in \mathbb{N}$ such that $f^{k_0}(U) \subset V_0(A)$. Clearly, $f^{k_0}(U)$ is a tripping region of $A(f)$ for f. Hence, $f^{k_0+k}(U) \subset f^{k_0}(U) \subset V_0(A)$ for every $k \in \mathbb{N}$.

Let $G(\alpha)$ be a generating set of the region $B\big(\widetilde{\alpha(f)}\big) \setminus \widetilde{\alpha(f)}$. By Lemma 2, $G(\alpha)$ is the generating set of the region $B\big(A(f)\big) \setminus A(f)$ as well. Since $G(\alpha)$ is a compact set, there is $n_0 \in \mathbb{N}$ such that $f^{n_0}\left(G(\alpha)\right) \subset f^{k_0}(U)$. It follows that $f^{n_0+k}\big(G(\alpha)\big) \subset f^{k_0+k}(U) \subset f^{k_0}(U) \subset V_0(A)$ for every $k \in \mathbb{N}$. As a consequence, $\cup_{k\geqslant n_0} f^k\big(G(\alpha)\big) \subset V_0(A)$. □

4. PROOF OF THEOREM 1

Suppose that homeomorphisms f_1, $f_2 \in SsH(M^n)$ are conjugate. Since a conjugacy mapping $M^n \to M^n$ is a homeomorphism, the sets $A(f_1)$, $A(f_2)$, as well as the sets $R(f_1)$, $R(f_2)$ have the same dynamical embedding.

To prove the inverse assertion, let us suppose for definiteness that the basic sets $\alpha(f_1)$, $\alpha(f_2)$ are trivial, while the sets $A(f_1)$, $A(f_2)$ have the same dynamical embedding. Keeping in mind that $A(f_1)$ and $A(f_2)$ are attracting sets, we see that there are neighborhoods δ_1 and δ_2 of $A(f_1)$ and $A(f_2)$, respectively, and a homeomorphism $h_0 : \delta_1 \to \delta_2$ such that

$$h_0 \circ f_1|_{\delta_1} = f_2 \circ h_0|_{\delta_1}, \quad f_1(\delta_1) \subset \delta_1, \quad h_0\big(A(f_1)\big) = A(f_2). \tag{4.1}$$

Without loss of generality, one can assume that $\delta_1 \subset B(A(f_1))$. Moreover, taking δ_1 smaller if one needs, we can assume that $\operatorname{clos}\delta_1$ is a trapping region for f_1 of the set $A(f_1)$. By (4.1), one gets

$$f_2(\operatorname{clos}\delta_2) = f_2 \circ h_0(\operatorname{clos}\delta_1) = h_0 \circ f_1(\operatorname{clos}\delta_1) \subset h_0(\delta_1) = \delta_2.$$

Thus, $\operatorname{clos}\delta_2$ is a trapping region for f_2 of the set $A(f_2)$. As a consequence, we get the following generalization of (4.1):

$$h_0 \circ f_1^k|_{\delta_1} = f_2^k \circ h_0|_{\delta_1}, \quad k \in \mathbb{N}, \quad f_1(\operatorname{clos}\delta_1) \subset \delta_1, \quad h_0\big(A(f_1)\big) = A(f_2). \tag{4.2}$$

By Lemma 2, there is the trapping region $T(\alpha_1)$ for f_1^{-1} of the set $\widetilde{\alpha(f_1)}$ consisting of pairwise disjoint open n-balls $b_1, \ldots, b_{l_1}$ such that each b_i contains a unique periodic point q_i from $\alpha(f_1)$. In addition, the region $B\big(\widetilde{\alpha(f_1)}\big) \setminus \widetilde{\alpha(f_1)}$ has the generating set $G(\alpha_1)$ consisting of pairwise disjoint closed n-annuli $a_1, \ldots, a_{l_1}$ such that $a_i = \operatorname{clos} f_1^{p_i}(b_i) \setminus b_i$ where $p_i \in \mathbb{N}$ is a minimal period of the periodic point q_i.

Due to Lemma 3, one can assume without loss of generality that $G(\alpha_1) \stackrel{\text{def}}{=} G_1 \subset \delta_1$. Hence,

$$A(f_1) \bigcup \Big(\cup_{k\geqslant 0} f_1^k(G_1)\Big) = A(f_1) \bigcup N^+ \subset \delta_1, \quad N^+ = \cup_{k\geqslant 0} f_1^k(G_1).$$

According to Lemma 2, G_1 is a generating set of the region $B(A(f_1)) \setminus A(f_1)$. Let us show that $h_0(G_1) \stackrel{\text{def}}{=} G_2$ is a generating set for the region $B\big(A(f_2)\big) \setminus A(f_2)$. Take a point $z_2 \in G_2$. There is a unique point $z_1 \in G_1$ such that $h_0(z_1) = z_2$. Note that $z_2 \notin A(f_2)$ since $z_1 \notin A(f_1)$. Since $G_1 \subset \big(B\big(A(f_1)\big) \setminus A(f_1)\big)$, $f_1^k(z_1) \to A(f_1)$ as $k \to \infty$. It follows from (4.2) that

$$f_2^k(z_2) = f_2^k \circ h_0(z_1) = h_0 \circ f_1^k(z_1) \to h_0\big(A(f_1)\big) = A(f_2) \quad \text{as} \quad k \to \infty.$$

Hence, $z_2 \in B\big(A(f_2)\big)$ and $G_2 \subset B\big(A(f_2)\big) \setminus A(f_2)$.

