Determination of the homotopy type of a Morse-Smale diffeomorphism on an orientable surface by a heteroclinic intersection

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Abstract

This paper is devoted to the study of homotopy types of orientation-preserving Morse-Smale diffeomorphisms on closed orientable surfaces. Since any Morse-Smale diffeomorphism has a finite set of periodic points, then, according to the Nielsen-Thurston classification, it is homotopic to either a periodic homeomorphism or an algebraically finite order homeomorphism. It follows from the results of V. Grines and A. Bezdenezhnykh that any gradient-like diffeomorphism is homotopic to a periodic homeomorphism. However, when the wandering set of a given diffeomorphism contains heteroclinic intersections, then the question of its homotopy type is remains open. In the present work, an algorithm for recognizing the homotopy type of a non-gradient-like Morse-Smale diffeomorphism by its heteroclinic intersection is proposed. The algorithm is based on the construction of a filtration for a diffeomorphism and calculation of the intersection index of saddle separatrices in the fundamental annuli of filtration elements. It is established that a Morse-Smale diffeomorphism is homotopic to a periodic homeomorphism if and only if the total intersection index over all homotopic annuli is equal to zero.

Keywords: Morse-Smale diffeomorphisms, Nielsen-Thurston theory, homotopy classes, homotopy types

1 Introduction and formulation of results

Let f be an orientation-preserving homeomorphism of a closed orientable surface S_p of genus $p \geq 1$. Denote by $\{f\}$ the set of all homeomorphisms of the surface S_p, that are homotopic to f (the *homotopy class* of homeomorphisms containing f).

According to the *Nielsen-Thurston classification* (see, for example, [11], [1] or [16, p. 284]), the set of all homotopy classes of homeomorphisms on S_p is represented as the union of four disjoint subsets T_1, T_2, T_3, T_4, distinguished by the following conditions:

1. if $\{f\} \in T_1$, then $\{f\}$ contains a periodic homeomorphism;
2. if $\{f\} \in T_2$, then $\{f\}$ contains a reducible non-periodic homeomorphism of algebraically finite order;
3. if $\{f\} \in T_3$, then $\{f\}$ contains a reducible homeomorphism that is not a periodic or homeomorphism of algebraically finite order;
4. if $\{f\} \in T_4$, then $\{f\}$ contains a pseudo-Anosov diffeomorphism.

Since the Nielsen-Thurston theory was originally described for surfaces of genus $p \geq 2$, it is worth mentioning the book [6], in which this theory is extended to the case of a torus and it is established that torus homemorphisms classes split only into three sets T_1, T_2 and T_4 (Theorem 13.1, p. 369).

The representatives of each homotopy class described above are called the *Thurston canonical forms*. S.Kh. Aranson and V.Z. Grines [1] conjectured that every homotopy class of a surface homeomorphism contains a structurally stable diffeomorphism. In the case when a structurally stable diffeomorphism has a finite non-wandering set, it is called a *Morse-Smale diffeomorphism*, the intersection points of its saddle separatrices are called *heteroclinic points*, a Morse-Smale diffeomorphism without heteroclinic points is called *gradient-like*.

For classes from the sets T_1 and T_2 the Aranson-Grines conjecture has been proved. Precisely, in each homotopy class from the set T_1 a gradient-like diffeomorphism was constructed and obtained a complete topological classification of such diffeomorphisms [3–5]. The realization of each homotopy class from the set T_2 by Morse-Smale diffeomorphisms with orientable heteroclinic intersections was proved in [8], in [9] a complete topological classification of such diffeomorphisms was obtained.

In paper [14] the existence in each homotopy class of T_4 of a structurally stable diffeomorphism whose non-wandering set consists of a finite number of source orbits and a unique one-dimensional attractor is announced. In [7] necessary and sufficient conditions for the topological conjugacy of two such diffeomorphisms are found.

By [1], any homeomorphism of homotopy type T_3 or T_4 has an infinite set of periodic points. Therefore, Morse-Smale diffeomorphisms can only belong to T_1 or T_2. Moreover, gradient-like diffeomorphisms are homotopic to some periodic homeomorphism [3–5], but the Morse-Smale diffeomorphism with heteroclinic intersections can belong to both T_1 and T_2.

In this paper, the authors propose an algorithm for recognizing that a given non-gradient-like diffeomorphism belongs to the Nielsen-Thurston set T_1 or T_2 by its heteroclinic intersection. Let's analyze this problem in more detail.

