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On periodically modulated rolls in the generalized Swift–Hohenberg equation: Galerkin' approximations

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ABSTRACT

We study in this paper the existence of periodically modulated in one variable and localized in another variable solutions to the cubic Swift–Hohenberg equation on the plane $\mathbb{R}^2$. In the first part we try to apply the method by Kirschgässner–Mielke to reduce the problem to the search of finite dimensional submanifolds with periodic orbits on them in some formal infinite-dimensional dynamical system generated by the stationary SH equation. It turns out that it cannot be done immediately due to properties of the spectrum for the linearized system at the localized roll (a one-dimensional pulse). In the second part we change roles of variables and formulate the problem as funding homoclinic orbits to an equilibrium of the formal infinite-dimensional system in the space of periodic functions in variable y. Staying apart the proof of the exact theorem on the existence of related center manifold, we exploit the Bubnov–Galerkin method to derive the Hamiltonian finite-dimensional ODEs with four or six degrees of freedom having the saddle type equilibrium whose homoclinic orbits correspond to approximate solution on needed type. The search for homoclinic orbits is performed by means of numerical methods.

1. Introduction

Nontrivial nonlinear patterns are ubiquitous in various physical systems. When studying PDEs on extended domains, for instance, on the whole plane, localized rolls are simplest non one-dimensional structures observed in many experiments including fluid currents, optical wave-guides, and so forth. Our aim here is to discuss the existence of periodically modulated localized rolls being, in a sense, the next candidate in complexity, after localized pulses, among observed regular patterns. As a typical pattern-forming model, we have chosen for the study the generalized Swift–Hohenberg equation being the variational type equation when considered on the whole plane $\mathbb{R}^2$ under proper conditions at infinity. Here by the generalized SH equation we understand the equation with the quadratic term that was added by Haken [1] into the original SH equation [2] for modeling the threshold character of the appearance of the Eckhaus instability. This equation is called sometimes as quadratic–cubic SH equation, other variants of SH equation are also widely investigated, for instance, the cubic–quintic equation [3,4] and also non-variational equations [5].

It is worth mentioning that PDEs with high powers of space derivatives are of great interest also for the soliton problem, because they allow non-one-dimensional stable static soliton solutions with finite energy. For standard equations, quadratic over gradients, such solitons are unstable against collapse that is the subject of so-called Hobbart–Derrick theorem [6,7]. Many examples are known in various models of field theory and condensed matter physics. Note first the so-called Skyrme model equation for meson field with stable three-dimensional (3D) topological solitons treated as hadrons [8,9]. Non-one-dimensional time-independent solitons appear for some standard non-linear equations like non-linear Schrödinger equation [10] and Landau–Lifshits equation [11], generalized by addition of fourth powers of gradients, whereas without such terms stable solitons are absent.

So, we focus on the Swift–Hohenberg equation with the cubic nonlinearity

$$u_t = \alpha u + \beta u^2 - u^3 - (1+\Delta)^2 u, \;\; \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{1}$$

that has been derived [2] in physics of convection near its Maxwell point but was proved to be a typical example for the pattern formation in different areas: Rayleigh–Bénard convection [2,12], nonlinear optics [13,14], granular media [15], chemical reactions [16], liquid crystals and solidification, see [17] and references there. Eq. (1) is of gradient type when studying it on

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bounded domains or on the whole space $\mathbb{R}^2$ with proper decay conditions at infinity and can be written in the form

$$\frac{\partial u}{\partial t} = -\frac{\delta \mathcal{F}}{\delta u}$$

with the functional $\mathcal{F}$ given as

$$\mathcal{F} = -\int_{\mathbb{R}^2} dxdy \left(\frac{1}{2}[(1+\Delta)u]^2 - \frac{\alpha}{2}u^2 - \frac{b}{3}u^3 + \frac{1}{4}u^4 \right).$$

Due to this form of the equation, its stationary (not depending on t) solutions are of the first importance. So, we come to our main object in this paper, i.e. the stationary generalized Swift–Hohenberg equation (SH equation, for brevity) on the plane $\mathbb{R}^2$ with coordinates (x, y)

$$(1+\Delta)^2 u = \alpha u + \beta u^2 - u^3. \quad (2)$$

It is well known that SH equation possesses different one-dimensional patterns as $x \in \mathbb{R}$: periodic, localized, i.e. pulses and kinks, and many others [3,18–24]. The temporally dependent equation (1) can also possess moving kinks in the form of traveling wave $u(x - ct)$ [20].

Multidimensional patterns of various structure were also found [3,5,25]. Among them radial localized solutions [23,24, 26,27], being the simplest two-dimensional patterns, were investigated in many details including the snaking phenomena for their bifurcation diagrams [4]. Also, localized patterns with the hexagonal structure were investigated [3], quasi-patterns based on interaction of two hexagonal lattices [21], etc. (see, the overview [28]). Recently other very interesting structure were discovered in numerical simulations [29].

For the stationary equation on the real line a pulse corresponds to a solution $u(x)$ that decays to the existing homogeneous state $u = 0$ as $|x| \to \infty$. The same solution, considered for the planar equation, when $(x, y) \in \mathbb{R}^2$, looks like a localized roll (or a ridge) extended in the variable y, but without any structure in y. The question about existence of such pulse type solutions, when $x \in \mathbb{R}$, is reduced to a more or less standard problem of the theory of smooth dynamical systems. Indeed, as is known (see [19,20,22]) the equation on the spatial domain $\mathbb{R}$ takes the form of the Euler–Lagrange–Poisson type equation [30]. Such equation can be transformed to a Hamiltonian system with two degrees of freedom after the change of variables $u = q_1$, $u' = q_2$, $-(u' + u''') = p_1$, $u + u'' = p_2$, here x plays the role of the temporal variable, then the Hamiltonian is $H = p_1 q_2 - p_2 q_1 + p^2/2 + \alpha q_1^2/2 + \beta q_1^3/3 - q_1^4/4$. It is worth remarking that the Hamiltonian system is, in addition, reversible with respect to the involution $\sigma : (q_1, q_2, p_1, p_2) \to (q_1, -q_2, -p_1, p_2)$. This implies that a solution of the system is transformed to another its solution (or the same solution), if one applies the involution and the change $x \to -x$.

This Hamiltonian system has the equilibrium at the origin $O = (0, 0, 0, 0)$ whose type depends on parameters α, β, but for negative α, that is assumed later on, the equilibrium is a saddle-focus (its eigenvalues are a complex quadruple $\pm\rho \pm i\sigma$, $\rho\sigma \neq 0$). In the phase space $\mathbb{R}^4$ such equilibrium has two smooth two-dimensional invariant manifolds passing through O, stable W^s and unstable W^u, which contain all orbits of the system tending O, as $x \to \infty$ (for W^s), and $x \to -\infty$ (for W^u). The existence of a pulse, a solution $u(x)$ that decays to zero as $|x| \to \infty$, is reformulated as the existence of a homoclinic orbit of the equilibrium O. Existence of such solutions was proved first in [19] via studying a bifurcation that occurs in the system, when α passes through zero (this is the so-called Hamiltonian Hopf bifurcation [31]). Similar results were obtained in [22] using reversibility and another normal form derived in [32]. In this system the bifurcation is accompanied by the creation of homoclinic orbits of the saddle-focus, if $|\beta| > \sqrt{27/38}$ [19,33]. As a consequence of the reversibility, two small (of the order $\sqrt{-\alpha}$) symmetric homoclinic orbits are born from the equilibrium [32]. As was discovered in [34], the intersections of W^s and W^u along both homoclinic orbits in the level $H = H(O)$, where they lie, are transversal. This implies, in particular, the existence of infinitely many multi-pulse homoclinic orbits of O, infinitely many saddle periodic orbits near the primary homoclinics and a complicated nearby orbit structure [35–38]. Because the homoclinic orbits are transverse, unstable manifold of the saddle-focus O intersects the stable manifold of some close saddle periodic orbit γ (in fact, infinitely many of them, see [36–38]), forming thereby a heteroclinic connection. Due to reversibility, a symmetric connection is formed by the intersection of the stable manifold of O and the unstable manifold of γ, therefore a heteroclinic contour is formed, this is a base of the snaking bifurcation diagrams described in several papers [4,39–41].

It is worth to remark that homoclinic orbits to O do not always exist in the one-dimensional SH equation [42], but only for parameters above some parabola-like curve in the parameter plane (α, β). Therefore, to find such orbits and, as a consequence, a heteroclinic contour rigorously, one should find its source. This is the codimension two point $(\alpha, \beta) = (0, \sqrt{27/38})$, where the Hamiltonian system has a doubly degenerate equilibrium O. As a partial structure of the phase portrait, investigated in detail near this equilibrium [19] for the truncated integrable normal form of sixth order (the least order in this case), the merging of the unstable manifold of the saddle-focus O and the stable manifold of a saddle periodic orbit γ lying in the same level of the Hamiltonian was proved (see Fig. 3(2) in [19]). In fact, in the full system for parameters (α, β), close to the codimension two point, this is not merging but they split but the splitting is exponentially small [39,40] and the existence of such heteroclinic connection is the source of snaking in this model, as in many other models. After this contour can be continued in parameters farther from the initial point using numerical methods.

The primary homoclinic orbits, which appear at the Hamiltonian Hopf bifurcation, are continued in parameters (α, β), and if one considers the temporally dependent equation (1), the related localized solutions can become temporally stable [42], though initially, just after their appearance at $\alpha \simeq -0$, they are temporally unstable [43]. Of course, finding such non-small homoclinic solutions requires drawing numerical methods to search them and verifying their temporal stability [42]. The form of the curve, obtained by a continuation of such homoclinic orbits, on the plane (L^2-norm versus parameter α) usually has a characteristic snake-like shape [39,40] (see Fig. 2, Fig. 1).