Take an orbit $\mathrm{Orb}_{f_2} \subset B\big(A(f_2)\big) \setminus A(f_2)$. Since this orbit intersects a trapping region of $A(f_2)$, $\mathrm{Orb}_{f_2} \cap \delta_2 \neq \emptyset$. Therefore, there exists a point $x_2 \in \mathrm{Orb}_{f_2} \cap \delta_2$. Since $h_0\big(A(f_1)\big) = A(f_2)$ and $x_2 \in B\big(A(f_2)\big) \setminus A(f_2)$, the orbit Orb_{f_1} of the point $x_1 = h_0^{-1}(x_2) \subset \delta_1$ under f_1 belongs to $B\big(A(f_1)\big) \setminus A(f_1)$. Hence, Orb_{f_1} intersects G_1 at some point $w_1 \in \delta_1$. Since x_1, $w_1 \in \mathrm{Orb}_{f_1}$, there is $k \in \mathbb{N}$ such that either $x_1 = f_1^k(w_1)$ or $w_1 = f_1^k(x_1)$. Suppose for definiteness that $w_1 = f_1^k(x_1)$. Using (4.1), one gets

$$w_2 = h_0(w_1) = h_0 \circ f_1^k(x_1) = h_0 \circ f_1^k \circ h_0^{-1}(x_2) = f_2^k(x_2) \in G_2 \cap \mathrm{Orb}_{f_2}.$$

Similarly, one can prove that, if Orb_{f_2} intersects the interior of G_2, then Orb_{f_2} intersects G_2 at a unique point, and if Orb_{f_2} intersects the boundary of G_2, then Orb_{f_2} intersects G_2 at two points. Thus, G_2 is a generating set for the region $B\big(A(f_2)\big) \setminus A(f_2)$.

Set

$$\cup_{k\geqslant 0} f_i^{-k}(G_i) \overset{\text{def}}{=} O^-(G_i), \quad \cup_{k\geqslant 0} f_i^{k}(G_i) \overset{\text{def}}{=} O^+(G_i), \quad i = 1,2.$$

We see that $O^-(G_i) \cup O^+(G_i)$ is invariant under f_i, $i = 1,2$. Given any point $x \in O^-(G_1) \cup O^+(G_1)$, there is $m \in \mathbb{Z}$ such that $x \in f_1^{-m}(G_1)$. Let us define the mapping

$$h : O^-(G_1) \cup O^+(G_1) \to O^-(G_2) \cup O^+(G_2)$$

as follows:

$$h(x) = f_2^{-m} \circ h_0 \circ f_1^m(x), \quad \text{where} \quad x \in f_1^{-m}(G_1).$$

Since G_1 and G_2 are generating sets, h is correct. It is easy to check that

$$h \circ f_1|_{O^-(G_1)\cup O^+(G_1)} = f_2 \circ h|_{O^-(G_1)\cup O^+(G_1)}.$$

It follows from (4.1) that

$$h : A(f_1) \cup O^-(G_1) \cup O^+(G_1) \to A(f_2) \cup O^-(G_2) \cup O^+(G_2)$$

is the homeomorphic extension of h_0 putting $h|_{A(f_1)} = h_0|_{A(f_1)}$. Moreover,

$$h \circ f_1^k|_{A(f_1)\cup O^-(G_1)\cup O^+(G_1)} = f_2^k \circ h|_{A(f_1)\cup O^-(G_1)\cup O^+(G_1)}, \quad k \in \mathbb{Z}.$$

By Lemma 2, G_i is a generating set for the region $B\left(\widetilde{\alpha(f_i)}\right) \setminus \widetilde{\alpha(f_i)} = B\big(A(f_i)\big) \setminus A(f_i)$ and $B\big(A(f_i)\big) \setminus A(f_i) = \cup_{k\in\mathbb{Z}} f_i^k(G_i)$, $i = 1,2$. Thus, one gets the conjugacy $h : M^n \setminus \widetilde{\alpha(f_1)} \to M^n \setminus \widetilde{\alpha(f_2)}$ from $f_1|_{M^n\setminus\widetilde{\alpha(f_1)}}$ to $f_2|_{M^n\setminus\widetilde{\alpha(f_2)}}$:

$$h \circ f_1^k|_{M^n\setminus\widetilde{\alpha(f_1)}} = f_2^k \circ h|_{M^n\setminus\widetilde{\alpha(f_1)}}, \quad k \in \mathbb{Z}. \tag{4.3}$$

