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Interaction of Interfacial Waves with an External Force: The Benjamin-Ono Equation Framework

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Abstract: This study aims to explore the complex interactions between an internal solitary wave and an external force using the Benjamin-Ono equation as the theoretical framework. The investigation encompasses both asymptotic and numerical approaches. By assuming a small amplitude for the external force, we derive a dynamical system that describes the behavior of the solitary wave amplitude and the position of its crest. Our findings reveal three distinct scenarios: (i) resonance between the solitary wave and the external force, (ii) oscillatory motion with closed orbits, and (iii) displacement from the initial position while maintaining the wave direction. However, through numerical simulations, we observe a different relationship between the amplitude of the solitary wave and its crest position. Specifically, for external forces of small amplitude, the simulations indicate the presence of an unstable spiral pattern. Conversely, when subjected to external forces of larger amplitudes, the solitary wave exhibits a stable spiral trajectory which resembles the classical damped mass-spring system.

Keywords: Benjamin-Ono equation; trapped waves; solitary waves

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Citation: Flamarion, M.V.; Pelinovsky, E. Interaction of Interfacial Waves with an External Force: The Benjamin-Ono Equation Framework. *Symmetry* **2023**, *15*, 1478. https://doi.org/10.3390/sym15081478

Academic Editor: Jaume Giné

Received: 1 July 2023
Revised: 22 July 2023
Accepted: 25 July 2023
Published: 26 July 2023

1. Introduction

Considerable research efforts have been devoted to studying weakly nonlinear models that describe the evolution of internal waves. Prominent among these models are the Korteweg-de Vries (KdV) equation, which applies to shallow water, the Intermediate Long Wave (ILW) equation, suitable for fluids of finite depth, and the Benjamin-Ono (BO) equation, which pertains to deep water dynamics [1–4]. These established models exhibit captivating characteristics, including the presence of periodic and solitary wave solutions that persist over time. However, it is essential to recognize that these model equations possess certain limitations that restrict their applicability to more generalized problems. Notably, they are valid only within specific depth ranges, thereby imposing a significant constraint on their practical utility.

The renowned Benjamin-Ono equation

$$u_t + c_0 u_x - \frac{3c_0}{2h_1} u u_x + \frac{c_0 h_1}{2\rho_r} \mathcal{H}[u_{xx}] = 0, \tag{1}$$

is commonly used to study the perturbed interface between two inviscid fluids of constant densities of a flat rigid lid and infinity depth. Here, h_1 is the thickness of the upper layer with density ρ_1, ρ_2 is the density of the lower fluid, $\rho_r = \rho_1/\rho_2 < 1$ is the ratio between the densities of the lighter fluid (upper layer) and the heavier one (lower layer) and c_0 is the linear speed given by

$$c_0^2 = gh_1\Big(\frac{1}{\rho_r} - 1\Big), \tag{2}$$

where g is the acceleration of gravity. More details of the geometry of the problem is depicted in Figure 1. The elevation of the interface in the position x and time t is denoted by $u(x,t)$ and $\mathcal{H}$ denotes the Hilbert transform defined as

$$\mathcal{H}[u(x,t)] = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{u(y,t)}{y-x}dy. \tag{3}$$

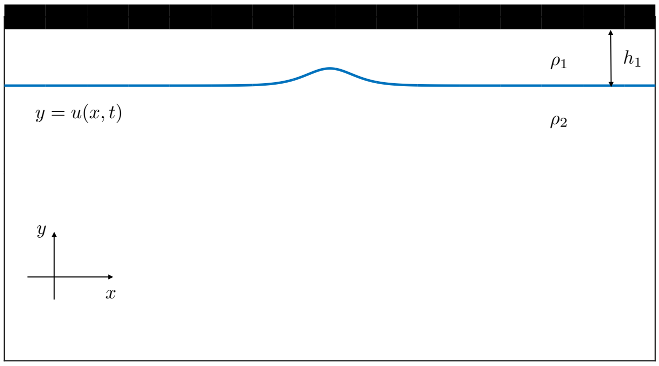

Figure 1. Sketch of the problem. The interface $y = u(x,t)$ at time t in Cartesian coordinates Oxy, which separates two inviscid fluids of constant densities ρ_1 and ρ_2. The upper layer, with a thickness of h_1, is bounded above by a rigid wall, while the lower layer extends infinitely in depth.