The goal of this paper is to understand, if other mechanisms exist, except for a heteroclinic contour structure proposed and numerically confirmed in [41,44], which will allow one to prove the existence of periodically modulated rolls numerically and rigorously.

The structure of the paper is as follows. In Section 2 we discuss the known structures observed in the stationary Swift–Hohenberg equation and method by Kirschgässner–Mielke to find periodically modulated localized solutions. This needs to understand the properties of the spectrum for the system linearized at the pulse solution. These properties are investigated in Section 3. The numerical investigations of the discrete spectrum are performed in Section 4. The reduction of the problem to solutions of the finite-dimensional differential system is done in Section 6. Also one finds there, using numerical methods, the homoclinic orbits to the equilibrium at the origin which correspond to periodically modulated rolls in this approximation. The discrepancies of two- and three-mode approximations are calculated and their graphs are plotted in Section 7. In the Addendum the details of the calculation of symmetric homoclinic orbits are discussed.

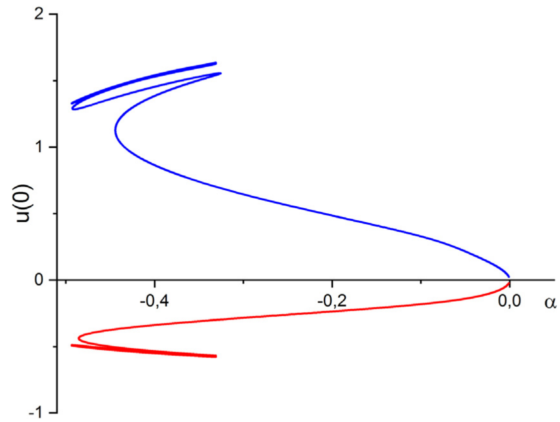

Fig. 1. Two different branches: dependence on α of pulse solutions, $\beta = 2.0$.

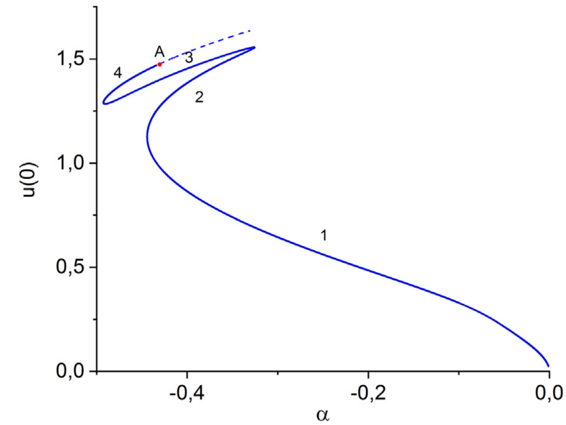

Fig. 2. Dependence on α of pulse solutions, $\beta = 2.0$.

2. The search for periodically modulated rolls

The question, we set up here, concerns transforming localized pulses $u(x)$ into periodically modulated in y rolls $u(x,y)$, $u(x, y+T) \equiv u(x,y)$, when Eq. (2) is considered on the whole plane $\mathbb{R}^2$. One may think that the method proposed by Kirschgässner [45] and developed further by Mielke [46] (see also, [47–49]) can be useful here. The method is as follows. Let us formally rewrite Eq. (2) as a system of two differential equations of the second order with the "time" y

$$\frac{\partial^2 u}{\partial y^2} = v - u - \frac{\partial^2 u}{\partial x^2},\ \frac{\partial^2 v}{\partial y^2} = \alpha u - v - \frac{\partial^2 v}{\partial x^2} + \beta u^2 - u^3.$$

Introducing new variables $p = \partial u/\partial y$, $q = \partial v/\partial y$ gives the first order differential system

$$\frac{\partial u}{\partial y} = p,\ \frac{\partial p}{\partial y} = v - u - \frac{\partial^2 u}{\partial x^2},\ \frac{\partial v}{\partial y} = q,\ \frac{\partial q}{\partial y} = \alpha u - v - \frac{\partial^2 v}{\partial x^2} + \beta u^2 - u^3.$$

In the vector form we have a differential equation of the form $X' = LX + N(X)$, where the $'$ denotes d/dy,

$$\frac{\partial}{\partial y}\begin{pmatrix} u \\ p \\ v \\ q \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(1+\frac{\partial^2}{\partial x^2}) & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha & 0 & -(1+\frac{\partial^2}{\partial x^2}) & 0 \end{pmatrix}\begin{pmatrix} u \\ p \\ v \\ q \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \beta u^2 - u^3. \end{pmatrix}. \tag{3}$$

Assume Eq. (2) has a localized in x solution $u_0(x)$, a pulse, at parameters (α_0, β_0). As is known from the previous studies [19, 34], this equation has localized pulses for all values (α, β) close enough to (α_0, β_0). This assertion relies on the assumption of a transverse intersection of W^s, W^u for the related Hamiltonian system with "time" x that was proved to be valid for small α [34], but needs in a numerical verification for non-small α like in [23].

Solutions to the system (3) of the form $(u_0(x,\alpha,\beta), 0, (u_0(x,\alpha,\beta) + u_0''(x,\alpha,\beta)), 0)$ are the formal "equilibria". The linearization of the system at such equilibrium leads to the linear system

$$\frac{\partial}{\partial y}\begin{pmatrix} u \\ p \\ v \\ q \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(1+\frac{\partial^2}{\partial x^2}) & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha + 2\beta u_0 - 3u_0^2 & 0 & -(1+\frac{\partial^2}{\partial x^2}) & 0 \end{pmatrix}\begin{pmatrix} u \\ p \\ v \\ q \end{pmatrix}. \tag{4}$$

Observe the useful fact for the further usage: the system (3) can be considered formally as being Hamiltonian, this is reached by introducing new variables

$$u = u,\ z = u_y,\ v = u + \Delta u,\ w = -\frac{\partial}{\partial y}[u + \Delta u],$$

with conjugate pairs (u, w), (z, v). In new variables the system casts

$$\begin{aligned} u_y &= z = \frac{\delta \mathcal{H}}{\delta w}, \\ z_y &= v - u - u_{xx} = \frac{\delta \mathcal{H}}{\delta v}, \\ w_y &= v + v_{xx} - \alpha u - \beta u^2 + u^3 = -\frac{\delta \mathcal{H}}{\delta u}, \\ v_y &= -w = -\frac{\delta \mathcal{H}}{\delta z} \end{aligned} \tag{5}$$

with the Hamiltonian

$$\mathcal{H} = \int_{-\infty}^{\infty} dx \left[zw - uv + \frac{1}{2}v^2 + \frac{\alpha}{2}u^2 + \frac{\beta}{3}u^3 - \frac{\beta}{4}u^4 + (\frac{\partial u}{\partial x})(\frac{\partial v}{\partial x}) \right],$$

where $\delta\mathcal{H}/\delta v$, etc., mean the variational derivatives of the functional $\mathcal{H}$ in its entering functions.

The property to be a Hamiltonian system says that the spectrum of the linearized system is invariant w.r.t. two involutions: complex conjugation and $\lambda \to -\lambda$. In particular, if there is a positive eigenvalue λ of some multiplicity in the spectrum, there exists also the negative eigenvalue $-\lambda$ with the same multiplicity.

The main assumption corroborated by numerical simulations is

There are values (α_0, β_0) such that the spectrum of the matrix linear differential operator in (4) has a pair of simple pure imaginary eigenvalues $\pm i\omega$, and the rest of spectrum in $\mathbb{C}$ does not intersect the imaginary axis.

Suppose, the rest of spectrum does not intersect the imaginary axis in $\mathbb{C}$ and, in addition, is separated out of the imaginary axis on a finite distance. The corresponding problem (3), derived from the elliptic PDE of the fourth order, is incorrect w.r.t. variable y, but it may become dynamically well defined on some finite-dimensional smooth invariant submanifolds of a center manifold type. If, in addition, the restriction of the system to this manifold gives a smooth Hamiltonian or reversible system, then this manifold can contain families of periodic orbits. For the initial equation this would give solutions periodic in y and localized in x.

Recall the formulation of the Mielke theorem [46]. A differential system in a real Banach space $X = X_1 \times X_2$ is studied

$$\dot{x}_1 - Ax_1 = f_1(t,x,\lambda),$$
$$\dot{x}_2 - Bx_2 = f_2(t,x,\lambda),\ x = (x_1,x_2),$$

where X_1 is finite-dimensional, the spectrum of the operator A belongs to the imaginary axis and the operator B, acting in the space X_2, is linear, possibly unbounded closed operator with a dense domain $D(B)$, its spectrum is separated from the imaginary axis. If the space X_2 is also finite-dimensional, then the existence of a center manifold for smooth f_1, f_2 is a standard theorem [50]. In order the theorem can be applied for finding solutions of elliptic PDEs, the following restrictions are imposed:

A 1. The space X_1 is finite-dimensional, eigenvalues of A are pure imaginary ones;

A 2. At some $\gamma \in [0,1)$ the closed operator B with the domain $D(B) \subset X_2$ has a closed fractional degree $B^\gamma : D(B^\gamma) \to X_2$. Denote $X_{2,\gamma}$ the Banach space $D(B^\gamma)$ with the norm $\|x_2\|_\gamma := \|x_2\| + \|B^\gamma x_2\|$. Assume also neighborhoods of the origin to exist $U_1' \subset X_1$, $U_2' \subset X_{2,\gamma}$, a dense subspace $V \subset X_2$, and a natural number $k \geq 1$ such that the inclusions hold

$$(f_1, f_2) \in C^k_{b,u}(\mathbb{R} \times U_1' \times U_2' \times \Lambda, X_1 \times V).$$

The region Λ of varying parameters is an open set in $\mathbb{R}^n$, containing the value λ_0, for which the equalities hold

$$f_i(t,0,\lambda_0) = 0,\ \frac{\partial}{\partial x_j} f_i(t,0,\lambda_0) = 0,\ i,j = 1,2,\ t \in \mathbb{R}.$$

A 3. There exists a Green function K, for which

$$\|B^\gamma K(t)\|_{V \to X_2} \leq \max\{1, |t|^{-\alpha}\} b e^{-\beta|t|}$$

for $t \neq 0$ and some $\alpha \in [0,1)$ and $b, \beta > 0$,

and for any $g \in C_b^1(\mathbb{R}, V)$ the equation $x_2' - Bx_2 = g$ has a unique solution $x_2 \in C_b^1(\mathbb{R}, X_2)$, defined as

$$x_2(t) = \int_{\mathbb{R}} K(s) g(t-s) ds.$$

These conditions hold, if the operator B defines a holomorphic semi-group e^{tB}, $t \geq 0$, then $\alpha = \gamma$ can be taken.