Recall that the sets $\alpha(f_1)$, $\alpha(f_2)$ are periodic sources $\{\alpha_j(f_1)\}_{j=1}^{l_1}$ and $\{\alpha_j(f_2)\}_{j=1}^{l_2}$, respectively. By Lemma 2, the generating set G_i consists of pairwise disjoint n-annuli $a_j(f_i)$, $i = 1,2$. Take an annulus $a_r(f_1) = a_r \subset G_1$ surrounding a source periodic point $\alpha_r(f_1)$ of minimal period p_r, $1 \leqslant r \leqslant l_1$. Then the set $\bigcup_{k\geqslant 0} f_1^{-kp_r}(a_r) \cup \{\alpha_r(f_1)\} = D_r^n$ is a closed n-ball. Since

$$M^n \setminus B\big(A(f_2)\big) = M^n \setminus \Big(A(f_2) \cup_{k\in\mathbb{Z}} f_2^k(G_2)\Big)$$

consists of the source periodic points $\alpha(f_2)$, the annulus

$$\bigcup_{k\geqslant 0} f_2^{-kp_r} \circ h(a_r) = \bigcup_{k\geqslant 0} h \circ f_1^{-kp_r}(a_r) = D_r^*$$

surrounds a unique source periodic point $\alpha_{j(r)}(f_2)$ of the same minimal period p_r. Moreover, $D_r^* \cup \{\alpha_{j(r)}(f_2)\}$ is a closed n-ball. Together with the existence of the homeomorphism $h : A(f_1) \cup O^-(G_1) \cup O^+(G_1) \to A(f_2) \cup O^-(G_2) \cup O^+(G_2)$, it implies the one-to-one correspondence $r \to j(r)$ inducing the one-to-one correspondence $j_0 : \big(\alpha_r(f_1)\big) \to \big(\alpha_{j(r)}(f_2)\big)$. Thus, $l_1 = l_2$. Since $\big(\alpha_r(f_1)\big)$ and $\big(\alpha_{j(r)}(f_2)\big)$ have the same period, one gets

$$j_0\left(f_1^k\big(\alpha_r(f_1)\big)\right) = f_2^k\left(j_0\big(\alpha_r(f_1)\big)\right) = f_2^k\left(\alpha_{j(r)}(f_2)\right), \quad 0 \leqslant k \leqslant p_r. \tag{4.4}$$

Put by definition, $h\big(\alpha_r(f_1)\big) = \alpha_{j(r)}(f_2)$. For sufficiently large $m \in \mathbb{N}$, both $f_1^{-mp_r}(D_r^n)$ and $f_2^{-mp_r}(D_r^*)$ can be embedded in arbitrary small neighborhoods of $\alpha_r(f_1))$ and $\big(\alpha_{j(r)}(f_2)\big)$, respectively, because $\widetilde{\alpha(f_1)}$ and $\widetilde{\alpha(f_2)}$ are repelling sets. Keeping in mind (4.4), it follows that $h : M^n \to M^n$ is a conjugacy from f_1 to f_2. This completes the proof. □

5. SOME APPLICATIONS

Following Smale [45, 46], we write $\sigma_1 \succ \sigma_2$ provided that $W^u(\sigma_1) \cap W^s(\sigma_2) \neq \emptyset$ where σ_1 and σ_2 are saddle periodic points. Next, we assume a surface M^2 to be closed and connected. Recall that a node is either a sink or a source.

Proof (*of Proposition* 1). is by induction on s_0. First, we consider the case $s_0 = 0$. We have to prove that $k = 1$. Suppose the contrary, namely, that $k \geqslant 2$. According to [13, 39] (see also [16, 17]), there are disjoint open sets U_i, $i = 1, \ldots, k$, such that each U_i is an attracting domain of Λ_i with no trivial basic sets. Moreover, the boundary ∂U_i consists of finitely many simple closed curves. Therefore, $M^2 \setminus \cup_{i=1}^{k} U_i$ is the disjoint union $\cup_{j\geqslant 1} K_j = G$ of compact connected sets K_j where $f^{-1}(G) \subset G$. Any iteration of f has at least k one-dimensional expanding attractors. Thus, without loss of generality, we can assume that $f^{-1}(K_j) \subset K_j$ for every K_j. In addition, we can assume that any periodic isolated point is fixed and the restriction of f on every invariant manifold of saddle isolated point preserves orientation.

Since $k \geqslant 2$ and M^2 is connected, there is a component of G, say K_1, and different sets U_l, U_r such that $\partial K_1 \cap \partial U_l \neq \emptyset$ and $\partial K_1 \cap \partial U_r \neq \emptyset$ where $U_l \cap U_r = \emptyset$. Any component of the boundary ∂K_1 is a circle. We see that there are at least two components of ∂K_1. Let us glue a disk to each boundary component of ∂K_1 to get a closed surface $\widetilde{K}_1$. Since $f^{-1}(K_1) \subset K_1$, one can extend $f|_{K_1}$ to an A-diffeomorphism $\widetilde{f} : \widetilde{K}_1 \to \widetilde{K}_1$ with a unique sink in each disk we glued. Note that, by construction, the nonwandering set $NW(\widetilde{f})$ of $\widetilde{f}$ consists of isolated nodal fixed points, and $NW(\widetilde{f})$ contains at least two sinks. According to [45], the surface $\widetilde{K}_1$ is the disjoint union of the stable manifolds of sinks and finitely many isolated sources (remark that the stable manifold of a source coincides with this source). This contradicts the connectedness of $\widetilde{K}_1$ because every stable manifold of a sink is homeomorphic to an open ball, and isolated sources do not separate the stable manifolds of two sinks. This contradiction proves that $k = 1$ provided that $s_0 = 0$.