One of the key issues in water wave research is the investigation of the interaction between solitary waves, which are symmetric waves with respect to their crest which propagates through space at a constant speed without a change in its shape or size, and a heterogeneous medium. Various frameworks have been employed to study this problem. Integrable models, such as the Korteweg-de Vries (KdV) and modified KdV (mKdV) equations, have been explored extensively [5–16], as well as nonintegrable models including the Whitham [17,18], and the Schamel equation [19]. However, to the best of our knowledge, the study of this phenomenon within the framework of the Benjamin-Ono (BO) equation has not been addressed in the existing literature. The inclusion of an external force in Equation (1) introduces intriguing physics problems. For instance, the external force commonly arises in two scenarios: (i) a pressure distribution is applied at the free surface of the upper layer [20,21], and (ii) the external force can represent bathymetry [20,22,23]. The latter case can effectively model flow over a mountain in the atmosphere when the upper layer extends to infinity. These two cases are depicted in Figure 2.

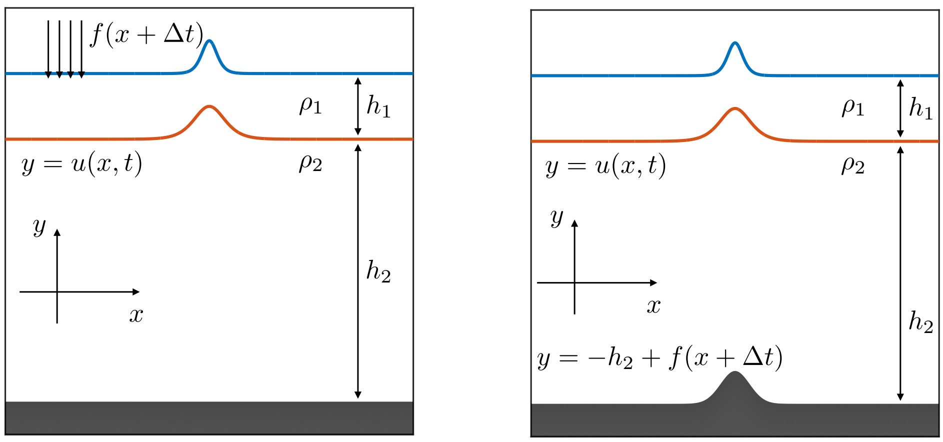

Figure 2. Sketch of the problem. (**Left**): A pressure distribution moving at a constant speed (Δ) applied to the free surface. (**Right**): A moving obstacle with a constant speed (Δ) located at the bottom.

The aim of this this study is to explore the interaction between an internal solitary wave and a localized and symmetric external force. To accomplish this, we investigate the forced Benjamin-Ono equation. By assuming a small amplitude for the external force, we derive a two-dimensional dynamical system that characterizes the position of the solitary wave crest and its amplitude. We then compare asymptotic results with fully numerical simulations conducted using pseudo-spectral methods. Our findings indicate that, at early times, the results exhibit qualitative agreement. However, the asymptotic theory predicts the presence of centers and closed orbits when the external force and the solitary wave approach resonance. In contrast, the fully numerical simulations suggest the occurrence of unstable spirals.

For reference, this article is organized as follows: The forced BO equation is presented in Section 2. In Section 3, we describe the asymptotic and numerical results. Then, we present the final considerations in Section 4.