Let $f: S_p \to S_p$ be an orientation-preserving Morse-Smale diffeomorphism. Since $\{f\} \in T_i$ if and only if $\{f^k\} \in T_i$, $k \neq 0$ (see, for example, [18–20]), then, without loss of generality, we can assume that the non-wandering set of the diffeomorphism f consists of fixed points and f preserves orientation on their invariant manifolds. Then the set Ω_1 of saddle points of the diffeomorphism f can be ordered: $\sigma_1, \dots, \sigma_{k_1}$ due to the Smale relation [13]. Precisely,

$$\text{if } W^u_{\sigma_j} \cap W^s_{\sigma_i} \neq \emptyset, \text{ then } i < j.$$

Denote by Ω_0, Ω_2 the sets of sinks and sources of the diffeomorphism f, respectively, and by k_0, k_2 the number of points in these sets. Let

$$A_{f,0} = \Omega_0,\ A_{f,i} = \Omega_0 \cup \bigcup_{j=1}^{i} W^u_{\sigma_j},\ i = 1, \dots, k_1,$$

$$R_{f,i} = \Omega_2 \cup \bigcup_{j=i+1}^{k_1} W^s_{\sigma_j},\ i = 0, \dots, k_1 - 1,\ R_{f,k_1} = \Omega_2.$$

It follows from [12], [15] that each of the sets $A_{f,i}\,(R_{f,i})$ is an *attractor (repeller)*, that is it has a *trapping neighbourhood* $M_{f,i}\,(N_{f,i})$, which is a compact surface with boundary, such that

$$f(M_{f,i}) \subset \operatorname{int} M_{f,i},\quad \bigcap_{n\in\mathbb{N}} f^n(M_{f,i}) = A_{f,i}$$

$$\left(f^{-1}(N_{f,i}) \subset \operatorname{int} N_{f,i}, \bigcap_{n\in\mathbb{N}} f^{-n}(N_{f,i}) = R_{f,i}\right).$$

Moreover, the attractor $A_{f,i}$ and the repeller $R_{f,i}$ are *dual*, i.e. $N_{f,i} = S_p \setminus \operatorname{int} M_{f,i}$. Since $A_{f,0} \subset A_{f,1} \subset \dots \subset A_{f,k_1}$, then, by [12], the trapping neighbourhoods can be chosen so that:

- $M_{f,i} \subset f(M_{f,i+1})$, $i = 0, \dots, k_1 - 1$,
- $\partial M_{f,i}$ does not contain heteroclinic points,
- each connected component v of the set $K_{f,i} = M_{f,i} \setminus \operatorname{int} f(M_{f,i})$ is diffeomorphic to 2-annulus $[0,1] \times \mathbb{S}^1$ by some diffeomorphism $h_v : [0,1] \times \mathbb{S}^1 \to v$ such that $h_v^{-1} f h_v|_{\{0\}\times\mathbb{S}^1}(0,s) = (1,s)$, $s \in \mathbb{S}^1$ (see Fig. 1).

Let $a_v = h_v([0,1] \times \{0\})$, $b_v = h_v(\{0\} \times \mathbb{S}^1)$. Let us orient the curve a_v by the direction on the interval $[0,1]$ from 0 to 1. Then the space of orbits $\hat{v} = v/f$ is diffeomorphic to the two-dimensional torus and the natural projection $p_v : v \to \hat{v}$ induces on the torus $\hat{v}$ generators

$$\hat{a}_v = p_v(a_v),\ \hat{b}_v = p_v(b_v).$$

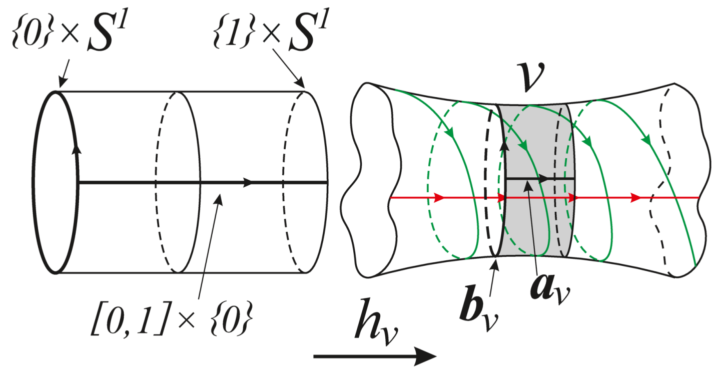

Fig. 1 Action of the diffeomorphism $h_v : [0,1] \times \mathbb{S}^1 \to v$.

Let v be a connected component of the set $K_{f,i}$ and $C_v = \bigcup\limits_{n\in\mathbb{Z}} f^n(v)$. Denote by $L^s_{f,v}$ $(L^u_{f,v})$ the union of stable (unstable) separatrices of saddles σ_j, $j \leqslant i\,(j > i)$, lying entirely in the set C_v (see Fig. 2).

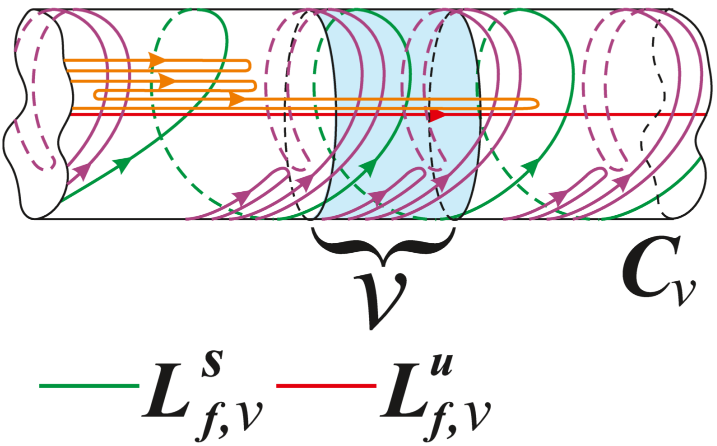

Fig. 2 An example of the set C_v, where the purple and orange separatrices do not belong to the sets $L^s_{f,v}$ and $L^u_{f,v}$ respectively.