Suppose the statement holds for $0, \ldots, s$ saddles. We have to prove this statement for $s_0 = s + 1$ saddles. Recall that, according to [45], the isolated saddles are endowed with the Smale partial order $\succ$. Since now the set of isolated saddles is not empty, there is a minimal saddle, say σ. Then the topological closure of $W^s(\sigma)$ is either a segment I with the endpoints being two sources or a circle S consisting of one source and $W^s(\sigma)$. In any of these cases, both I and S are repelling sets. Let us consider these cases.

The segment I has a neighborhood $U(I) = U$ homeomorphic to a disk such that $\operatorname{clos} U \subset f(U)$. Note that σ is inside of U. One can change f inside of U by replacing $\operatorname{clos} W^s(\sigma)$ with a unique source. One gets a diffeomorphism with k expanding attractors and s saddles. By the inductive assumption, $k \leqslant s + 1 \leqslant s_0 < s_0 + 1$.

Similarly, the circle S has a neighborhood $U(S)$ homeomorphic to an annulus such that $\operatorname{clos} U(S) \subset f\big(U(S)\big)$. Note that σ belongs to $U(S)$. The manifold $M_1^2 = M^2 \setminus U(S)$ has two boundary components M_1 and M_2, each homeomorphic to a circle. One can attach two disks D_1^2 and D_2^2 along their boundaries to M_1 and M_2, respectively, to get a closed surface $\tilde{M}^2$. This surface either is connected or consists of two connected surfaces. Since S is a repelling set, one can extend f on $\tilde{M}^2$ to get a diffeomorphism $\tilde{f} : \tilde{M}^2 \to \tilde{M}^2$ with k expanding attractors and s saddles. If $\tilde{M}^2$ is connected, then the inductive assumption implies $k \leqslant s + 1 \leqslant s_0$. Let us consider the case where $\tilde{M}^2$ consists of two connected closed surfaces $\tilde{M}_1^2$, $\tilde{M}_2^2$. Suppose that $\tilde{M}_i^2$ contains k_i expanding attractors and s_i isolated saddles, $i = 1, 2$. Obviously, $k = k_1 + k_2$ and $s_1 + s_2 = s$. By the inductive assumption, $k_i \leqslant s_i + 1$, $i = 1, 2$. Hence, $k \leqslant (s_1 + 1) + (s_2 + 1) = s_1 + s_2 + 2 = s + 2 = s_0 + 1$. This concludes the proof. □

Proof (*of Theorem* 2). Let us consider a structurally stable diffeomorphism $f : M^2 \to M^2$ with the nonwandering set consisting of $k \geqslant 2$ one-dimensional expanding attractors $\Lambda_1, \ldots, \Lambda_k$, and isolated source periodic orbits, and $k-1$ saddle periodic points $\sigma_1, \ldots, \sigma_{k-1}$. Each Λ_i has a neighborhood U_i that is an attracting region of Λ_i. Then $M^2 \setminus (\cup_{i=1}^{k-1} U_i)$ is the disjoint union $G = \cup_{j\geqslant 1} K_j$ of compact connected sets where $f^{-1}(G) \subset G$. Note that any positive iteration of f has at least k one-dimensional expanding attractors. Obviously, any iteration of f has the same number $k-1$ of saddle periodic points. Due to Proposition 1, any positive iteration of f has no more than k one-dimensional expanding attractors. Hence, any positive iteration of f has exactly the same number k of expanding attractors. This implies that every attractor Λ_i is C-dense [5, 43]. As a consequence, each unstable manifold $W^u(\cdot) \subset \Lambda_i$ is dense in Λ_i [5, 13].

Take a connected component K of the set G. The boundary ∂K is the disjoint union of circles $c_1, \ldots$. By construction, these circles belong to the boundaries of the attracting regions $U_1, \ldots, U_k$. Therefore, one can glue a disk d_j to each circle c_j extending f to d_j with a sink inside of d_j. If K is without isolated saddles, then $K \cup_j d - J$ is a 2-sphere with a unique source and a unique sink [14]. Therefore, if K is without isolated saddles, then K is a disk with a unique source. We will call such a set K a disk with no saddles. Now, take a neighborhood U of some Λ_i. Suppose that all components of the boundary ∂U are attached to components of G that are disks with no saddles. Then the union of U and these disks give a closed surface with exactly one expanding attractor Λ_i. This contradicts either the connectedness of M^2 or the inequality $k \geqslant 2$. Thus, given any neighborhood U_i of Λ_i, the boundary ∂U_i has a common part with the boundary ∂K_j of some component $K \subset G$ which contains at least one isolated saddle.

Let K be a component of G containing a saddle σ and U a neighborhood of some Λ_i such that $\partial K \cap \partial U \neq \emptyset$. Let us show that $W^u(\sigma) \cap W^s(\Lambda_i) \neq \emptyset$. Suppose the contrary. We know that $W^u(\sigma) \setminus \{\sigma\}$ belongs to stable manifolds of isolated periodic points lying in K. Then there is a saddle $\sigma_1 \in K$ such that $\sigma_1 \prec \sigma$, and the topological closure of $W^s(\sigma_1)$ is either a segment I with the endpoints being two sources or a circle S consisting of one source and $W^s(\sigma_1)$. In any case, both I and S are repelling sets. Therefore, f can be changed inside of K so that a diffeomorphism obtained has $k-2$ isolated saddles and k one-dimensional expanding attractors. This contradicts Proposition 1. Thus, $W^u(\sigma) \cap W^s(\Lambda_i) \neq \emptyset$.