2. The Forced BO Equation

Our focus of study lies in exploring the interaction between internal solitary waves and an external force field. To accomplish this, we examine the Benjamin-Ono (BO) equation in its dimensionless form, incorporating an external force function $f(x)$ and a constant speed Δ

$$u_t + uu_x + \mathcal{H}[u_{xx}] = f_x\Big(x + \Delta t\Big). \tag{4}$$

This equation encompasses terms such as the convective nonlinearity, dispersion through the Hilbert transform $\mathcal{H}$, and the external force acting on the wave field represented by $u(x,t)$.

Our objective is to delve into the intricate dynamics of solitary waves when subjected to the influence of this external force field. For numerical purposes, it is convenient to consider Equation (4) in the moving frame associated with the external force. This transformation is achieved by introducing the new variables $x' = x + \Delta t$ and $t' = t$. Within this new coordinate system, Equation (4) can be expressed as

$$u_t + \Delta u_x + uu_x + \mathcal{H}[u_{xx}] = f_x(x), \tag{5}$$

wherein the effect of the external force is accounted for solely in terms of its spatial derivative $f_x(x)$.

In this context, it is crucial to note that the mass of the system, represented by the integral of the wave field over space, is preserved. This conservation of mass is mathematically expressed by

$$M(t) = \int_{-\infty}^{+\infty} u(x,t)dx. \tag{6}$$

Furthermore, the momentum of the system, denoted as $P(t)$, is balanced by the external force as

$$\frac{dP}{dt} = \int_{-\infty}^{+\infty} u(x,t)f_x(x)dx, \text{ where } P(t) = \int_{-\infty}^{+\infty} u^2(x,t)dx. \tag{7}$$

These mass and momentum formulas, as stated in Equations (6) and (7), hold significant importance, particularly in evaluating the accuracy and dependability of numerical methods employed for solving the BO Equation (5). By utilizing these formulas, one can assess the precision of the numerical methodologies utilized and gain confidence in the reliability of the obtained results.

In the absence of an external force present, the BO Equation (5) admits a two-parameter (a, λ) family of periodic waves as solutions [1]. These waves are described by the following expressions [1]

$$u(x,t)=\frac{A}{1-B\cos\left(\frac{2\pi}{\lambda}(x-ct)\right)},\ \text{where}\ \ c=\Delta+\frac{a}{4},\ A=\frac{32\pi^2}{a\lambda^2}\ \text{and}\ B=\left[1-\left(\frac{8\pi}{a\lambda}\right)^2\right]^{1/2}. \tag{8}$$

As $\lambda\to\infty$, it reduces to the solitary wave solution [1]

$$u(x,t)=\frac{al^2}{(x-ct)^2+l^2},\ \text{where}\ \ c=\Delta+\frac{a}{4}\ \text{and}\ |l|=\frac{4}{a}. \tag{9}$$

Here, a represents the solitary wave amplitude, c represents its speed, and l characterizes the solitary wavenumber.

In what follows, the external force is chosen as

$$f(x)=b\exp\left(-\frac{x^2}{w^2}\right), \tag{10}$$

where b is the amplitude of the external force and w is its width. In the next section, we investigate the interaction of the solitary waves (11) and the external force (10).

3. Results

3.1. Asymptotic Results

In this section, our objective is to deduce the governing equations for the interaction between solitary waves and an external force, considering the force to have a small amplitude. To achieve this objective, we introduce a minor positive parameter represented as ϵ and replace the external force f in Equation (5) with ϵf. Moreover, we assume that the wave field closely resembles a solitary wave, characterized by parameters that slowly vary over time [24–26]. Mathematically, the solitary wave can be described using the following expressions

$$u(\mathbf{\Phi},T)=\frac{a(T)l(T)^2}{\mathbf{\Phi}^2+l(T)^2},\ \text{where}\ \ \mathbf{\Phi}=x-X(T)\ \text{and}\ X(T)=x_0+\frac{1}{\epsilon}\int_0^T c(T)dT, \tag{11}$$

where the solitary wave initial position is denoted as x_0, and the functions a and c are established based on the interaction between the wave field and the external field. To facilitate our analysis, we introduce the concept of slow time by introducing a new variable, namely $T=\epsilon t$. We aim to find a solution by utilizing an asymptotic expansion in the following form

$$\begin{aligned} u(\mathbf{\Phi},T)&=u_0+\epsilon u_1+\epsilon^2u_2+\cdots,\\ c(T)&=c_0+\epsilon c_1+\epsilon^2c_2+\cdots. \end{aligned} \tag{12}$$

At the first-order of the asymptotic theory, it immediately follows that the solutions u_0 and q_0 are precisely defined in accordance with Equation (11).