Under these conditions the following theorem is valid

Теорема **1** (*Mielke*)**.** *Suppose assumptions A1, A2, A3 hold. Then neighborhoods of the origin $U_1 \subset X_1$, $U_2 \subset X_{2,\gamma}$, a neighborhood $\Lambda_0 \subset \Lambda$ of the point λ_0 and a function $h \in C_b^k(\mathbb{R} \times U_1 \times \Lambda_0, U_{2,\gamma})$, exist with the following properties:*

1. *The set $M_\lambda := \{(t, x_1, h(t,x_1,\lambda)) | (t,x_1) \in \mathbb{R} \times U_1\}$ is a local integral manifold of the system.*
2. *Each solution of the system staying in $U_1 \times U_{2,\gamma}$ for all $t \in \mathbb{R}$, belongs to M_λ.*
3. *For all $t \in \mathbb{R}$ the equalities hold:*

$$h(t,0,\lambda_0) = 0,\ \frac{\partial}{\partial x_1} h(t,0,\lambda_0) = 0.$$

Observe that one of the essential requirements of the theorem is that the spectrum of the operator B is separated out from the imaginary axis.

It is worth remarking that this idea was realized in many cases [51–53], and for a more simple situation for the nonlinear elliptic equation $\Delta u - u + u^3 = 0$ considered on the whole plane $\mathbb{R}^2$ with coordinates (x,y) [54]. That equation has a pair of symmetric pulse solutions $u_\pm(x) = \pm\sqrt{2}/\cosh(x)$, they correspond to homoclinic loops of the saddle equilibrium for the related Hamiltonian system with one degree of freedom. This latter system describes solutions independent on y, in this case the initial elliptic equation becomes the Duffing type equation being integrable. For the elliptic equation on the plane, the linearization on such pulse solution gives a linear second order equation with a parameter being the wave number of the periodically modulated roll, i.e. the standard Schrödinger type linear differential equation with a potential [55]

$$\mathcal{L} = -\frac{d^2\varphi}{dx^2} + (1 - 3u_0^2(x))\varphi = \lambda^2 \varphi.$$

The differential operator $\mathcal{L}$ has, as a discrete spectrum, a unique eigenvalue $\lambda^2 = -3$ with its eigenfunction $h(x) = c/\cosh^2(x)$, the rest of the spectrum is continuous and coincides with the semi-axis $[1,\infty)$. Thus, the conditions on the linearized equation hold and the eigenvalue presents a boundary of the possible periods for the modulated roll (see also [56]).

For the case under consideration the situation is more involved, since we need previously to find both the family of localized solutions and after that those values (α_0, β_0) along the family for which the related localized solution has in the spectrum a pair of imaginary eigenvalues. Also, we need to verify that the continuous spectrum is separated out from the imaginary axis.

Thus, in order to apply the Mielke's theorem, one needs to investigate the location of the spectrum for the linearized operator at the pulse solution.

3. Examination of the spectrum

The linearization of the system (5) at the equilibrium state $(u_0(x), 0, 0, u_0(x) + u_0''(x))$ of the pulse type gives the linear system with the "time" y

$$\begin{aligned} u_y &= z = \tfrac{\delta\mathcal{H}}{\delta w}, \\ z_y &= v - u - u_{xx} = \tfrac{\delta\mathcal{H}}{\delta v}, \\ w_y &= v + v_{xx} - (\alpha + 2\beta u_0(x)u - 3u_0^2(x))u = -\tfrac{\delta\mathcal{H}}{\delta u}, \\ v_y &= -w = -\tfrac{\delta\mathcal{H}}{\delta z} \end{aligned} \tag{6}$$

Separation of variables $\exp[\sigma y](\psi(x), z(x), w(x), \chi(x))$ leads to the system for finding four functions $(\psi(x), z(x), w(x), \chi(x))$, namely, the spectral problem with the spectral parameter σ. Periodic in y solutions to this equation correspond to pure imaginary $\sigma = \pm i\omega$, ω is the wave number in y. This system is transformed to a system of two second order ordinary differential equations with respect to the pair of real functions $(\psi(x), \chi(x))$

$$\begin{aligned} -\psi'' - \psi + \chi &= \sigma^2 \psi, \\ -\chi'' - \chi + (\alpha + 2\beta u_0(x) - 3u_0^2(x))\psi &= \sigma^2 \chi, \end{aligned} \tag{7}$$

with evident conditions of the localization

$$\lim_{|x|\to\infty} \psi(x) = 0, \quad \lim_{|x|\to\infty} \chi(x) = 0. \tag{8}$$

Pure imaginary σ gives negative $\sigma^2 = -\omega^2$.

The natural functional set-up for the problem (7)–(8) is the eigenvalue problem for the differential operator $\mathcal{L}$ in the left hand side of (7) acting in the space of vector-functions $(\psi(x), \chi(x))$ and instead of conditions of localization (8) to seek for solutions to the system which belong to the space $L_2(\mathbb{R}) \times L_2(\mathbb{R})$. Then it is naturally to start with the space $C_0^2(\mathbb{R}) \times C_0^2(\mathbb{R})$ of functions $(\psi(x), \chi(x))$ with compact supports and then extend this differential operator till an operator $\mathcal{L}$ in $L_2(\mathbb{R}) \times L_2(\mathbb{R})$.

The differential operator l acting on a vector-function $y(x) = (\psi(x), \chi(x))^\top$ is as follows

$$l(y) = P_0 y'' + P_2 y,\ P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ P_2 = \begin{pmatrix} 1 & -1 \\ -\alpha - g(x) & 1 \end{pmatrix},$$
$$g(x) = 2\beta u_0(x) - 3u_0^2(x) g(x). \tag{9}$$

We use the standard inner product in $L_2(\mathbb{R}) \times L_2(\mathbb{R})$

$$\langle\langle(\psi, \chi), (\psi_1, \chi_1)\rangle\rangle = \int_{-\infty}^{\infty} [\psi(x)\psi_1^*(x) + \chi(x)\chi_1^*(x)]dx,$$

where asterisk means the complex conjugation. Let us calculate the conjugate operator l^*. Using the inner product and integration by parts we get

$$l^* = P_0 y_1'' + P_2^\top y_1,\ P_2^\top = \begin{pmatrix} 1 & -\alpha - g(x) \\ -1 & 1 \end{pmatrix},\ y_1(x) = \begin{pmatrix} \psi_1(x) \\ \chi_1(x) \end{pmatrix}.$$

Thus, we see that the operator l is not symmetric.

So, the boundary value problem (7)–(8) with the spectral parameter $\kappa = \sigma^2$ is not self-adjoined. So, we need to investigate this spectral problem taking into account that κ can be a complex number. Because of sufficiently fast (exponential) decay $|u_0(x)| \to 0$, as $|x| \to \infty$, the continuous spectrum of the problem is defined by the limiting operator with constant coefficients [57]. We need to find negative eigenvalues κ at some (α_0, β_0), they correspond to pure imaginary σ. The discrete spectrum of the problem always contains the point $\kappa = 0$, since a pair of functions $\psi_0(x) = u_0'(x)$, $\chi_0(x) = u_0'(x) + u_0'''(x)$ provides a localized solution to the linearized equation (7) with boundary conditions (8).

One can separate out the spectrum from zero, if the operator is restricted on the subspace of even functions or, in other words, consider solutions in the space $L_2(\mathbb{R}_+) \times L_2(\mathbb{R}_+)$ with the boundary conditions $\psi'(0) = \chi'(0) = 0$ at the left end. Observe that eigenvalues just correspond to those solutions of the system (7) for which both $\psi(x), \chi(x)$ belong to $L_2(\mathbb{R})$.

Let us apply the theory of linear ordinary differential equations to study the spectral problem. To distinguish the continuous spectrum, we remark that the continuous spectra of the operators $\mathcal{L}$ and $\mathcal{L}_0$

$$\mathcal{L} = \begin{pmatrix} \frac{d^2}{dx^2} + 1 & -1 \\ -\alpha - g(x) & \frac{d^2}{dx^2} + 1 \end{pmatrix} = \begin{pmatrix} \frac{d^2}{dx^2} + 1 & -1 \\ -\alpha & \frac{d^2}{dx^2} + 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -g(x) & 0 \end{pmatrix} = \mathcal{L}_0 + \mathcal{L}_1, \tag{10}$$

coincide, due to the rapid decay of $u_0(x)$ [57], where $\mathcal{L}_0$ is the limiting operator as $|x| \to \infty$.