Since f is a structurally stable diffeomorphism, all intersections $W^u(\sigma) \cap W^s(x)$, $x \in \Lambda_i$, are transversal. It follows from $W^u(\sigma) \cap W^s(\Lambda_i) \neq \emptyset$ that there is $x \in \Lambda_i$ such that $W^u(\sigma)$ intersects transversally the stable manifold $W^s(x)$. Recall that the attractor Λ_i is C-dense. Since any unstable manifold $W^u(\cdot) \subset \Lambda_i$ is dense in Λ_i, the topological closure of $W^u(\sigma)$ contains Λ_i, $\operatorname{clos} W^u(\sigma) \supset \Lambda_i$.

Clearly, if f_1 and f_2 are conjugate, then $\cup_{j=1}^{j=k-1} W^u(\sigma_j^{(1)})$, $\cup_{j=1}^{j=k-1} W^u(\sigma_j^{(2)})$ have the same dynamical embedding. Suppose that the sets $\cup_{j=1}^{j=k-1} W^u(\sigma_j^{(1)})$ and $\cup_{j=1}^{j=k-1} W^u(\sigma_j^{(2)})$ have the same dynamical embedding. It follows from above that

$$\operatorname{clos}\left(\cup_{j=1}^{j=k-1} W^u\big(\sigma_j^{(i)}\big)\right) \supset \cup_{j=1}^{j=k} \Lambda_j^{(i)}, \quad i = 1, 2.$$

Since $A(f_i) = \cup_{j=1}^{j=k-1} W^u\big(\sigma_j^{(i)}\big) \bigcup \left(\cup_{j=1}^{j=k} \Lambda_j^{(i)}\right)$, we see that

$$\operatorname{clos} A(f_1) = \operatorname{clos}\left(\cup_{j=1}^{j=k-1} W^u\big(\sigma_j^{(1)}\big)\right), \quad \operatorname{clos} A(f_2) = \operatorname{clos}\left(\cup_{j=1}^{j=k-1} W^u\big(\sigma_j^{(2)}\big)\right).$$

Therefore, the sets $A(f_1)$ and $A(f_2)$ have the same dynamical embedding. As a consequence of Theorem 1, we have that f_1 and f_2 are conjugate. This completes the proof. $\square$

Consider $f \in MS(\mathbb{P}^2, 1, 1, 1)$ with a unique saddle $\sigma(f)$. By definition, f is conjugate in some neighborhood of $\sigma(f)$ to a linear diffeomorphism with a saddle hyperbolic fixed point [41]. It easy to check that up to conjugacy there are exactly four such mappings :

$$T_1 = \begin{cases} \bar{x} = \dfrac{1}{2}x \\ \bar{y} = 2y, \end{cases} \qquad T_2 = \begin{cases} \bar{x} = -\dfrac{1}{2}x \\ \bar{y} = 2y, \end{cases} \qquad T_3 = \begin{cases} \bar{x} = \dfrac{1}{2}x \\ \bar{y} = -2y, \end{cases} \qquad T_4 = \begin{cases} \bar{x} = -\dfrac{1}{2}x \\ \bar{y} = -2y. \end{cases}$$

We'll say that the saddle $\sigma(f)$ is of type T_1, T_2, T_3, T_4, respectively, see Fig. 5.

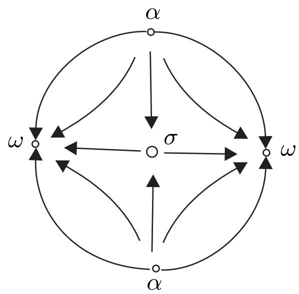


Fig. 5. Phase portrait for $f \in MS(\mathbb{P}^2, 1, 1, 1)$: the diametrically opposite points are identified.

Proof (*of Proposition* 2). Take $f \in MS(\mathbb{P}^2, 1, 1, 1)$ with a unique saddle $\sigma(f) = \sigma$. The attracting set $A(f)$ is a closed curve consisting of an unstable manifold $W^u(\sigma)$ of a unique saddle σ and a sink ω. A neighborhood U of $A(f)$ is homeomorphic to a Möbius band. Since U contains only two fixed points, the saddle σ and the sink ω, the dynamics of $f|_U$ depends completely on the local dynamics of f at σ which is defined by one of the types T_i, $i = 1, 2, 3, 4$. Due to Corollary 1, diffeomorphisms $f_1,\ f_2 \in MS(\mathbb{P}^2, 1, 1, 1)$ are conjugate if and only if the types of their saddles coincide.