The momentum balance equation at the first-order is

$$\frac{1}{2}\frac{d}{dT}\int_{-\infty}^{\infty}u_0^2(\mathbf{\Phi})d\mathbf{\Phi}=\epsilon\int_{-\infty}^{\infty}u_0(\mathbf{\Phi})\frac{df}{d\mathbf{\Phi}}(\mathbf{\Phi}+X)d\mathbf{\Phi}. \tag{13}$$

Therefore, replacing the Formula (11) into the Equation (13) yields the two-dimensional dynamical system

$$\begin{aligned} \frac{da}{dt} &= \int_{-\infty}^{\infty} \left[\frac{al^2}{\mathbf{\Phi}^2 + l^2} \right] \frac{df}{d\mathbf{\Phi}} (\mathbf{\Phi} + X) d\mathbf{\Phi}, \\ \frac{dX}{dt} &= \Delta + \frac{a}{4}. \end{aligned} \tag{14}$$

When the external force extends far beyond the scope of the solitary wave wavelength, it becomes feasible to approximate the solitary waves as delta functions. This approximation allows us to simplify the dynamical system that governs the amplitude and position of the crest of the solitary wave. Phase portraits for the dynamical system are depicted in Figure 3. In doing so, we arrive at the following simplified form, which encapsulates the essence of the solitary wave behavior

$$\begin{aligned} \frac{da}{dt} &= \frac{df}{dX}(X), \\ \frac{dX}{dt} &= \Delta + \frac{a}{4}. \end{aligned} \tag{15}$$

From Equation (15) we have that the crest position of a solitary wave is describes the oscillator

$$\frac{d^2 X}{dt^2} = \frac{1}{4} f(X). \tag{16}$$

This is similar to what happens to the forced KdV equation [24,27] and the forced mKdV equation [8,26].

Equilibrium points in the dynamical system (15) exist exclusively when Δ assumes a negative value. The magnitude of the solitary wave amplitude (a_0) and the location of its peak (X_0) are

$$a_0 = -\frac{a}{4} \text{ and } X_0 = 0 \text{ for } \Delta < 0. \tag{17}$$

When the disturbance and the solitary wave exhibit the same polarity, the equilibrium position is categorized as a center. Centers correspond to stable solitary waves that remain steady. The interaction between the solitons and the external force concludes once the waves have traversed the external force region.

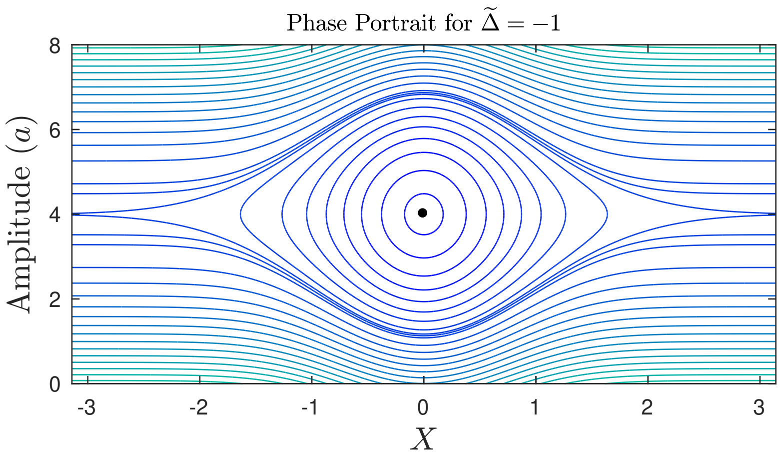

Figure 3. Phase portraits for the dynamical system (15). Dot corresponds to the center point.