Let $\hat{L}^s_{f,v} = p_v(L^s_{f,v}), \hat{L}^u_{f,v} = p_v(L^u_{f,v})$. Then each connected component of these sets is a knot on the torus $\hat{v}$, of homotopy type $\langle 1, d^s_{f,v}\rangle$, $\langle 1, d^u_{f,v}\rangle$ in the chosen generators $\hat{a}_v$, $\hat{b}_v$ (see Fig. 3). Let

$$\xi_{f,v} = d^s_{f,v} - d^u_{f,v}.$$

By [17, Chapter 2.C] the number $\xi_{f,v}$ does not depend on the choice of the basis $\hat{a}_v, \hat{b}_v$. A component v will be called a *heteroclinic annulus* if the set $L^u_{f,v}$ contains at least one unstable separatrix of the saddle σ_{i+1}, intersecting the

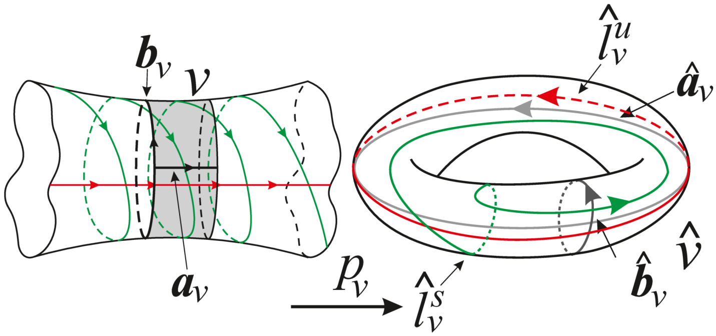

Fig. 3 The figure shows the location of knots $\hat{l}^s_v \in \hat{L}^s_{f,v}, \hat{l}^u_v \in \hat{L}^u_{f,v}$ on the torus $\hat{v}$.

set $L^s_{f,v}$. Note that there are at most two heteroclinic annuli in each set $K_{f,i}$. The heteroclinic annulus v will be called (see Fig. 4):

- *contractible*, if the curve b_v is homotopic to zero on the surface S_p;
- *trivial*, if $\xi_{f,v} = 0$;
- *essential*, if v is neither contractible nor trivial.

Denote by $\mathcal{V}_f$ the set of essential heteroclinic annuli v. Let us define *heteroclinic intersections index* ξ_f of the diffeomorphism f. If the set $\mathcal{V}_f$ is empty, then we set $\xi_f = 0$. Otherwise, we introduce the following equivalence relation on the set $\mathcal{V}_f$: components $v \subset K_{f,i}$, $v' \subset K_{f,i'}$ will be called *equivalent*, if the curves b_v, $b_{v'}$ are homotopic (see an example of a diffeomorphism containing annuli of different equality classes in Fig. 4). Denote by $[v]$ the equivalence class of the annulus v and by $[\mathcal{V}_f]$ the set of equivalence classes. Assuming that the curves b_v, $b_{v'}$ are consistently oriented for the equivalent annuli v, v', we set

$$\xi_{f,[v]} = \sum_{v \in [v]} \xi_{f,v}, \ \ \xi_f = \sum_{[v] \in [\mathcal{V}_f]} |\xi_{f,[v]}|.$$

Theorem 1 *Let f be an orientation-preserving Morse-Smale diffeomorphism on a surface. Then $\{f\} \in T_1$ ($\{f\} \in T_2$) $\Leftrightarrow \xi_f = 0$ ($\xi_f \neq 0$)*[1].

The figure 4 shows the phase portrait of the Morse-Smale diffeomorphism on a two-dimensional torus with six saddle points. It denotes all sets $K_{f,i}$, $i = 0, \dots, 6$, as well as all heteroclinic annuli v_i, $i = 0, \dots, 6$. In this example, the annulus v_1 is trivial, the annuli v_2, v_3, v_4 are essential, belong to the same equivalence class and the annuli v_5, v_6 are contractible.

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[1] For Morse-Smale diffeomorphisms with a finite set of heteroclinic orbits given on a two-dimensional torus, the theorem was proved in [10].