Choose any type $T_i \in \{T_1, T_2, T_3, T_4\}$. Let B be a Möbius band with the middle closed curve c_0. There is a mapping $f_0 : B \to B$ with the attracting set c_0 such that the nonwandering set of f_0 consists of a hyperbolic sink $\omega \in c_0$ and a hyperbolic saddle $\sigma \in c_0$ with $W^u(\sigma) = c_0 \setminus \{\omega\}$. Note that the set $\mathbb{P}^2 \setminus B$ is a 2-disk D^2. Since c_0 is an attracting set, one can extend f_0 to f with a hyperbolic source in D^2. This gives $f \in MS(\mathbb{P}^2, 1, 1, 1)$, as desired. $\square$

Proof (*of Theorem* 3). 1) Since f has a unique saddle, both $W^u_{\sigma_f} \cup \{\omega_f\}$ and $W^s_{\sigma_f} \cup \{\alpha_f\}$ are topologically embedded spheres denoted by S^{k_1} and S^{k_2}, respectively. According to [32], $k_1 = k_2 = k$, and the complements $M^{2k}(1,1,1) \setminus \left(W^u_{\sigma_f} \cup \{\omega_f\}\right)$, $M^{2k}(1,1,1) \setminus \left(W^s_{\sigma_f} \cup \{\alpha_f\}\right)$ are homeomorphic to an open $2k$-ball (see also, [33]). Thus, we have the embedding

$$\varphi_u(f) : S^k \to W^u_{\sigma_f} \cup \{\omega_f\} \subset M^{2k}(1,1,1), \quad \varphi_s(f) : S^k \to W^s_{\sigma_f} \cup \{\alpha_f\} \subset M^{2k}(1,1,1).$$

Since the codimension of S^k equals $k \geqslant 4$, $\varphi_u(f)(S^k)$ and $\varphi_s(f)(S^k)$ are locally flat spheres [10]. Hence, $\varphi_u(f)$ and $\varphi_s(f)$ are basic embeddings.

2) According to the theorem of approximation by Haefliger [20], we can assume without loss of generality that $\varphi(S^k)$ is a smoothly embedded k-sphere. Hence, there is a tubular neighborhood T^{2k} of $\varphi(S^k)$ that is the total space of a locally trivial fiber bundle $p : T^{2k} \to \varphi(S^k)$ with the base $S^k_0 = \varphi(S^k)$ and a fiber k-disk D^k [22]. Let $\vartheta_{ns} : S_0 \to S_0$ be a Morse – Smale diffeomorphism with a unique sink ω_0 and a unique source N, the so-called "north-south" diffeomorphism. The fiber $p^{-1}(N)$ is an open k-disk. Let $\psi_N : p^{-1}(N) \to p^{-1}(N)$ be the mapping with a unique hyperbolic sink at N such that $\operatorname{clos} \psi_N(p^{-1}(N)) \subset \psi_N(p^{-1}(N))$ and $\cap_{j \geqslant 0} \psi^j_N\big(p^{-1}(N)\big) = \{N\}$. Since p is a locally trivial fiber bundle, one can extend ψ_N and ϑ_{ns} to get the mapping $f_0 : T^{2k} \to T^{2k}$ such that a) N is a hyperbolic saddle with k-dimensional local stable and unstable manifolds, and ω_0 is a hyperbolic sink; b) given any point $a \in T^{2k} \setminus p^{-1}(N)$, $f^l_0(a)$ tends to ω_0 as $l \to \infty$; moreover, $S_0 = \cap_{l \geqslant 0} f^l_0(T^{2k})$.

It was proved in [32] that the boundary ∂T^{2k} of T^{2k} is a $(2k-1)$-sphere, say S^{2k-1}. Moreover, S^{2k-1} bounds the ball $B^{2k} = M^{2k}(1,1,1) \setminus T^{2k}$. Take a point $a_0 \in B^{2k}$. Since B^{2k} is a ball, one can extend f_0 to B^{2k} to get a mapping $f : M^{2k}(1,1,1) \to M^{2k}(1,1,1)$ with a unique hyperbolic source at a_0. It follows from (a) and (b) that we get the desired Morse – Smale diffeomorphism $f \in MS^{2k}(1,1,1)$ with the sink $\omega_0 = \omega_f$, the saddle $N = \sigma_f$, and the source $a_0 = \alpha_f$.