The level curves of the stream function (streamlines) serve as a representation of the solutions of the system (15). They are represented by the Hamiltonian

$$\mathbf{\Psi}(X,a) = -f(X) + \Delta a + \frac{a^2}{8}. \tag{18}$$

To examine the phase portrait of system (15) with the external force (10), we introduce a rescaling of variables. The coordinate X is scaled relative to w, while the amplitude a is scaled relative to $b^{1/2}$. Here, $b > 0$ represents the external force amplitude. This rescaling results in the emergence of a new parameter

$$\widetilde{\Delta} = \frac{\Delta}{\sqrt{b}}. \tag{19}$$

With these new scalings, the stream function becomes

$$\mathbf{\Psi}(X,a) = -e^{-X^2} + \tilde{\Delta} a + \frac{a^2}{8}. \tag{20}$$

Figure 1 illustrates typical phase portraits of system (15). The phase portrait is symmetric with respect to the lines $X = 0$ and $a = 4$. It is crucial to emphasize that the presence of a closed orbit in the phase portrait signifies the existence of a solitary wave. This solitary wave is effectively confined without any radiation, which arises due to its interaction with the external force. For more analytical results the readers are referred to [28].

3.2. Numerical Results

Solutions of Equation (5) are studied numerically via a pseudospectral method using periodic boundary conditions [29]. For this purpose, we choose the computational domain with uniformly spaced points. To avoid the effects of spatial periodicity, the domain is taken to be sufficiently large. The time advancement is computed through the Runge-Kutta fourth-order method. The accuracy of the numerical method is verified by choosing a solitary wave of amplitude $a = 1$ and evolving it until $t = 3000$, while monitoring the total mass (6) and momentum (7) in the absence of an external force. We observe that the retained mass and momentum are accurate up to a precision of 10^{-13}. Details of the spatial and temporal resolution of a similar numerical method can be found in [30]. In order to compare the numerical solutions with the asymptotic methods developed in the previous section, we set the amplitude of the external force (10) as $b = 1$, and its width as $w = 10$. This choice of parameters ensures that the external force is broad, allowing us to compare the numerical results with the developed asymptotic theory (dynamical system (15)) in the previous section.

To verify the asymptotic results described in the previous section, we conduct a series of simulations using the BO Equation (5). We choose the initial solitary wave amplitude as $a = 1$ and the speed deviation $\Delta = -a/4$. With these parameters, the dynamical system (15) predicts steady solutions, and small perturbations of these values result in closed orbits or trapped waves without radiation.

Initially, we consider $\epsilon = 0.01$. In this case, we observe that the solitary wave oscillates back and forth over the external force for extended periods, as shown in Figure 4(Top). The fluctuation in the amplitude was found to be of the order of $\mathcal{O}(10^{-2})$, indicating that the a solitary wave amplitude remains nearly unchanged over time. Although the results closely match the asymptotic theory predictions at small times, the fully numerical computations revealed a behavior resembling an unstable spiral in the amplitude vs. crest position space and a resonant harmonic oscillator in the crest position vs. time space, as illustrated in Figure 4(Bottom).

Next, we increase the value of the parameter to $\epsilon = 0.1$. In this case, the solitary wave remains trapped at the external force for long durations, as depicted in Figure 5(Top). However, there were significant differences compared to the previous case. The amplitude oscillations were much larger, as shown in Figure 5(Bottom-left). Moreover, the oscillations in the crest position vs. time space were more pronounced, see Figure 5(Bottom-right). Nevertheless, the solitary wave continue to remain trapped at the external force.