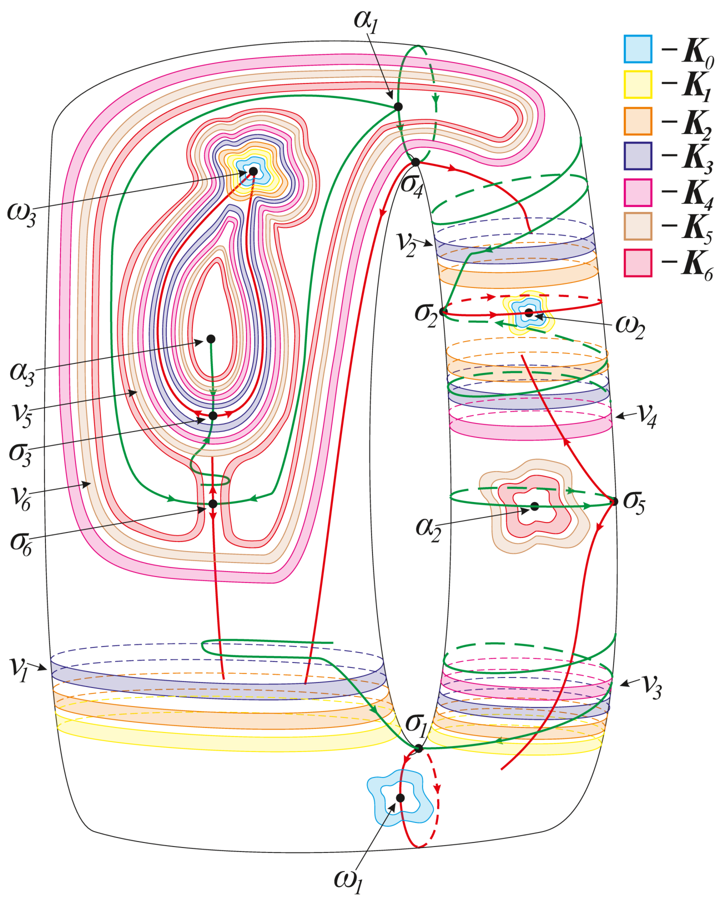

Fig. 4 Heteroclinic annuli of Morse-Smale diffeomorphism on a torus

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2 Morse-Smale diffeomorphisms without essential heteroclinic annuli

Denote by $MS(S_p)$ the set of orientation-preserving Morse-Smale diffeomorphisms $f : S_p \to S_p$. In this section, we will prove the following result.

Lemma 1 *If an orientation-preserving Morse-Smale diffeomorphism f on a surface has no essential heteroclinic annuli, then $\{f\} \in T_1$.*

The proof of the lemma 1 follows directly from the statements 1, 2, 3, proved below.

Statement 1 If a diffeomorphism $f \in MS(S_p)$ has no heteroclinic annuli, then it is gradient-like and, therefore, by [4], $\{f\} \in T_1$.

Proof Assuming the opposite, we find saddle points σ_k, σ_q, $k > q$ such that $W^u_{\sigma_k} \cap W^s_{\sigma_q} \neq \emptyset$ and does not exist saddle point σ_r such that $W^u_{\sigma_k} \cap W^s_{\sigma_r} \neq \emptyset$ and $W^u_{\sigma_r} \cap W^s_{\sigma_q} \neq \emptyset$. Let $k*$ be the smallest of such k. Then the set K_{k*-1} contains a connected component v such that C_v entirely contains the unstable separatrix of the saddle σ_k and the stable separatrix of the saddle σ_q. That is, v is a heteroclinic annulus, which contradicts the assumption. □

Statement 2 Let $f \in MS(S_p)$. Then there exists a diffeotopy $f_t : S_p \to S_p$ such that $f_0 = f$, $f_1 \in MS(S_p)$ and the diffeomorphism f_1 has no contractible heteroclinic annuli.

Proof Let v be a contractible heteroclinic annulus of the diffeomorphism f. Denote by $d_v \subset S_p$ the two-dimensional disc bounded by the curve b_v. Without loss of generality, we assume that $f(b_v) \subset d_v$ (otherwise, we can proceed to consider the diffeomorphism f^{-1}).

According to Alexander's lemma [17, Chapter 2.A, Lemma 5] the diffeomorphism $f|_{b_v} : b_v \to f(b_v)$ can be extended to the diffeomorphism $g : d_v \to f(d_v)$, which has a single fixed point, by the formula $g(tx) = tf(x)$ for $x \in b_v$. Let us set $h = f^{-1}g : d_v \to d_v$ and extend its by identity mapping to $S_p \setminus d_v$. Then, according to [17, Chapter 2.D, Exercise 6], there is an ambient isotopy $H_t : S_p \to S_p, t \in [0, 1]$ such that $H_0 = id$, $H_1 = h$, $H_t|_{S_p \setminus d_v} = id|_{S_p \setminus d_v}$. Thus, the diffeomorphism fH_1 is a Morse-Smale diffeomorphism diffeotopic to the diffeomorphism f and has fewer contractible heteroclinic annuli than the diffeomorphism f.