3) The last statement immediately follows from Corollary 1. $\square$

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. Afraimovich, V. S. and Hsu, S.-B., *Lectures on Chaotic Dynamical Systems,* AMS/IP Stud. Adv. Math., vol. 28, Providence, R.I.: AMS, 2003.
2. Akin, E., *The General Topology of Dynamical Systems,* Grad. Stud. Math., vol. 1, Providence, R.I.: AMS, 1993.
3. Akin, E., Hurley, M., and Kennedy, J. A., Dynamics of Topologically Generic Homeomorphisms, *Mem. Amer. Math. Soc.,* 2003, vol. 164, no. 783, 138 pp.
4. Anosov, D. V., Geodesic Flows on Closed Riemannian Manifolds of Negative Curvature, *Proc. Steklov Inst. Math.,* 1967, vol. 90, pp. 1–235; see also: *Tr. Mat. Inst. Steklova,* 1967, vol. 90, pp. 3–210.
5. Anosov, D. V., On a Class of Invariant Sets of Smooth Dynamical Systems, in *Proc. of the 5th Internat. Conf. on Nonlinear Oscillations: Vol. 2,* Kiev: Math. Inst. Ukrainian Acad. Sci., 1970, pp. 39–45 (Russian).
6. Anosov, D. V. and Zhuzhoma, E. V., Nonlocal Asymptotic Behavior of Curves and Leaves of Laminations on Universal Coverings, *Proc. Steklov Inst. Math.,* 2005, no. 2(249), pp. 1–219; see also: *Tr. Mat. Inst. Steklova,* 2005, vol. 249, pp. 3–239.
7. Aranson, S. Kh., Belitsky, G. R., and Zhuzhoma, E. V., *Introduction to the Qualitative Theory of Dynamical Systems on Surfaces,* Transl. Math. Monogr., vol. 153, Providence, R.I.: AMS, 1996.
8. Jiang, B., Ni, Y., and Wang, Sh., 3-Manifolds That Admit Knotted Solenoids As Attractors, *Trans. Amer. Math. Soc.,* 2004, vol. 356, no. 11, pp. 4371–4382.
9. Bowen, R., Topological Entropy and Axiom A, in *Global Analysis: Proc. Sympos. Pure Math. (Berkeley, Calif., 1968): Vol. 14,* Providence, R.I.: AMS, 1970, pp. 23–41.
10. Daverman, R. J. and Venema, G. A., *Embeddings in Manifolds,* Grad. Stud. Math., vol. 106, Providence, R.I.: AMS, 2009.
11. Eells, J., Jr. and Kuiper, N. H., Manifolds Which Are Like Projective Planes, *Inst. Hautes Études Sci. Publ. Math.,* 1962, No. 14, pp. 5–46.
12. Franks, J., Anosov Diffeomorphisms, in *Global Analysis: Proc. Sympos. Pure Math. (Berkeley, Calif., 1968): Vol. 14,* Providence, R.I.: AMS, 1970, pp. 61–93.
13. Grines, V., Medvedev, T., and Pochinka, O., *Dynamical Systems on* 2- *and* 3-*Manifolds,* Dev. Math., vol. 46, New York: Springer, 2016.
14. Grines, V. Z., Gurevich, E. Ya., Zhuzhoma, E. V., and Pochinka, O. V., Classification of Morse – Smale Systems and the Topological Structure of Underlying Manifolds, *Russian Math. Surveys,* 2019, vol. 74, no. 1, pp. 37–110; see also: *Uspekhi Mat. Nauk,* 2019, vol. 74, no. 1(445), pp. 41–116.
15. Grines, V., Medvedev, T., Pochinka, O., and Zhuzhoma, E., On Heteroclinic Separators of Magnetic Fields in Electrically Conducting Fluids, *Phys. D,* 2015, vol. 294, pp. 1–5.
16. Grines, V. Z. and Zhuzhoma, E. V., The Topological Classification of Orientable Attractors on an n-Dimensional Torus, *Russian Math. Surveys,* 1979, vol. 34, no. 4, pp. 163–164; see also: *Uspekhi Mat. Nauk,* 1979, vol. 34, no. 4, pp. 185–186.
17. Grines, V. and Zhuzhoma, E., On Structurally Stable Diffeomorphisms with Codimension One Expanding Attractors, *Trans. Amer. Math. Soc.,* 2005, vol. 357, no. 2, pp. 617–667.
18. Grobman, D. M., Homeomorphism of Systems of Differential Equations, *Dokl. Akad. Nauk SSSR,* 1959, vol. 128, no. 5, pp. 880–881 (Russian).
19. Grobman, D. M., Topological Classification of Neighborhoods of a Singularity in n-Space, *Mat. Sb. (N. S.),* 1962, vol. 56(98), pp. 77–94 (Russian).