So, we need first to study the spectrum of the operator $\mathcal{L}_0$. It is a matrix differential operator with constant coefficients acting in the space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. The Fourier transform

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp[-\xi x] dx,$$

applied to both functions $\psi(x), \chi(x)$ gives as image the matrix operator

$$G = \begin{pmatrix} \xi^2 - 1 & -1 \\ -\alpha & \xi^2 - 1 \end{pmatrix} \tag{11}$$

and $\det(G - \sigma^2 E) = 0$ gives the biquadratic characteristic equation $P(\sigma) = [\xi^2 - 1 - \sigma^2]^2 - \alpha = 0$ with respect to σ. This equation has solutions $\sigma^2 = \xi^2 - 1 \pm i\sqrt{-\alpha}$. This gives the relation for the searching a continuous spectrum $\sigma = \Lambda + i\Omega$, $\sigma^2 = \Lambda^2 - \Omega^2 + 2i\Lambda\Omega$, where

$$\Lambda = \pm\sqrt{\frac{\sqrt{(\xi^2-1)^2 - \alpha} + \xi^2 - 1}{2}},$$
$$\Omega = \pm\sqrt{\frac{\sqrt{(\xi^2-1)^2 - \alpha} - \xi^2 + 1}{2}}. \tag{12}$$

Thus, in the complex plane $\sigma = \Lambda + i\Omega$ we have two hyperbolas $\Lambda\Omega = \sqrt{-\alpha}/2$ as the location of continuous spectrum of $\mathcal{L}_0$. In particular, if we are interested in searching for solutions with $\sigma = \pm i\omega$, we have to set $\sigma^2 = -\omega^2$ and the spectrum curves show the absence of such points for the matrix differential operator $\mathcal{L}_0$.

Concerning the spectrum of the operator $\mathcal{L}$ the following assertion holds

Proposition 1. *For any negative α and real κ differential operator $\mathcal{L}$ has either a bounded inverse operator or the related κ is an eigenvalue of the multiplicity one or two. The continuous spectrum of the operator $\mathcal{L}$ coincides with the spectral curve $\Lambda\Omega = \sqrt{-\alpha}/2$ of the operator $\mathcal{L}_0$.*

Proof. Let us introduce new coordinates $Y = (q_1, q_2, p_1, p_2)^\top$ with $q_1 = \psi$, $q_2 = \psi'$, $p_1 = -\chi'$, $p_2 = \chi$. Then the system (7) casts as

$$Y' = (A_\kappa + V(x))Y,\ A_\kappa = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -\alpha & 0 & 0 & 1-\kappa \\ 0 & 0 & -1 & 0 \end{pmatrix},$$
$$V(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -g(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{13}$$

with the exponentially decaying function $g(x)$. The eigenvalues of the matrix A are quadruple of complex numbers (12) with nonzero Λ, Ω. This implies that the system $Y' = (A_\kappa + V(x))Y$ possesses an exponential dichotomy of solutions on both semi-axes $x \in \mathbb{R}_+$ and $x \in \mathbb{R}_-$ [58]. The types of the dichotomy are such that all solutions of the system with initial conditions $Y = Y_0$ at $x_0 = 0$, which belong to some two-dimensional subspace U_+, exponentially decay to zero as $x \to \infty$, but all solutions with other initial conditions exponentially increase. For solutions, as $x \to -\infty$, there is a unique two-dimensional subspace U_- such that solutions with initial conditions $Y_0 \in U_-$ at $x_0 = 0$ exponentially decay to zero as $x \to -\infty$ but all other solutions increase exponentially. These properties of solutions to the system say that only three opportunities can be realized: (1) subspaces U_+ and U_- intersect each other at the origin $Y_0 = 0$ only (they are two transversal 2-planes in Y). This means the system to possess an exponential dichotomy of the type $(2, 2)$ on all $\mathbb{R}$, and therefore for any bounded on $\mathbb{R}$ function $f(x)$ there is a unique bounded solution of the system $Y' = (A_\kappa + V(x))Y + f(x)$. This solution is represented via the Green function that exists here for this equation, so $\mathcal{L}$ has a bounded inverse operator. These values of κ belong to the resolvent set and for them the unique solution to the system with $f(x) \equiv 0$ is the zeroth solution. In particular, all $\kappa = -\sigma^2$ are such for $g(x) = 0$; (2) subspaces U_+ and U_- intersect each other along a one-dimensional subspace, thus there is a one-parametric family of solutions to the system which decay exponentially fast as $|x| \to \infty$, any such solution gives an eigenfunction corresponding to the eigenvalue κ; (3) subspaces U_+ and U_- coincide, then all solutions with initial conditions on this 2-plane decay exponentially fast as $|x| \to \infty$, any such solution gives the eigenfunction corresponding to the double eigenvalue κ. This eigenfunction belongs to $L^2(\mathbb{R})$ due to exponential decaying as $|x| \to \infty$.

For the operator $\mathcal{L}_0$ with constant coefficients for all κ we have eigenvalues of the matrix A being complex quadruple (see (12)). Thus, the system $Y' = A_\kappa Y$ has no nonzero bounded solutions and the limiting system (7), as $|x| \to \infty$, has no solutions $(\psi(x), \chi(x))$ which belong to $L_2(\mathbb{R}) \times L_2(\mathbb{R})$. □

So, the problem, we are interested in, consists in searching those $\kappa < 0$ which correspond to the cases (2) or (3). This is done using numerical methods and will be shown in the next Section 4.

4. Numerical study of the discrete spectrum

In this section we present details of searching for eigenvalues for the spectral problem (7)–(8). The idea is presented in the proof of Proposition 1. We found several branches of pulse solutions, two of them are presented in Fig. 1, calculations were performed for $\alpha < 0$ and $\beta = 2$. These curves have a characteristic snake-like shape. We use later on the first family of pulse solutions $u_0(x)$ (see Fig. 2).

We fix $\beta = 2$ and move along the fixed curve decreasing α, starting at $\alpha \simeq -10^{-3}$, and calculate at these α the discrete spectrum of the related linear differential operator, using the algorithm described in the previous section. More precisely, we search for symmetric solutions to the system (7), i.e. functions $(\psi(x), \chi(x))$, $\psi(-x) = \psi(x)$, $\chi(-x) = \chi(x)$, $\psi'(0) = 0$, $\chi'(0) = 0$, which decay exponentially as $|x| \to \infty$. To this end, we consider solutions with asymptotics $\psi(x) \to 0$, $\psi'(x) \to 0$, $\chi(x) \to 0$, $\chi'(x) \to 0$, as $x \to \infty$. Because of exponentially fast decay of the function g (see Section 3), as $|x| \to \infty$, and the representation for eigenvalues (12) for the limiting system, which have a complex conjugate pair with negative real parts, there is at $x = 0$ a two-dimensional subspace of initial conditions whose related solutions of the system (7) decay exponentially to zero. We need to find such subspace.

We deal in fact with the non-autonomous reversible linear system (13). It transforms to itself under the changes $Y \to G(Y)$ and $x \to -x$, where G is the involution of the phase space $\mathbb{R}^4$. Finding the initial subspace for decaying solutions cannot be done directly, since the system is non-autonomous. We address here to the numerical methods. To that end, we take $L > 0$ large enough, where one can regard $g(L) \simeq 0$ and choose two independent vectors Y_1^0, Y_2^0 in the stable linear subspace of the limiting autonomous system (see details in the Addendum). Using them as initial vectors, we calculate numerically solutions of the system (13) till $x = 0$. We get there two linearly independent vectors and norm them, since they can be large enough in the norm. Using the reversibility, we have simultaneously solutions with initial vectors $G(Y_1^0), G(Y_2^0)$ at $x = -L$ where G is the reversing involution of the system (13). We continue them till $x = 0$ and therefore we get in the 4-dimensional space $\mathbb{R}_0^4$, corresponding to $x = 0$, four vectors and calculate the determinant of the matrix composed of these four vectors. This determinant (at fixed parameters α, β) depends on the parameter ω and we search zeros of the determinant, i.e. spectral points.

If the determinant equals zero, two subspaces obtained in $\mathbb{R}_0^4$ can intersect each other either along one-dimensional subspace or they coincide, the first case gives simple eigenvalues and they are double for the second case. In principle, one can exist several such values of ω which correspond to different eigenvalues and each of them can be continued in α. For instance, at $\alpha = -10^{-3}$ we found eight spectral points (they are colored by red, green, blue and violet). When continuing in α, these points turned out be connected by spectral curves in pairs, related curves are colored with the same colors, the results are plotted in Fig. 3 (their magnification near the point $\alpha = 0$, $\omega^2 = 1$ is shown on Fig. 4). The most interesting among them is the red curve being the continuation of the red spectral point at $\alpha = -10^{-3}$. When we move along this curve decreasing α, we reach the point A (it corresponds to the point A on the curve Fig. 2), where two spectral lines coalesce. After that we move along the same curve in Fig. 2, but in the backward direction.

A characteristic property of the curve in Fig. 2 is that at fold points, where two different pulses coalesce, the spectral curves also branches in α and the number of spectral points change. At Fig. 3 we see the existence at a fixed negative α two real eigenvalues κ, negative and positive. The positive eigenvalue corresponds to the pair of imaginary eigenvalues of the linear system (6).

The curve Fig. 2 shows that at a fixed value $\alpha < 0$ (and $\beta = 2$) several pulses can exist, for instance, at $\alpha = -0.4$ there are four pulses (in fact, there are more, since on the plot only a part of the curve has shown). This suggests that at some fold points two different pulses can coalesce.

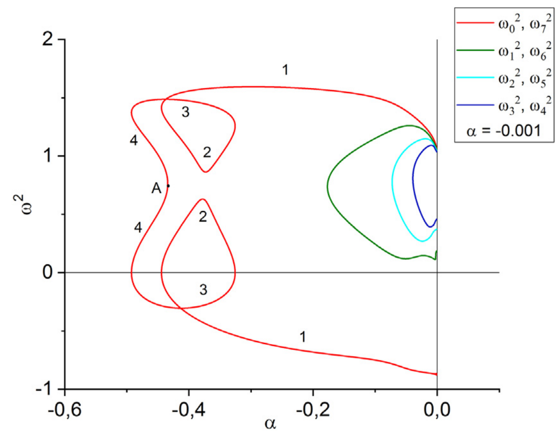

Fig. 3. Spectrum of localized u_0 solutions along the first family.