Repeating the arguments, we will get a diffeomorphism $f_1 \in MS(S_p)$, that has no contractible heteroclinic annuli. □

Let $f \in MS(S_p)$ and $v \subset K_{f,i}$ be its heteroclinic annulus. Let us consider the set $E_{f,v}$ of diffeomorphisms $g : S_p \to S_p$ such that $\Omega_f = \Omega_g$, g coincides with f in some neighborhood of the attractor $A_{f,i}$ and on the set $f(N_{f,i})$. Then the saddle separatrices of the diffeomorphisms f and g coincide in a neighborhood of the saddle points, which allows us to establish a one-to-one correspondence between them and the objects $L^u_{g,v}$, $L^s_{g,v}$, $\hat{L}^u_{g,v}$, $\hat{L}^s_{v}$, $d^u_{g,v}$, $d^s_{g,v}$, $\xi_{g,v}$ are similar to the objects of the diffeomorphism f.

Lemma 2 *Let $g \in E_{f,v}$ and $\xi_{g,v} = 0$. Then there is a diffeotopy $g_t : S_p \to S_p$ such that $g_t \in E_{f,v}$, $g_0 = g$ and $\hat{L}^s_{g_1,v} \cap \hat{L}^u_{g_1,v} = \emptyset$.*

Proof Note that the diffeomorphism $h_v : [0,1] \times \mathbb{S}^1 \to v$ can be extended to the diffeomorphism $h_v : \mathbb{R} \times \mathbb{S}^1 \to C_v$, conjugating the diffeomorphism $h_v^{-1} f h_v(r,s) = (r+1,s)$ with diffeomorphism f. Therefore, without loss of generality, we assume that $C_v = \mathbb{R} \times \mathbb{S}^1$, $f|_{C_v}(s,r) = (s, r+1)$, $v = [0,1] \times \mathbb{S}^1$ and $\hat{v} = \mathbb{T}^2$. Then the projection $p_v : v \to \hat{v}$ is the restriction of the projection $p : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{T}^2$, given by the formula $p(r,s) = (r \pmod 1, s)$. Without loss of generality, we assume that g coincides with f on the sets $U^- = (-\infty, 1) \times \mathbb{S}^1$ and $U^+ = [r, +\infty) \times \mathbb{S}^1$ for some $r > 0$.

Since the stable (unstable) manifolds of different periodic points of the diffeomorphism do not intersect, the set $\hat{L}^s_{g,v}$ ($\hat{L}^u_{g,v}$) consists of pairwise disjoint knots on the torus $\mathbb{T}^2$, that have the same homotopy types (see, for example, [17, Chapter C., exercise.7]). Since $\xi_{g,v} = 0$, then all knots of the sets $\hat{L}^s_{g,v}$ $\hat{L}^u_{g,v}$ have the same homotopy types. Moreover, if $\hat{L}^s_{g,v} \cap \hat{L}^u_{g,v} = \emptyset$, then $g_t = g$ and the lemma proven. Suppose $\hat{L}^s_{g,v} \cap \hat{L}^u_{g,v} \neq \emptyset$.

According to [17, Chapter 5 C, Theorem 8; Theorem 13] there exists a diffeomorphism $\hat{h} : \mathbb{T}^2 \to \mathbb{T}^2$ isotopic to the identity and such that $\hat{h}(\hat{L}^s_{g,v}) \cap \hat{L}^u_{g,v} = \emptyset$. We choose an open cover $D = \{\hat{D}_1, \dots, \hat{D}_q\}$ of the torus $\mathbb{T}^2$ by two-dimensional disks so that $p^{-1}(\hat{D}_j) \cap \{r_j, r_j+1\} \times \mathbb{S}^1 = \emptyset$ for some $r_j \in \mathbb{R}$. According to [2, Fragmentation lemma] there exist diffeomorphisms $\hat{w}_1, \dots, \hat{w}_q : \hat{v} \to \hat{v}$ that are smoothly isotopic to the identity and have the following properties:

i) for every $j \in \{1, \dots, q\}$ there is a smooth isotopy $\{\hat{w}_{j,t}\}$ that is identical outside $\hat{D}_j$ and connects identity map and $\hat{w}_j$;

ii) $\hat{h} = \hat{w}_1 \dots \hat{w}_q$.

Choose q disks $D_1, \dots, D_q$ one in each of the sets $p^{-1}(\hat{D}_1), \dots, p^{-1}(\hat{D}_q)$ so that $D_j \subset \tilde{v}_j$, with $\tilde{v}_j = (r_j, r_j+1) \times \mathbb{S}^1$, $r < r_1$ and $r_j + 1 < r_{j+1}$ for $j \in \{1, \dots, q-1\}$. Let $w_{j,t} : C_{\tilde{v}_j} \to C_{\tilde{v}_j}$ be a diffeomorphism that coincides with $(p|_{\tilde{v}_j})^{-1} \hat{w}_{j,t} p$ on $\tilde{v}_j$ and coincides with the identity map outside $\tilde{v}_j$. Let

$$g_t = g w^{-1}_{q,t} \dots w^{-1}_{1,t}.$$

By construction $g_t \subset E_{f,v}$ and $g_0 = g$. Let us show that $\hat{L}^u_{g_1,v} = \hat{L}^u_{g,v}$, $\hat{L}^s_{g_1,v} = \hat{h}(\hat{L}^s_{g,v})$.