20. Haefliger, A., Plongements différentiable de variétés dans variétés, *Comment. Math. Helv.,* 1961/1962, vol. 36, pp. 47–82.
21. Hartman, Ph., On the Local Linearization of Differential Equations, *Proc. Amer. Math. Soc.,* 1963, vol. 14, no. 4, pp. 568–573.
22. Hirsch, M. W., *Differential Topology,* Grad. Texts in Math., vol. 33, New York: Springer, 1976.
23. Hirsch, M. W. and Pugh, C. C., Stable Manifolds and Hyperbolic Sets, in *Global Analysis: Proc. Sympos. Pure Math. (Berkeley, Calif., 1968): Vol. 14,* Providence, R.I.: AMS, 1970, pp. 133–163.
24. Hirsch, M. W., Pugh, C. C., and Shub, M., *Invariant Manifolds,* Lecture Notes in Math., vol. 583, New York: Springer, 1977.
25. Ma, J. and Yu, B., The Realization of Smale Solenoid Type Attractors in 3-Manifolds, *Topology Appl.,* 2007, vol. 154, no. 17, pp. 3021–3031.
26. Kuznetsov, S. P., Example of a Physical System with a Hyperbolic Attractor of the Smale – Williams Type, *Phys. Rev. Lett.,* 2005, vol. 95, no. 14, 144101, 4 pp.
27. Leontovich, E. A. and Mayer, A. G., On Trajectories Determining Qualitative Structure of Sphere Partition into Trajectories, *Dokl. Akad. Nauk SSSR,* 1937, vol. 14, no. 5, pp. 251–257 (Russian).
28. Leontovich, E. A. and Maier, A. G., On a Scheme Determining the Topological Structure of a Decomposition into Trajectories, *Dokl. Akad. Nauk SSSR,* 1955, vol. 103, no. 4, pp. 557–560 (Russian).
29. Mañé, R., A Proof of the C^1 Stability Conjecture, *Publ. Math. Inst. Hautes Études Sci.,* 1987, vol. 66, pp. 161–210.
30. Manning, A., There Are No New Anosov Diffeomorphisms on Tori, *Amer. J. Math.,* 1974, vol. 96, no. 3, pp. 422–429.
31. Zhuzhoma, E. V. and Medvedev, V. S., Global Dynamics of Morse – Smale Systems, *Proc. Steklov Inst. Math.,* 2008, vol. 261, no. 1, pp. 112–135; see also: *Tr. Mat. Inst. Steklova,* 2008, vol. 261, pp. 115–139.
32. Medvedev, V. S. and Zhuzhoma, E. V., Morse – Smale Systems with Few Non-Wandering Points, *Topology Appl.,* 2013, vol. 160, no. 3, pp. 498–507.
33. Zhuzhoma, E. V. and Medvedev, V. S., Continuous Morse – Smale Flows with Three Equilibrium Positions, *Sb. Math.,* 2016, vol. 207, no. 5, pp. 702–723; see also: *Mat. Sb.,* 2016, vol. 207, no. 5, pp. 69–92.
34. Medvedev, V. S. and Zhuzhoma, E. V., Two-Dimensional Attractors of A-Flows and Fibered Links on Three-Manifolds, *Nonlinearity,* 2022, vol. 35, no. 5, pp. 2192–2205.
35. Milnor, J., On Manifolds Homeomorphic to the 7-Sphere, *Ann. of Math. (2),* 1956, vol. 64, no. 2, pp. 399–405.
36. Newhouse, S., On Codimension One Anosov Diffeomorphisms, *Amer. J. Math.,* 1970, vol. 92, no. 3, pp. 761–770.
37. Nikolaev, I. and Zhuzhoma, E., *Flows on* 2*-Dimensional Manifolds: An Overview,* Lect. Notes in Math., vol. 1705, Berlin: Springer, 1999.
38. Palis, J., On Morse – Smale Dynamical Systems, *Topology,* 1968, vol. 8, no. 4, pp. 385–404.
39. Plykin, R. V., On the Geometry of Hyperbolic Attractors of Smooth Cascades, *Russian Math. Surveys,* 1984, vol. 39, no. 6, pp. 85–131; see also: *Uspekhi Mat. Nauk,* 1984, vol. 39, no. 6(240), pp. 75–113.
40. Poincaré, H., Sur les courbes définies par les équations différentielles: 4, *J. Math. Pures Appl.,* 1886, vol. 2, pp. 151–218.
41. Pugh, C. C., On a Theorem of P. Hartman, *Amer. J. Math.,* 1969, vol. 91, no. 2, pp. 363–367.
42. Pugh, C. C., Walker, R. B., and Wilson, F. W., Jr., On Morse – Smale Approximations: A Counterexample, *J. Differential Equations,* 1977, vol. 23, no. 1, pp. 173–182.
43. Robinson, C., *Dynamical Systems: Stability, Symbolic Dynamics, Chaos,* 2nd ed., Stud. Adv. Math., vol. 28, Boca Raton, Fla.: CRC, 1998.
44. Robinson, C. and Williams, R., Finite Stability Is Not Generic, in *Proc. of Symp. on Dynamical Systems (Brazil, 1971),* M. M. Peixoto (Ed.), New York: Acad. Press, 1973, pp. 451–462.
45. Smale, S., Morse Inequalities for a Dynamical System, *Bull. Amer. Math. Soc.,* 1960, vol. 66, pp. 43–49.
46. Smale, S., Differentiable Dynamical Systems, *Bull. Amer. Math. Soc.,* 1967, vol. 73, no. 6, pp. 747–817.
47. Strogatz, S., *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering,* 2nd ed., Boca Raton, Fla.: CRC, 2015.
48. Williams, R. F., The "DA" Maps of Smale and Structural Stability, in *Global Analysis: Proc. Sympos. Pure Math. (Berkeley, Calif., 1968): Vol. 14,* Providence, R.I.: AMS, 1970, pp. 329–334.
49. Williams, R. F., Expanding Attractors, *Inst. Hautes Études Sci. Publ. Math.,* 1974, No. 43, pp. 169–203.
50. Zhuzhoma, E. V. and Isaenkova, N. V., On Zero-Dimensional Solenoidal Basic Sets, *Sb. Math.,* 2011, vol. 202, no. 3–4, pp. 351–372; see also: *Mat. Sb.,* 2011, vol. 202, no. 3–4, pp. 47–68.
51. Zhuzhoma, E. V. and Medvedev, V. S., Necessary and Sufficient Conditions for the Conjugacy of Smale Regular Homeomorphisms, *Sb. Math.,* 2021, vol. 212, no. 1, pp. 57–69; see also: *Mat. Sb.,* 2021, vol. 212, no. 1, pp. 63–77.