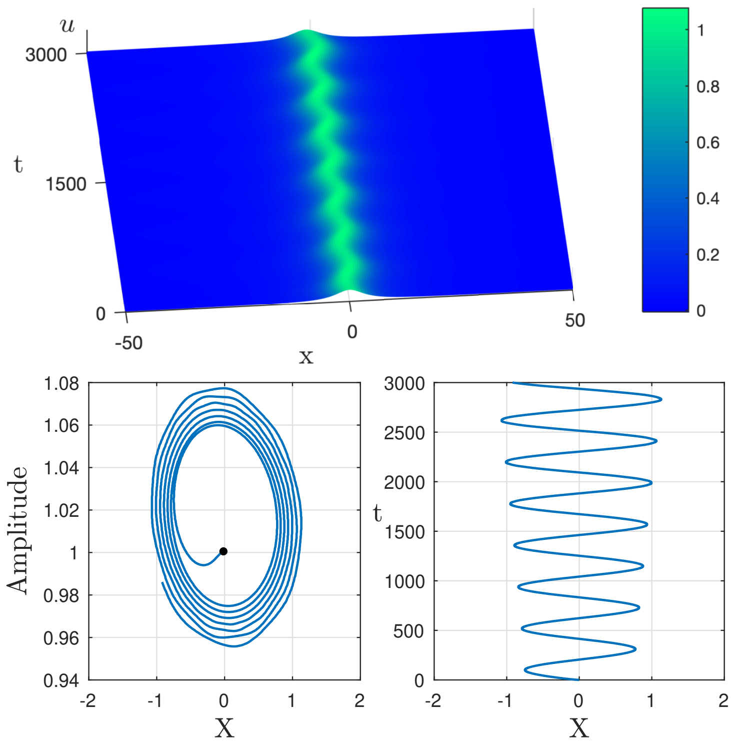

Figure 4. Top: The trapped solitary wave under the influence of external force. **Bottom** (**left**): The solitary wave amplitude against crest position. **Bottom** (**right**): The crest position over time. Parameters: $a = 1$, $w = 10$, $\Delta = -0.25$ and $\epsilon = 0.01$.

Lastly, we further increase the parameter to $\epsilon = 0.5$. In this scenario, we expect an increase in radiation and the possibility of the solitary wave moving past the external force at earlier times. However, the increased value of ϵ causes a substantial increase in the amplitude of the solitary wave, as depicted in Figure 6(Top). The resulting amplitude of the solitary wave becomes much larger than that of the external force, effectively rendering the presence of the external force negligible in the dynamics. The fully numerical computations revealed a behavior resembling a stable spiral in the amplitude vs. crest position space and a damped harmonic oscillator in the crest position vs. time space, as shown in Figure 6(Bottom). Furthermore, minimal radiation was observed in the interaction between the solitary wave and the external force, as depicted in Figure 7.