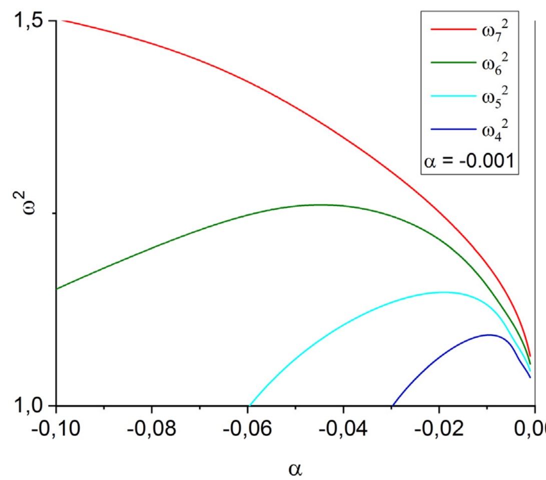

Fig. 4. Branches of modulated rolls in the plane (α, ω^2) at the fine scale, $\beta = 2.0$.

5. Changing roles

Now we shall try to permute the roles of variables and consider x as the temporal variable and formally the differential equation be considered in the space of periodic in y functions. To this end, let us rewrite Eq. (2) as a system of two differential equations of the second order with the "time" x

$$\frac{\partial^2 u}{\partial x^2} = v - u - \frac{\partial^2 u}{\partial y^2},\ \frac{\partial^2 v}{\partial x^2} = \alpha u - v - \frac{\partial^2 v}{\partial y^2} + \beta u^2 - u^3.$$

Introducing new variables $p = \partial u/\partial x$, $q = \partial v/\partial x$ gives the first order differential system

$$\frac{\partial u}{\partial x} = p,\ \frac{\partial p}{\partial x} = v - u - \frac{\partial^2 u}{\partial y^2},\ \frac{\partial v}{\partial x} = q,\ \frac{\partial q}{\partial x} = \alpha u - v - \frac{\partial^2 v}{\partial y^2} + \beta u^2 - u^3.$$

In the vector form we have a differential equation of the form $X' = LX + N(X)$, where "the prime" denotes d/dx,

$$\frac{\partial}{\partial x}\begin{pmatrix} u\\ p\\ v\\ q\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -(1+\frac{\partial^2}{\partial y^2}) & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \alpha & 0 & -(1+\frac{\partial^2}{\partial y^2}) & 0\end{pmatrix}\begin{pmatrix} u\\ p\\ v\\ q\end{pmatrix} + \begin{pmatrix} 0\\ 0\\ 0\\ \beta u^2 - u^3.\end{pmatrix}.$$

This formal system is considered in the space of periodic in y functions of the period $2\pi/\omega$, where $\omega > 0$ becomes an additional parameter. To fix the period 2π of the functions, we scale the variable y, $\xi = \omega y$, then $\frac{\partial}{\partial y}$ becomes $\omega\frac{\partial}{\partial \xi}$ and the system casts

$$\frac{\partial}{\partial x}\begin{pmatrix} u\\ p\\ v\\ q\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -(1+\omega^2\frac{\partial^2}{\partial \xi^2}) & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \alpha & 0 & -(1+\omega^2\frac{\partial^2}{\partial \xi^2}) & 0\end{pmatrix}\begin{pmatrix} u\\ p\\ v\\ q\end{pmatrix} + \begin{pmatrix} 0\\ 0\\ 0\\ \beta u^2 - u^3.\end{pmatrix}. \quad (14)$$

Observe the system (14) have a four-dimensional invariant plane of solutions which do not depend on y (or $\omega = 0$). The limit $\omega = 0$ is singular (derivatives in ξ are lost), so this 4-plane is the slow manifold of the problem, if ω assumes to be small but we shall not suppose this.

The system on this 4-plane coincides with the system corresponding to the one-dimensional SH equation. So, it does have homoclinic orbits of the equilibrium O at the origin as $\alpha < 0$. We need to construct homoclinic orbits to O not lying in this 4-plane for $\omega \neq 0$. The system (14) is $O(2)$-invariant, since it is invariant with respect to the shift $X(x, \xi + s) \to X(x, \xi)$, $s \in [0, 2\pi]$, and also is invariant w.r.t. the reflection $X(x, \xi) \to X(x, -\xi)$. The 4-plane is the fixed set of this action. Thus, the group $G = O(2)$ acts only on the fast variables being transversal to the 4-plane. Below we restrict the system on the subspace of even 2π-periodic in ξ functions.

Let us examine the linearization at the equilibrium $X = 0$ existing at all parameters. At any α we have the linear system

$$\frac{\partial}{\partial x}\begin{pmatrix} u\\ p\\ v\\ q\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0\\ -(1+\omega^2\frac{\partial^2}{\partial \xi^2}) & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \alpha & 0 & -(1+\omega^2\frac{\partial^2}{\partial \xi^2}) & 0\end{pmatrix}\begin{pmatrix} u\\ p\\ v\\ q\end{pmatrix}. \quad (15)$$

Since we work in the space of even 2π-periodic in ξ functions, all components of the vector X are Fourier series in $\cos(n\xi)$ with coefficients being functions in x. The related spectral problem for the vector function $\exp[\lambda x]Z(\xi)$, $Z(\xi) = \sum_{n\geq 0} A_n\cos(n\xi)$ reduces to the infinite system of two second order linear equations in variable ξ with the spectral parameter λ and the parameter $\omega \geq 0$

$$\lambda^2 u_n = -u_n + n^2\omega^2 u_n + v_n,\ \lambda^2 v_n = \alpha u_n - v_n + n^2\omega^2 v_n.$$

So, the spectrum is given by roots of the equation $(1 + \lambda^2 - n^2\omega^2)^2 = \alpha$. Thus, for $\alpha = 0$ we conclude that all eigenvalues are double, they are pure imaginary for $|n| < [1/\omega]$ and reals for $|n| > [1/\omega]$. Double eigenvalues have 2-dimensional Jordan boxes of the normal form.

For α negative, the spectrum consists of infinite number of simple eigenvalues forming complex quadruples $\pm\sqrt{n^2\omega^2 - 1 \pm i\sqrt{-\alpha}}$,. For $n = 0$ this quadruple corresponds to the saddle-focus on the invariant 4-plane, other quadruples correspond to transverse coordinates to this 4-plane.

To avoid dealing with slow–fast problems, when ω is small, we assume ω to belong to the interval $(1/2, 1)$, then only modes with $n = 0$ and $n = 1$ give pure imaginary eigenvalues at $\alpha = 0$. Higher modes with $n \geq 2$ give double real eigenvalues for the equilibrium $X = 0$. Thus, we expect to have a local center manifold C corresponding to these double imaginary (if $\alpha = 0$) or closest to them quadruples of eigenvalues (if $\alpha < 0$ and small). The dimension of C in the space of even in ξ vector functions is eight for this set of ω. This sub-manifold contains 4-plane (its piece near $X = 0$, if more precisely). So, if we shall prove the existence of this center manifold C, using the method due to Kirchgässner and Mielke [45,46,59] and more global construction for α not small [48,49], and if this sub-manifold be sufficiently smooth to apply results of the bifurcation theory, then we shall be able to prove the existence of homoclinic orbits arising from homoclinic orbit on the 4-plane. We hope to present further details somewhere.

Now suppose the center manifold C to exist and satisfies all necessary requirements of smoothness. For sufficiently small negative α the problem for the eight-dimensional system on C near the equilibrium O becomes local: at $\alpha = 0$ the system has two pairs of pure imaginary double eigenvalues $\pm i\rho_1$, $\pm i\rho_2$ and for negative α small enough the system possesses two quadruples of complex eigenvalues. This bifurcation problem is of codimension two but we have just two parameters α and ω. So, the bifurcation problem reminds the Hamiltonian Hopf Bifurcation (HHB), but in a more involved set-up, since the degenerate system has two pairs of pure imaginary eigenvalues instead of one for the ordinary HHB. The problem can simplify, if we assume, as for our case, the system has an invariant 4-plane corresponding to the first double pair. We regard the problem of investigating this two-parametric bifurcation to be very interesting and hope to return to its study later. Now we try to model the problem of finding periodically modulated rolls reducing it to the finite-dimensional problem via Bubnov–Galerkin approximations.

6. Bubnov–Galerkin method

In this section we exploit the Bubnov–Galerkin method. Here we keep in mind that solutions we seek are periodic in y with some unknown period, so one can expand the solution in the Fourier series in periodic variable y. In fact, since the equation is nonlinear, we scarcely can use this method for proving the proper solutions. Nevertheless, if we shall find needed solutions to the approximating system, this gives a numerical evidence that they do exist. The approximating system, that will be derived, can also have other solutions which may corroborate the existence of solutions of other types for the SH equation, for instance, doubly periodic in x and y, and so forth.