Since the separatrices $L^u_{g_1,v}$ are obtained from the local separatrices $L^u_{g_1,v} \cap U^-$ by iteration due to the map g_1 and $L^u_{g_1,v} \cap U^- = L^u_{g,v} \cap U^-$, and $\hat{L}^u_{g_1,v} = \hat{L}^u_{g,v}$. We set $g_j = g w_q^{-1} \dots w_j^{-1}$, where $w_\iota = w_{\iota,1}$ for all $\iota \in \{1, \dots, q\}$. By construction $g_j \in E_{f,v}$, $g_j = g_{j+1} w_j^{-1}$ and $g_{q+1} = g$. Then, to prove that $\hat{L}^s_{g_1,v} = \hat{h}(\hat{L}^s_{g,v})$, it suffices to show that $\hat{L}^s_{g_j,v} = \hat{w}_j(\hat{L}^s_{g_{j+1},v})$.

Let $\tilde{U}^+ = (r_q + 1, \infty) \times \mathbb{S}^1$. We choose natural numbers k_j^-, k_j^+ so that $\tilde{v}_j^- = g_j^{-k_j^-}(\tilde{v}_j) \subset U^-$, $\tilde{v}_j^+ = g_j^{k_j^+}(\tilde{v}_j) \subset \tilde{U}^+$. Let $k_j = k_j^- + k_j^+$. Since the separatrices $L^s_{g_j,v}$ are obtained from the local separatrices $L^s_{g_{j+1},v} \cap \tilde{U}^+$ by iteration due to the map g_j^{-1} and $L^s_{g_j,v} \cap \tilde{U}^+ = L^s_{g_{j+1},v} \cap \tilde{U}^+$, then (see Fig. 5)

$$\hat{L}^s_{g_j,v} = p g_j^{-k_j}(L^s_{g_{j+1},v} \cap \tilde{v}_j^+) = p g_j^{-k_j^-} w_j g_{j+1}^{-k_j^+}(L^s_{g_{j+1},v} \cap \tilde{v}_j^+) =$$

$$= p g_j^{-k_j^-} w_j g_j^{k_j^-} g_{j+1}^{-k_j}(L^s_{g_{j+1},v} \cap \tilde{v}_j^+) = p g_j^{-k_j^-} w_j g_j^{k_j^-} (p|_{\tilde{v}_j^-})^{-1} p g_{j+1}^{-k_j}(L^s_{g_{j+1},v} \cap \tilde{v}_j^+) =$$

$$= p w_j (p|_{\tilde{v}_j})^{-1}(\hat{L}^s_{g_{j+1},v}) = \hat{w}_j(\hat{L}^s_{g_{j+1},v}).$$

□

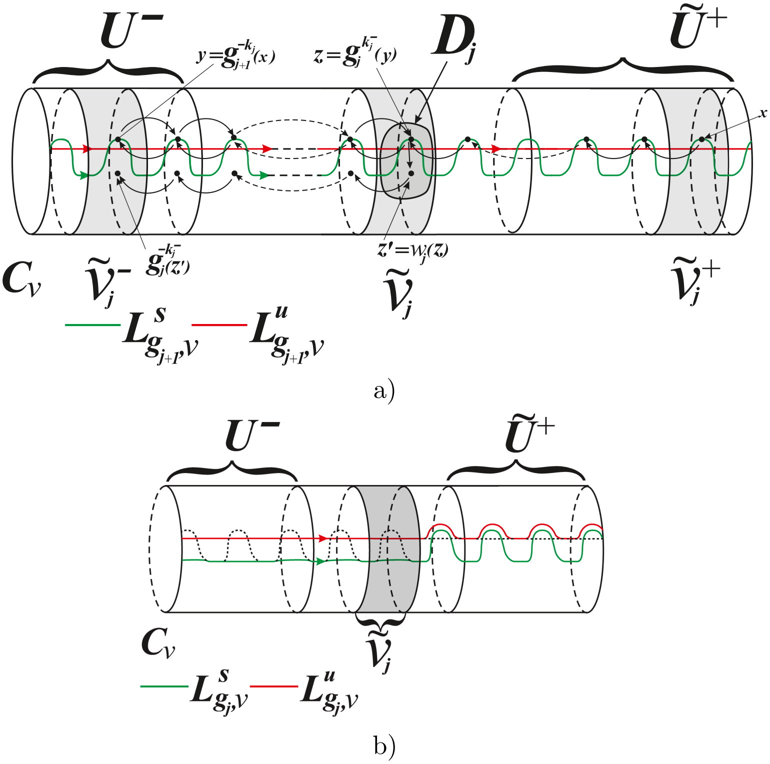

Fig. 5 Figure (a) illustrates step by step the formula:
$x \underset{g_{j+1}^{-k_j}}{\rightarrow} y \underset{g_j^{k_j^-}}{\rightarrow} z \underset{w_j}{\rightarrow} z' \underset{g_j^{-k_j^-}}{\rightarrow} g_j^{-k_j^-}(z')$. Figure b) shows separatrices $L^s_{g_j,v}, L^u_{g_j,v}$.