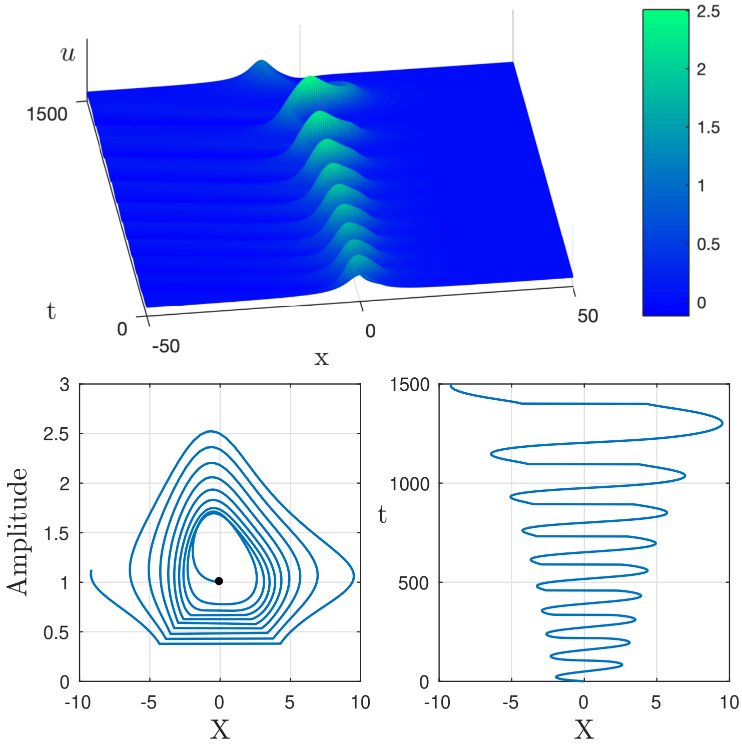

Figure 5. Top: The trapped solitary wave under the influence of external force. **Bottom** (**left**): The solitary wave amplitude against crest position. **Bottom** (**right**): The crest position over time. Parameters: $a = 1$, $w = 10$, $\Delta = -0.25$ and $\epsilon = 0.1$.

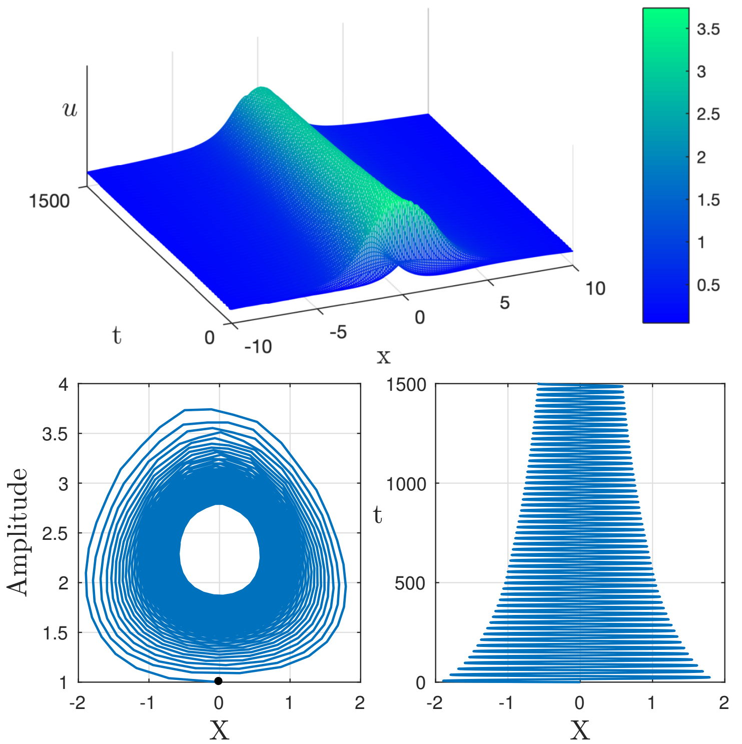

Figure 6. Top: The trapped solitary wave under the influence of external force. **Bottom** (**left**): The solitary wave amplitude against crest position. **Bottom** (**right**): The crest position over time. Parameters: $a = 1$, $w = 10$, $\Delta = -0.25$ and $\epsilon = 0.5$.