To derive the approximating system of differential equations and perform numerical simulations, we shall search for even in y

solutions and stay only three modes in the Fourier expansion

$u = u_0(x)/2 + u_1(x)\cos\omega y + u_2(x)\cos 2\omega y + \cdots$,

where ω is unknown so far wave number. Then for the functions $u_i(x)$, $i = 0, 1, 2$, we get the following system of ordinary differential equations of the fourth order

$$\begin{aligned}
&(\Delta_0+1)^2u_0 = \alpha u_0 + \beta(\frac{1}{2}u_0^2 + u_1^2 + u_2^2)\\
&\quad -(\frac{1}{4}u_0^3 + \frac{3}{2}u_0u_1^2 + \frac{3}{2}u_0u_2^2 + \frac{3}{2}u_1^2u_2),\\
&(\Delta_1+1)^2u_1 = \alpha u_1 + \beta u_1(u_0+u_2)\\
&\quad -\tfrac{3}{4}u_1(u_0^2+u_1^2+2u_0u_2+2u_2^2),\\
&(\Delta_2+1)^2u_2 = \alpha u_2 + \beta(u_0u_2 + \frac{1}{2}u_1^2)\\
&\quad -\frac{3}{4}(u_0u_1^2 + u_0^2u_2 + 2u_1^2u_2 + u_2^3),
\end{aligned} \tag{16}$$

here $\Delta_k = \dfrac{d^2}{dx^2} - k^2\omega^2,\ k = 0, 1, 2.$

The one-mode and two-mode approximations are obtained from this system, if we set $u_1 = u_2 = 0$ or $u_2 = 0$, respectively. We intend to compare the conclusions concerning the existence of periodically modulated in y and localized in x solutions in these systems.

It is more convenient for the study to reduce these equations to the form of a system of the Euler–Lagrange equations, and further to a Hamiltonian system, where solutions we search for correspond to homoclinic orbits of the equilibrium state at the origin. We re-scale first the equation for u_0 : $u_0 \to \sqrt{2}u_0$, after that we come to the system of Euler–Lagrange differential equations that is transformed the system (16) to the Hamiltonian form with the Hamiltonian

$$\begin{aligned}
&H = p_1q_2 - p_2q_1 + [p_3q_4 - (1-\omega^2)p_4q_3]\\
&\quad + [p_5q_6 - (1-4\omega^2)p_6q_5] +\\
&\tfrac{1}{2}(p_2^2+p_4^2+p_6^2) + \tfrac{\alpha}{2}(q_1^2+q_3^2+q_5^2)\\
&\quad + \tfrac{\beta}{\sqrt{2}}[\tfrac{1}{3}q_1^3 + q_1(q_3^2+q_5^2) + \tfrac{1}{\sqrt{2}}q_3^2q_5] -\\
&\frac{1}{16}[2q_1^4 + 3q_3^4 + 3q_5^4 + 12q_1^2(q_3^2+q_5^2) + 12\sqrt{2}q_1q_3^2q_5 + 12q_3^2q_5^2]
\end{aligned}$$

and the standard symplectic 2-form $dq \wedge dp = \sum_i dq_i \wedge dp_i$, $q = (q_1,\ldots,q_6)^\top$, $p = (p_1,\ldots,p_6)^\top$, $H = H(q,p)$. Thus, we come to the Hamiltonian system $q' = H_p,\ p' = -H_q$, depending on three parameters α, β, ω

$$\begin{aligned}
&q_1' = q_2,\\
&q_2' = p_2 - q_1,\\
&q_3' = q_4,\\
&q_4' = p_4 - (1-\omega^2)q_3,\\
&q_5' = q_6,\\
&q_6' = p_6 - (1-4\omega^2)q_5,\\
&p_1' = p_2 - \alpha q_1 - \tfrac{\beta}{\sqrt{2}}(q_1^2+q_3^2+q_5^2) + \tfrac{1}{2}q_1^3\\
&\quad + \tfrac{3}{2}q_1(q_3^2+q_5^2) + \tfrac{3\sqrt{2}}{4}q_3^2q_5,\\
&p_2' = -p_1,\\
&p_3' = (1-\omega^2)p_4 - \alpha q_3 - \beta q_3(\sqrt{2}q_1 + q_5)\\
&\quad + \tfrac{3}{2}q_3(q_1^2 + \tfrac{1}{2}q_3^2 + q_5^2 + \sqrt{2}q_1q_5),\\
&p_4' = -p_3,\\
&p_5' = (1-4\omega^2)p_6 - \alpha q_5 - \beta(\sqrt{2}q_1q_5 + \tfrac{1}{2}q_3^2)\\
&\quad + \tfrac{3}{4}(\sqrt{2}q_1q_3^2 + 2q_1^2q_5 + 2q_3^2q_5 + q_5^3),\\
&p_6' = -p_5.
\end{aligned} \tag{17}$$

The system (17) has an invariant symplectic four-dimensional plane $q_3 = q_4 = q_5 = q_6 = p_3 = p_4 = p_5 = p_6 = 0$. The restriction of the system (17) on this 4-plane gives the Hamiltonian system that corresponds to the Swift–Hohenberg equation on the spatial domain $\mathbb{R}$ and for $\alpha < 0$ localized pulses match to homoclinic orbits of the saddle-focus O. But for the system (17) we are interested in homoclinic orbits to the equilibrium O not lying in this 4-dimensional subspace, because only they are related with modulated in y localized rolls. For such solution, if it exists, at least one of the coordinate function $q_j(x), p_j(x), j \geq 3$, has not vanish identically.

The system (17) is, in addition, reversible w.r.t. the linear involution $L(q,p) = (S_eq, S_op)$, $S_eq = (q_1, -q_2, q_3, -q_4, q_5, -q_6)$, $S_op = (-p_1, p_2, -p_3, p_4, -p_5, p_6)$. The fixed point set $Fix(L)$ of the involution L is the 6-dimensional plane $q_2 = q_4 = q_6 = p_1 = p_3 = p_5 = 0$. The equilibrium O ($L(O) = O$) at the origin of the Hamiltonian system (17) is symmetric, i.e. $O \in Fix(L)$.

In order to have homoclinic orbits of O, the equilibrium should have eigenvalues with positive and negative real parts. The linearization matrix at the equilibrium O consists of three independent (4×4)-blocks and its characteristic polynomial is the product of three biquadratic polynomials

$$[(\lambda^2+1)^2 - \alpha][(\lambda^2+1-\omega^2)^2 - \alpha][(\lambda^2+1-4\omega^2)^2 - \alpha].$$

So, if $\alpha = 0$ and $\omega \in (1/2, 1)$ the equilibrium has two double pure imaginary eigenvalues $\pm i$, $\pm i\sqrt{1-\omega^2}$ and a pair of real double eigenvalues $\pm\sqrt{4\omega^2-1}$, all of them are non semi-simple ones, i.e. with 2×2 Jordan boxes. Hence, for this set of parameters the local center manifold C exists and is eight-dimensional. For negative α roots of the characteristic polynomial are four complex quadruples

$$\begin{aligned}
&\pm\sqrt{\frac{\sqrt{1-\alpha}-1}{2}} \pm i\sqrt{\frac{\sqrt{1-\alpha}+1}{2}},\\
&\pm\sqrt{\frac{\sqrt{(\omega^2-1)^2-\alpha}+\omega^2-1}{2}} \pm i\sqrt{\frac{\sqrt{(\omega^2-1)^2-\alpha}+1-\omega^2}{2}},\\
&\pm\sqrt{\frac{\sqrt{(4\omega^2-1)^2-\alpha}+4\omega^2-1}{2}}\\
&\pm i\sqrt{\frac{\sqrt{(4\omega^2-1)^2-\alpha}+1-4\omega^2}{2}}.
\end{aligned}$$

Thus, in this approximation the situation is similar as for the full system (4). For negative α the squares of distances of these quadruples from the imaginary axis in the complex plane $\mathbb{C}$ of eigenvalues are ordered as follows

$$\begin{aligned}
\frac{1}{2}[\sqrt{1-\alpha}-1] &< \frac{1}{2}[\sqrt{(\omega^2-1)^2-\alpha}+\omega^2-1]\\
&< \frac{1}{2}[\sqrt{(4\omega^2-1)^2-\alpha}+4\omega^2-1].
\end{aligned}$$

So, the leading stable direction (and unstable one, as well) of the equilibrium at the origin is two-dimensional and coincides with the invariant plane corresponding to the pair of eigenvalues with negative real parts of the first quadruple. We expect that homoclinic orbits of this equilibrium will approach to the equilibrium along this direction (a generic case).

We search for symmetric homoclinic orbits Γ to O, their symmetricity means the invariance $L(\Gamma) = \Gamma$ or, equivalently, $\Gamma \cap Fix(L)$ consists of the unique point. This implies that we need to find the point of intersection of $W^u(O)$ with the fixed 6-plane $Fix(L)$. If for some values of parameters $(\alpha_0, \beta_0, \omega_0)$ the intersection of these two six-dimensional submanifolds in $\mathbb{R}^{12}$ is nonempty and transverse at the intersection point, the symmetric homoclinic orbit will preserve for all close values of parameters. Thus, transversal homoclinic orbits have the unique continuations in parameters and no their branching can occur.

Let us recall some useful facts of the theory of homoclinic solutions in Hamiltonian systems. Suppose a smooth (autonomous) Hamiltonian system with n degrees of freedom be given and smooth function H be its Hamiltonian. Suppose p be an equilibrium of a saddle type, this means no eigenvalues on the imaginary axis to exist for the linearization matrix at p. For a Hamiltonian system the spectrum of the linearization matrix at the equilibrium p is invariant w.r.t. the symmetry $\lambda \to -\lambda$ in $\mathbb{C}$, hence stable $W^s(p)$ and unstable $W^u(p)$ manifolds of p are smooth submanifolds of the same dimension, they both belong to the level set $H = H(p)$ due to invariance of H. This level set is a smooth submanifold (of the dimension $2n-1$) near every its point, except for critical points where $dH = 0$, i.e. equilibria of the vector field X_H. Near a critical point the level $H = H(p)$ looks like a cone. Thus, the intersection of stable $W^s(p)$ and unstable $W^u(p)$ manifolds in $H = H(p)$, if it exists, is generically transversal. A transversal homoclinic orbit to p is preserved under varying parameters entering smoothly into H, but a homoclinic orbit can be destroyed, if at some specific value of the parameter this homoclinic orbit becomes non-transversal or this homoclinic orbit gets stuck at some periodic orbit lying in the same level of H.