Statement 3 Let $f \in MS(S_p)$ and $\xi_{f,v} = 0$ for all heteroclinic annuli v of the diffeomorphism f. Then there exists a diffeotopy $f_t : S_p \to S_p$ such that $f_0 = f$, $f_1 \in MS(S_p)$ and the diffeomorphism f_1 has no heteroclinic annuli.

Proof We numerate heteroclinic annuli of the diffeomorphism f: $v_1, \dots, v_m$ so that $j_1 < j_2$, if $v_{j_1} \subset K_{f,i_{j_1}}$, $v_{j_2} \subset K_{f,i_{j_2}}$ and $i_{j_1} < i_{j_2}$. By the virtue of lemma 2 there exists a diffeotopy $g_t \in E_{f,v_1}$ such that $g_0 = f$ and $\hat{L}^s_{g_1,v_1} \cap \hat{L}^u_{g_1,v_1} = \emptyset$. Let us show that the diffeomorphism g_1 has no heteroclinic annuli $v \subset K_i$ for $i \leqslant i_1$ and $\xi_{g_1,v} = 0$ for any heteroclinic annuli v of the diffeomorphism g_1.

Indeed, the condition $\hat{L}^s_{g_1,v_1} \cap \hat{L}^u_{g_1,v_1} = \emptyset$ implies that the annulus v_1 is not a heteroclinic annulus of the diffeomorphism g_1. Since all unstable saddle manifolds of σ_i, $i < i_1$, belong to the set A_{i_1}, and the diffeomorphisms f and g_1 coincide in some neighborhood of the attractor A_i, then the diffeomorphism g_1 does not have new heteroclinic annuli $v \subset K_i$ for $i \leqslant i_1$.

Since $g_t \in E_{f,v_1}$, then the unstable separatrices of any saddle σ_k, $k > i_1$ of the diffeomorphisms f and g_1 coincide on the set $f(N_{f,i_1})$, and stable separatrices change continuously in this set. Then for any heteroclinic annulus $v \subset K_i$, $i > i_1$ of the diffeomorphism g_1 the knots $\hat{L}^u_{g_1,v_1}$ and $\hat{L}^u_{f,v_1}$ coincide, and the knots $\hat{L}^s_{g_1,v_1}$ and $\hat{L}^s_{f,v_1}$ are homotopic. So, we have $\xi_{g_1,v} = \xi_{f,v} = 0$.

Repeating the arguments for the diffeomorphism g_1 etc., we will get a Morse-Smale diffeomorphism without heteroclinic annuli, which completes the proof. □

3 Morse-Smale diffeomorphisms with essential heteroclinic annuli

Let us define the function $\nu : [0,1] \to [0,1]$ (see Fig. 6) by the formula:

$$\nu(t) = \begin{cases} 0, & t = 0, \\ \frac{1}{1+\exp\left(\frac{\frac{1}{2}-t}{t^2(t-1)^2}\right)}, & 0 < t < 1, \\ 1, & t = 1. \end{cases}$$

We define the Dehn twist $D : [0,1] \times \mathbb{S}^1 \to [0,1] \times \mathbb{S}^1$ on the annulus $[0,1] \times \mathbb{S}^1$

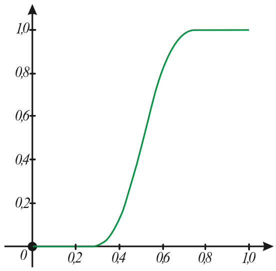

Fig. 6 Graph of the function $\nu : [0,1] \to [0,1]$.

by the formula

$$D(t, e^{i\phi}) = \left(t, e^{i(\phi + 2\pi\nu(t))}\right).$$

Let $v \subset K_{f,i}$ be the essential heteroclinic annulus of the diffeomorphism $f \in MS(S_p)$. We set $D_v = h_v D^{\xi_{f,v}} h_v^{-1} : v \to v$. Then the map $\hat{D}_v = p_v D_v p_v^{-1}$ induces the map $\hat{D}_{v*}$ in the fundamental group of the torus $\hat{v}$. The map $\hat{D}_{v*}$ is given by the matrix $\hat{D}_{v*} = \begin{pmatrix} 1 & \xi_{f,v} \\ 0 & 1 \end{pmatrix}$. Let us continue the diffeomorphism D_v with the identity map onto the ambient surface S_p and set $g_v = f D_v$. By construction $g_v \in E_{f,v}$.

Lemma 3 $\xi_{g_v,v} = 0$.