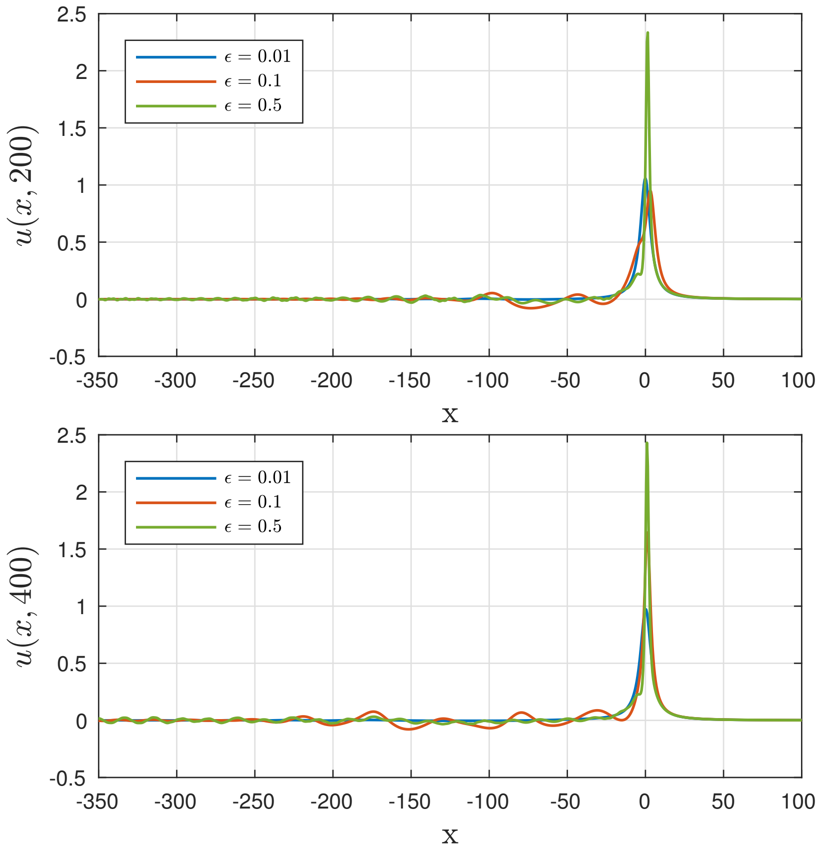

Figure 7. Comparison between the trapped waves for different values of the parameter ϵ at different times. Parameters: $a = 1$, $w = 10$, $\Delta = -0.25$.

4. Conclusions

In this work, our focus was to examine the interactions between internal solitary waves and an external force. Utilizing asymptotic expansion techniques, we derived a simplified model that describes the position and amplitude of the solitary waves. However, when comparing our numerical results with the asymptotic model, we found agreement only in the early stages. Interestingly, while the asymptotic predictions indicated steady solutions and perfect trapping, the fully numerical solutions showed the emergence of unstable and stable spirals, depending on the magnitude of the external force. This discrepancy underscores the significance of considering higher-order terms in the asymptotic expansion. We believe that incorporating these higher-order terms will enable the asymptotic theory to not only predict the positions of solitary waves but also capture the occurrence of unstable spirals, which closely resemble the outcomes of the numerical simulations. Therefore, a logical next step would involve further investigation and comparison of numerical and asymptotic solutions, incorporating higher-order terms, as part of our future research direction.

It is important to note that the conditions assumed in the studied Benjamin-Ono (BO) equation, such as unidirectional and long waves with a small degree of nonlinearity propagating in infinite depth, may differ significantly from real-world sea conditions. Therefore, further verification is necessary to project these findings onto the dynamics of sea waves. Nonetheless, the consistency of our results with previous studies [6,7,24] and their qualitative alignment reinforce the generality of the problem under investigation and the validity of our conclusions. It is plausible to anticipate that these findings extend beyond the BO equation, encompassing other related frameworks and corresponding phenomena.

Gaining a thorough understanding of the behavior and distinct traits of ocean internal waves holds immense significance for multiple scientific disciplines, encompassing oceanography, geophysics, and marine ecology. Dedicated researchers and scientists delve into the intricacies of these waves to unravel their impact on marine ecosystems and climate patterns. By deepening our knowledge of internal waves, we can gain valuable insights into the intricate workings of the ocean and its profound influence on shaping the Earth environment.

Author Contributions: Conceptualization, E.P. and M.V.F.; methodology and analysis, E.P. and M.V.F.; writing, M.V.F. All authors have read and agreed to the published version of the manuscript.

Funding: E.P. is supported by support by the Russian Science Foundation, grant 22-17-00153.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Informed consent was obtained from all subjects involved in the study.

Data Availability Statement: Not applicable.

Acknowledgments: M.V.F. is grateful to IMPA for hosting him as visitor during the 2023 Post-Doctoral Summer Program. The authors are grateful to the anonymous referees for helpful comments and suggestions which improved the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

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