The system (17) does have homoclinic orbits on the invariant 4-plane [19,22] and these orbits are transversal (within this 4-plane) for majority values of the parameter α. This transversality follows from [34] for small negative α, but it is a numerically checkable fact for not small α. So, the branching of such homoclinic orbit for the full system (17) can occur, if either the transversality loses within this 4-plane or the tranversality within the invariant 4-plane preserves, but the tangency arises due to transverse variables. In the former case the branching occurs near the fold points on the snake-like curve (see Figs. 2 and 1). In the latter case the homoclinic orbit on the 4-plane persists, but the branching gives one more homoclinic orbit in the transverse directions to the 4-plane, an example of such branching is demonstrated in Section 6.1.

Take ω as the governing parameter in our case. So, in order at fixed values of α_0, β_0 a homoclinic orbit to p, not lying wholly in 4-plane, would arise at some parameter ω, this orbit should become at this ω non-transversal in the direction transverse to 4-plane. To find such special (spectral) values of ω, we linearize the system (17) with fixed α_0, β_0 at a symmetric homoclinic solution lying in the invariant 4-plane and seek those ω at which the intersection of stable and unstable manifolds within the level $H = H(O) = 0$ be non-transversal.

The linearization of (17) at the homoclinic solution in the invariant 4-plane, corresponding to the pulse $u_0(x)$, looks as follows

$$\begin{aligned}
\xi_1' &= \xi_2,\\
\xi_2' &= \eta_2 - \xi_1,\\
\xi_3' &= x_4,\\
\xi_4' &= \eta_4 - (1-\omega^2)\xi_3,\\
\xi_5' &= \xi_6,\\
\xi_6' &= \eta_6 - (1-4\omega^2)\xi_5,\\
\eta_1' &= \eta_2 - (\alpha + \beta\sqrt{2}u_0 - \tfrac{3}{2}u_0^2)\xi_1,\\
\eta_2' &= -\eta_1,\\
\eta_3' &= (1-\omega^2)\eta_4 - (\alpha + \beta\sqrt{2}u_0 - \tfrac{3}{2}u_0^2)\xi_3,\\
\eta_4' &= -\eta_3,\\
\eta_5' &= (1-4\omega^2)\eta_6 - (\alpha + \beta\sqrt{2}u_0 - \tfrac{3}{2}u_0^2)\xi_5,\\
\eta_6' &= -\eta_5.
\end{aligned} \tag{18}$$

Here we keep into account that for such homoclinic orbit one has $q_1(x) = \sqrt{2}u_0(x)$, $q_2(x) = \sqrt{2}u_0'(x)$, $p_1(x) = -\sqrt{2}(u_0'(x) + u_0'''(x))$, $p_2(x) = \sqrt{2}(u_0(x) + u_0''(x))$, and $q_i = p_i = 0$ for $i \geq 3$). We see that the linearized system breaks into three independent subsystems of the fourth order, first of which corresponds to the linearization on the localized solution for the system derived for the SH equation on $\mathbb{R}$. These three subsystems can be transformed to the three independent linear second order differential systems

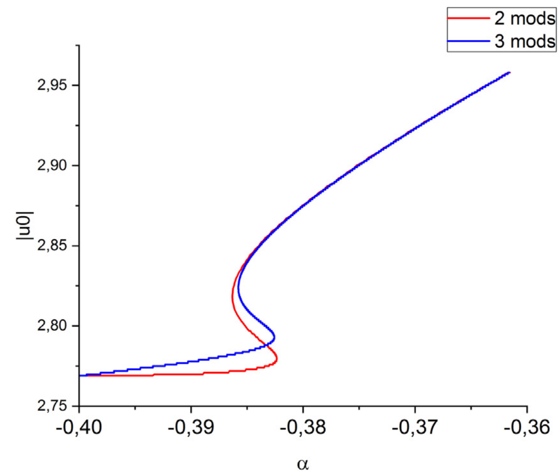

Fig. 5. Dependence of the pulse solution on α.

w.r.t. three pairs of functions (ξ_1, η_2), (ξ_3, η_4), (ξ_5, η_6)

$$\begin{aligned}
&\xi_1'' + \xi_1 = \eta_2,\ \ \eta_2'' + \eta_2 = (\alpha + \beta\sqrt{2}u_0 - \tfrac{3}{2}u_0^2)\xi_1,\\
&\xi_3'' + (1-\omega^2)\xi_3 = \eta_4,\\
&\eta_4'' + (1-\omega^2)\eta_4 = (\alpha + \beta\sqrt{2}u_0 - \tfrac{3}{2}u_0^2)\xi_3,\\
&\xi_5'' + (1-4\omega^2)\xi_5 = \eta_6,\\
&\eta_6'' + (1-4\omega^2)\eta_6 = (\alpha + \beta\sqrt{2}u_0 - \tfrac{3}{2}u_0^2)\xi_5,
\end{aligned}$$

where each subsystem coincides with the system (7) for $\sigma^2 = 0$, $\sigma^2 = (i\omega)^2$ and $\sigma^2 = (2i\omega)^2$, respectively. Observe the following property of this obtained system: if we find a nontrivial solution $\xi_3(x), \eta_4(x)$, decaying to zero as $|x| \to \infty$ to the second pair of equations at some ω_*^2, then the set of functions $\xi_5(x) = \xi_3(x)$, $\eta_6(x) = \eta_4(x)$ gives a solution of the third pair of equations with $\omega = \omega_*/2$. In particular, if we fix somehow ω varying within the interval $(1/2, 1)$, then the branching at the wave number $\omega/2$ is impossible.

We know from our simulations and results of [19,34] that in the invariant 4-plane the Hamiltonian system for variables (q_1, q_2, p_1, p_2) (not depending on ω) has transversal homoclinic orbits to the saddle-focus O for small enough $\alpha < 0$ and $\beta > \sqrt{27/38}$. Transversality of such homoclinic orbit implies an exponential dichotomy of the first linear subsystem in (18) on the whole $\mathbb{R}$. Second and third subsystems do have at some (spectral) points ω bounded solutions being intersections at $x = 0$ of the subspace of solutions decaying at $x = -\infty$ with the subspace of solutions decaying at $x = +\infty$ (see Proposition 1). These intersections just say on tangency of stable and unstable manifolds of the point O.

When we have found a spectral point ω_* for the linear system, we change other parameters of the system (17), in our case that was α (we kept $\beta = 2$ in our simulations) and construct numerically close homoclinic orbit branching from that on the 4-plane. We do it for the 2-mode and 3-mode subsystems, the comparison of found homoclinics can be seen in Figs. 5 and 6. Our numerical findings of homoclinic solutions to O for the approximating system (17) allow us to present the pictures of expected periodically modulated rolls for the initial equation (2) (see Fig. 7)

In the next Section we present the numerical simulations for finding homoclinic orbits and their comparison when taking into account two- and three-mode approximations. In fact, we search for the branching of needed symmetric homoclinic orbits from symmetric homoclinic orbits on the invariant 4-plane.

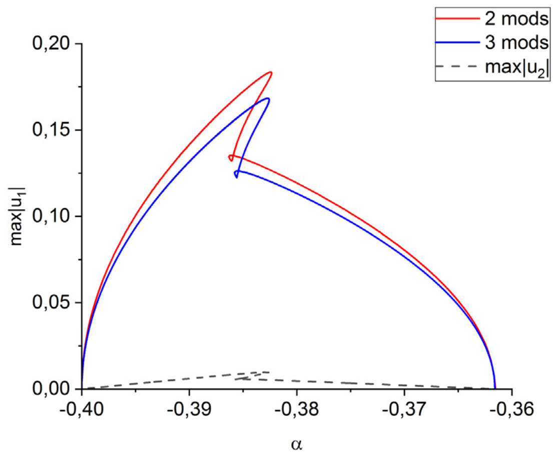

Fig. 6. Comparison of curves of homoclinic solutions in α for two- and three-mode systems.

6.1. Illustration for the 2 DOF Hamiltonian

The idea formulated above can be illustrated on the case of the least dimension, namely, a Hamiltonian system in two degrees of freedom in $\mathbb{R}^4$ with symplectic coordinates (x_1, x_2, y_1, y_2) that has a saddle equilibrium at the origin. Assume the system possesses an invariant symplectic 2-plane with the restriction on the plane being a system with a homoclinic orbit to a saddle equilibrium. Denote $H(x_1, x_2, y_1, y_2)$ the Hamiltonian that takes the form

$$H(x_1, x_2, y_1, y_2) = h(x_1, y_1) + \frac{1}{2}(x_2^2 h_{20} + 2x_2y_2h_{11} + y_2^2h_{02}),$$

where functions h_{ij} depend on all four variables. We assume the point $O = (0, 0, 0, 0)$ be an equilibrium of a saddle type (its eigenvalues are $(\lambda_1, \lambda_2, -\lambda_1, -\lambda_2)$, $\lambda_i \neq 0$) and the system has invariant symplectic plane $x_2 = y_2 = 0$, the restriction of the Hamiltonian on this plane is $h(x_1, y_1)$. Let $x_1^0(t), y_1^0(t)$ be a homoclinic orbit Γ to O, i.e. $|x_1^0(t)| \to 0$, $|y_1^0(t)| \to 0$ as $|t| \to \infty$. The linearization of the system at Γ is

$$\begin{aligned} \dot\xi_2 &= h_{11}^0\xi_2 + h_{02}^0\eta_2, \\ \dot\eta_2 &= -h_{20}^0\xi_2 - h_{11}^0\eta_2, \end{aligned} \tag{19}$$

where functions h_{ij}^0 are calculated along the homoclinic orbit: $h_{ij}^0(x_1^0(t), 0, y_1^0(t), 0)$. Because O is a saddle, we have the inequality at the limit $|t| \to \infty$: $h_{20}^0h_{02}^0 - (h_{11}^0)^2 < 0$. Thus the linear non-autonomous asymptotically autonomous system (19) has in the extended phase space $\mathbb{R}^2 \times \mathbb{R}$ on the section cross-section $\mathbb{R}_0^2 = \mathbb{R}^2 \times \{0\}$ two one-dimensional subspaces l_+, l_-, of which the first consists of those initial vectors, whose related solutions decay to zero at $t \to \infty$, but for the second this does at $t \to -\infty$. These two subspaces can be either transversal or coincide, in the first case the homoclinic orbit Γ is transversal in $\mathbb{R}^4$ (stable and unstable two-dimensional manifolds of O intersect each other transversely in the level $H = H(O)$), in the second case Γ is tangent homoclinic orbit. It is worth remarking that the tangency guarantees the tangency of stable and unstable manifolds in the linear approximation. To understand the nearby structure of the flow, one needs to know what type of tangency is: quadratic, cubic or higher degenerate, etc. (see details in [60–63]).