Proof The knots $\hat{L}^u_{f,v}$ and $\hat{L}^s_{f,v}$ of the diffeomorphism f have homotopy types $\langle 1, d^u_{f,v}\rangle$ and $\langle 1, d^s_{f,v}\rangle$ respectively on the torus $\hat{v}$. Since the global unstable manifold of the saddle point is obtained from local iteration by direct map and in a neighborhood $f(N_{f,i})$ separatrices $L^u_{g_v,v}$ and $L^u_{f,v}$ are coincide, then $L^u_{g_v,v} \cap v = L^u_{f,v} \cap v$. Since the global stable manifold of the saddle points is obtained from the local manifold due to reversed iteration and in the neighborhood of the attractor $A_{f,i}$ manifolds $L^s_{g_v,v}$ and $L^s_{f,v}$ are coincide, then $L^s_{g_v,v} \cap v = D_v^{-1}(L^s_{f,v} \cap v)$. Since the map $\hat{D}_v : \hat{v} \to \hat{v}$ acts on the fundamental group of the torus $\hat{v}$ by the matrix $\hat{D}^{-1}_{v*} = \begin{pmatrix} 1 & -\xi_{f,v} \\ 0 & 1 \end{pmatrix}$, then

$$\langle 1, d^s_{g_v,v} \rangle = \langle 1, d^s_{f,v} \rangle \begin{pmatrix} 1 & -\xi_{f,v} \\ 0 & 1 \end{pmatrix} = \langle 1, d^u_{f,v} \rangle$$

and $d^s_{g_v,v} = d^u_{f,v} = d^u_{g_v,v}$. Consequently $\xi_{g_v,v} = 0$. □

Proof of the theorem 1. We state the set $\mathcal{V}_f$ is not empty for the diffeomorphism $f \in MS(S_p)$. Due to the virtue of the statement 2, without loss of generality, we can assume that the diffeomorphism f does not contain contractible heteroclinic annuli. Let us numerate the essential heteroclinic annuli of the diffeomorphism f: $v_1, \dots, v_m$ so that $j_1 < j_2$, if $v_{j_1} \subset K_{f,i_{j_1}}$, $v_{j_2} \subset K_{f,i_{j_2}}$ and $i_{j_1} < i_{j_2}$. Let us consider the diffeomorphism $g_1 = fD_{v_1}$ and show that it has exactly one essential heteroclinic annulus v_1 less than the diffeomorphism f and the condition $\xi_{g_1,v_{\tilde{j}}} = \xi_{f,v_{\tilde{j}}}$, $\tilde{j} > 1$ holds.

Indeed, it follows from the lemma 2 that $\xi_{g_1,v_1} = 0$ and, consequently, v_1 is not an essential heteroclinic annulus for the diffeomorphism g_1. Let us show that the diffeomorphism g_1 has no new heteroclinic annuli. Assume the opposite: the diffeomorphism g_1 has a heteroclinic annulus v, that is distinct from the heteroclinic annulus of the diffeomorphism f. Then this annulus cannot contain heteroclinic points generated by the unstable separatrices of the saddles σ_i, $i < i_1$, since all unstable manifolds of such saddles belongs to A_{i_1}, and diffeomorphisms f and g_1 coincide in some neighborhood of the attractor A_i. Let the annulus v contain heteroclinic points generated by the unstable separatrices of the saddle σ_{k*}, $k_* > i_1 + 1$. Since the unstable separatrices of the saddle σ_{k*} are coincide on the set $f(N_{f,i})$ for both diffeomorphisms f and g_1, then, there exists a saddle point σ_{q*}, $q* \leqslant i_1$, whose stable separatrix l^s intersects with the domain C_{v_1}, intersects with the unstable separatrix l^u of the saddle σ_{k*} and entirely belongs to the domain C_v. Since $\xi_{f,v_1} \neq 0$, then the unstable separatrix of the saddle σ_{i_1+1}, which belongs to C_{v_1} intersects with each separatrix of the set $L^s_{v_1}$. Then there is no stable separatrix, which intersects with C_{v_1} and entirely belongs to C_v. Thus we have a contradiction. The same reasoning implies that $\xi_{g_1,v_{\tilde{j}}} = \xi_{f,v_{\tilde{j}}}$, $\tilde{j} > 1$.

Repeating the arguments consequentially for each heteroclinic annulus, we will get the diffeomorphism $g_m = fD_{v_1} \dots D_{v_m} \in MS(S_p)$ such that $\xi_{g_m,v} = 0$ for each heteroclinic annulus v of the diffeomorphism g_m. By the lemma 1 the diffeomorphism g_m is diffeotopic to the identity map. Then the diffeomorphism f is diffeotopic to the composition of Dehn twists and, since heteroclinic annuli are pairwise disjoint, it is isotopic to the identity map if and only if $\xi_f = 0$. □

Discussion

The questions of realization of a structurally stable representative in each homotopy class of the third T_3 and fourth T_4 Nielsen-Thurston types remain open. Also of interest is the question of determining the homotopy class of a diffeomophrism having a non-trivial basis set, possibly by a heteroclinic intersection.

Data availability statement. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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