7. Discrepancies

In this section we present the discrepancies for two- and three-mode approximations to estimate the adequateness of the Bubnov–Galerkin method. We found that they are of the order 10^{-2} and 10^{-3}, see, Fig. 8.

Recall that the discrepancy of some approximating solution $U_a(x, y)$ for Eq. (2) is the function

$$err(x, y) = (1 + \Delta)^2U_a - \alpha U_a + \beta U_a^2 - U_a^3.$$

We keep into account that function U_a is the sum $u_0(x)/2 + u_1(x)\cos\omega y + u_2(x)\cos 2\omega y$ and functions $u_i(x)$ satisfy Eqs. (16). Denote err_i, $i = 1, 2$, the function err, when we hold in U_a two ($u_2 = 0$) and three modes, respectively. Then we get

$$\begin{aligned} &err_3(x, y) = \beta[u_1u_2\cos 3\omega y + \tfrac{1}{2}u_2^2\cos 4\omega y] \\ &\quad -[\tfrac{1}{4}u_1(u_1^2 + 3u_2^2 + 6u_0u_2)\cos 3\omega y + \\ &\tfrac{3}{4}u_2(u_0u_2 + u_1^2)\cos 4\omega y + \tfrac{3}{4}u_1u_2^2\cos 5\omega y + \tfrac{1}{4}u_2^3\cos 6\omega y], \\ &err_2(x, y) = \tfrac{1}{2}\beta u_1^2\cos 2\omega y - \tfrac{1}{4}u_1(3u_0u_1\cos 2\omega y + u_1^2\cos 3\omega y), \end{aligned}$$

We plot the related functions in Fig. 8. We see the rather good approximations: the order of err_2 is 10^{-2} and for err_3 is 10^{-3}.

8. Conclusion

We investigate solutions to the stationary Swift–Hohenberg equation on the plane being periodic in some direction and localized in other transverse direction. Here these directions are along variables y and x. An ideal intention we would like to reach is some mathematical tool that gives the rigorous basement for the existence of such solutions, similar to what was done in [48,49, 56,64]. Unfortunately, we were not succeeded in this direction so far, but rather presented some numerical corroborations for them. Our work with the approximate system (17) and calculations of homoclinic orbits show that this system plausibly reflects the behavior on the center manifold (if it exists) of the formal system (3). So, the branching of homoclinic solutions from those on the invariant 4-plane that corresponds to the one-dimensional SH equation can be considered as one more mechanism of a formation periodically modulated rolls, in addition to the found in papers [4,22,41,44].

9. Addendum: Asymptotics of solution as $x \to -\infty$

Taking into account that $u_0(x) \to 0$ as $x \to -\infty$ we get the differential system for ψ, χ

$$\begin{aligned} &\psi'' + (1 - \omega^2)\psi - \chi = 0, \\ &-\alpha_0\psi + \chi'' + (1 - \omega^2)\chi = 0. \end{aligned} \tag{20}$$

The characteristic equation

$$(\lambda^2 + 1 - \omega^2)^2 - \alpha_0 = 0$$

has roots

$$\lambda^2 = \omega^2 - 1 \pm \imath\sqrt{-\alpha_0}.$$

Recall that pulses exist only if $\alpha_0 < 0$. So, we need to find two roots with $Re(\lambda) > 0$. Such roots are two complex conjugate numbers

$$\lambda_\pm = \Lambda \pm \imath\Omega = \sqrt{\frac{\sqrt{(\omega^2 - 1)^2 - \alpha_0} + \omega^2 - 1}{2}} \pm \imath\sqrt{\frac{\sqrt{(\omega^2 - 1)^2 - \alpha_0} - \omega^2 + 1}{2}}$$

Therefore, the complex solution of the system (20) has the form

$$\psi(x) = e^{\Lambda x}(\cos\Omega x + \imath\sin\Omega x), \quad \chi(x) = (\lambda_+^2 + 1 - \omega^2)\psi(x).$$

Here we take into account that $\lambda_+^2 = \omega^2 - 1 + \imath\sqrt{-\alpha_0}$, and therefore

$$\begin{aligned} &\psi(x) = e^{\Lambda x}(\cos\Omega x + \imath\sin\Omega x), \\ &\chi(x) = \imath\sqrt{-\alpha_0}e^{\Lambda x}(\cos\Omega x + \imath\sin\Omega x). \end{aligned}$$

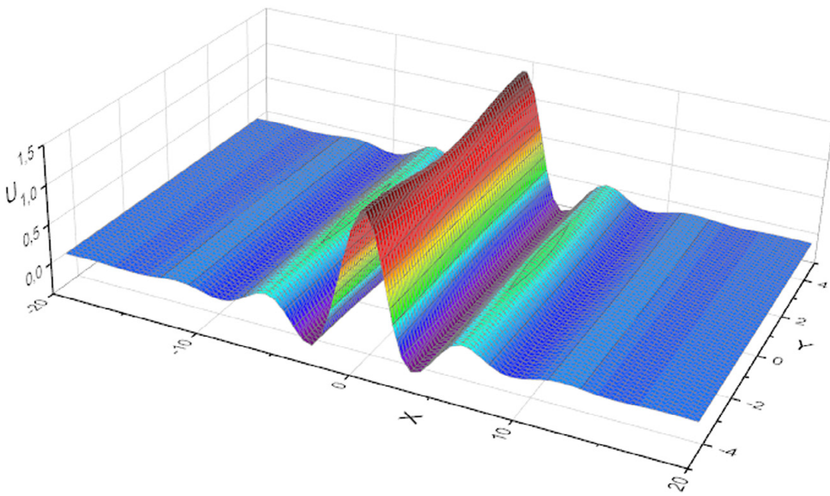

Fig. 7. The constructed shape of modulated rolls at $\alpha = -0.394265, \omega = 0.707015$.

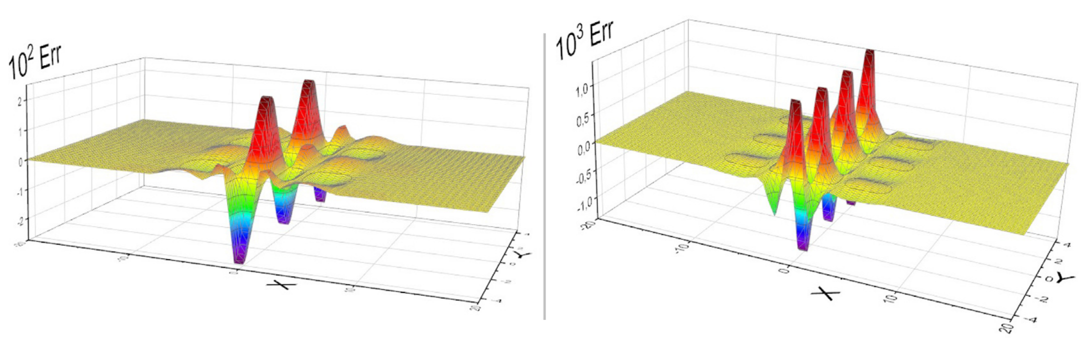

Fig. 8. Discrepancies for 2-mode (left) and 3-mode (right) approximations.

To solve the Cauchy problem we need to calculate derivatives

$$\psi'(x) = e^{\Lambda x}(\Lambda \cos \Omega x - \Omega \sin \Omega x + \imath(\Lambda \sin \Omega x + \Omega \cos \Omega x)),$$
$$\chi'(x) = \imath\sqrt{-\alpha_0}e^{\Lambda x}(\Lambda \cos \Omega x - \Omega \sin \Omega x + \imath(\Lambda \sin \Omega x + \Omega \cos \Omega x)).$$

Thus, two linear independent solutions decaying at infinity $x = -L$, $L \gg 1$, can be singled out by conditions

$$\begin{pmatrix} \psi \\ \psi' \\ \chi \\ \chi' \end{pmatrix}_1^{(-\infty)} = \begin{pmatrix} \cos \Omega L \\ \Lambda \cos \Omega L - \Omega \sin \Omega L \\ -\sqrt{-\alpha_0} \sin \Omega L \\ -\sqrt{-\alpha_0}(\Lambda \sin \Omega L + \Omega \cos \Omega L) \end{pmatrix} \tag{21}$$

and

$$\begin{pmatrix} \psi \\ \psi' \\ \chi \\ \chi' \end{pmatrix}_2^{(-\infty)} = \begin{pmatrix} \sin \Omega L \\ \Lambda \sin \Omega L + \Omega \cos \Omega L \\ \sqrt{-\alpha_0} \cos \Omega L \\ \sqrt{-\alpha_0}(\Lambda \cos \Omega L - \Omega \sin \Omega L) \end{pmatrix}. \tag{22}$$

Here a multiplier $\exp(\Lambda L)$ has been cut being inessential for our purposes.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: N. Kulagin reports financial support was provided by Russian Science Foundation.

Data availability

No data was used for the research described in the article.

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